Judicial precedent as a dynamic rationale
for axiomatic bargaining theory

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Axiomatic bargaining theory (e.g., Nash's theorem) is static. We attempt to provide a dynamic justification for the theory. Suppose a judge or arbitrator must allocate utility in an (infinite) sequence of two-person problems; at each date, the judge is presented with a utility possibility set in $\mathbb{R}_+^2$. He/she must choose an allocation in the set, constrained only by Nash's axioms, in the sense that a penalty is paid if and only if a utility allocation is chosen at date T that is inconsistent, according to one of the axioms, with a utility allocation chosen at some earlier date. Penalties are discounted with t and the judge chooses any allocation, at a given date, that minimizes the penalty he/she pays at that date. Under what conditions will the judge's chosen allocations converge to the Nash allocation over time? We answer this question for three canonical axiomatic bargaining solutions—Nash, Kalai–Smorodinsky, and "egalitarian"—and generalize the analysis to a broad class of axiomatic models.

KEYWORDS. Axiomatic bargaining theory, judicial precedent, dynamic foundations, Nash's bargaining solution.

JEL CLASSIFICATION. C70, C78, K4.

1. INTRODUCTION

Axiomatic bargaining theory is timeless. In Nash's (1950) original conception, the apparatus is meant to model a bargaining problem between two individuals, each of whom initially possesses an endowment of objects, and von Neumann–Morgenstern (vNM) preferences over lotteries on the allocation of these objects to the two individuals. An impasse point is defined as the pair of utilities each receives if no trade takes place, that is, if no bargain is reached (here, particular vNM utility functions are employed). Nash quickly passes to a formulation of the problem in utility space, where a bargaining problem becomes a convex, compact, comprehensive utility possibilities set, containing the

impasse point. He then imposes the axioms of Pareto efficiency, symmetry, independence, and scale invariance, and proves that the only "solution" that satisfies these axioms on an unrestricted domain of problems is the Nash solution—for any problem, the utility point that maximizes the product of the individual gains from the threat point.[1]

We say the theory is timeless, because of the independence axiom, for this axiom requires consistency in bargaining behavior between pairs of problems. What kind of experience might lead the bargainers to respect the independence axiom? Presumably, if they bargained for a sufficiently long period of time, facing many different problems, they might come across a pair of problems that are related as the premise of the independence axiom requires: problem S is contained in problem Q (as utility possibilities sets), the bargainers faced problem Q last year and chose allocation $q \in Q$, and it so happens that $q \in S$. It is certainly reasonable, they reason, to agree upon q when facing S this year, because of something like Le Chatelier's principle. ("If we chose q when all those allocations in $Q \setminus S$ were available, we effectively had decided to restrict our bargaining to S last year anyway, so let's choose $q \in S$ again now.") But if this is the way that bargainers might "learn" how independence bears on decisions, then Nash's theory seems quite unrealistic. For with an unrestricted domain of problems, how often will bargainers face two problems that are related as the premise of the independence axiom requires? Almost never.

Notice that the same argument of timelessness does not apply to the scale invariance axiom, even though that axiom compares the behavior of the solution on pairs of problems, because that axiom is meant to model the idea that only von Neumann–Morgenstern preferences count, not their particular representation as utility functions. While the independence axiom can be viewed as a behavioral axiom, the scale invariance axiom is an informational axiom.

The other axioms—symmetry and Pareto—are also behavioral but not timeless in our sense. It is not a mystery why bargainers should learn to cooperate (Pareto) or that two bargainers with the same preferences (and the same strengths) and the same endowments should end up at a symmetric allocation. Thus, the critique we are proposing of Nash bargaining theory is that one of the behavioral axioms (independence) has no apparent justification via some kind of learning through history, in the presence of another axiom (unrestricted domain), which essentially precludes that learning could ever take place.

Our goal in this article is to replace the timelessness of axiomatic bargaining theory with a dynamic approach in which decision makers learn from history. Indeed, there is, we think, an obvious judicial practice, which provides a way to render the theory dynamic. Suppose a judge or a court or an arbitrator faces a number of cases over time. There is a constitution that prescribes what the judicial decision must be in certain clear and polar cases. But most cases do not fit the specifications of these constitutionally

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[1]Axiomatic bargaining theory has two major applications: one to bargaining and the other to distributive justice. Of course, Nash (1950) pioneered the first interpretation, and the second was pioneered by Thomson and Lensberg (1989), who showed that many of the classical bargaining solutions (Nash, Kalai–Smorodinsky, egalitarian) could be characterized by sets of axioms with ethical interpretations. See Roemer (1996) for a history of the subject in its two variants.

described cases, so judges rely on judicial precedent or case law: they look for a case in the past that is similar in important respects, or related, to the one at hand and decide the present case in like manner. Thus judicial precedent is a procedure that provides a link to the past that is similar to the links between problems that the independence axiom—and, indeed, from a formal viewpoint, the scale invariance axiom—impose.

Of course, there is a possibility that the case being considered at present time, i, has two precedent cases j and k, each of which is related to i in some important way, but which were decided differently. In general, the judge cannot decide the present case in a way to satisfy both precedents, and we will represent this conflict in our formal model.[2]

Imagine, then, that there is a *domain* of "cases" D, which is some set of Nash-type bargaining problems (convex, compact, comprehensive sets in $\mathbb{R}^2_+$). Suppose that the domain is rich enough that there are pairs of cases that are related by the scale invariance axiom, and pairs of cases that are related by the independence axiom; there are also some symmetrical cases in D. At each date $t = 1, 2, 3, \ldots$, a case is drawn randomly by Nature, according to some probability distribution on D. This infinite sequence of cases is called a history. The judge must decide each case sequentially (here, how to choose a feasible utility allocation) and he is restricted to obey the Nash axioms. What does this mean? If the case is symmetric, he must choose a symmetric point in the case or pay a penalty of 1; for every case, he must choose a Pareto efficient point or pay a penalty of 1. If a case is related to a prior case in the history by the scale invariance or independence axiom, and he does not choose the allocation in the present case that is consistent with his prior choice according to the salient axiom, he must pay a penalty of δ^t if the prior case appeared t periods ago, where $0 < \delta < 1$ is a given discount factor. (Thus, paying a penalty of 1 if a Pareto efficient point is not chosen in the case at hand is just a special case of this rule, because $\delta^0 = 1$.) If a case comes up that is not symmetric and is not related to any prior case by scale invariance or independence, he can choose any Pareto efficient point with zero penalty. At each date, the judge must choose an allocation that minimizes his penalty. In general, at a given date, he may end up paying penalties with respect to a number of cases in the past that are precedents, and so his penalty would be a sum of the form $\sum_{i \in P} \delta^t$ for some set of nonnegative integers P.

Now suppose that we consider a domain D where Nash's theorem is true: that is, any solution $\varphi : D \to \mathbb{R}^2_+$ that satisfies $\varphi(i) \in i$ for all $i \in D$ that satisfies the Nash axioms on D is, in fact, the Nash solution on D, denoted N. Call such a domain a Nash domain. (The simplest Nash domain consists of precisely one symmetric set. Any solution on this domain must obey the symmetry and Pareto axioms. Thus any solution obeying the axioms coincides with N on this domain.) Our question is this: When is it the case that a judge who plays by the above rules and faces an infinite history of cases, will converge over time almost surely to prescribing the Nash solution to the cases he faces?

To be precise, consider a superdomain $\mathfrak{H}^D$ of all possible histories over a given Nash domain, D, endowed with the product probability measure induced on histories by the

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[2]Real judges tend to decide which precedent fits the case at hand more closely and arguments revolve around the proximity of various precedents to the case at hand, but we will not follow this tack.

given probability measure on D. When would the judge almost surely converge to prescribing the Nash solution as time passes on histories in $\mathfrak{H}^D$? We prove, under some simple additional assumptions, that convergence to the Nash solution occurs almost surely for every set of histories $\mathfrak{H}^D$, where D is a finite Nash domain that satisfies a specific condition if and only if $0 < \delta \leq \frac{1}{3}$, that is, if and only if history is discounted at a sufficiently high rate. (Recall that the discount factor δ and the discount rate r are related by the formula $\delta = 1/(1+r)$.) This is our dynamic justification of Nash's theorem. However, we also show that there are Nash domains for which convergence to the Nash solution does not occur almost surely. In that sense, we can say that the Nash characterization theorem is dynamically imperfect. In contrast, we show that Kalai and Smorodinsky's (1975) characterization of their alternative solution, as well as Kalai's (1977) characterization of the egalitarian solution, are dynamically perfect in the sense that for every finite domain on which the theorem is true, almost sure convergence to the solution is obtained for appropriate values of δ.

We extend the results to more general penalty systems and to a general class of axiomatic theorems. The rest of the paper is structured as follows. Section 2 introduces the axiomatic framework. Sections 3 and 4 successively deal with the Nash solution, the Kalai–Smorodinsky solution, and the egalitarian solution. Section 5 shows how such results can be generalized and applied to any characterization theorem in a general axiomatic framework. Section 6 considers the possibility for the judge to make decisions not only on the basis of penalties currently incurred, but also on the basis of future possible penalties. Section 7 concludes.

2. Framework and axioms

A *domain* $D = \{i, j, k, \ldots\}$ contains *problems*, namely, subsets of $\mathbb{R}_+^2$ that are compact, convex, and comprehensive.[3] We restrict attention throughout the paper to finite domains. For simplicity, we also restrict attention to sets that have a nonempty intersection with $\mathbb{R}_{++}^2$. Let ∂i denote the *upper frontier* of i, i.e.,[4]

$$\partial i = \{x \in i \mid \nexists y \in i, y \gg x\},$$

and let $\partial^* i$ denote the subset of Pareto efficient points of i:

$$\partial^* i = \{x \in i \mid \nexists y \in i, y > x\}.$$

Let $I(i)$ denote the vector of *ideal points*, i.e.,

$$I(i) = \big(\max\{x_1 \in \mathbb{R}_+ \mid \exists x_2, (x_1, x_2) \in i\}, \max\{x_2 \in \mathbb{R}_+ \mid \exists x_1, (x_1, x_2) \in i\}\big).$$

For any $\alpha \in \mathbb{R}_{++}^2$, a set j is an α-*rescaling* of i if

$$j = \{x \in \mathbb{R}_+^2 \mid \exists y \in i, x_1 = \alpha_1 y_1, x_2 = \alpha_2 y_2\}.$$

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[3]A set i is comprehensive when for all $x \in i$ and all $y \leq x$, one has $y \in i$.
[4]Vector inequalities are denoted $\geq$, $>$, and $\gg$.

A *solution* $\varphi : D \to \mathbb{R}_+^2$ is a mapping such that for all $i \in D$, $\varphi(i) \in i$. The following axioms appear in the landmark theorems by Nash (1950), Kalai and Smorodinsky (1975), and Kalai (1977).

WEAK PARETO (WP). For all $i \in D$, $\varphi(i) \in \partial i$.

SYMMETRY (Sym). For all $i \in D$, if i is symmetric, then $\varphi_1(i) = \varphi_2(i)$.

SCALE INVARIANCE (ScInv). For all $i, j \in D$, if j is an α-rescaling of i for some $\alpha \in \mathbb{R}_{++}^2$, then

$$\varphi(j) = (\alpha_1 \varphi_1(i), \alpha_2 \varphi_2(i)).$$

NASH INDEPENDENCE (Ind). For all $i, j \in D$, if $i \subseteq j$ and $\varphi(j) \in i$, then $\varphi(i) = \varphi(j)$.

MONOTONICITY (Mon). For all $i, j \in D$, if $i \subseteq j$, then $\varphi(i) \le \varphi(j)$.

INDIVIDUAL MONOTONICITY (IMon). For all $i, j \in D$ and $p \in \{1, 2\}$, if $i \subseteq j$ and $I_p(i) = I_p(j)$, then $\varphi_{3-p}(i) \le \varphi_{3-p}(j)$.

Consider a domain D and an infinite number of periods $t = 1, 2, \dots$. A *history H* is a sequence of problems and chosen points

$$H = ((i_1, x_1), (i_2, x_2), \dots)$$

such that at every period t, $x_t \in i_t$. At each t, a random process picks $i_t \in D$. For any given $i \in D$, the probability that $i_t = i$ may depend on the previous part of the history $((i_1, x_1), \dots, (i_{t-1}, x_{t-1}))$. We assume throughout the paper that the random process is *regular* in the sense that it never ascribes a zero probability (or a probability converging to zero) to any given problem, i.e., if for every $i \in D$, there exists $\pi_i > 0$ such that for every $t \in \mathbb{N}$ and for every past history $((i_1, x_1), \dots, (i_{t-1}, x_{t-1}))$, the probability that $i_t = i$ is at least π_i.

At each period t, the judge chooses $x_t \in i_t$. His objective at each period is to minimize the penalty for this period, which is the sum of penalties incurred for a violation of each axiom. Each violation of an axiom implies a penalty of 1 unit. However, the penalty for violating an axiom involving a reference to past problems is discounted by a factor $\delta \in (0, 1)$: the farther back in the past the reference problem is, the lower is the penalty. Let r denote the corresponding discount rate: $\delta = 1/(1 + r)$.

To avoid any ambiguity, it is useful to specify what a violation of an axiom is exactly. Choosing $x_t \in i_t$ may entail the following penalties.

- WP: Penalty of 1 if $x_t \notin \partial i_t$.

- Sym: Penalty of 1 if i_t is symmetric and $\varphi_1(i_t) \ne \varphi_2(i_t)$.

- ScInv: Penalty of δ^s if i_t is an α-rescaling of i_{t-s} and $\varphi(i_t) \ne (\alpha_1 \varphi_1(i_{t-s}), \alpha_2 \varphi_2(i_{t-s}))$.

- Ind: Penalty of δ^s if $\varphi(i_t) \neq \varphi(i_{t-s})$ and either $[i_t \subseteq i_{t-s}, \varphi(i_{t-s}) \in i_t]$ or $[i_{t-s} \subseteq i_t, \varphi(i_t) \in i_{t-s}]$.

- IMon: Penalty of δ^s if for some $p \in \{1, 2\}$, either $i_{t+1} \subseteq i_t$, $I_p(i_{t+1}) = I_p(i_t)$, and $\varphi_{3-p}(i_{t+1}) \not\subseteq \varphi_{3-p}(i_t)$ or $i_t \subseteq i_{t+1}$, $I_p(i_{t+1}) = I_p(i_t)$, and $\varphi_{3-p}(i_t) \not\subseteq \varphi_{3-p}(i_{t+1})$.

- Mon: Penalty of δ^s if either $i_t \subseteq i_{t-s}$ and $\varphi(i_t) \not\subseteq \varphi(i_{t-s})$ or $i_{t-s} \subseteq i_t$ and $\varphi(i_{t-s}) \not\subseteq \varphi(i_t)$.

One restriction of this system of penalties is that the violation of any axiom that involves the past always counts less than the violation of any axiom that does not refer to the past. We examine more general systems of penalties in Section 5.

Given a domain D and a random process to select problems, we say that the judge *converges almost surely* to the solution φ if with probability 1 there is a date T such that for all $t \geq T$, the judge chooses $\varphi(i_t)$.

3. Nash

The *Nash solution*, denoted N, is defined by

$$N(i) = \{x \in i \mid \forall y \in i, \, x_1 x_2 \geq y_1 y_2\}.$$

The domain D is called a *Nash domain* if Nash's theorem holds on D, i.e., if $N(\cdot)$ is the only solution that satisfies WP, Sym, ScInv, and Ind on D.

We are interested in domains that satisfy the following condition.

CONDITION C_N. For all $i \in D$, there exists a sequence $j_1, \ldots, j_n \in D$ such that $j_1 = i$, j_n is symmetric, and for all $t = 1, \ldots, n - 1$, either

(i) $j_t \subseteq j_{t+1}$ and $N(j_{t+1}) \in j_t$ or

(ii) $\exists \alpha \in \mathbb{R}^2_{++}$, j_{t+1} is an α-rescaling of j_t.

Call such a sequence a *special chain beginning at i*.

PROPOSITION 1. *Domain D is a Nash domain if it satisfies Condition C_N. The converse is not true.*

PROOF. If: Let φ be any solution on D that satisfies Nash's axioms. Let $i \in D$. By Condition C_N, there is a special chain $j_1, \ldots, j_n$ beginning at i. By Sym and WP, $\varphi(j_n) = N(j_n)$. One can now roll back along the special chain to i, and at each step, $\varphi(j_k) = N(j_k)$ either by Ind (case (i)) or by ScInv (case (ii)). For $k = 1$, we have $\varphi(i) = N(i)$. It follows that $\varphi = N$ on D.

Converse: Let $D = \{i, j, k, l, m\}$ for

$$i = \mathrm{co}\{(0, 0), (3, 0), (2, 2), (0, 4)\}$$

$$j = \mathrm{co}\{(0, 0), (3, 0), (2, 2), (0, 3)\}$$

$$k = \text{co}\{(0,0), (2,0), (2,2), (0,4)\}$$
$$l = \text{co}\{(0,0), (4,0), (4,4), (0,8)\}$$
$$m = \text{co}\{(0,0), (8,0), (0,8)\}.$$

By WP and Sym, $\varphi(j) = N(j) = (2,2)$ and $\varphi(m) = N(m) = (4,4)$. By Ind, due to $l \subseteq m$, $\varphi(l) = N(l) = (4,4)$. By ScInv, as k is a rescaling of l, $\varphi(k) = N(k) = (2,2)$.

Now consider i. There is no special chain that begins at i. It is not symmetric, it is not the rescaling of another set, and it is not included in another set for which the Nash point is in i.

Yet one must have $\varphi(i) = N(i) = (2,2)$. By WP, $\varphi(i)$ must belong either to the segment $(3,0)(2,2)$ or to the segment $(2,2)(0,4)$. Suppose one took $\varphi(i)$ from a point x of the segment $(3,0)(2,2)$ different from $(2,2)$. Then, as $j \subseteq i$, by Ind one should have $\varphi(j) = x$, a contradiction. Suppose one took $\varphi(i)$ from a point y of the segment $(2,2)(0,4)$ different from $(2,2)$. Then, as $k \subseteq i$, by Ind one should have $\varphi(k) = y$, a contradiction. Therefore, Nash's theorem holds on D even though Condition C_N does not hold. □

We can now study the convergence of the judge's decisions toward the Nash solution. The following proposition states that with probability 1 the judge's decisions will exactly coincide with the Nash solution within a finite number of periods. The argument is that when Condition C_N holds for D, with probability 1, there will be some finite time at which all the elements of D appear in a row, each preceded by the special chain beginning at it, in reverse order: $j_n, j_{n-1}, \ldots, j_2, i$. When encountering j_n, the judge will choose $N(j_n)$ to avoid the penalties for violation of WP and Sym, and this will induce him to choose the Nash point in the subsequent problems to avoid the penalties for violation of Ind or ScInv. This happens, however, only if earlier possible "mistakes," and the related penalties, are not overwhelming. Therefore, this requires the past to be sufficiently discounted. When the past is strongly discounted, however, one may fear that once this particular sequence is past, the judge may err again when confronted with an arbitrary following sequence of problems. We prove, however, that the particular sequence of special chains is powerful enough to impose the Nash solution on all subsequent problems.

THEOREM 1. *The judge converges almost surely to the Nash solution on every domain satisfying Condition C_N if and only if $\delta \leq \frac{1}{3}$.*

PROOF. If: Let D be a domain satisfying Condition C_N. Recall that by assumption, D is finite and the random process is regular.

Step 1. Enumerate the problems in D as $1, 2, \ldots, M$. For each problem i, define the special chain beginning at i as $i, j_2(i), \ldots, j_{n(i)}(i)$. Consider the sequence of problems

$$j_{n(1)}(1), j_{n(1)-1}(1), \ldots, 1, j_{n(2)}(2), \ldots, 2, j_{n(3)}(3), \ldots, 3, \ldots, j_{n(M)}(M), \ldots, M.$$

At every period, the probability that this sequence will occur at the next period is, by the assumption that the random process is regular, at least

$$\pi_{j_{n(1)}(1)} \pi_{j_{n(1)-1}(1)} \cdots \pi_1 \pi_{j_{n(2)}(2)} \cdots \pi_2 \cdots \pi_{j_{n(M)}(M)} \cdots \pi_M > 0.$$

Therefore, with probability 1, this sequence occurs at a finite date T.

If $N(j_{n(1)}(1))$ is not chosen, the penalty is at least 1, since either WP or Sym is violated. If, however, $N(j_{n(1)}(1))$ is chosen, this entails at most two violations with respect to all previous choices—namely, for any previous date, a violation of ScInv and/or Ind. So the worst penalty that can be incurred is $2\sum_{t=1}^{T-1}\delta^t$. As $\delta > 0$,

$$2\sum_{t=1}^{T-1}\delta^t < 2\sum_{t=1}^{\infty}\delta^t = 2\frac{\delta}{1-\delta}.$$

Since $\delta \leq \frac{1}{3}$, one has

$$2\frac{\delta}{1-\delta} \leq 1,$$

so the judge will choose $N(j_{n(1)}(1))$. Indeed, this argument shows that any symmetric problem will be assigned the Nash point by the judge when it occurs.

Step 2. Now consider a later element $j_{n(1)-k}$ in the sequence, for $k = 1,\ldots,n(1)-1$. If the judge does not choose $N(j_{n(1)-k}(1))$, he violates either ScInv or Ind with respect to the previous date, so the penalty is at least δ. If he does choose $N(j_{n(1)-k}(1))$, he is penalized at most

$$2\sum_{t=k+1}^{T}\delta^t < 2\sum_{t=k+1}^{\infty}\delta^t = 2\frac{\delta^{k+1}}{1-\delta}.$$

As one has

$$2\frac{\delta^{k+1}}{1-\delta} \leq \delta^k \leq \delta,$$

the judge chooses $N(j_{n(1)-k}(1))$. In this way, we see that we have the Nash choice on the whole sequence.

Step 3. Now let the element that occurs after this sequence be i. If the judge does not choose $N(i)$, he violates two axioms with respect to the previous occurrence of i in the sequence—namely, ScInv and Ind. The penalty is, therefore, at least $2\delta^t$ for some $1 \leq t \leq \sum_{j=2}^{M} n(j) + 1$. (The lowest penalty is when $i = 1$.) Let $Q = \sum_{j=2}^{M} n(j) + 1$. Alternatively, if he chooses $N(i)$, he at most violates ScInv and Ind with respect to all problems preceding the sequence (from the beginning of the history until $Q + 1$ periods before) and is, therefore, penalized by no more than

$$2\sum_{t=Q+1}^{\infty}\delta^t = 2\frac{\delta^{Q+1}}{1-\delta}.$$

So he will choose $N(i)$ as long as

$$2\frac{\delta^{Q+1}}{1-\delta} \leq 2\delta^Q.$$

This is equivalent to $\delta \leq \frac{1}{2}$, which holds true.

Step 4. Assume that the judge has chosen the Nash point for S periods after the end of the sequence (in the previous step we showed this to be true for $S = 1$). Let the element that occurs at $S + 1$ be i. If the judge does not choose $N(i)$, he violates at least ScInv and Ind with respect to the previous occurrence of i in the sequence, and the penalty is, therefore, at least $2\delta^t$ for some $S + 1 \leq t \leq S + Q$. If he chooses $N(i)$, he at most violates ScInv and Ind with respect to all problems preceding the sequence (from the beginning of the history until $S + Q + 1$ periods before), and is penalized by no more than

$$2 \sum_{t=S+Q+1}^{\infty} \delta^t = 2\frac{\delta^{S+Q+1}}{1-\delta}.$$

So he will choose $N(i)$ as long as

$$2\frac{\delta^{S+Q+1}}{1-\delta} \leq 2\delta^{S+Q},$$

which is equivalent to $\delta \leq \frac{1}{2}$.

By induction he chooses Nash henceforth.

Only if: Suppose $\frac{1}{3} < \delta < 1$. Let $D = \{i, j\}$, as described in Figure 1. The problem j is symmetric.

The fact that $\delta > \frac{1}{3}$ is equivalent to

$$\frac{1}{2}\left(\frac{1-\delta}{\delta}\right) < 1.$$

Let T be an integer that satisfies

$$T > \frac{\ln\left(1 - \frac{1}{2}\left(\frac{1-\delta}{\delta}\right)\right)}{\ln \delta}.$$

There is a positive probability (at least π_i^T) that history starts with T occurrences of i. Suppose the judge picks point x in i for $t = 1, \ldots, T$.
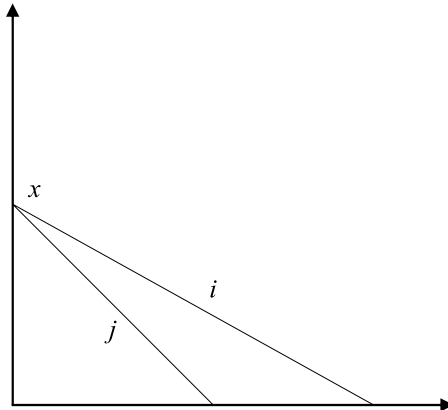

FIGURE 1. Example: the domain $D = \{i, j\}$.

Let j occur at $t = T + 1$. If the judge chooses $N(j)$, he violates ScInv and Ind with respect to the previous T periods and the penalty is $2\sum_{t=1}^{T}\delta^t$. If he chooses x, he violates Sym and the penalty is 1. If he chooses another point, the penalty is $1 + 2\sum_{t=1}^{T}\delta^t$. This last option is, therefore, dominated by $N(j)$. We have

$$2\sum_{t=1}^{T}\delta^t > 1 \quad \Leftrightarrow \quad 2\delta\frac{1-\delta^T}{1-\delta} > 1$$

$$\Leftrightarrow \quad T > \frac{\ln\left(1 - \frac{b}{c+d}\left(\frac{1-\delta}{\delta}\right)\right)}{\ln\delta},$$

which is true by assumption. Therefore, the judge picks x.

Consider a period $S > T + 1$ and assume that x has been chosen at all times before (we know this to be true for $S = T + 2$). If i occurs, x is picked again without any penalty, while any other point costs a penalty. If j occurs, picking x costs 1, while picking $N(j)$ costs $2\sum_{t=1}^{S-1}\delta^t > 2\sum_{t=1}^{T}\delta^t$. So, again x is chosen.

By induction, at no period in the future can the Nash point be chosen.   □

Note that the result holds only if, as assumed in this paper, $\delta > 0$. When $\delta = 0$ the judge is tied only by WP and Sym, and this is clearly insufficient to make him converge to the Nash solution.

REMARK 1. Theorem 1 remains true if we assume that the judge takes his office at a certain point in time, after an arbitrary history has unfolded, and feels bound by the previous decisions and the attached penalties. No matter how far from the Nash solution the antecedent decisions have been, he will converge almost surely to the Nash solution under the conditions of the theorem.

REMARK 2. These results depend on the judge being myopic. For instance, in the second part of the proof of Theorem 1, the judge could anticipate that j will occur at some date and that the only way not to incur any penalty is to take $N(i)$ right from the beginning. More on this issue will be said in Section 6.

The main limitation of Theorem 1 is that it applies only to domains for which Condition C_N holds. By Proposition 1, this is a strict subset of the set of Nash domains. It is easy to weaken Condition C_N in such a way that Theorem 1 remains valid over the corresponding larger set of domains, but the next proposition shows that Theorem 1 does not generalize to the full set of *all* Nash domains. Moreover, this problem is independent of the particular system of penalties adopted. (This result does not even require the random process to be regular.)

PROPOSITION 2. *There exist Nash domains such that, whatever δ, whatever the value of the penalty attached to each axiom, and whatever the random process, convergence to N does not occur almost surely on such domains.*

The proof involves a tedious example with a 10-problem domain and is available as a supplementary file on the journal website.[5] This negative result is due to the particular way in which Ind may work in the characterization of the Nash solution for some domains. Observe that in the example given in the proof of Proposition 1, one must have $\varphi(i) = N(i)$ because $j, k \subseteq i$ and the constraints known about $\varphi(j)$ and $\varphi(k)$ force $\varphi(i)$ to belong to two different segments of ∂i, the intersection of which is $\{N(i)\}$. This is the static form of the axiomatic analysis. In the dynamic setting in which the judge operates, this kind of constraint may be too weak to force him to choose $N(i)$.[6] This does not happen in this particular example because the constraints on $\varphi(j)$ and $\varphi(k)$ are $\varphi(j) = \varphi(k) = N(i)$, so that, given the shape of these sets, a violation of Ind in j or k would occur if the judge chose any non-Nash point in i. The proof, therefore, requires a more complicated example in which the constraints on the smaller sets are less precise so that the judge may pick points other than the Nash point in these sets and then also pick non-Nash points in the large set.

One can see from the example that proves Proposition 2 that the failure of convergence is not a convergence to another solution, but an oscillation between several solutions. One may then wonder if a stronger form of failure can occur, namely, convergence to another solution. The answer is, fortunately for the Nash approach, negative. (We assume again that the random process is regular.)

PROPOSITION 3. *If $\delta \leq \frac{1}{3}$ and convergence to a particular solution φ occurs with positive probability in a Nash domain, then $\varphi = N$.*

PROOF. Let D be a Nash domain and assume that convergence to a particular solution φ occurs with positive probability. This means that there is a set H of histories, occurring with positive probability, such that for every history $h \in H$, there is a finite T_h such that for all $t \geq T_h$, $\varphi(i_t)$ is chosen in every i_t.

Define the subset of H:

$H_0 = \{h \in H \mid \exists i, j \in D, \text{ the sequence } (i, j) \text{ occurs only a finite number of times in } h\}$.

Subset H_0 is a set of histories of measure zero because the process is regular. Thus, the set of histories $H' = H \setminus H_0$ is not empty (it has the same mass as H) and for every $h \in H'$, for every $i, j \in D$, the sequence (i, j) occurs an infinite number of times. A fortiori, note that every i also occurs an infinite number of times.

We now prove that φ obeys all the Nash axioms on D; since D is a Nash domain, it must be that $\varphi = N$.

First, φ must satisfy Sym, because $\delta \leq \frac{1}{3}$ and, therefore, as shown in the proof of Theorem 1, the judge always selects the Nash point (which is symmetric) in symmetric sets.

Second, suppose φ does not satisfy WP. Let $h \in H'$ and let date t be the first date in h at which $\varphi(i_t) \notin \partial i_t$. By the argument of the previous paragraph, i_t is not symmetric. If

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[5]http://econtheory.org/supp/588/supplement.pdf.

[6]When Ind imposes a penalty on the judge only when the past set is the larger set, this constraint simply vanishes.

the judge selects $\varphi(i_t)$, the penalty is at least 1 (for a violation of WP). If the judge selects a point in ∂i_t, he does not violate WP or Sym, but may at worst violate ScInv and Ind with respect to all $t - 1$ periods, so that the penalty is less than

$$2(\delta + \delta^2 + \cdots) = 2\frac{\delta}{1 - \delta}.$$

The penalty is therefore less than 1, as $\delta \leq \frac{1}{3}$. Therefore, the judge will never choose $\varphi(i_t)$. As $h \in H'$, i_t occurs an infinite number of times, which contradicts the assumption that convergence to φ occurs in every $h \in H'$.

Third, suppose that φ violates ScInv with respect to a particular pair (i, j). Let $h \in H'$. As φ selects the Nash point in symmetric sets, and i and j occur infinitely many times in h, i and j are not symmetric if convergence to φ is obtained in h. Moreover, h contains infinitely many occurrences of (i, j). When such a sequence occurs, the fact that the combination of $\varphi(i)$ and $\varphi(j)$ violates ScInv implies that choosing $\varphi(j)$ costs at least δ. Choosing a point $x \in \partial j \setminus \varphi(j)$ costs less than

$$2(\delta^2 + \delta^3 + \cdots) = 2\frac{\delta^2}{1 - \delta},$$

which is less than δ as $\delta \leq \frac{1}{3}$. Therefore, it is impossible for the judge to choose $\varphi(j)$ from j when (i, j) occurs. Convergence to φ cannot occur in h, a contradiction.

Fourth, φ must satisfy Ind. Suppose that it violates it with respect to a particular pair (i, j). As φ selects the Nash point in symmetric sets, necessarily one of them is not symmetric, say j. One can then repeat the rest of the argument developed for ScInv and derive a contradiction.   □

4. Other solutions

We now examine how similar results can be obtained for the other two classical solutions of bargaining theory, the Kalai–Smorodinsky solution and the egalitarian solution. They reveal interesting differences with the Nash solution. One difference is that special chains can now be found that exactly delineate the domains for which the characterization theorems hold true. Another difference is that the theorem that characterizes the egalitarian solution has a smaller number of axioms.

4.1 *Kalai–Smorodinsky*

The *Kalai–Smorodinsky solution* is denoted KS. One has

$$KS(i) = \{x \in \partial i \mid x_1/x_2 = I_1(i)/I_2(i)\}.$$

A domain D is called a *Kalai–Smorodinsky* domain if the Kalai–Smorodinsky theorem holds on D, i.e., if $KS(\cdot)$ is the only solution satisfying WP, Sym, ScInv, and IMon on D.

Condition C_{KS}. For all $i \in D$, there exists a sequence $j_1, \ldots, j_n$ such that $j_1 = i$, j_n is symmetric and for all $t = 1, \ldots, n - 1$, either

(i) $j_t \subseteq j_{t+1}$ (or $j_t \supseteq j_{t+1}$), $I(j_t) = I(j_{t+1})$, and $\mathrm{KS}(j_{t+1}) \in \partial j_t$ or

(ii) $\exists \alpha \in \mathbb{R}_{++}^2$, j_{t+1} is an α-rescaling of j_t.

Again, and without risk of confusion with the previous section, let us call such a sequence a *special chain beginning at i*.

PROPOSITION 4. *A domain D is a Kalai–Smorodinsky domain if and only if it satisfies Condition C_{KS}.*

The proof of this proposition is tedious and is available as a supplementary file on the journal website.[7] This result makes it possible to obtain the following theorem.

THEOREM 2. *The judge converges to KS almost surely on all Kalai–Smorodinsky domains if and only if $\delta \leq \frac{1}{3}$.*

The proof closely mimics the proof of Theorem 1, with IMon replacing Ind.

4.2 *Egalitarian solution*

The *egalitarian solution* is denoted E. One has

$$E(i) = \{x \in \partial i \mid x_1 = x_2\}.$$

A domain D will be called an *E domain* if the egalitarian solution is the only solution satisfying WP, Sym, and Mon on D. This egalitarian theorem is a variant of Theorem 1 in Kalai (1977) and can be found in Thomson and Lensberg (1989, Theorem 2.5) and Peters (1992, Theorem 4.31).

CONDITION C_E. For all $i \in D$, there exists a sequence $j_1, \ldots, j_n$ such that $j_1 = i$, j_n is symmetric and for all $t = 1, \ldots, n-1$, $E(j_t) = E(j_{t+1})$ and either $j_t \subseteq j_{t+1}$ or $j_{t+1} \subseteq j_t$.

Again the sequence $j_1, \ldots, j_n$ will be called a special chain beginning at i.

PROPOSITION 5. *A domain D is an E domain if and only if it satisfies Condition C_E.*

PROOF. If: Let φ be any solution on D satisfying the axioms of the egalitarian theorem. Let $i \in D$. By Condition C_E there is a special chain $j_1, \ldots, j_n$ beginning at i. By Sym, $E(j_n)$ is chosen from j_n and one rolls back along the special chain by applying Mon. This implies $\varphi(i) = E(i)$.

Only if: Let D_+ be the subset of D containing the problems i with a special chain. We must show that $D_+ = D$ if the egalitarian theorem holds on D. Suppose that there exists a problem $k \in D \setminus D_+$. Let

$$Z = \{x \in \mathbb{R}_+^2 \mid \exists i \in D_+, x = E(i)\}.$$

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[7]http://econtheory.org/supp/588/supplement.pdf.

Construct a monotone path P from zero for which the intersection with the 45° line coincides with Z on $\mathbb{R}^2_{++}$. More precisely, P is the graph of an increasing function f such that $f(0) = 0$ and

$$\{x \in P \mid x_1 = x_2\} = Z \cup \{0\}.$$

Let φ be defined by, for all $i \in D$, $\{\varphi(i)\} = P \cap \partial i$. By construction φ satisfies WP and Mon. It satisfies Sym because all symmetric problems are in D_+ and φ coincides with E on D_+. But $\varphi \neq E$ unless $D = D_+$, which proves the "only if" part of the proposition. $\square$

The next theorem displays a more favorable threshold for δ thanks to the presence, in the egalitarian theorem, of fewer axioms that involve a reference to past decisions.

THEOREM 3. *The judge converges to E almost surely on all E domains if and only if $\delta \leq \frac{1}{2}$.*

The "if" part is a corollary of Theorem 4. The converse is an immediate adaptation of the second part of the proof of Theorem 1.

5. GENERALIZATION

The similarity between the results of the previous sections suggests an underlying common structure. In this section, we provide a general result that covers more theorems and other frameworks than the bargaining model. Consider an abstract setting in which a problem is a subset i of a general set $\mathcal{O}$ of options and a solution φ, defined on a domain D, has to pick an element of this set: $\varphi(i) \in i$.

The axioms of a characterization theorem have two general forms and are labelled $1k$ for $k = 1, \ldots, K_1$ and $2k$ for $k = 1, \ldots, K_2$, respectively. The first type of axiom, "unary" axioms, requires the solution to be chosen from a specific subset of i whenever i is of a particular sort. Let D_1^k be a subset of the domain D and let G_1^k be a correspondence from D to $2^{\mathcal{O}}$ such that for all $i \in D$, $G_1^k(i) \subseteq i$.

AXIOM 1K. *For all $i \in D$, if $i \in D_1^k$, then $\varphi(i) \in G_1^k(i)$.*

The second type of axiom, "binary" axioms, requires the points chosen by the solution for two sets i, j to stand in a particular relation whenever these two sets are themselves related in a specific way. Let D_2^k be a subset of D^2 that contains the pair (i, i) for all $i \in D$ and let G_2^k be a correspondence from D^2 to $2^{\mathcal{O} \times \mathcal{O}}$ such that for all $(i, j) \in D^2$, $G_2^k(i, j) \subseteq i \times j$. Moreover, we impose that for all $i \in D$, $G_2^k(i, i)$ contains no (x, y) such that $x \neq y$.

AXIOM 2K. *For all $(i, j) \in D^2$, if $(i, j) \in D_2^k$, then $(\varphi(i), \varphi(j)) \in G_2^k(i, j)$.*

Let us illustrate these general formulations with the axioms introduced in Section 2. Weak Pareto and Symmetry are of the first kind. For Weak Pareto, $D_1^k = D$ and $G_1^k(i) = \partial i$. For Symmetry, D_1^k is the subset of symmetric problems and $G_1^k(i)$ is the intersection of i with the 45° line.

The other axioms are of the second kind. For Scale Invariance, D_2^k is the subset of pairs such that one set is a rescaling of the other, and $G_2^k(i, j)$ is the set of pairs in $i \times j$ such that one point is the rescaling of the other in the same proportion as for the sets i, j. For Nash Independence, D_2^k is the subset of pairs (i, j) such that $i \subseteq j$, and $G_2^k(i, j)$ is the set of pairs $(x, y) \in i \times j$ such that if $y \in i$, then $x = y$:

$$G_2^k(i, j) = \{(x, y) \in i \times j \mid x = y \text{ or } y \notin i\}.$$

For Monotonicity, D_2^k is also the subset of pairs (i, j) such that $i \subseteq j$, and $G_2^k(i, j)$ is the set of pairs $(x, y) \in i \times j$ such that $x \leq y$, and so on.

The binary axioms used in the previous sections all satisfy the restriction that for all $i \in D$, $G_2^k(i, i)$ contains no (x, y) such that $x \neq y$. This restriction is not needed in static axiomatics because by definition, $\varphi(i)$ is only one element of i. But in the sequential framework of the judge, it is possible for him to choose different elements of i at different occurrences of i. It is then important that binary axioms give him incentives to choose consistently.

One could imagine other types of axioms, involving a greater number of problems, such as

$$\varphi(i) = \varphi(j) \implies \varphi(i \cup j) = \varphi(i).$$

This would require defining a system of penalties when the judge violates such an axiom that involves two problems treated at two different periods in the past. This extension is left for future research.

Let D be given, with a set of K_1 unary axioms and K_2 binary axioms. A special chain for a solution φ beginning at i in D is a sequence of problems $j_1, \ldots, j_n \in D$ such that $j_1 = i$ and

(i) for a subset $K^* \subseteq \{1, \ldots, K_1\}$, $j_n \in \bigcap_{k \in K^*} D_1^k$ and $\bigcap_{k \in K^*} G_1^k(j_n) = \{\varphi(j_n)\}$

(ii) for all $t = 1, \ldots, n - 1$, there is a subset $K^{**} \subseteq \{1, \ldots, K_2\}$ such that $(j_t, j_{t+1}) \in \bigcap_{k \in K^{**}} D_2^k$ and

$$\left\{x \in j_t \mid (x, \varphi(j_{t+1})) \in \bigcap_{k \in K^{**}} G_2^k(j_t, j_{t+1})\right\} = \{\varphi(j_t)\}.$$

What these conditions say is simple: for any solution φ' that satisfies all the axioms, $\varphi'(j_n) = \varphi(j_n)$ is imposed by the unary axioms, while for all pairs (j_t, j_{t+1}), $\varphi'(j_t) = \varphi(j_t)$ is imposed by the binary axioms if $\varphi'(j_{t+1}) = \varphi(j_{t+1})$. One then sees that by rolling back the sequence from j_n to j_1, $\varphi'(i) = \varphi(i)$ is imposed by the combination of all the axioms.

Note that in condition (ii) one could incorporate constraints on $\varphi'(j_t)$ imposed by unary axioms in conjunction with binary axioms. One could also consider additional constraints from binary axioms based on the symmetrical situation: $(j_{t+1}, j_t) \in \bigcap_{k \in K^{**}} D_2^k$ and the set

$$\left\{x \in j_t \mid (\varphi(j_{t+1}), x) \in \bigcap_{k \in K^{**}} G_2^k(j_{t+1}, j_t)\right\} = \{\varphi(j_t)\}.$$

Such possibilities were actually used in the special chains defined in the previous sections. We ignore them here because it does not alter the results obtained in this section, but it does complicate the presentation.

By the "rolling back" argument, we have obtained the first part of the following result.

PROPOSITION 6. *A solution φ is the only one that satisfies all the $K_1 + K_2$ axioms on a finite domain D if for all $i \in D$, there is a special chain for φ beginning at i. The converse does not hold in general.*

We do not need to prove the second part of this statement because from Proposition 1 we already know that the converse is not true in general. Indeed, in general there are many other ways to force a precise value of $\varphi(i)$ than by a special chain beginning at i, and it is somewhat surprising that we could obtain the converse for the Kalai–Smorodinsky and the egalitarian solutions.

Let us assume that the minimal (undiscounted) penalty for the violation of any axiom in the judge's court is a and that for a binary axiom, the average penalty is b. The key number in the following theorem is the ratio of penalties $K_2 b/a$, which is a lower bound for the "interest rate" with which the judge discounts the past. The critical interest rate never increases when a penalty that involves a unary axiom increases. Indeed, a greater weight for these axioms reinforces the right choice when a set j_n occurs and never encourages the judge to preserve past "mistakes." The role of the binary axioms and their penalties is more subtle. The critical interest rate increases with a penalty for a binary axiom if it is greater than another penalty, because this raises b without altering a, but r decreases if the penalty for a binary axiom is lower than all other penalties, because $K_2 b$ and a then increase by the same increment. This pattern can be explained as follows. When a binary axiom has heavy relative weight, this may give too much influence to past mistakes. However, when its associated penalty is small relative to the others, it is good to increase it so as to force the judge to take account of the good decisions that have been made under the stronger pressure of the other axioms.

THEOREM 4. *Assume that for every $i \in D$ there is a special chain for φ beginning at i. The judge converges almost surely to the solution φ if*

$$r \geq \frac{K_2 b}{a}. \tag{1}$$

PROOF. The structure of the proof is similar to the proof of Theorem 1. The quantity $K_2 b_2$ is the greatest penalty that the judge may incur for a violation of binary axioms. The second inequality in (1) is equivalent to

$$K_2 b \frac{\delta}{1 - \delta} \leq a.$$

Step 1. Enumerate the problems in D as $1, 2, \ldots, M$. For each problem i, define the special chain beginning at i as $i, j_2(i), \ldots, j_{n(i)}(i)$. Consider the sequence of problems

$$j_{n(1)}(1), j_{n(1)-1}(1), \ldots, 1, j_{n(2)}(2), \ldots, 2, j_{n(3)}(3), \ldots, 3, \ldots, j_{n(M)}(M), \ldots, M.$$

With probability 1, this sequence occurs at a finite date T.

If $\varphi(j_{n(1)}(1))$ is not chosen, the penalty is at least a, since a unary axiom is violated. If, however, $\varphi(j_{n(1)}(1))$ is chosen, this entails at most K_2 violations of binary axioms with respect to all previous choices. So the worst penalty that can be incurred is $K_2 b \sum_{t=1}^{T-1} \delta^t$. As $\delta > 0$,

$$K_2 b \sum_{t=1}^{T-1} \delta^t < K_2 b \sum_{t=1}^{\infty} \delta^t = K_2 b \frac{\delta}{1-\delta}.$$

Since by (1),

$$K_2 b \frac{\delta}{1-\delta} \leq a,$$

the judge will choose $\varphi(j_{n(1)}(1))$. For the same reason, the judge will choose $\varphi(j_{n(t)}(t))$ for $t = 2, \ldots, M$.

Step 2. Consider another element $j_{n(1)-k}(1)$, $k = 1, \ldots, n(1) - 1$, in the sequence. If the judge does not choose $\varphi(j_{n(1)-k}(1))$, he violates at least one binary axiom with respect to the previous date, so the penalty is at least $a\delta$. If he does choose $\varphi(j_{n(1)-k}(1))$, he is penalized at most

$$K_2 b \sum_{t=k+1}^{T} \delta^t < K_2 b \sum_{t=k+1}^{\infty} \delta^t = K_2 b \frac{\delta^{k+1}}{1-\delta}.$$

As

$$K_2 b \frac{\delta^{k+1}}{1-\delta} \leq a\delta,$$

the judge chooses $\varphi(j_{n(1)-k}(1))$. Therefore, φ is chosen throughout the sequence.

Step 3. Let the element that occurs after this sequence be i. If the judge does not choose $\varphi(i)$, he violates all binary axioms with respect to the previous occurrence of i in the sequence. The penalty is, therefore, $K_2 b \delta^t$ for some $1 \leq t \leq Q = \sum_{j=2}^{M} n(j) + 1$. Alternatively, if he chooses $\varphi(i)$, he at most violates K_2 binary axioms with respect to all problems preceding the sequence (from the beginning of the history until $Q + 1$ periods before) and is, therefore, penalized by not more than

$$K_2 b \sum_{t=Q+1}^{\infty} \delta^t = K_2 b \frac{\delta^{Q+1}}{1-\delta}.$$

So he will choose $\varphi(i)$ as long as

$$K_2 b \frac{\delta^{Q+1}}{1-\delta} \leq K_2 b \delta^Q,$$

which is satisfied because

$$\frac{\delta}{1-\delta} \leq \frac{a}{K_2 b} \leq 1.$$

Step 4. Assume that the judge has chosen the φ point for S periods after the end of the sequence. Let the element that occurs at $S+1$ be i. If the judge does not choose $\varphi(i)$, he violates K_2 binary axioms with respect to the previous occurrence of i in the sequence and the penalty is, therefore, $K_2 b \delta^t$ for some $S+1 \leq t \leq S+Q$. If he chooses $\varphi(i)$, he at most violates K_2 binary axioms with respect to all problems preceding the sequence (from the beginning of the history until $S+Q+1$ periods before) and is penalized by less than

$$K_2 b \sum_{t=S+Q+1}^{\infty} \delta^t = K_2 b \frac{\delta^{S+Q+1}}{1-\delta}.$$

So he will choose $\varphi(i)$ as long as

$$K_2 b \frac{\delta^{S+Q+1}}{1-\delta} \leq K_2 b \delta^{S+Q}.$$

This is equivalent to the condition obtained in Step 3.   □

It seems difficult to obtain a converse to Theorem 4 because the counterexamples constructed in the previous sections rely on the specifics of the models and solutions under consideration.

REMARK 3. In the previous sections, we assumed $a = b = 1$, in which case the premise in Theorem 4 becomes $r \geq K_2$ or, equivalently, $\delta \leq 1/(1 + K_2)$. This explains why the upper bound for δ with the egalitarian solution ($\frac{1}{2}$ for $K_2 = 1$) differs from the bound for the Nash and Kalai–Smorodinsky solutions ($\frac{1}{3}$ for $K_2 = 2$).

REMARK 4. We noticed in Section 2 that the assumption that the violation of a binary axiom never counts for more than δ, which is less than the penalty for a unary axiom, appears restrictive. It is difficult to escape this pattern, though. The more general system of penalties considered in this section allows for a relative penalty for binary axioms, b/a, that is as large as one wishes. However, the inequality $r \geq K_2 b/a$ implies that one always has $K_2 b \delta < a$, because

$$\delta \leq \frac{1}{1 + \frac{K_2 b}{a}} < \frac{a}{K_2 b}.$$

The inequality $K_2 b \delta < a$ means that all the binary axioms together always have a lower *discounted* penalty than any unary axiom. It is intuitive that this must hold if one wants the judge always to make the right choice in every set j_n of a special chain.

6. FORESIGHT

It is against the philosophy of our approach to endow the judge with foresight, according to which he would compute the effect of his present decision on penalties he is likely to incur in the future, because our approach is one of bounded rationality and learning, not full rationality. In addition, foresight is not an important aspect of the doctrine of real-world judicial precedent, because the judges typically focus on consistency with past

judgments rather than on the constraints their current decisions will impose on future related cases, so our approach is not far-fetched.

Even with foresight, however, the problem does not become trivial if the judge has to live with an arbitrary set of precedents that he inherits upon taking office and that will determine penalties he incurs in the future. It is then possible that historical errors will continue to influence his decisions and prevent convergence to the "correct" solution.

In this section, we present an example to show that this can indeed occur when the judge has foresight. We adopt the framework of Section 3 (focusing on the Nash solution in the axiomatic bargaining model) and assume that the judge knows the probability law that governs the occurrence of successive problems. He discounts the future penalties with a factor β. Suppose he starts his job at time 0, after an arbitrary sequence of decisions have been made for periods $-T, \ldots, -1$. He faces a problem i_0 and devises a conditional strategy

$$x_0, x_1(i_1), x_2(i_1, i_2), \ldots, x_t(i_1, \ldots, i_t), \ldots.$$

When a particular history of problems $i_1, i_2, \ldots$ is realized, he must pay the total discounted penalty $\sum_{t \geq 0} \beta^t p_t$, where p_t is the penalty paid in t for violations of unary axioms in t and violations of binary axioms in t with respect to past decisions (with the discount factor δ). Knowing the probability of occurrence of all possible histories, he can then compute the expected value of $\sum_{t \geq 0} \beta^t p_t$ for a given conditional strategy and select the conditional strategy that minimizes this quantity. When history unfolds, he has to follow only the conditional strategy. Note that the conditional strategy and the computation of the expected value of $\sum_{t \geq 0} \beta^t p_t$ can incorporate the fact that the observed sequence of problems up to t may alter the probability of occurrence of future problems for $t + 1, t + 2, \ldots$.

What has been done in the previous sections corresponds to the special case in which $\beta = 0$. The judge then only has to choose x_t so as to minimize p_t, and it suffices that he does so sequentially for the actual sequence of problems, ignoring the counterfactual problems.

Consider for a moment that history does start at period 0, i.e., there is no arbitrary sequence of precedents. If the domain satisfies the chain condition (i.e., a special chain begins at every member) and the random process is regular, then the only way to avoid penalties in the future is to follow the solution characterized by the axioms. Whenever $\beta > 0$, the judge always follows the solution.

We now show that, in contrast, when an arbitrary sequence of precedents encumbers the judge's decisions, a positive β may not suffice to converge to the solution. Consider the example of Theorem 1. Suppose that the past history consists of T times x (it does not matter whether i or j was the set). Let j occur at period 0.

Suppose the judge knows that the history that will occur beginning at $t = 0$ is an infinite sequence of j's. In a moment, we will calculate the condition under which it would minimize his total discounted penalty to continue playing x forever and hence never converge to the Nash solution. Now if this condition holds, it must be the case that not knowing what the sequence will be except that j has occurred at $t = 0$, his best strategy

is to play x forever, because the largest total penalty he can ever incur is when the sequence beginning at $t = 0$ is an infinite sequence of j's. (He would never pay a penalty when i occurs in a history under this strategy, but he pays a penalty whenever j occurs.) Additionally, under this special history, if it is rational for him to stick to playing x, then it must be the optimal conditional strategy as well.

Let us prove that if the judge thinks that only j will occur from $t = 0$ on, at period 0 he adopts the strategy to retain x forever. If he retains x, he pays an expected penalty of $1/(1 - \beta)$. If he switches to $N(j)$ he pays

$$2 \sum_{t=1}^{T} \delta^t + \beta 2 \sum_{t=2}^{T+1} \delta^t + \cdots = 2\delta \frac{1 - \delta^T}{1 - \delta} \frac{1}{1 - \beta\delta}.$$

The latter is greater than the former if

$$T > \frac{\ln\left(1 - \frac{1}{2}\frac{1-\beta\delta}{1-\beta}\frac{1-\delta}{\delta}\right)}{\ln \delta}.$$

This formula requires

$$\frac{1}{2}\frac{1-\beta\delta}{1-\beta}\frac{1-\delta}{\delta} < 1,$$

which is true if

$$\beta < \frac{1 - \frac{1-\delta}{\delta}\frac{1}{2}}{1 - (1-\delta)\frac{1}{2}} \quad \text{and} \quad \delta > \frac{1}{3}.$$

The condition on δ is the same as in the counterexample of Theorem 1, which is interesting because it shows that the presence of foresight does not radically alter the constraints on δ.

To illustrate, one obtains a lack of convergence with, e.g., $\delta = 0.8$, $\beta = 0.95$, and $T = 5$.

7. CONCLUSION

An interesting fact is that, in all the results of this paper, we get convergence to the solution precisely when discounting the future is large. This is somewhat counterintuitive: one might think that convergence to the solution occurs only for intermediate values of the discount rate, because even if the past decisions must be easily forgotten when they are bad, they must also retain some force when they are good. As it turns out, for the latter concern it is enough if the past is not completely ignored ($\delta > 0$). This can be understood by the fact that when convergence takes place, the good decisions are typically more recent than the bad decisions. Forgetting the latter is then at least as important as remembering the former and is obtained with a low δ.

However, the analysis of general systems of penalties in Section 5 shows that it is indeed bad for convergence if *some* binary axiom induce too low a penalty relative to the other axioms. This indeed creates the risk that the recent good decisions are binding only through this "feeble" axiom and their influence on the current decision may be

overwhelmed by the previous bad decisions that may bind through other axioms. It is in this mechanism that the intuition that the past must retain some power is vindicated.

The approach proposed in this paper may suggest a ranking of characterization theorems. Suppose we have a set $\mathfrak{T}$ of axiomatic theorems of the type we discuss here and for each theorem $\tau \in \mathfrak{T}$, we prove that in the benchmark case, almost sure convergence to the appropriate solution occurs if and only if $\delta \in (0, \delta_\tau]$. This provides a way to rank the axiomatic theorems in terms of plausibility: the greater is δ_τ, the more plausible is the theorem, in the sense that the dynamic version of the theorem (as developed here) holds for a larger set of discount factors. Thus, we say that the egalitarian theorem is more plausible than Nash's or Kalai and Smorodinsky's theorem.

To be precise, we are saying that if we observe societies that abide by an egalitarian constitution and societies that abide by a Nash constitution, and discount factors vary across societies randomly, then it is more likely that we will observe allocations that look like the egalitarian solution in the egalitarian societies than allocations that look like the Nash solution in Nash societies, because $(0, \frac{1}{3}] \subset (0, \frac{1}{2}]$.

An issue that we did not explore in this paper is the speed of convergence. Almost sure convergence is obtained in our results with the help of a particular sequence of problems, all special chains for all members of the domain in a row, which is a rather unlikely event. For a domain with n problems, each having a special chain of average length m, this requires a particular arrangement of nm problems, with $n!$ acceptable permutations of this arrangement. The expected number of periods needed for one of these arrangements to occur is large. For $n = 10$, $m = 2$, and assuming a random process with independent and identically distributed draws and equiprobable problems, the expected number of periods is around 7.6×10^{26}.[8] Convergence can nevertheless occur in other cases, for instance, if all special chains occur in a sequence, but without a repetition of problems (i.e., if a special chain has appeared, its elements do not appear again in the arrangement; if two or more sets share the end of their special chains but not the beginning, one chain is followed by the remaining part of the other chain). The length of the special sequence of problems is then reduced from nm to n. In the above example with $n = 10$, $m = 2$, and assuming that there are four special chains, the expected time of convergence is reduced to 1.7×10^{17}, still a large number but significantly less so. The adaptation of our proofs of almost sure convergence to this shorter sequence is straightforward. This, however, provides only a very rough upper bound of the expected time of convergence. We leave this issue for future research. A related issue, also left for future research, is the computation of the probability of convergence, which may be high without being equal to 1 when δ is greater than the threshold identified here.

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[8]The probability that one of the acceptable arrangements starts at any given time is $p = n!/n^{nm}$ and the expected time of occurrence is $(1 - p)/p^2$.

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