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von Neumann–Morgenstern farsightedly stable sets in two-sided matching

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We adopt the notion of von Neumann–Morgenstern (vNM) farsightedly stable sets to determine which matchings are possibly stable when agents are farsighted in one-to-one matching problems. We provide the characterization of vNM farsightedly stable sets: a set of matchings is a vNM farsightedly stable set if and only if it is a singleton subset of the core. Thus, contrary to the vNM (myopically) stable sets (Ehlers 2007), vNM farsightedly stable sets cannot include matchings that are not in the core. Moreover, we show that our main result is robust to many-to-one matching problems with substitutable preferences: a set of matchings is a vNM farsightedly stable set if and only if it is a singleton set and its element is in the strong core.

Keywords. Matching problem, von Neumann–Morgenstern stable sets, farsighted stability.

JEL classification. C70, C78.

1. Introduction

Gale and Shapley (1962) propose the simple two-sided matching model, known as the marriage problem, in which matchings are one-to-one. There are two disjoint sets of agents—men and women—and the problem is to match agents from one side of the market with agents from the other side, where each agent has the possibility to remain

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single. They show that the core is nonempty. A matching is in the core if there is no subset of agents who, by forming all their partnerships only among themselves (and having the possibility of becoming single), can all obtain a strictly preferred set of partners.[1] Recently, Ehlers (2007) characterizes the von Neumann–Morgenstern (hereafter, vNM) stable sets in one-to-one matching problems. A set of matchings is a vNM stable set if this set satisfies two conditions: (internal stability) no matching inside the set is dominated by a matching belonging to the set; (external stability) any matching outside the set is dominated by some matching belonging to the set. Ehlers (2007) shows that the set of matchings in the core is a subset of any vNM stable set.

The notions of core and of vNM stable set are myopic notions since the agents cannot be farsighted in the sense that individual and coalitional deviations cannot be countered by subsequent deviations.[2] An interesting contribution is Diamantoudi and Xue (2003), who investigate farsighted stability in hedonic games (of which one-to-one matching problems are a special case) by introducing the notion of the coalitional largest farsighted conservative stable set, which coincides with the largest consistent set of Chwe (1994).[3] The largest consistent set is based on the indirect dominance relation, which captures the fact that farsighted agents consider the end matching that their move(s) may lead to. Diamantoudi and Xue (2003) show that in hedonic games with strict preferences, core partitions are always contained in the largest consistent set. However, we show by means of an example that the largest consistent set may contain more matchings than those matchings that are in the core. Based on the indirect dominance relation, Diamantoudi and Xue (2003) define the farsighted (or abstract) core as the set of matchings that are not indirectly dominated. But the farsighted core is too exclusive because it does not take into account the credibility of the dominating matching and, hence, it can often be empty. The farsighted core exists only when the core contains a unique matching and no other matching indirectly dominates the matching in the core.

In this paper, we adopt the notion of von Neumann–Morgenstern farsightedly stable sets to determine which matchings are possibly stable when agents are farsighted. This concept is studied by Chwe (1994), who introduces the notion of indirect dominance into the standard definition of vNM stable sets. Thus, a set of matchings is a vNM farsightedly stable set if no matching inside the set is indirectly dominated by a matching belonging to the set (internal stability) and any matching outside the set is indirectly dominated by some matching belonging to the set (external stability).

Our main result is the characterization of vNM farsightedly stable sets in one-to-one matching problems. We show that a set of matchings is a vNM farsightedly stable set if and only if it is a singleton set and its element is in the core. Thus, contrary to the vNM (myopically) stable sets, vNM farsightedly stable sets cannot include matchings that are not in the core. In other words, we provide an alternative characterization of the core in

[1]We refer to Roth and Sotomayor (1990) for a comprehensive overview on two-sided matching problems.

[2]Harsanyi (1974) argues that the von Neumann–Morgenstern definition of stable sets is unsatisfactory because it neglects the destabilizing effect of *indirect* dominance relations.

[3]Other approaches to farsightedness in coalition and/or network formation are suggested by the work of Xue (1998) or Herings et al. (2004, 2009).

one-to-one matching problems. We also show that our main result is robust to many-to-one matching problems with substitutable preferences: a set of matchings is a vNM farsightedly stable set if and only if it is a singleton set and its element is in the strong core. Finally, we show that if the preferences satisfy the top coalition property, then the unique vNM farsightedly stable set, the farsighted core, and the largest consistent set consist of the unique core element.

It is straightforward to see that if a matching belongs to the core, then it indirectly dominates all other matchings and hence it must be a vNM farsightedly stable set. This by itself does not characterize the vNM stable sets in a matching problem. Indeed, we need to tackle the question of whether there can be matchings that do not belong to the core, but do belong to some vNM farsightedly stable set. This question is not trivial, as in general the core matchings can be indirectly dominated by matchings that do not belong to the core.

Our characterization raises the question of why the idea of vNM farsightedly stable sets makes sense in the case where the core contains more than one element, as any vNM farsightedly stable set is indirectly dominated by any other vNM farsightedly stable set. We should keep in mind that the idea of a vNM (farsightedly) stable set is a set-valued concept and, as such, is fundamentally different from a non-cooperative equilibrium concept: any deviation from a vNM farsightedly stable set, even to another stable set, is deterred because there is a (farsighted) path leading back to the initial stable set. As Diamantoudi and Xue (2007) write, a vNM (farsightedly) stable set of matchings is "free from inner contradictions and accounts for every element it excludes." In fact, vNM farsightedly stable sets can be interpreted as the set of social outcomes consistent with a certain stable standard of behavior (Greenberg 1990). In particular, agents do not deviate if there *exists* a path leading back to the solution set in which the deviators are not better off and, as such, they behave optimistically in the face of (Knightian) uncertainty (Xue 1998). We show that any matching that belongs to the core, and no other set of matchings, satisfies the requirements of an optimistic stable standard of behavior: for any deviation there is a path leading back to this core matching. If the core contains more than one element, each one of them is then a stable standard of behavior.

The paper is organized as follows. Section 2 introduces one-to-one matching problems and standard notions of stability. Section 3 defines vNM farsightedly stable sets. Section 4 provides the characterization of vNM farsightedly stable sets in one-to-one matching problems. Section 5 deals with many-to-one matching problems. Section 6 concludes.

2. One-to-one matching problems

A one-to-one matching problem consists of a set of N agents divided into a set of men, $M = \{m_1, \ldots, m_r\}$, and a set of women, $W = \{w_1, \ldots, w_s\}$, where possibly $r \neq s$. We sometimes denote a generic agent by i, a generic man by m, and a generic woman by w. Each agent has a complete and transitive preference ordering over the agents on the other side of the market and the prospect of being alone. Preferences are assumed to be strict. Let P be a preference profile specifying for each man $m \in M$ a strict preference ordering over $W \cup \{m\}$ and for each woman $w \in W$ a strict preference ordering over $M \cup \{w\}$: $P = \{P(m_1), \dots, P(m_r), P(w_1), \dots, P(w_s)\}$, where $P(i)$ is agent i's strict preference ordering over the agents on the other side of the market and himself (or herself). For instance, $P(w) = m_4, m_1, w, m_2, m_3, \dots, m_r$ indicates that woman w prefers m_4 to m_1 and she prefers to remain single rather than to marry anyone else. We denote by R the weak orders associated with P. We write $m \succ_w m'$ if woman w strictly prefers m to m', write $m \sim_w m'$ if w is indifferent between m and m', and write $m \succeq_w m'$ if $m \succ_w m'$ or $m \sim_w m'$. Similarly, we write $w \succ_m w'$, $w \sim_m w'$, and $w \succeq_m w'$. A one-to-one matching problem is simply a triple (M, W, P).

A *matching* is a function $\mu : N \to N$ satisfying the following properties: (i) $\forall m \in M$, $\mu(m) \in W \cup \{m\}$; (ii) $\forall w \in W$, $\mu(w) \in M \cup \{w\}$; and (iii) $\forall i \in N$, $\mu(\mu(i)) = i$. We denote by $\mathcal{M}$ the set of all matchings. Given a matching μ, an agent i is said to be unmatched or single if $\mu(i) = i$. A matching μ is *individually rational* if each agent is acceptable to his or her mate, i.e., $\mu(i) \succeq_i i$ for all $i \in N$. For a given matching μ, a pair $\{m, w\}$ (possibly $m = w$) is said to form a *blocking pair* if they are not matched to one another but prefer one another to their mates at μ, i.e., $w \succ_m \mu(m)$ and $m \succ_w \mu(w)$. A matching μ is *stable* (or *pairwise stable*) if it is not blocked by any individual or any pair of agents.

We extend each agent's preference over the agent's potential partners to the set of matchings in the following way. We say that agent i prefers μ' to μ if and only if agent i prefers his or her mate at μ' to his or her mate at μ, $\mu'(i) \succ_i \mu(i)$. Abusing notation, we write this as $\mu' \succ_i \mu$. A coalition S is a subset of the set of agents N.[4] For $S \subseteq N$, $\mu(S) = \{\mu(i) : i \in S\}$ denotes the set of mates of agents in S at μ. A matching μ is *blocked by a coalition* $S \subseteq N$ if there exists a matching μ' such that $\mu'(S) = S$ and for all $i \in S$, $\mu' \succ_i \mu$. If S blocks μ, then S is called a *blocking coalition* for μ. Note that if a coalition $S \subseteq N$ blocks a matching μ, then there exists a pair $\{m, w\}$ (possibly $m = w$) that blocks μ. The *core* of a matching problem consists of all matchings that are not blocked by any coalition. An alternative way to define the core of a matching problem is by means of the domination relation.

Definition 1. Given a matching μ, a coalition $S \subseteq N$ is said to be able to enforce a matching μ' over μ if the following conditions hold: (i) $\mu'(i) \notin \{\mu(i), i\}$ implies $\{i, \mu'(i)\} \subseteq S$ and (ii) $\mu'(i) = i \neq \mu(i)$ implies $\{i, \mu(i)\} \cap S \neq \varnothing$.

In other words, this enforceability condition implies both that any new match in μ' that does not exist in μ should be between players in S, and that to destroy an existing match in μ, one of the two players involved in that match should belong to coalition S.[5] Notice that the concept of enforceability is independent of preferences. Furthermore, the fact that coalition $S \subseteq N$ can enforce a matching μ' over μ implies that there exists a sequence of matchings $\mu^0, \mu^1, \dots, \mu^K$ (where $\mu^0 = \mu$ and $\mu^K = \mu'$) and a sequence of disjoint pairs $\{m_0, w_0\}, \dots, \{m_{K-1}, w_{K-1}\}$ (possibly for some $k \in \{0, 1, \dots, K-1\}$,

[4]Throughout the paper, we use the notation $\subseteq$ for weak inclusion and $\subsetneq$ for strict inclusion.

[5]Notice that this enforceability condition is similar to the enforceability condition defined in Roth and Sotomayor (1990). That is, a coalition S can enforce the set of marriages in the matching μ' that concerns its members if and only if every man in S is married to a woman in S and vice versa.

$m_k = w_k$) such that for any $k \in \{1, \dots, K\}$, the pair $\{m_{k-1}, w_{k-1}\} \in S$ can enforce the matching μ^k over μ^{k-1}.

Definition 2. A matching μ is directly dominated by μ', or $\mu < \mu'$, if there exists a coalition $S \subseteq N$ of agents such that $\mu' \succ_i \mu \; \forall i \in S$ and S can enforce μ' over μ.

Definition 2 gives us the definition of *direct dominance*. The direct dominance relation is denoted by $<$. A matching μ is in the core if there is no subset of agents who, by rearranging their partnerships only among themselves and possibly dissolving some partnerships of μ, can all obtain a strictly preferred set of partners. Formally, a matching μ is in the core if μ is not directly dominated by any other matching $\mu' \in \mathcal{M}$.[6] Given a profile P, we denote the set of matchings in the core by $C(P)$. Gale and Shapley (1962) prove that the core is nonempty. Sotomayor (1996) provides a nonconstructive elementary proof of the existence of stable marriages.

Another concept used to study one-to-one matching problems is the vNM stable set (von Neumann and Morgenstern 1944), a set-valued concept that imposes both internal and external stability. A set of matchings is a vNM stable set if (internal stability) no matching inside the set is directly dominated by a matching belonging to the set, and (external stability) any matching outside the set is directly dominated by some matching belonging to the set.

Definition 3. A set of matchings $V \subseteq \mathcal{M}$ is a vNM stable set if the following conditions are met:

(i) For all $\mu \in V$, there does not exist $\mu' \in V$ such that $\mu' > \mu$.

(ii) For all $\mu' \notin V$, there exists $\mu \in V$ such that $\mu > \mu'$.

Definition 3 gives us the definition of a vNM stable set $V(<)$. Ehlers (2007) studies the properties of the vNM stable sets in one-to-one matching problems using a different enforceability notion. Given a matching μ, a coalition $S \subseteq N$ is said to be able to enforce a matching μ' over μ if $\mu'(S) = S$. He shows that the core is a subset of any vNM stable set when using this last enforceability notion.[7] Example 1 illustrates his main result.

[6]Setting $|S| \leq 2$ in the definition of the core, we obtain the concept of pairwise stability defined in Gale and Shapley (1962) that is equivalent to the core in one-to-one matchings (due to the fact that the existence of any blocking coalition induces the existence of a blocking pair as already mentioned before).

[7]The notion of enforceability used by Ehlers (2007) is very strong. Let μ be the matching where all agents are single and let μ' be the matching where $\mu'(m_1) = w_1$, $\mu'(m_2) = w_2$, $\mu'(m_3) = w_3, \dots$ (assuming $|M| = |W|$). Let $S = \{m_1, w_1\}$. Then, according to the enforceability notion of Ehlers (2007), S can enforce μ' over μ. In particular, S not only enforces being matched together, but its members can also change the matching structure for all other agents in an arbitrary way. To avoid this arbitrariness, we use another enforceability notion. Notice that under our enforceability notion, the vNM stable set in Definition 3 contains the vNM stable set defined in Ehlers (2007).

Example 1 (Ehlers 2005). Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Let P be such that

$P(m_1)$	$P(m_2)$	$P(m_3)$	$P(w_1)$	$P(w_2)$	$P(w_3)$
w_1	w_2	w_3	m_2	m_3	m_1
w_2	w_3	w_1	m_3	m_1	m_2
m_1	m_2	m_3	w_1	w_2	w_3
w_3	w_1	w_2	m_1	m_2	m_3

Let

$$\mu = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_1 & w_2 & w_3 \end{pmatrix}, \qquad \mu' = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_2 & w_3 & w_1 \end{pmatrix}, \qquad \mu'' = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_3 & w_1 & w_2 \end{pmatrix}.$$

It can be shown that the core contains a unique matching $C(P) = \{\mu'\}$ and that the unique vNM stable set (when using the notion of enforceability of Ehlers 2007) is $V(<) = \{\mu, \mu', \mu''\}$. ◊

In Example 1, the matchings μ and μ'' belong to the unique vNM stable set because μ' does not directly dominate either μ or μ'' even though μ and μ'' are not individually rational matchings (either all women or all men prefer to become single). However, farsighted women may decide first to become single in the expectation that further marriages form, leading to μ'. Women prefer μ' to μ and once everybody is divorced, men and women prefer μ' to the situation where everybody is single. A similar reasoning can be made for μ'' with the roles of men and women reversed. Then we may say that (i) μ' farsightedly dominates μ, (ii) μ' farsightedly dominates μ'', and (iii) $V(<) = \{\mu, \mu', \mu''\}$ is not a reasonable candidate for being a vNM farsightedly stable set.

In what follows, we use the notion of enforceability given in Definition 1 unless otherwise mentioned.

3. von Neumann–Morgenstern farsighted stability

The *indirect dominance* relation is first introduced by Harsanyi (1974), but is later formalized by Chwe (1994). It captures the idea that coalitions of agents can anticipate the actions of other coalitions. In other words, the indirect dominance relation captures the fact that farsighted coalitions consider the end matching that their deviations may lead to. A matching μ' *indirectly dominates* μ if μ' can replace μ in a sequence of matchings, such that at each matching along the sequence, all deviators are strictly better off at the end matching μ' compared to the status quo they face. Formally, indirect dominance is defined as follows.

Definition 4. A matching μ is indirectly dominated by μ', or $\mu \ll \mu'$, if there exists a sequence of matchings $\mu^0, \mu^1, \ldots, \mu^K$ (where $\mu^0 = \mu$ and $\mu^K = \mu'$) and a sequence of coalitions $S^0, S^1, \ldots, S^{K-1}$ such that for any $k \in \{1, \ldots, K\}$, the following conditions are met:

(i) For all $i \in S^{k-1}$, $\mu^K \succ_i \mu^{k-1}$.

(ii) Coalition S^{k-1} can enforce the matching μ^k over μ^{k-1}.

Definition 4 gives us the definition of indirect dominance. The indirect dominance relation is denoted by $\ll$. Direct dominance is obtained by setting $K = 1$ in Definition 4. Obviously, if $\mu < \mu'$, then $\mu \ll \mu'$.

Diamantoudi and Xue (2003) investigate farsighted stability in hedonic games (of which one-to-one matching problems are a special case), introducing the notion of the coalitional largest farsighted conservative stable set, which coincides with the *largest consistent set* of Chwe (1994).

Definition 5. The set $Z(\ll) \subseteq \mathcal{M}$ is a consistent set if $\mu \in Z(\ll)$ if and only if $\forall \mu', S$ such that S can enforce μ' over μ, $\exists \mu'' \in Z(\ll)$, where $\mu' = \mu''$ or $\mu' \ll \mu''$ such that $\mu(i) \not\prec_i \mu''(i)$ for some $i \in S$. The largest consistent set $\Gamma(P)$ is the consistent set that contains any consistent set.

Interestingly, Diamantoudi and Xue (2003) show that in hedonic games with strict preferences, (i) any partition belonging to the core indirectly dominates any other partition and (ii) core partitions are always contained in the largest consistent set.[8] Thus, in one-to-one matching markets, for all $\mu' \neq \mu$ with $\mu \in C(P)$, we have that $\mu \gg \mu'$ and $C(P) \subseteq \Gamma(P)$. However, the largest consistent set may contain more matchings than those matchings that are in the core as is shown in Example 2.

Example 2. Let $M = \{m_1, m_2, m_3\}$ and $W = \{w_1, w_2, w_3\}$. Let P be such that

$P(m_1)$	$P(m_2)$	$P(m_3)$	$P(w_1)$	$P(w_2)$	$P(w_3)$
w_1	w_2	w_2	m_2	m_1	m_3
w_2	w_1	w_1	m_3	m_2	m_1
w_3	w_3	w_3	m_1	m_3	m_2
m_1	m_2	m_3	w_1	w_2	w_3

Let

$$\mu^1 = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_2 & w_1 & w_3 \end{pmatrix}, \qquad \mu^2 = \begin{pmatrix} m_1 & m_2 & m_3 \\ w_1 & w_2 & w_3 \end{pmatrix}.$$

Note that μ^1 is the *unique* element in the core of this matching problem and it belongs to the largest consistent set, $\Gamma(P)$. Indeed, since $\mu^1 \in C(P)$, we have that $\mu^1 \gg \widehat{\mu}$ for all $\widehat{\mu} \neq \mu^1$. We show now that $\{\mu^1, \mu^2\}$ is a consistent set and hence μ^2 also belongs to $\Gamma(P)$. To do so, we have to show that $\forall \mu', S$ such that S can enforce μ' over μ^2, $\exists \mu \in \{\mu^1, \mu^2\}$, where $\mu' = \mu$ or $\mu' \ll \mu$, such that $\mu^2(i) \not\prec_i \mu(i)$ for some $i \in S$. The only profitable deviation from μ^2 is the one in which $\{m_3, w_1\}$ get married at μ', leaving w_3 and m_1 single. Since $\mu^1 \in C(P)$, we have $\mu^1 \gg \mu'$, and note that one of the deviating players, m_3, is not

[8] The largest consistent set always exists, is nonempty, and satisfies external stability (i.e., any matching outside the set is indirectly dominated by some matching belonging to the set). But a consistent set does not necessarily satisfy the external stability condition. Only the largest consistent set is guaranteed to satisfy external stability.

better off at μ^1 compared to μ^2; i.e., $\mu^1(m_3) \sim_{m_3} \mu^2(m_3)$. Consequently, $\{\mu^1, \mu^2\}$ is a consistent set and the largest consistent set may contain matchings that do not belong to the core. $\Diamond$

Another way to introduce farsighted stability is to replace direct by indirect dominance in the definition of the core. Diamantoudi and Xue (2003) define the *farsighted core* (or abstract core) as

$$C(P, \ll) = \{\mu \in \mathcal{M} \mid \nexists \mu' \in \mathcal{M} \text{ such that } \mu' \gg \mu\}.$$

Since $\mu < \mu'$ implies $\mu \ll \mu'$, it must be that $C(P, \ll) \subseteq C(P)$. The farsighted core $C(P, \ll)$ is too exclusive because it does not consider the credibility of the dominating alternative and, hence, it can often be empty. For instance, it follows immediately from the result that any partition belonging to the core indirectly dominates any other partition, that $C(P, \ll)$ is empty when there are at least two elements in $C(P)$. But even when $C(P)$ is a singleton, $C(P, \ll)$ can be empty. In Example 2, the matching in which the men (m_1, m_2, m_3) are matched to (w_1, w_2, w_3) indirectly dominates the core stable matching where the men (m_1, m_2, m_3) are matched to (w_2, w_1, w_3) and hence the farsighted core is empty. The farsighted core exists only when the core is unique and no other matching indirectly dominates the core. But this requires that one needs to restrict the preferences for the farsighted core to exist, and this severely limits its usefulness for general (strict) preferences.

Now we give the definition of a vNM farsightedly stable set put forth by Chwe (1994).

Definition 6. A set of matchings $V \subseteq \mathcal{M}$ is a vNM farsightedly stable set with respect to P if the following conditions are met:

(i) For all $\mu \in V$, there does not exist $\mu' \in V$ such that $\mu' \gg \mu$.

(ii) For all $\mu' \notin V$, there exists $\mu \in V$ such that $\mu \gg \mu'$.

Definition 6 introduces the notion of a vNM farsightedly stable set $V(\ll)$. Part (i) in Definition 6 is the internal stability condition: no matching inside the set is indirectly dominated by a matching belonging to the set. Part (ii) is the external stability condition: any matching outside the set is indirectly dominated by some matching belonging to the set.[9]

As shown in Greenberg (1990) and Xue (1998), the vNM farsightedly stable set assumes *optimistic behavior*. To see this, suppose that players behave consistently with indirect dominance and start bargaining, given a matching $\mu \in \mathcal{M}$ as the status quo. Let $\mathcal{M}(\mu) = \{\mu\} \cup \{\mu' \in \mathcal{M} \mid \mu' \gg \mu\}$ denote the set of all possible matchings that can be

[9] Diamantoudi and Xue (2007) extend the notion of the equilibrium binding agreement (EBA) (see Ray and Vohra 1997) with unrestricted coalitional deviations by using the vNM stable set with the indirect dominance relationship. They study whether the agents reach efficient agreements when they can negotiate openly and form coalitions. They show that, while the extended notion of the EBA facilitates the attainment of efficient agreements, inefficient agreements can arise, even if utility transfers are possible. However, no characterization of the vNM stable set with the indirect dominance relationship is provided.

reached by means of this bargaining process. Following Greenberg (1990), a standard of behavior σ is a mapping $\sigma(\cdot)$ that assigns to each status quo μ, a subset of $\mathcal{M}(\mu)$ as the set of possible agreement points. Now consider the stability condition on σ,

$$\mu' \in \sigma(\mu) \quad \Longleftrightarrow \quad \mu' \not\prec_S \mu'' \text{ for all } \mu'' \in \sigma(\widehat{\mu}) \text{ and } S \text{ can enforce } \widehat{\mu} \text{ over } \mu',$$

where $\mu' \not\prec_S \mu''$ means that there exists $i \in S$ such that $\mu' \succsim_i \mu''$.[10] Xue (1998) shows that the set $\bigcup_{\mu \in \mathcal{M}} \sigma(\mu)$ is a vNM farsightedly stable set $V(\ll)$. Indeed, a vNM farsightedly stable set can be interpreted as a set of matchings that are supported by an *optimistic* stable standard of behavior, while the largest consistent set can be seen as a *conservative* stable standard of behavior (see Chwe 1994). In both stable standards of behavior, when a coalition deviates, it deviates from a matching to another matching from which several other matchings might occur. The difference between the two solution concepts is that, in the vNM farsightedly stable set, coalitions deviate if some possible matching makes them better off, while in the largest consistent set, coalitions deviate only if all possible matchings make them better off. Chwe (1994) shows that the vNM farsightedly stable sets are contained in the largest consistent set. That is, if $V(\ll)$ is a vNM farsightedly stable set, then $V(\ll) \subseteq \Gamma(P)$.

We next reconsider the above examples to show that matchings outside the core, that belong either to the vNM stable set, $V(<)$, or to the largest consistent set, $\Gamma(P)$, do not survive the stability requirements imposed by introducing farsightedness into the concept of vNM stable sets.

Example 1 continued. Remember that $C(P) = \{\mu'\}$ and that $V(<) = \{\mu, \mu', \mu''\}$ is the unique vNM stable set (when using the notion of enforceability of Ehlers 2007). It is easy to verify that $\mu' \gg \mu$ and $\mu' \gg \mu''$. Let $\mu^0 = \mu$, $\mu^1 = \varnothing$ (all agents are single), $\mu^2 = \mu'$, $S^0 = \{w_1, w_2, w_3\}$, and $S^1 = N$. We have (i) $\mu^2 \succ \mu^0\ \forall i \in S^0$ and S^0 can enforce μ^1 over μ^0, and (ii) $\mu^2 \succ \mu^1\ \forall i \in S^1$ and S^1 can enforce μ^2 over μ^1. Thus, $\mu^2 \gg \mu^0$ or $\mu' \gg \mu$. Similarly, it is easy to verify that $\mu' \gg \mu''$. Hence, $\{\mu, \mu', \mu''\}$ cannot be a vNM farsightedly stable set; neither can $\{\mu, \mu'\}$ or $\{\mu', \mu''\}$ be a vNM farsightedly stable set since internal stability is violated. Moreover, μ does not indirectly dominate μ' and μ'' does not indirectly dominate μ'. This implies that the sets $\{\mu, \mu''\}$, $\{\mu\}$, or $\{\mu''\}$ cannot be vNM farsightedly stable sets, as they violate the external stability condition. In fact, $V(\ll) = \{\mu'\}$ is the unique vNM farsightedly stable set. $\diamondsuit$

Example 2 continued. Remember that μ^2 belongs to the largest consistent set but does not belong to the core. First, we show that $\{\mu^2\}$ cannot be a vNM farsightedly stable set since the external stability condition would be violated. Indeed, μ^2 does not indirectly dominate the matching μ^3 where men (m_1, m_2, m_3) are matched to (w_2, w_3, w_1). Second, we show that a set composed of μ^2 and other matching(s) cannot be a vNM farsightedly stable set. Tedious calculations show that there are only three other matchings that are not indirectly dominated by μ^2 and vice versa: these are the ones in which men (m_1, m_2, m_3) are matched to (m_1, w_3, w_1), (w_3, m_2, w_1), and (m_1, m_2, w_1), respectively.

[10] Note that σ is not necessarily unique.

These three matchings are the only ones that can belong to a vNM farsightedly stable set containing μ^2, but none of these indirectly dominates the matching μ^3, violating external stability. Thus, the largest consistent set may contain more matchings than those matchings that belong to the vNM farsightedly stable sets. ♢

4. Main results

4.1 *Characterization results*

From Definition 6, we have that for $V(\ll)$ to be a singleton vNM farsightedly stable set, only external stability needs to be verified. That is, the set $\{\mu\}$ is a vNM farsightedly stable set if and only if, for all $\mu' \neq \mu$, we have that $\mu \gg \mu'$.

To show our main results, we use Lemma 1, which shows that an individually rational matching μ indirectly dominates μ' if and only if there does not exist a pair $\{i, \mu'(i)\}$ that blocks μ. In other words, an individually rational matching μ does not indirectly dominate another matching μ' if and only if there exists a pair $\{i, \mu'(i)\}$ that blocks μ.

Lemma 1. *Consider any two matchings $\mu', \mu \in \mathcal{M}$ such that μ is individually rational. Then $\mu \gg \mu'$ if and only if there does not exist a pair $\{i, \mu'(i)\}$ such that both i and $\mu'(i)$ prefer μ' to μ.*

The proof of this lemma, as well as all other proofs, can be found in the Appendix.

Lemma 2. *Consider any two matchings $\mu', \mu \in \mathcal{M}$ such that μ' is individually rational. Then $\mu \gg \mu'$ implies that μ is also individually rational.*

The next theorem shows that every matching in the core is a vNM farsightedly stable set.

Theorem 1. *If μ is a matching in the core, $\mu \in C(P)$, then $\{\mu\}$ is a vNM farsightedly stable set, $\{\mu\} = V(\ll)$.*

Since μ is in the core, there is no pair of players matched in any other matching μ' such that they both prefer μ' to μ. Then Lemma 1 applies and μ indirectly dominates any other matching μ'. Thus, it follows that if $\mu \in C(P)$, then $\{\mu\}$ is a vNM farsightedly stable set.[11] But, a priori, there may be other vNM farsightedly stable sets of matchings. We now show that the only possible vNM farsightedly stable sets are singleton sets whose elements are in the core.

Theorem 2. *If $V(\ll) \subseteq \mathcal{M}$ is a vNM farsightedly stable set of matchings, then $V(\ll) = \{\mu\}$ with $\mu \in C(P)$.*

[11] Diamantoudi and Xue (2003) are the first to show that in hedonic games (of which marriage problems are a special case) with strict preferences, any partition belonging to the core indirectly dominates any other partition. Here, we provide an alternative proof of their result for one-to-one matching problems with strict preferences.

We here provide a sketch of the proof. Because of Theorem 1, it is clear that if $V(\ll)$ has more than one element and has a nonempty intersection with the core $C(P)$, then internal stability is violated, since any element of the core indirectly dominates any other matching. Second, if $V(\ll)=\{\mu\}$ with $\mu\notin C(P)$, then there exists a deviating coalition that can enforce a new matching in which all coalition members are better off. Then this new matching cannot be indirectly dominated by μ and hence external stability is violated. A third possibility is that $V(\ll)$ has more than one element and has an empty intersection with $C(P)$. In this case, we pick any element μ_1 of $V(\ll)$ and construct a deviation to a matching μ_1' such that no blocking pair of μ_1 blocks μ_1'. We first show that μ_1' always exists. Next, to satisfy external stability, there must be a $\mu_2\in V(\ll)$ such that $\mu_2\gg\mu_1'$. We then show that internal stability cannot be satisfied whenever $\mu_2\gg\mu_1'$: either $\mu_2\gg\mu_1$ or $\mu_1\gg\mu_2$. In particular, we prove first that if μ_2 does not contain any blocking pair of μ_1, then, by Lemma 1, either $\mu_1\gg\mu_2$ (if μ_1 is individually rational) or $\mu_2\gg\mu_1$ (if μ_1 is not individually rational). Second, if μ_2 does contain some blocking pair of μ_1 (and hence $\mu_1\not\gg\mu_2$), we prove that $\mu_2\gg\mu_1$, because μ_2 also indirectly dominates the matching that we obtain from μ_1 by means of the deviation of the blocking pairs that are still matched at μ_2. Therefore, we can conclude that there does not exist a vNM farsightedly stable set $V(\ll)$ containing more than one matching and satisfying $V(\ll)\cap C(P)=\varnothing$. We refer the reader to the Appendix for a detailed proof. Hence, while Ehlers (2007) shows that the set of matchings in the core is a subset of vNM (myopically) stable sets, vNM farsightedly stable sets cannot include matchings that are not in the core.[12]

Notice that our results also hold if only pairwise deviations are allowed in the definition of indirect dominance. Indeed, as mentioned before, the fact that coalition $S\subseteq N$ can enforce a matching μ' over μ implies that there exists a sequence of matchings $\mu^0,\mu^1,\dots,\mu^K$ (where $\mu^0=\mu$ and $\mu^K=\mu'$) and a sequence of disjoint pairs $\{m_0,w_0\},\dots,\{m_{K-1},w_{K-1}\}$ (possibly for some $k\in\{0,\dots,K-1\}$, $m_k=w_k$) such that for any $k\in\{1,\dots,K\}$, the pair $\{m_{k-1},w_{k-1}\}$ can enforce the matching μ^k over μ^{k-1}. Hence, any deviation of a blocking coalition S from μ to μ' can be replaced by a sequence of pairwise deviations of the disjoint blocking pairs contained in S.

4.2 *Discussion*

Konishi and Ray (2003) study a model of dynamic coalition formation where individuals are farsighted and evaluate the desirability of a move in terms of its consequences on the entire discounted stream of payoffs. Contrary to ours, their model is in spirit closer to non-cooperative game theory. They model the formation of coalitions by means of an intertemporal Markovian process of coalition formation (PCF). A *PCF is an equilibrium* (EPCF) if a coalitional move from one state to another, as specified by the PCF, yields

[12]The notion of bargaining set for one-to-one matching problems defined by Klijn and Massó (2003) is a first attempt to reflect the idea that agents are not myopic. The bargaining set is the set of matchings that have no justified objection. Klijn and Massó show that the set of core stable matchings is a subset of the bargaining set. So, contrary to the vNM farsightedly stable set, the bargaining set can contain matchings outside the core.

its members higher discounted future payoffs instead of remaining inactive under the ongoing state. Thus, Konishi and Ray (2003) introduce farsightedness by letting the importance of future payoffs, according to some PCF, vary. In the limit, when the discount factor approaches 1, agents only care about the payoffs received in the (set of) state(s) an EPCF converges to. Note that one can consider the set of absorbing states of a deterministic EPCF as a stable set for that EPCF: once in an absorbing state, an EPCF stays there (internal stability) and for each state that is not absorbing, there exists a path to some absorbing state (external stability).

Konishi and Ray show that, for a class of games that can be characterized by a characteristic function, the class of deterministic[13] EPCFs with a unique limit state characterizes the core, provided the discount factor approaches 1. When the discount factor is high enough, (i) for any element of the core, there exists a deterministic EPCF with that core element as its unique limit (Theorem 4.1); (ii) any deterministic EPCF with a unique limit must be such that the limit belongs to the core (Theorem 4.2).

We argue that the idea of an EPCF with a set of absorbing states when discounting vanishes is closely related to the idea of farsightedly stable sets. In fact, indirect dominance can be considered to be the limit of such a process of coalition formation in which people care only about the end state. In particular, if an absorbing state of an EPCF is reachable from a given matching μ for a range of discount factors close to 1, then the absorbing state indirectly dominates matching μ. We can now relate our results to those of Konishi and Ray (2003). Our Theorem 1 is very similar to Theorem 4.1 in Konishi and Ray (2003): any core matching is a singleton farsightedly stable set or, in the terms of Konishi and Ray (2003), there exists an EPCF that has this core matching as its unique limit when the discount factor approaches 1. Theorem 4.2 provides the converse: if there is a deterministic EPCF with a unique limit when the discount factor approaches 1, then this limit must belong to the core. In our setting, this is stated as follows: if there is a matching that indirectly dominates all other matchings, and hence forms a singleton vNM farsightedly stable set, then this matching must be in the core. But our Theorem 2 says more. The fact that a singleton vNM farsightedly stable set must be a core element is not enough to characterize all farsightedly stable sets in the matching problem. Our Theorem 2 says that there can be no other vNM farsightedly stable sets. Their Theorems 4.1 and 4.2 remain silent about other (deterministic) EPCF when the discount factor approaches 1. Our Theorem 2 may prove useful in understanding and characterizing all EPCF of the matching problem when discounting vanishes.[14]

[13]Konishi and Ray (2003) define a PCF to be deterministic if, for all states, the probability of moving from one state to another is degenerate. A state is absorbing if the probability of moving to any other state is equal to zero. A PCF is absorbing if, for each state, there is a positive probability to move, after a finite amount of steps, to an absorbing state. A PCF has a unique limit if it is absorbing and possesses a single absorbing state.

[14]Roth and Vande Vate (1990) demonstrate that, starting from an arbitrary matching, the process of allowing randomly chosen blocking pairs to match converges to a matching in the core with probability 1 in the marriage problem. Relative to this, Jackson and Watts (2002) show that if preferences are strict, then the set of stochastically stable matchings coincides with the core. But both papers propose a dynamic process in which blocking pairs form and disappear based on the improvement that the resulting matching offers relative to the current matching.

We show that the vNM farsightedly stable set, the farsighted core, and the largest consistent set can lead to very different conclusions in one-to-one matching problems. We characterize the vNM farsightedly stable sets to be singleton sets that contain a core element. One interesting question is whether one can find restrictions on the preferences such that all the above concepts agree. Note that they coincide only if the core has a unique element. One popular restriction that guarantees uniqueness of the core is the *top-coalition property,* introduced by Banerjee et al. (2001).

Given a nonempty set of agents $T \subseteq N$, a coalition $S \subseteq T$ is a *top coalition* of T if, for every $i \in S$, he or she is matched to his or her most preferred partner in T. A matching problem (M, W, P) satisfies the top-coalition property if, for any nonempty set of players $T \subseteq N$, there exists a top coalition of T. This implies that for any group of agents who can match, there are always two agents that top rank each other (or prefer to remain single). Banerjee et al. (2001) show that when this condition is satisfied, the core has a unique element that is equal to the top coalition partition $\mu^* = \{S_1, S_2, \dots, S_k\}$ with $S_i = (m_i, \mu^*(m_i))$ for all $i \in M$ such that coalition S_1 is a top coalition of N, coalition S_2 is a top coalition of $N \setminus S_1, \dots,$ and coalition S_k is a top coalition of $N \setminus \bigcup_{j<k} S_j$ and $N \setminus \bigcup_{j\leq k} S_j = \varnothing$. Note that if m_i is single in μ^*, then $\mu^*(m_i) = m_i$.

We now introduce Lemma 3 to show that when preferences satisfy the top-coalition property, the vNM farsightedly stable set, the farsighted core, and the largest consistent set coincide with the core. Lemma 3 asserts that the top-coalition partition is not indirectly dominated by any other matching.

Lemma 3. *If (M, W, P) satisfies the top-coalition property, then $\nexists\ \mu \in \mathcal{M}$ such that $\mu \gg \mu^*$, where μ^* is the top coalition partition.*

From Lemma 3, we immediately obtain the following corollary.

Corollary 1. *If (M, W, P) satisfies the top-coalition property and μ^* is the unique core element, then $\{\mu^*\}$ is the unique vNM farsightedly stable set, the farsighted core, and the largest consistent set.*

The next example shows that the vNM stable set with the direct dominance relation based on the notion of enforceability of Definition 1 can contain elements outside the core even when the preferences satisfy the top-coalition property. Thus, the top-coalition property is not enough to guarantee that direct and indirect dominance coincide.

Example 3. Let $M = \{m_1, m_2\}$ and $W = \{w_1, w_2\}$. Let P be such that

$P(m_1)$	$P(m_2)$	$P(w_1)$	$P(w_2)$
w_1	w_1	m_1	m_2
w_2	w_2	m_2	m_1
m_1	m_2	w_1	w_2

Let

$$\mu^* = \begin{pmatrix} m_1 & m_2 \\ w_1 & w_2 \end{pmatrix}, \qquad \mu' = \begin{pmatrix} m_1 & m_2 \\ w_2 & w_1 \end{pmatrix}.$$

Note that μ^* is the *unique* element in the core of this matching problem. But $\mu^* \not> \mu'$, since m_2 would block the deviation of the grand coalition from μ' to μ^*. Hence, $\mu' \in V(<)$. In fact, $V(<) = \{\mu^*, \mu'\}$. $\Diamond$

However, notice that μ^* is the unique element of the vNM stable set if we use the Ehlers' (2007) notion of enforceability, because then the coalition $S = \{m_1, w_1\}$ can enforce the matching μ^* from μ', with $\mu^* \succ_i \mu'$ $\forall i \in S$. Indeed, it can be shown that if (M, W, P) satisfies the top-coalition property, then the vNM stable set with the direct dominance relation based on the Ehlers' (2007) notion of enforceability coincides with the core. But, in general, the core is a subset of any vNM stable set (see Theorem 2 of Ehlers 2007).

5. Many-to-one matching problems

A many-to-one matching problem consists of a set of N agents divided into a set of hospitals, $H = \{h_1, \ldots, h_r\}$, and a set of medical students, $I = \{i_1, \ldots, i_s\}$, where possibly $r \neq s$. Each hospital $h \in H$ has a strict, transitive, and complete preference relation $P(h)$ over the set of all subsets of I, including the empty set, which represents the prospect of having all its positions unfilled. Each medical student $i \in I$ has a strict, transitive, and complete preference relation $P(i)$ over $H \cup \{i\}$. Preferences profiles are $(r+s)$-tuples of preference relations; we denote them by $P = (P(h_1), \ldots, P(h_r); P(i_1), \ldots, P(i_s))$.

Given a hospital's preference relation, $P(h)$, the sets of medical students that h prefers to the empty set are called acceptable; thus we allow that hospital h may prefer not to enroll any medical student rather than to enroll unacceptable subsets of students. Similarly, given a medical student's preference relation $P(i)$, the hospitals preferred by i to the possibility of being unemployed are called acceptable; in this case, we are allowing that student i may prefer to remain unemployed rather than to work for an unacceptable hospital. It turns out that only acceptable partners matter, so we write preference relation concisely as lists of acceptable partners. For example, $P(h_i) = \{i_1, i_3\}, \{i_2\}, \{i_1\}, \{i_3\}$ indicates that $\{i_1, i_3\} \succ_{h_i} \{i_2\} \succ_{h_i} \{i_1\} \succ_{h_i} \{i_3\} \succ_{h_i} \varnothing$ and $P(i_j) = h_2, h_1, i_j$ indicates that $h_2 \succ_{i_j} h_1 \succ_{i_j} i_j$. We denote by R the weak orders associated with P. So $h_i \succeq_i h_j$ if $h_i \sim_i h_j$ or $h_i \succ_i h_j$, and similarly for $R(h)$. A many-to-one matching problem is simply (H, I, P).

A matching μ is a mapping from the set $H \cup I$ into the set of all subsets of $H \cup I$ such that for all $i \in I$ and $h \in H$, (i) either $|\mu(i)| = 1$ and $\mu(i) \subseteq H$ or else $\mu(i) = i$, (ii) $\mu(h) \in 2^I$, and (iii) $\mu(i) = \{h\}$ if and only if $i \in \mu(h)$. Given (H, I, P), we denote by $\mathcal{M}$ the set of all matchings. Let P be a preference profile. Given a set $S \subseteq I$, let $\mathrm{Ch}(S, P(h))$ denote hospital h's most preferred subset of S according to its preference ordering $P(h)$. We call $\mathrm{Ch}(S, P(h))$ the choice set of S according to $P(h)$. That is, $S' = \mathrm{Ch}(S, P(h))$ if and only if $S' \subseteq S$ and $S' \succ_h S''$ for all $S'' \subseteq S$ with $S' \neq S''$.

Definition 7. A hospital h's preference ordering $P(h)$ satisfies substitutability if for any set $S \subseteq I$ containing students i and i' $(i \neq i')$, if $i \in \mathrm{Ch}(S, P(h))$, then $i \in \mathrm{Ch}(S \setminus \{i'\}, P(h))$. A preference profile P is substitutable if, for each hospital h, the preference ordering $P(h)$ satisfies substitutability.

That is, if h has *substitutable* preferences, then if its preferred set of students from S includes i, so does its preferred set of students from any subset of S that still includes i. We assume that hospitals' preferences satisfy the property of *substitutability*.[15]

Let P be a preference profile and let μ be a matching. We say that μ is *individually rational* if $\mu(i) \succeq_i i$ for all $i \in I$ and if $\mu(h) = \mathrm{Ch}(\mu(h), P(h))$ for all $h \in H$. That is, μ is individually rational if no agent can unilaterally improve over its assignment in μ (students by choosing to remain unemployed and hospitals by firing some of their students). A student–hospital pair (i, h) *blocks* μ if $i \notin \mu(h)$, $i \in \mathrm{Ch}(\mu(h) \cup \{i\}, P(h))$, and $h \succ_i \mu(i)$; i.e., if i and h are not matched through μ, hospital h wants to enroll i (possibly after firing some of its current students in $\mu(h)$) and student i prefers hospital h over her current match $\mu(i)$. A pair $(S, h) \in 2^I \times H$ *blocks** μ if $h \succ_i \mu(i)$ for all $i \in S$ and there is $S' \subseteq \mu(h)$ such that $[S' \cup S] \succ_h \mu(h)$. In words, (S, h) blocks* μ if hospital h is willing to enroll the students in S (possibly after firing some of its current students in $\mu(h)$) and all students i in S prefer h over their current match $\mu(i)$. This notion of blocking was used by Echenique and Oviedo (2004).

Definition 8. A matching μ is stable if it is individually rational and there is no student–hospital pair that blocks μ. A matching μ is stable* if it is individually rational and there is no pair (S, h) that blocks* μ.

Given a preference profile P, we denote the set of stable matchings by $\Sigma(P)$ and denote the set of stable* matchings by $\Sigma^*(P)$. Echenique and Oviedo (2004) show that $\Sigma^*(P) \subseteq \Sigma(P)$. Given a preference profile P, the *strong core* is the set of matchings μ for which there is no $H' \subseteq H$, $I' \subseteq I$ with $H' \cup I' \neq \varnothing$, and $\mu' \in \mathcal{M}$ such that (i) for all $i \in I'$ and $h \in H'$, $\mu'(i) \in H'$ and $\mu'(h) \subseteq I'$; (ii) for all $i \in I'$ and $h \in H'$, $\mu'(i) \succeq_i \mu(i)$ and $\mu'(h) \succeq_h \mu(h)$; and (iii) there is $j \in H' \cup I'$ with $\mu'(j) \succ_j \mu(j)$. We denote by $C_w(P)$ the strong core. According to this definition, members of the deviating coalition need only to be weakly better off and one member needs to be strictly better off.[16] Echenique and Oviedo (2004) show that the set of stable* matchings equals the strong core: $\Sigma^*(P) = C_w(P)$. However, the strong core may not coincide with the core (see Roth and Sotomayor 1990).[17]

Let $\mathcal{S}^{\mu(h)}$ denote the power set of the set $\mu(h)$. We now adapt the definition of enforceability to many-to-one matching problems.

[15] A hospital's preferences over group of students are *responsive* if, for any two assignments that differ in only one student, it prefers the assignment containing the more preferred student. Note that responsive preferences have the substitutability property.

[16] On the contrary, in the definition of the core, all members of the deviating coalition should be strictly better off.

[17] Roth and Sotomayor (1990) show that if hospitals' preferences are substitutable, then the set of stable matchings, $\Sigma(P)$, equals the strong core. In one-to-one matching problems with strict preferences, the set of stable matchings coincides with the core, which is equal to the strong core.

Definition 9. Given a matching μ, a coalition $S \subseteq N$ is said to be able to enforce a matching μ' over μ if the following conditions hold: (i) $\mu'(h) \notin \mathcal{S}^{\mu(h)} \cup \{h\}$ implies $\mu'(h) \setminus \mu(h) \cup \{h\} \subset S$ and (ii) $\mu'(h) \in \mathcal{S}^{\mu(h)} \cup \{h\}$, $\mu'(h) \neq \mu(h)$, implies either h or $\mu(h) \setminus \mu'(h)$ or h together with a nonempty subset of $\mu(h) \setminus \mu'(h)$ should be in S.

Condition (i) says that any new match in μ' that contains different partners than in μ should be such that h and the different partners of h belong to S. Condition (ii) states that so as to leave some (or all) positions of one existing match in μ unfilled, either h or the students leaving such positions or h and some nonempty subset of such students should be in S.[18]

We now provide a condition that characterizes indirect dominance in many-to-one matching problems. Similarly to one-to-one matching problems, we have that an individually rational matching μ indirectly dominates μ' if and only if there does not exist a pair $(S, h) \in 2^I \times H$, with $S \subseteq \mu'(h)$, that *blocks** μ. In other words, an individually rational matching μ does not indirectly dominate another matching μ' if and only if there exists a pair (S, h) that *blocks** μ.

Lemma 4. *Consider any two individually rational matchings $\mu', \mu \in \mathcal{M}$. Then $\mu \gg \mu'$ if and only if there does not exist a pair $(S, h) \in 2^I \times H$, with $S \subseteq \mu'(h)$, that blocks* μ; i.e., such that $h \succ_i \mu(i)$ for all $i \in S$ and there is $S' \subseteq \mu(h)$ such that $[S' \cup S] \succ_h \mu(h)$.*

In the following discussion, we extend our characterization of the vNM farsightedly stable set for one-to-one matching problems to many-to-one matching problems with substitutable preferences. Indeed, when preferences are substitutable, Roth and Sotomayor (1990) show that the set of stable matchings (that coincides with the strong core and with the set of stable* matchings) is always nonempty. We now show that the only possible vNM farsightedly stable sets are singleton sets whose elements are the stable* matchings.

Theorem 3. *In a many-to-one matching problem with substitutable preferences, a set of matchings is a vNM farsightedly stable set if and only if it is a singleton set and its element belongs to the strong core $C_w(P)$.*

The proof of Theorem 3 follows the proof of Theorem 2 but replacing now Lemma 1 by Lemma 4 and proving that μ_1' always exists. Thus, our characterization of the vNM farsightedly stable set for one-to-one matching problems extends to many-to-one matching problems with substitutable preferences. This result contrasts with Ehlers (2007), who shows that there need not be any relationship between the vNM stable sets of a many-to-one matching problem with responsive preferences and its associated

[18] Roth and Sotomayor (1990) use another definition of enforceability so as to define when a matching weakly dominates another matching. The only difference is that Condition (i) in Roth and Sotomayor (1990) says that any new match in μ' that contains different partners than in μ should be such that h and the partners of h in μ' (that is, $\mu'(h)$ instead $\mu'(h) \setminus \mu(h)$) belong to S. Since in the definition of indirect dominance, we impose that all the deviators are strictly better off, here we require only the different partners of h to belong to S.

one-to-one matching problem. We also show that if there is a matching of a many-to-one matching problem with substitutable preferences that is in the core but not in the strong core, then this matching is never a vNM farsightedly stable set.

6. Conclusion

We characterize the vNM farsightedly stable sets in one-to-one matching problems: a set of matchings is a vNM farsightedly stable set if and only if it is a singleton set and its element is in the core. Thus, we provide an alternative characterization of the core in one-to-one matching problems. Finally, we show that our main result is robust to many-to-one matching problems with substitutable preferences: a set of matchings is a vNM farsightedly stable set if and only if it is a singleton set and its element is in the strong core.

Appendix

Proof of Lemma 1. Let $B(\mu', \mu)$ be the set of men and women who are strictly better off in μ than in μ'. Accordingly, let $I(\mu', \mu)$ and $W(\mu', \mu)$ be the set of men and women who are indifferent between μ and μ', and worse off in μ than in μ', respectively.

($\Rightarrow$) Assume, to the contrary, that $\mu \gg \mu'$ and that there is a pair $\{i, \mu'(i)\}$ such that both prefer μ' to μ. For μ to indirectly dominate μ', it must be that i or $\mu'(i)$ get divorced along the path from μ' to μ. But both i and $\mu'(i)$ belong to $W(\mu', \mu)$, and then they never divorce. Hence $\mu \not\gg \mu'$, a contradiction.

($\Leftarrow$) We prove $\Leftarrow$ by showing that $\mu \gg \mu'$ if the above condition is satisfied. Assume that for all pairs $\{i, \mu'(i)\}$ such that $\mu'(i) \neq \mu(i)$, either i or $\mu'(i)$ or both belong to $B(\mu', \mu)$. Notice that every agent i single in μ' that accepts a match with someone else in μ also belongs to $B(\mu', \mu)$ since μ is individually rational. Next construct the following sequence of matchings from μ' to μ: μ^0, μ^1, μ^2 (where $\mu^0 = \mu'$, $\mu^1 = \{\mu^1(i) = i, \mu^1(\mu'(i)) = \mu'(i)$ for all $i \in B(\mu', \mu)$, and $\mu^1(j) = \mu'(j)$ otherwise$\}$, and $\mu^2 = \mu$). Also construct the following sequence of coalitions S^0, S^1 with $S^0 = B(\mu', \mu)$ and $S^1 = B(\mu', \mu) \cup \{\mu(i)$ for $i \in B(\mu', \mu)\}$. Then coalition S^0 can enforce μ^1 over μ^0 and coalition S^1 can enforce μ^2 over μ^1. Moreover, $\mu^2 \succ \mu^0$ for S^0 and $\mu^2 \succ \mu^1$ for S^1 because every mate of $i \in B(\mu', \mu)$ in μ^2 (in μ) also prefers his or her mate in μ^2 to being single in μ^1. Indeed, for every $i \in B(\mu', \mu)$, either $\mu^2(i) \in B(\mu', \mu)$, and hence both prefer μ^2 to μ^1, or $\mu^2(i) \in W(\mu', \mu)$. In this last case, $\mu^2(i)$ must have lost his or her mate in μ^0 and $\mu^0(\mu^2(i))$ must belong to $B(\mu', \mu)$ since otherwise $\mu^0(\mu^2(i))$ and $\mu^2(i)$ would form a blocking pair of μ^2, and this, by assumption, is not possible. Hence $\mu^2(i)$ must be single in μ^1. Then since μ^2 is individually rational, $\mu^2(i)$ must prefer accepting his or her mate in μ^2 to remaining single at μ^1. So, we have that $\mu \gg \mu'$. $\square$

Proof of Lemma 2. Suppose not. Then there exists $i \in N$ that prefers to be single than to be married to $\mu(i)$ in μ. Since $\mu \gg \mu'$ and μ' is individually rational, we have that i was either single at μ' or matched to $\mu'(i) \succ_i i$. But then in the sequence of moves between μ' and μ, the first time i has to move she/he was either matched with $\mu'(i)$ or single and, hence, i cannot belong to a coalition S^{k-1} that can enforce the matching μ^k over μ^{k-1} and such that all members of S^{k-1} prefer μ to μ^{k-1}, contradicting the fact that $\mu \gg \mu'$. □

Proof of Theorem 1. We only need to verify condition (ii) in Definition 6: for all $\mu' \neq \mu$, we have that $\mu \gg \mu'$. Since $\mu \in C(P)$, we know that $\forall \mu' \neq \mu$, $\nexists\, i \in M$ and $j \in W$ such that $\mu'(i) = j$ and $\mu' \succ \mu$ for both i and j. Since μ is individually rational, we have from Lemma 1 that $\mu \gg \mu'$. □

Proof of Theorem 2. Notice that if $V(\ll) \subseteq C(P)$, then $V(\ll)$ is a vNM farsightedly stable set only if $V(\ll)$ is a singleton set $\{\mu\}$ with $\mu \in C(P)$. From Theorem 1, we know that for all $\mu' \neq \mu$, $\mu \gg \mu'$. Suppose now that $V(\ll) \subsetneq C(P)$. Then, either $V(\ll) \cap C(P) \neq \varnothing$ or $V(\ll) \cap C(P) = \varnothing$.

Suppose first that $V(\ll) \cap C(P) \neq \varnothing$. Let $\mu \in V(\ll) \cap C(P)$, and $\mu' \in V(\ll)$ with $\mu' \notin C(P)$. Then, by Theorem 1, we have that $\mu \gg \mu'$, violating the internal stability condition.

Suppose now that $V(\ll) \cap C(P) = \varnothing$. Then we show that $V(\ll)$ is not a vNM farsightedly stable set because either the internal stability condition (condition (i) in Definition 6) or the external stability condition (condition (ii) in Definition 6) is violated.

Assume first that $V(\ll) = \{\mu\}$ is a singleton. Since $\mu \notin C(P)$, there exists a deviating coalition S in μ and a matching $\mu' \in \mathcal{M}$ such that $\mu' \succ_i \mu$ for all $i \in S$ and S can enforce μ' over μ. Then $\mu \not\gg \mu'$ and the external stability condition is violated.

Assume now that $V(\ll)$ contains more than one matching that does not belong to $C(P)$. Take any matching $\mu_1 \in V(\ll)$. Since $\mu_1 \notin C(P)$, there exists at least a pair of agents $\{i, j\}$ such that $\mu_1(j) \neq i$ (or a single agent $\{i\}$) and a matching $\mu_1' \in \mathcal{M}$ such that $\mu_1' \succ \mu_1$ for both i and j (or $\mu_1' \succ \mu_1$ for i), and $\{i, j\}$ (or i) can enforce μ_1' over μ_1, i.e., such that $\mu_1'(j) = i$ (or $\mu_1'(i) = i$). Let $S(\mu_1)$ be the set of blocking pairs of μ_1. Consider the deviation from μ_1 to μ_1' of the subset of blocking pairs $S'(\mu_1) \subseteq S(\mu_1)$, where $S'(\mu_1)$ contains the maximum number of blocking pairs and is such that the subset $S(\mu_1) \setminus S'(\mu_1)$ does not contain any blocking pair of μ_1'. We first prove that such a matching always exists.

Claim 1. *For any matching $\mu_1 \notin C(P)$, there always exists a matching $\mu_1' \in \mathcal{M}$ that directly dominates μ_1 and that can be enforced over μ_1 by the blocking pairs $S'(\mu_1)$, and such that μ_1' is not blocked by any pair in $S(\mu_1) \setminus S'(\mu_1)$.*

Proof. We prove the existence of μ_1' by construction. Define M' to be the set of men who belong to some blocking pair of μ: $M' = \{i \in M \mid \exists j \in W \cup \{i\}$ such that $\{i, j\} \in S(\mu_1)\}$. Equally, define W' to be the set of women who belong to some blocking pair of μ_1: $W' = \{j \in W \mid \exists i \in M \cup \{j\}$ such that $\{i, j\} \in S(\mu_1)\}$.

Consider the following restricted matching problem in which each agent only ranks those agents with whom he or she can form a deviating blocking pair. That is, the preferences of each $i \in M'$ are only over the set $W_i' = \{j \in W' \cup \{i\}$ such that $\{i, j\} \in S(\mu_1)\}$.

For each $j \in W'$, her preferences are restricted to the set $M'_j = \{i \in M' \cup \{j\}$ such that $\{i, j\} \in S(\mu_1)\}$. Let $P'(\mu_1)$ denote these restricted preferences. Then the matching problem $\{M', W', P'(\mu_1)\}$ has at least one stable matching (since it is a marriage market), call it μ'.

We define μ'_1 and $S'(\mu_1)$ as follows. Consider first the agents in M' and W'. Every pair $\{i, \mu'(i)\}$ in μ' (possibly $i = \mu'(i)$) such that both prefer their partner in μ' to their partner in μ_1 belongs to $S'(\mu_1)$ and, hence, both i and $\mu'(i)$ move from μ_1 to μ'_1 becoming a couple (as in μ'). Every single agent at μ' preferring being married at μ_1 rather than being single at μ' does not belong to $S'(\mu_1)$, but to $S(\mu_1) \setminus S'(\mu_1)$. Every pair $\{i, \mu_1(i)\}$ in μ_1 with $\mu_1(i) = \mu'(i)$ is such that i (and/or $\mu_1(i)$) belongs to $S(\mu_1) \setminus S'(\mu_1)$ when i (and/or $\mu_1(i)$) belongs to M' (belongs to W'). Consider now all agents who do not belong to either M' or W'. They do not belong to a pair of $S'(\mu_1)$ (they do not move themselves, although they can lose their match in the move from μ_1 to μ'_1 if they were initially matched to some of the deviating players in some of the pairs of $S'(\mu_1)$). Clearly, the subset $S(\mu_1) \setminus S'(\mu_1)$ does not contain any blocking pair of μ'_1, because otherwise μ' would not be a stable matching for the restricted matching problem $\{M', W', P'(\mu_1)\}$. Since the set of single agents in any stable matching is always the same, then $S'(\mu_1)$ contains the maximum possible number of blocking pairs such that $S(\mu_1) \setminus S'(\mu_1)$ does not contain any blocking pair of μ'_1. $\triangleleft$

Now, for $V(\ll)$ to be a vNM farsightedly stable set, we need the following conditions to be satisfied:

(i) For any other matching $\mu_2 \in V(\ll)$, $\mu_2 \neq \mu_1$, it should be that $\mu_1 \not\gg \mu_2$ and $\mu_2 \not\gg \mu_1$.

(ii) For all $\mu' \notin V(\ll)$, there should exist $\mu \in V(\ll)$ such that $\mu \gg \mu'$ (in particular, we need that there exists a matching $\mu_2 \in V(\ll)$ such that $\mu_2 \gg \mu'_1$ for each matching, like μ'_1, that can be enforced by any subset of blocking pairs of any matching in $V(\ll)$).

We show that a $V(\ll)$ containing more than one matching, none of them in $C(P)$, is not a vNM farsightedly stable set because one of the above conditions is not satisfied.

Let $\mu_1 \in V(\ll)$. We know by Claim 1 that there exists a matching μ'_1 that can be enforced from μ_1 by the blocking pairs in $S'(\mu_1)$ and such that $\mu'_1 > \mu_1$ (and $\mu_1 \not\gg \mu'_1$). Notice that $\mu'_1 \notin V(\ll)$. By condition (ii) in Definition 6, there exists a matching $\mu_2 \in V(\ll)$ such that $\mu_2 \gg \mu'_1$. Condition (i) in Definition 6 implies that $\mu_1 \not\gg \mu_2$ and $\mu_2 \not\gg \mu_1$. We prove that condition (i) is violated. Two cases should be considered.

1. Assume that μ_2 does not contain any blocking pair of μ_1; i.e., $\nexists\{i, \mu_2(i)\}$ such that $\{i, \mu_2(i)\} \in S(\mu_1)$. If μ_1 is individually rational, we have by Lemma 1 that $\mu_1 \gg \mu_2$, violating condition (i) in Definition 6. Otherwise, if μ_1 is not individually rational, consider the deviation from μ_1 to μ''_1, where any agent i who prefers being single to being married to $\mu_1(i)$, divorces from $\mu_1(i)$, while the other agents do not move. Then $\mu''_1 > \mu_1$ (and $\mu_1 \not\gg \mu''_1$). By condition (ii) in Definition 6, there exists a matching $\mu_2 \in V(\ll)$ such that $\mu_2 \gg \mu''_1$. But then we also have that $\mu_2 \gg \mu_1$, since the

agents who divorce from μ_1 to μ_1'' never marry someone else and become worse off than being single. Hence, the internal stability condition is violated and $V(\ll)$ is not a vNM farsightedly stable set. So $V(\ll)$ cannot contain nonindividually rational matchings.

2. Assume that μ_2 contains some blocking pair(s) of μ_1; that is, $\exists\{i, \mu_2(i)\}$ such that $\{i, \mu_2(i)\} \in S(\mu_1)$. Notice that, in this case, we have by Lemma 1 that $\mu_1 \not\gg \mu_2$. Let $S''(\mu_1) \subseteq S(\mu_1)$ be the set of blocking pairs of μ_1 that are still matched in μ_2. Consider the deviation from μ_1 to μ_1'', where only the blocking pairs $\{i, j\} \in S''(\mu_1) \subseteq S(\mu_1)$ get married. Then $\mu_1'' > \mu_1$ (and $\mu_1 \not\gg \mu_1''$) and $\mu_1' > \mu_1''$ (if $\mu_1'' \neq \mu_1'$). Since $\mu_2 \gg \mu_1'$, we also have that $\mu_2 \gg \mu_1''$, because the only difference between μ_1' and μ_1'' is that at μ_1', the rest of the blocking pairs of μ_1 (who are not still matched at μ_1'' and who divorce from μ_1' to μ_2) get married. Hence, μ_1'' does not contain any blocking pair of μ_2. Then, since $\mu_2 \gg \mu_1''$, we also have that $\mu_2 \gg \mu_1$, violating condition (i) in Definition 6. □

Proof of Lemma 3. Suppose to the contrary that there exists a matching μ with $\mu \neq \mu^*$ and such that $\mu \gg \mu^*$. Take the lowest j such that S_j is the top coalition of $N \setminus \bigcup_{l<j} S_l$ and m_j is better off in μ. Then m_j must be matched in μ to $\mu^*(m_l)$ for some $l < j$. Otherwise S_j would not be a top coalition of $N \setminus \bigcup_{l<j} S_l$. Moreover, $\mu^*(m_l)$ must be worse off in μ compared to μ^*, because, otherwise, $(m_j, \mu^*(m_l))$ would be a blocking pair of μ^*. But then m_l must be better off in μ, because, otherwise, $(m_l, \mu^*(m_l)) = S_l$ blocks μ. But then m_l is better off in μ and $l < j$, a contradiction. □

Proof of Lemma 4. ($\Rightarrow$) Assume to the contrary that $\mu \gg \mu'$ and that there exists a pair $(S, h) \in 2^I \times H$ that blocks* μ. That is, the pair (S, h) is such that $h \succ_i \mu(i)$ for all $i \in S$, $S \subseteq \mu'(h)$, and there is $S' \subseteq \mu(h)$ ($S' \subseteq \mu'(h)$) such that $[S' \cup S] \succ_h \mu(h)$. At no step along the path between μ' and μ does any $i \in [S' \cup S]$ leave h. So, along the path between μ' and μ, hospital h must at some point get rid of any $i \in S$. Since μ' is individually rational and $[S' \cup S] \succ_h \mu(h)$, then $\mu'(h) \succ_h \mu(h)$ and h never initiates a move at μ' so as to go to μ. Hence, some or all of the students in $\mu'(h) \setminus [S' \cup S]$ who prefer μ to μ' leave h. Since μ' is individually rational, any intermediate matching obtained once some students in $\mu'(h) \setminus [S' \cup S]$ leave h between μ' and the matching in which h is only matched to $[S' \cup S]$, are all preferred by h to this last matching in which h is matched to $[S' \cup S]$. So at any step along the path between μ' and the matching in which h is only matched to $[S' \cup S]$, h is in a better position compared to μ. But then h never has an incentive to get rid of any $i \in S$. Hence $\mu \not\gg \mu'$, a contradiction.

($\Leftarrow$) We prove $\Leftarrow$ by construction. In the first step, let anyone (student or hospital) get rid of all matches in μ' if they are better off at μ. After this step, only hospitals that are (weakly) worse off at μ compared to μ' may still have some students they are matched to (called it S_h with $S_h \subseteq \mu'(h)$ for some h). In the second step, let these hospitals get rid of all their matches (all $i \in S_h$). They want to do so, since, by assumption, they are better off at μ compared to being matched only to S_h. After the second step, everyone is alone. In the third step, allow all matches necessary to obtain μ. This is possible since μ is individually rational. $\square$

Proof of Theorem 3. First, we prove that (i) if μ is in the strong core, $\mu \in C_w(P)$, then $\{\mu\}$ is a vNM farsightedly stable set, $\{\mu\} = V(\ll)$. Second, we prove that (ii) if $V(\ll) \subseteq \mathcal{M}$ is a vNM farsightedly stable set of matchings, then $V(\ll) = \{\mu\}$ with $\mu \in C_w(P)$.

(i) We only need to verify condition (ii) in Definition 6: for all $\mu' \neq \mu$, we have that $\mu \gg \mu'$. Since $\mu \in C_w(P) = \Sigma^*(P)$, we know that $\forall \mu' \neq \mu$, there does not exist a pair $(S, h) \in 2^I \times H$, with $S \subseteq \mu'(h)$, such that $h \succ_i \mu(i)$ for all $i \in S$, and there is $S' \subseteq \mu(h)$ ($S' \subseteq \mu'(h)$) such that $[S' \cup S] \succ_h \mu(h)$. Since μ is individually rational, we have from Lemma 4 that $\mu \gg \mu'$.

(ii) The proof runs exactly along the same lines as the proof of Theorem 2 by simply proving that $V(\ll)$ cannot contain only matchings that do not belong to the strong core.

Indeed, assume now that $V(\ll)$ contains more than one matching[19] that does not belong to the strong core $C_w(P)$. Take any matching $\mu_1 \in V(\ll)$, where $\mu_1 \notin C_w(P)$. Let $S(\mu_1)$ be the set of blocking* pairs of μ_1. That is, $S(\mu_1)$ contains the pairs $(S, h) \in 2^I \times H$, such that $h \succ_i \mu_1(i)$ for all $i \in S$, and there is $S' \subseteq \mu_1(h)$ such that $[S' \cup S] \succ_h \mu_1(h)$. Consider the deviation from μ_1 to μ_1' of the subset of blocking* pairs $S'(\mu_1) \subseteq S(\mu_1)$, where $S'(\mu_1)$ contains the maximum number of blocking* pairs and is such that the subset $S(\mu_1) \setminus S'(\mu_1)$ does not contain any blocking* pair of μ_1'. We now establish formally that μ_1' exists. We do so by making use of the property of substitutable preferences, which allows us to make use of the fact that a stable matching exists.

Claim 2. *For any matching $\mu_1 \notin C_w(P)$, there always exists a matching $\mu_1' \in \mathcal{M}$, which directly dominates μ_1, that can be enforced over μ_1 by the blocking* pairs $S'(\mu_1)$ and such that μ_1' is not blocked* by any pair in $S(\mu_1) \setminus S'(\mu_1)$.*

Proof. Define H' to be the set of hospitals that belong to some blocking* pair of μ: $H' = \{h \in H \mid \text{either } \mu_1(h) \notin \text{Ch}(\mu_1(h), P(h)) \text{ or } \exists S \subseteq I \text{ such that } (h, S) \in S(\mu_1)\}$. Equally, define I' to be the set of students who belong to some blocking* pair of μ: $I' = \{i \in I \mid \text{either } i \succ \mu_1(i) \text{ or } \exists h \in H \text{ such that } (h, S) \in S(\mu_1) \text{ with } i \in S\}$.

Consider the following restricted matching problem in which each agent ranks only those agents with whom she can form a deviating blocking* pair. That is, the preferences of each $h \in H'$ are only over the set $I_h' = \{S \subseteq I' \text{ such that } (h, S) \in S(\mu_1)\}$. For each $i \in I'$, her preferences are restricted to the set $H_i' = \{h \in H' \cup \{i\} \text{ such that } (h, S) \in S(\mu_1) \text{ with } i \in S\}$. Let $P'(\mu_1)$ denote these restricted preferences. Notice that the restricted preferences $P'(\mu_1)$ also satisfy the substitutability property and, therefore, the matching problem $\{H', I', P'(\mu_1)\}$ has at least one stable matching, call it μ'. Then define μ_1' and

[19]Of course, if $V(\ll) = \{\mu\}$ with $\mu \notin C_w(P)$, then there exists a matching μ' and a pair $(S, h) \in 2^I \times H$, with $S \subseteq \mu'(h)$, such that $h \succ_i \mu(i)$ for all $i \in S$, and there is $S' \subseteq \mu(h)$ such that $[S' \cup S] \succ_h \mu(h)$. Then $\mu \not\gg \mu'$ and the external stability condition is violated.

$S'(\mu_1)$ as follows. All agents who do not belong to either H' or I' do not belong to $S'(\mu_1)$ (they do not move themselves, although they can lose their match in the move from μ_1 to μ'_1 if they were initially matched to some of the deviating players in some blocking pair of $S'(\mu_1)$). Now consider the people in H' and I'. Every pair $\{h, \mu'(h)\}$ of μ' belongs to $S'(\mu_1)$ and, hence, both h and $\mu'(h)$ move from μ_1 to μ'_1 to their match in μ', with $\mu'_1(h) = [S' \cup \mu'(h)]$ and $S' \subseteq \mu_1(h)$ such that $[S' \cup \mu'(h)] \succ_h \mu_1(h)$. Every single student (every hospital that has some places unfilled) at μ' who prefers to be unemployed (that have some places unfilled) at μ' rather than to be in a hospital (hiring some students) at μ_1 belongs to $S'(\mu_1)$ and we let them become unemployed (leaving some places unfilled) in the move from μ_1 to μ'_1. Every single student (hospital) at μ' who prefers to be employed (prefers hiring some students) at μ_1 rather than to be unemployed (to have some places unfilled) at μ' does not belong to $S'(\mu_1)$, but to $S(\mu_1) \setminus S'(\mu_1)$. Every pair $\{h, \mu_1(h)\}$ of μ_1 with $\mu_1(h) = \mu'(h)$ is such that h (and/or $\mu_1(h)$) belongs to $S(\mu_1) \setminus S'(\mu_1)$ when h (and/or $\mu_1(h)$) belongs to H' (belongs to I'). Clearly, the subset $S(\mu_1) \setminus S'(\mu_1)$ does not contain any blocking* pair of μ'_1, because, otherwise, μ' would not be a stable matching for the matching problem $\{H', I', P'(\mu_1)\}$. $\triangleleft$

Once we show the existence of μ'_1, the proof follows the proof of Theorem 2, but now Lemma 1 is replaced by Lemma 4. $\square$

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