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Balancing supply and demand under bilateral constraints

Olivier Bochet
University of Bern and Maastricht University

Rahmi İlkılıç
Bilkent University

Hervé Moulin
Rice University

Jay Sethuraman
Columbia University

In a moneyless market, a nondisposable homogeneous commodity is reallocated between agents with single-peaked preferences. Agents are either suppliers or demanders. Transfers between a supplier and a demander are feasible only if they are *linked*. The links form an arbitrary bipartite graph. Typically, supply is short in one segment of the market, while demand is short in another.

Our *egalitarian transfer* solution generalizes Sprumont's (1991) and Klaus et al.'s (1998) uniform allocation rules. It rations only the long side in each market segment, equalizing the net transfers of rationed agents as much as permitted by the bilateral constraints. It elicits a truthful report of *both* preferences and links: removing a feasible link is never profitable to either one of its two agents. Together with efficiency and a version of equal treatment of equals, these properties are characteristic.

Keywords. Bipartite graph, bilateral trade, strategy-proofness, equal treatment of equals, single-peaked preferences.

JEL classification. C72, C78, D61, D63, D71.

Olivier Bochet: olivier.bochet@vwi.unibe.ch
Rahmi İlkılıç: rahmi.ilkilic@bilkent.edu.tr
Hervé Moulin: moulin@rice.edu
Jay Sethuraman: jay@ieor.columbia.edu

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1. Introduction

Balancing demand and supply cannot always be achieved by prices and cash transfers. Rationing is the normal allocation method for emergency aid supplies, assigning patients to hospitals, assigning students to schools, assigning workload among coworkers, etc.

In Sprumont's (1991) original rationing model, a given amount of a single nondisposable commodity is allocated between agents with single-peaked preferences. An allocation is efficient if and only if individual shares are all on the same side of individual peaks (all consume less or all consume more than they wish). The striking result is that the most egalitarian profile of shares, in the sense of Lorenz dominance (de Frutos and Massó 1995), defines a revelation mechanism that is uniquely fair and incentive compatible, in the strong sense of strategy-proofness (truthtelling is a dominant strategy). This profile is known as the *uniform rationing* solution.

A natural two-sided version of Sprumont's model has agents initially endowed with some commodity, so that someone endowed with less (resp. more) than her peak is a potential demander (resp. supplier), and the simultaneous presence of demanders and suppliers creates an opportunity to trade. The corresponding solution gives their peak consumption to agents on the short side of the market, while those on the long side are uniformly rationed (see Klaus et al. 1998, Barberà and Jackson 1995). We generalize that model to a considerable extent by assuming that the commodity can only be transferred between certain pairs of agents. Such constraints are typically logistical (e.g., which supplier can reach which demander in an emergency situation (Özdamar et al. 2004) or which worker can handle which job request), but could be subjective as well (as when a hospital chooses to refuse a new patient by declaring "red status" (New Jersey Hospital Association 2009)[1]).

Our model allows an arbitrary pattern of feasibility for transfers between suppliers and demanders, represented by a bipartite graph. This complicates the analysis of efficient (Pareto optimal) allocations, because short demand and short supply typically coexist in the same market (see a numerical example in Section 2). We use network flow techniques (Ahuja et al. 1993) to show that in the relevant subset of efficient allocations, the market splits into two segments across which no trade occurs: one segment where demanders are rationed while the corresponding suppliers unload their ideal (peak) transfer and another segment where demanders receive their ideal transfer while suppliers are rationed.

We identify a unique *egalitarian* efficient allocation, and show that the corresponding revelation mechanism possesses unique fairness and incentive-compatibility properties (discussed more below). Our egalitarian allocation admits a compact definition as the *Lorenz dominant* element within the set of efficient allocations where all agents are weakly rationed (they get at most their ideal trade).[2] This definition relies on the

[1] New Jersey Hospital Association (2009) notes that "Diversions may be overridden by the emergency physician in charge when medical judgement indicates that the diverting hospital can handle a certain patient better than the alternative hospital."

[2] In particular, this allocation is lexicographically optimal. Therefore, if we interpret the peaks as capacity constraints, it coincides with the allocation proposed by Megiddo (1974, 1977) for a general network with multiple sources and sinks.

intimate connection between the solution we propose and the egalitarian solution, introduced by Dutta and Ray (1989), of a supermodular cooperative game. Supermodularity implies that the core of the game has a Lorenz dominant element, which Dutta and Ray call the egalitarian solution of the game. We show that the set of Pareto efficient allocations where all agents are weakly rationed can be expressed as the intersection of the cores of two supermodular games. Hence this set has a Lorenz dominant element, which is precisely the egalitarian allocation (Theorem 1 in Section 6).

We propose a centralized organization of the market—a clearinghouse—that prescribes an allocation that is efficient with respect to (reported) preferences and (reported) feasible links between agents.[3] We insist on two strong incentive-compatibility properties: *strategy-proof* report of individual preferences and *link monotonicity*, stating that an agent can never benefit by unilaterally "closing" one of his feasible links to the other side of the market. For fairness, we require *constrained equal treatment of equals*, i.e., our rule must treat two agents with identical preferences on trades (hence on the same side of the market) as equally as possible, given the bilateral feasibility constraints: We cannot make their two transfers more equal without altering the transfer to or from some other agent. Our main result (Theorem 2 in Section 8) is that these three requirements characterize our egalitarian solution.

We already mentioned that our model generalizes that in Klaus et al. (1998), where transfers are possible between every supplier–demander pair. These authors use some different properties to single out uniform rationing of the long side. Back to arbitrary bilateral constraints, if the peaks are identically 1 on both sides, we are in the setting of the (random) matching model with dichotomous preferences, studied by Bogomolnaia and Moulin (2004), where now the flow between a supplier–demander pair is thought of as the probability that this pair is matched. That paper focuses as well on the egalitarian solution and the incentives to truthful revelation of feasible links by individuals (as we do) or by coalitions of agents.

Our analysis is also related to the design of exchange mechanisms in networks, for which Kranton and Minehart (2001) propose an ascending price mechanism that is strategy-proof and efficient. See also Corominas-Bosch (2004) for a bargaining model between agents on a network. A common feature with our work is the decomposition of a graph into several submarkets that simultaneously clear and where a different price prevails in each submarket. In our model, however, monetary transfers are not allowed, so the market does not clear.

Finally our paper (Bochet et al. 2010) considers the related model where the peak supplies are treated as hard constraints, not as agents. This is a direct generalization of Sprumont's model. We characterize there a rule similar to our egalitarian transfer rule by means of efficiency, strategy-proofness, and a constrained version of equal treatment of equals. While there are similarities between the two models, there are some important

[3] In the patients allocation example, there is evidence that decentralized diversion is wasteful, and some attempts at centralization are being developed. REDDINET (http://www.reddinet.com) is a medical communications network linking hospitals in several California counties for the purpose of improving the efficiency of patients' allocation.

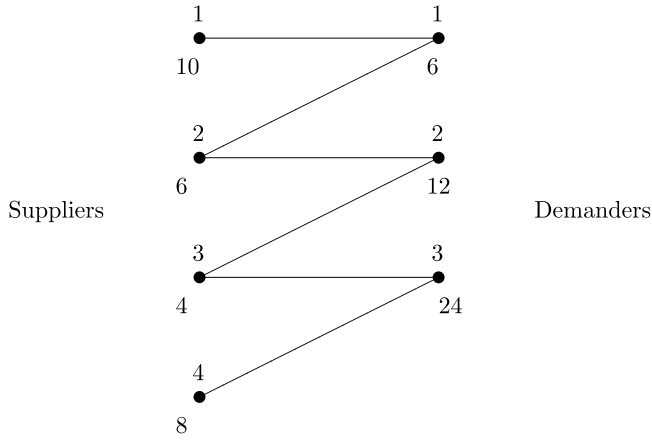


FIGURE 1. Short supply and short demand co-exist.

differences as well: for example, the egalitarian solution is not link monotonic in that setting.

In the next section, we present two motivating examples. We introduce the model in Section 3 and the maximal-flow formulation in Section 4. In Section 5, we characterize the set of Pareto optimal allocations. The egalitarian mechanism is defined in Section 6, and its properties are analyzed in Section 7. Section 8 states our characterization result and Section 9 gives some concluding comments. Those proofs not presented in the main text are given in the Appendix.

2. TWO SIMPLE EXAMPLES

EXAMPLE 1 (Short supply and short demand coexist). Short supply and short demand typically coexist in two independent segments of the market. This is illustrated in Figure 1. Supplier 1 can transfer only to demander 1, whose demand is short against 1's long supply. The two demanders 2 and 3 are similarly captive of suppliers 2, 3, and 4, whose supply is short against their long demand. Note that decentralized trade may fall short of efficiency. Indeed demander 1 and supplier 2 achieve their ideal consumption by a bilateral transfer of 6 units. However, after this transfer, supplier 1 is unable to trade, and demanders 2 and 3 have to share a short supply of 12 against their long demand of 36. It is more efficient to transfer 6 units from supplier 1 to demander 1, and let suppliers 2, 3, and 4 send their 18 units to demanders 2 and 3.

The first market segment contains the *long* supplier 1 and the *short* demander 1. Alternatively, demanders 2 and 3 compete for transfers from suppliers 2, 3, and 4. These agents form the short supply/long demand segment. Our egalitarian solution rations the long side of the market in each of the two segments. Consider the efficient profile of net transfers $(x, y) = ((6, 6, 4, 8), (6, 8, 10))$ (x for suppliers, y for demanders). Here demanders 2 and 3 equally split the transfer from supplier 3—their only common link. However, the profile $((6, 6, 4, 8), (6, 9, 9))$ is feasible and Lorenz dominates (x, y): it is our egalitarian solution. ◊

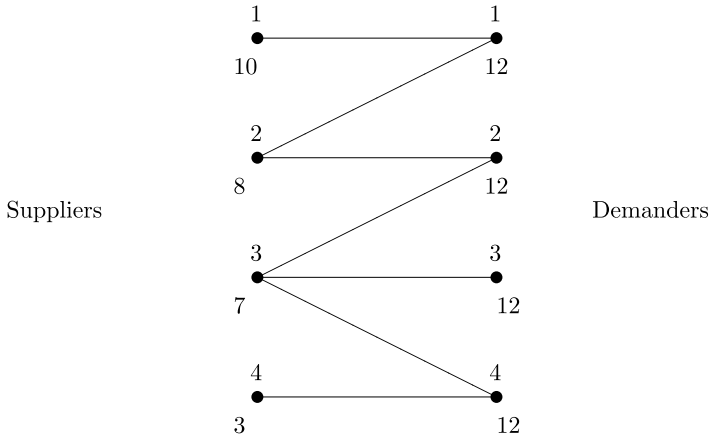


FIGURE 2. Agents on the short side are not treated identically.

Another implication of the bilateral constraints is that agents with identical preferences cannot always be treated equally.

EXAMPLE 2 (Identical preferences, different transfers). This is illustrated in Figure 2. There is a single market segment with a long demand, so the suppliers unload their peak transfer. The bilateral constraints restrict the (nonnegative) transfers y_i to the four demanders as follows:

$$10 \le y_1 \le 12; \qquad 6 \le y_2 \le 12; \qquad y_3 \le 7$$

$$\sum_1^4 y_i = 28; \qquad y_1 + y_2 \ge 18 \quad \Leftrightarrow \quad y_3 + y_4 \le 10.$$

Absent the bilateral constraints, we can achieve $y_i = 7$, $i = 1, 2, 3, 4$. Under these constraints, the most egalitarian profile is $y_1 = 10$, $y_2 = 8$, $y_3 = y_4 = 5$. ◊

3. TRANSFERS WITH BILATERAL CONSTRAINTS

We have a set S of suppliers with generic element i and a set D of demanders with generic element j. A set of transfers of the single commodity from suppliers to demanders results in a vector $(x, y) \in \mathbb{R}_+^S \times \mathbb{R}_+^D$, where x_i (resp. y_j) is supplier i's (resp. demander j's) *net transfer*, with $\sum_S x_i = \sum_D y_j$.

The commodity can only be transferred between certain pairs of supplier i and demander j. The bipartite graph G, a subset of $S \times D$, represents these constraints: $ij \in G$ means that a transfer is possible between $i \in S$ and $j \in D$. We assume throughout that the graph G is connected; otherwise, we can treat each connected component of G as a separate problem.

We use the following notation. For any subsets $T \subseteq S$ and $C \subseteq D$, the restriction of G is $G(T, C) = G \cap \{T \times C\}$ (not necessarily connected). The set of demanders that are compatible with the suppliers in T is $f(T) = \{j \in D \mid G(T, \{j\}) \neq \varnothing\}$. The set of suppliers

that are compatible with the demanders in C is $g(C) = \{i \in S \mid G(\{i\}, C) \neq \varnothing\}$. For any subsets $T \subseteq S$ and $C \subseteq D$, $x_T := \sum_{i\in T} x_i$ and $y_C := \sum_{j\in C} y_j$.

A transfer of goods from S to D is realized by a G-flow φ, i.e., a vector $\varphi \in \mathbb{R}_+^G$. We write $x(\varphi)$, $y(\varphi)$ for the transfers implemented by φ, namely

$$\text{for all } i \in S: \quad x_i(\varphi) = \sum_{j\in f(i)} \varphi_{ij}; \qquad \text{for all } j \in D: \quad y_j(\varphi) = \sum_{i\in g(j)} \varphi_{ij}.$$

We say that the net transfers (x, y) are *feasible* if they are implemented by some G-flow. We write $\Phi(G)$ for the set of feasible flows and write $\mathcal{A}(G)$ for the set of feasible net transfers. We define $\mathcal{A}(G(S', D'))$ similarly for any $S' \subseteq S$, $D' \subseteq D$. These sets are described as follows.

Lemma 1. *For any $S' \subseteq S$ and $D' \subseteq D$, the three following statements are equivalent.*

(i) $(x, y) \in \mathcal{A}(G(S', D'))$

(ii) for all $T \subseteq S'$, $x_T \leq y_{f(T)}$ *and* $x_{S'} = y_{D'}$

(iii) for all $C \subseteq D'$, $y_C \leq x_{g(C)}$ *and* $y_{D'} = x_{S'}$.

The proof is a standard application of the marriage lemma; see, e.g., Ahuja et al. (1993).

4. Maximal flow under capacity constraints

Assume, *in this section only*, that each supplier $i \in S$ has a (hard) capacity constraint s_i, i.e., cannot send more than s_i units of the commodity. Similarly each demander $j \in D$ cannot receive more than d_j units.

We write $\Phi(G, s, d)$ for the set of feasible flows φ such that $x(\varphi) \leq s$ and $y(\varphi) \leq d$, and write $\mathcal{A}(G, s, d)$ for the corresponding set of feasible constrained transfers.

The problem of finding the maximal feasible flows between suppliers and demanders thus constrained is well understood. We can apply the celebrated max-flow/min-cut theorem to the oriented capacity graph $\Gamma(G, s, d)$ obtained from G by adding a source σ connected to all suppliers, and a sink τ connected to all demanders; by orienting the edges from source to sink; by setting the capacity of an edge in G to infinity, that of an edge σi, $i \in S$, to s_i, and that of $j\tau$, $\tau \in D$, to d_j. A σ-τ cut (or simply a cut) in this graph is a subset X of nodes that contains σ but not τ. The capacity of a cut X is the total capacity of the edges that are oriented from a node in X to a node outside of X (such edges are said to be *in the cut*).

We illustrate this construction next.

Example 3 (Canonical flow representation). Figure 3 shows the canonical flow representation of Example 1 and Figure 1. The maximum flow from σ to τ is bounded by the capacity of any σ-τ cut, in particular, the minimum capacity σ-τ cut. The max-flow/min-cut theorem says that the maximum σ-τ flow has value equal to the capacity

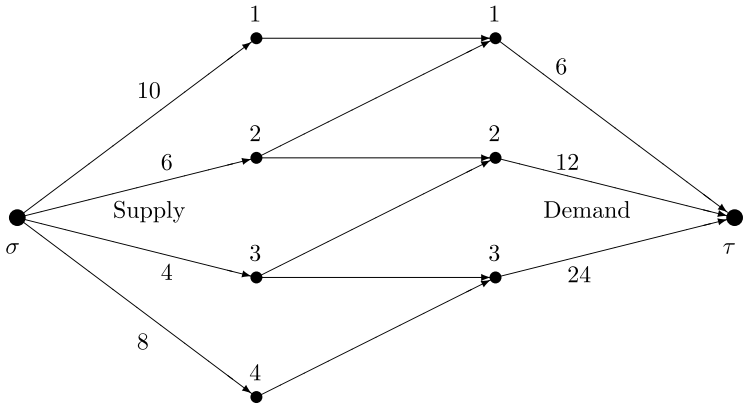


FIGURE 3. The max-flow problem.

of the minimum σ-τ cut. In Figure 3, the minimum capacity cut contains supplier 1 and demander 1 only (and σ), and has a capacity of 24. This is the maximum flow. Note that in the subset of efficient allocations where the long side always gets rationed, any allocation involves a net transfer of 24. This implies that supplier 1 unloads only 6 units on demander 1: Agents in the minimum cut are in the market segment with long supply; agents outside the minimum cut belong to the segment with long demand. ◊

These observations are summarized as follows: If we fix a maximum flow from σ to τ and a minimum-capacity σ-τ cut, then every edge in the cut must carry a flow equal to its capacity; moreover, every edge that is oriented from a node outside of the cut to a node inside the cut should carry zero flow. This leads to a key decomposition result.

LEMMA 2. *(i) There exists a partition S_+, S_- of S, and a partition D_+, D_- of D, where at most one of $S_+ = D_- = \varnothing$ or $S_- = D_+ = \varnothing$ is possible, with the properties*

$$\begin{aligned} &G(S_-, D_-) = \varnothing, \qquad D_+ = f(S_-), \qquad S_+ = g(D_-) \\ &s_{S'} \le d_{f(S')\cap D_-} \quad \text{for all } S' \subseteq S_+; \qquad d_{D'} \le s_{g(D')\cap S_-} \quad \text{for all } D' \subseteq D_+. \end{aligned} \tag{1}$$

(ii) The maximal flow is $s_{S_+} + d_{D_+}$. The flow $\varphi \in \Phi(G, s, d)$, with net transfers x, y is maximal if and only if

$$\varphi = 0 \quad \text{on } G(S_+, D_+), \qquad x = s \quad \text{on } S_+, \qquad y = d \quad \text{on } D_+.$$

(iii) The profile of transfers $(x, y) \in \mathcal{A}(G, s, d)$ is achieved by a maximal flow if and only if

$$x_S = y_D = s_{S_+} + d_{D_+}.$$

PROOF. We apply the max-flow/min-cut to $\Gamma(G, s, d)$. The max-flow from σ to τ is clearly finite and so must be the capacity of a minimum σ-τ cut. Fix a min-cut, and

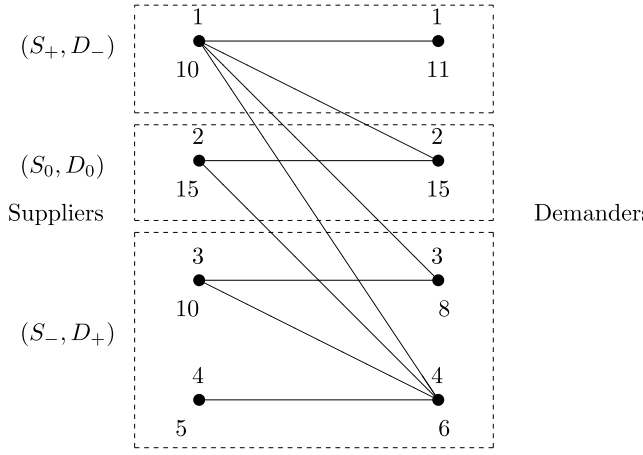


FIGURE 4. Decomposition with a balanced subgraph.

let X and Y be the set of suppliers and demanders, respectively, in that min-cut. Then we claim that $Y = f(X)$. If there exists demander $j \in Y$ such that $j \notin f(X)$, then the cut's capacity can be reduced by deleting the demander j; if, however, there exists demander $j \notin Y$ such that $j \in f(X)$, then the cut has infinite capacity.

Set $S_- = X$, $D_+ = Y$, $S_+ = S \setminus X$, and $D_- = D \setminus Y$. By construction, $G(S_-, D_-) = \varnothing$, $D_+ = f(S_-)$, and $S_+ = g(D_-)$. The capacity of the cut $\sigma \cup X \cup Y$ is, by definition, $s_{S\setminus X} + d_Y$, which equals $s_{S_+} + d_{D_+}$. Moreover, in any maximum flow, the edges oriented from S_+ to D_+ are backward edges in the cut, so they must carry zero flow. The edges from σ to S_+ and the edges from D_+ to τ are the edges in the cut, so these edges carry flow equal to their respective capacities. This establishes (ii) of the lemma. Parts (i) and (iii) follow from Lemma 1. □

The inequalities (1) express that the supply from S_+ is short with respect to the demanders in D_-, whereas the demand in D_+ is short with respect to the supply in S_-.

EXAMPLE 4 (Several possible decompositions). In general, the decomposition is not unique as there are several minimum cuts, all with identical capacities. If there is a unique min-cut, as, for instance, in Figure 3, the decomposition of the market into two segments is unique too (this holds true for an open and dense set of vectors (s, d)). If it is not unique, there is a partition S_+, S_- (resp. D_+, D_-) where S_- (resp. D_-) is the largest possible and one where it is the smallest. In Figure 4, there are two ways to decompose the demand and the supply sides. One possible decomposition is $D_- = \{1, 2\}$, $D_+ = \{3, 4\}$, $S_+ = \{1, 2\}$, $S_- = \{3, 4\}$; the other is $D'_- = \{1\}$, $D'_+ = \{2, 3, 4\}$, $S'_+ = \{1\}$, $S'_- = \{2, 3, 4\}$. ◊

In contrast, a familiar graph-theoretical result—the Gallai–Edmonds decomposition (see Ore 1962, and Bogomolnaia and Moulin 2004, Bochet et al. 2010 for applications)—determines a unique partition of the market, but in up to *three* segments. In one segment, supply is overdemanded and the corresponding demanders must be rationed; in

the second segment, supply is underdemanded and these suppliers transfer less than their ideal share; in the third segment, supply exactly balances demand. In Figure 4, the three segments of this decomposition are depicted as (S_+, D_-), (S_-, D_+), and (S_0, D_0), respectively.

5. Pareto optimality

We now have a bipartite graph G between S and D as before, but we replace the hard capacity constraint of the previous section by a soft ideal consumption. Each supplier i has *single-peaked preferences*[4] R_i (with corresponding indifference relation I_i) over her *net transfer* x_i, with peak s_i, and each demander j has single-peaked preferences R_j (I_j) over her net transfer y_j, with peak d_j. We write $\mathcal{R}$ for the set of single-peaked preferences over $\mathbb{R}_+$ and write $\mathcal{R}^{S\cup D}$ for the set of preference profiles.

The feasible net transfer $(x, y) \in \mathcal{A}(G)$ is Pareto optimal if for any other $(x', y') \in \mathcal{A}(G)$, we have

$$\{\text{for all } i, j\colon x'_i\, R_i\, x_i \text{ and } y'_j\, R_j\, y_j\} \quad \Rightarrow \quad \{\text{for all } i, j\colon x'_i\, I_i\, x_i \text{ and } y'_j\, I_j\, y_j\}.$$

We write $\mathcal{PO}(G, R)$ for the set of Pareto optimal net transfers.

Proposition 1. *Fix the economy (G, R), and two partitions S_+, S_- and D_+, D_- corresponding to the profile of peaks (s, d) at R (as in Lemma 2).*

(i) If the G-flow φ implements Pareto optimal net transfers (x, y), then transfers occur only between S_+ and D_-, and between S_- and D_+:

$$\varphi_{ij} > 0 \quad \Rightarrow \quad ij \in G(S_+, D_-) \cup G(S_-, D_+).$$

(ii) $(x, y) \in \mathcal{PO}(G, R)$ if and only if $(x, y) \in \mathcal{A}(G)$ and

$$\begin{aligned} x \geq s \quad &\text{on } S_+, \qquad y \leq d \quad \text{on } D_-, \quad \text{and} \quad x_{S_+} = y_{D_-} \\ x \leq s \quad &\text{on } S_-, \qquad y \geq d \quad \text{on } D_+, \quad \text{and} \quad x_{S_-} = y_{D_+}. \end{aligned}$$

An important feature of the Pareto set is that it depends only on the profile of peaks s, d, and not on the full preference profile R. The same is true of our egalitarian solution. To emphasize this important simplification, we speak of a *transfer problem* (S, D, G, s, d) or simply (G, s, d), keeping in mind the underlying single-peaked preferences.

The following subset of $\mathcal{PO}(G, R)$ will play an important role:

$$\mathcal{PO}^*(G, s, d) = \mathcal{PO}(G, R) \cap \{(x, y) \mid x \leq s;\, y \leq d\}.$$

[4] Writing P_i for agent i's strict preference, we have for every x_i, x'_i that $x_i < x'_i \leq s_i \Rightarrow x'_i\, P_i\, x_i$ and $s_i \leq x_i < x'_i \Rightarrow x_i\, P_i\, x'_i$.

By Proposition 1, this is the set of efficient allocations where the short side gets its optimal transfer:

$$x = s \quad \text{on } S_+, \qquad y \le d \quad \text{on } D_-, \quad \text{and} \quad y_{D_-} = s_{S_+}$$
$$x \le s \quad \text{on } S_-, \qquad y = d \quad \text{on } D_+, \quad \text{and} \quad x_{S_-} = y_{D_+}.$$

Moreover by Lemma 2, the net transfers in $\mathcal{PO}^*(G, s, d)$ are precisely those implemented by all the maximal flows of the capacity graph $\Gamma(G, s, d)$.

We focus on allocations in $\mathcal{PO}^*(G, s, d)$, because under the Voluntary Trade (requiring $x_i R_i 0$, $y_j R_j 0$ for all i, j; see Section 8) property, they are the only allocations that are Pareto optimal for any choice of preferences in $\mathcal{R}$ with peaks (s, d).

6. The egalitarian transfer solution

We give two definitions of our egalitarian solution: the first is a constructive algorithm; the second is based on the fact that, within the subset $\mathcal{PO}^*$ of Pareto optimal allocations, this allocation equalizes individual shares in the strong sense of Lorenz dominance (defined below).

We fix a problem (G, s, d) such that $s_i, d_j > 0$ for all i, j (clearly if $s_i = 0$ or $d_j = 0$, we can ignore supplier i or demander j altogether). We independently define our solution for the suppliers and for the demanders.

The definition for suppliers is by induction on the number of agents $|S| + |D|$. Consider the parameterized capacity graph $\Gamma(\lambda)$, $\lambda \ge 0$: the only difference between this graph and $\Gamma(G, s, d)$ is that the capacity of the edge σi, $i \in S_-$, is $\min\{\lambda, s_i\}$, which we denote $\lambda \wedge s_i$. (In particular, the edge from j to τ still has capacity d_j.) We set $\alpha(\lambda)$ to be the maximal flow in $\Gamma(\lambda)$. Clearly α is a piecewise linear, weakly increasing, strictly increasing at 0, and concave function of λ, reaching its maximum when the total σ-τ flow is d_{D_+}. Moreover, each breakpoint is one of the s_i (type 1) and/or is associated with a subset of suppliers X such that

$$\sum_{i \in X} \lambda \wedge s_i = \sum_{j \in f(X)} d_j. \tag{2}$$

Then we say it is of type 2. In the former case, the associated supplier reaches his peak and so cannot send any more flow. In the latter case, the group of suppliers in X is a *bottleneck* in the sense that they are sending enough flow to satisfy the collective demand of the demanders in $f(X)$ and these are the only demanders to which they are connected; any further increase in flow from any supplier in X would cause some demander in $f(X)$ to accept more than his peak demand.

If the given problem does not have any type-2 breakpoint, then the egalitarian solution obtains by setting each supplier's allocation to his peak value. Otherwise, let λ^* be the first type-2 breakpoint of the max-flow function; by the max-flow min-cut theorem, for every subset X satisfying (2) at λ^*, the cut $C^1 = \{\sigma\} \cup X \cup f(X)$ is a minimal cut in $\Gamma(\lambda^*)$, providing a certificate of optimality for the maximum-flow in $\Gamma(\lambda^*)$. If there are

several such cuts, we pick the one with the largest X^* (its existence is guaranteed by the usual supermodularity argument). The egalitarian solution obtains by setting

$$x_i = \min\{\lambda^*, s_i\} \quad \text{for } i \in X^*, \qquad y_j = d_j \quad \text{for } j \in f(X^*),$$

and assigning to other agents their egalitarian share in the reduced problem $(G(S \setminus X^*, D \setminus f(X^*)), s, d)$; that is, we construct $\Gamma^{S\setminus X^*, D\setminus f(X^*)}(\lambda)$ for $\lambda \geq 0$ by changing in $\Gamma(G(S \setminus X^*, D \setminus f(X^*)), s, d)$ the capacity of the edge σi to $\lambda \wedge s_i$, and look for the first type-2 breakpoint λ^{**} of the corresponding max-flow function. An important fact is that $\lambda^{**} > \lambda^*$. Indeed, there exists a subset X^{**} of $S \setminus X^*$ such that

$$\sum_{i\in X^{**}} \lambda^{**} \wedge s_i = \sum_{j\in f(X^{**})\setminus f(X^*)} d_j.$$

If $\lambda^{**} \leq \lambda^*$ we can combine this with equation (2) at X^* as

$$\sum_{i\in X^*\cup X^{**}} \lambda^* \wedge s_i \geq \sum_{i\in X^*} \lambda^* \wedge s_i + \sum_{i\in X^{**}} \lambda^{**} \wedge s_i = \sum_{j\in f(X^*\cup X^{**})} d_j,$$

contradicting our choice of X^* as the largest subset of S_- satisfying (2) at λ^*.

The solution thus obtained recursively is the egalitarian allocation for the suppliers. A similar construction works for demanders: We consider the parameterized capacity graph $\Delta(\mu)$, $\mu \geq 0$, with the capacity of the edge τj, $j \in D$, set to $\mu \wedge d_j$. We look for the first type-2 breakpoint μ^* of the maximal flow $\beta(\mu)$ of $\Delta(\mu)$ and for the largest subset of demanders Y such that

$$\sum_{j\in Y} \mu \wedge d_j = \sum_{i\in g(Y)} s_i$$

etc. Combining these two egalitarian allocations yields the egalitarian allocation $(x^e, y^e) \in \mathbb{R}_+^{S\cup D}$ for the overall problem.

We now illustrate the algorithm by revisiting the examples of Section 2.

EXAMPLE 5 (Example 1 revisited). In Example 1, the egalitarian allocation is $(x', y') = ((6, 6, 4, 8), (6, 9, 9))$. In Example 3, we saw that there is a unique min-cut given by $C^1 = \{\sigma\} \cup \{X\} \cup \{f(X)\}$, where $X = \{\text{supplier } 1\}$. Agents in the minimum cut form the partition (S_-, D_+) whereas $S_+ = \{\text{suppliers } 2, 3, 4\}$ and $D_- = \{\text{demanders } 2, 3\}$. We start with (S_-, D_+). The algorithm looks for λ_1 such that $\min\{s_1, \lambda_1\} = 6$, giving $\lambda_1 = 6$. For the other segment, the descending algorithm stops at $\lambda_2 = 9$. Indeed $\min\{d_2, \lambda_2\} + \min\{d_3, \lambda_2\} = s_2 + s_3 + s_4$. ◊

EXAMPLE 6 (Example 2 revisited). Recall that there is a single segment in which the demand is long. The algorithm first stops at $\lambda_1 = 10$. Indeed, $\min\{d_1, \lambda_1\} = s_1$. The algorithm next stops at $\lambda_2 = 8$ since $\min\{d_2, \lambda_2\} = s_2$. Finally, the algorithm stops at $\lambda_3 = 5$ since $\min\{d_3, \lambda_3\} + \min\{d_4, \lambda_3\} = s_3 + s_4$. ◊

We turn now to the Lorenz dominant position of our solution inside $\mathcal{PO}^*(G,s,d)$. For any $z\in\mathbb{R}^N$, write z^* for the *order statistics* of z, obtained by rearranging the coordinates of z in increasing order. For $z,w\in\mathbb{R}^N$, we say that z *Lorenz dominates* w, written z *LD* w, if for all k, $1\le k\le n$,

$$\sum_{a=1}^{k} z^{*a}\ge\sum_{a=1}^{k} w^{*a}.$$

Lorenz dominance is a partial ordering, so not every set—even convex and compact—admits a Lorenz dominant element. Alternatively, in a convex set A, there can be at most one Lorenz dominant element. The appeal of a Lorenz dominant element in A is that it maximizes over A *any* symmetric and concave collective utility function $W(z)$ (see, e.g., Moulin 1988).

THEOREM 1. *The allocation* (x^e,y^e) *is the Lorenz dominant element in* $PO^*(G,s,d)$.

We note that our solution is not Lorenz dominant in the entire Pareto set.

EXAMPLE 7 (Example 1 continued). The egalitarian allocation is $(x^e,y^e)=((6,6,4,8),(6,9,9))$. The allocation $(x'',y'')=((10,6,4,8),(10,9,9))$, where supplier 1 improves to his peak at the expense of demander 1, is also Pareto optimal by Proposition 1. It Lorenz dominates (x^e,y^e). ◊

7. PROPERTIES OF THE EGALITARIAN TRANSFER RULE

We introduce the incentives and equity properties that form the basis of our characterization result in the next section. Those properties bear on the profile of individual preferences R; therefore, instead of a transfer *problem* (G,s,d), we consider now a transfer *economy* (G,R). We use the notation $s[R_i]$, $d[R_j]$ for the peak transfer of supplier i and demander j.

DEFINITION. Given the agents (S,D), a rule ψ selects for every economy $(G,R)\in 2^{S\times D}\times\mathcal{R}^{S\cup D}$ a feasible allocation $\psi(G,R)\in\mathcal{A}(G)$.

We first define five incentive and monotonicity properties for an abstract rule ψ; then we define two equity properties. *Link monotonicity* requires that an agent on either side of the market weakly benefits from the access to new links. As discussed in the Introduction, this ensures that no agent has an incentive to close a feasible link; equivalently, it is a dominant strategy to reveal all feasible links to the manager.

LINK MONOTONICITY. For any economy $(G,R)\in 2^{S\times D}\times\mathcal{R}^{S\cup D}$ and any $i\in S$, $j\in D$, we have $\psi_k(G\cup\{ij\},R)\,R_k\,\psi_k(R,G)$ for $k=i,j$.

PROPOSITION 2. *The egalitarian transfer rule is link-monotonic.*

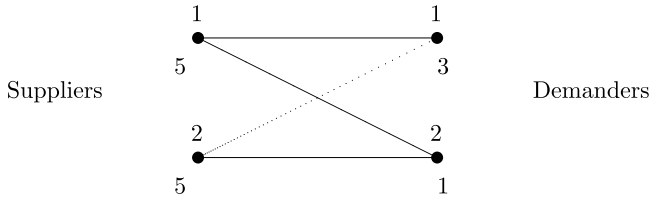


FIGURE 5. A new link may hurt noninvolved agents.

PROOF. We fix a supplier $i \in S$ and show that her allocation x_i increases weakly from the addition of link ij to G. In the algorithm defining the egalitarian solution for suppliers, we denote by λ^k, $k = 1, 2, \ldots$, the kth type-2 breakpoints of the max-flow function with corresponding bottleneck sets X^k: hence λ^k is the first type-2 breakpoint of the max-flow over the graph $G(S \setminus \bigcup_{t=1}^{k-1} X^t, D \setminus f(\bigcup_{t=1}^{k-1} X^t))$ with capacity $\lambda^k \wedge s_i$ on the link σi. Recall that λ^k increases strictly in k. We say that supplier i is *of order* k if

$$\{i \in X^k \text{ and/or } \lambda^{k-1} < s_i \leq \lambda^k\} \quad \Leftrightarrow \quad x_i = \lambda^k \wedge s_i > \lambda^{k-1}$$

(with the convention $\lambda^0 = 0$).

Compare the algorithms that define our solution at G and $G' = G \cup \{ij\}$, assuming that i is of order k. Clearly, the first $k-1$ steps of the algorithm are unchanged at G'; in particular, $\lambda^t = \lambda'^t$ for $t = 1, \ldots, k-1$. Moreover $\lambda^k \leq \lambda'^k$ because the right-hand term in (2) increases weakly while the left-hand term stays put. Distinguish two cases: If $s_i \leq \lambda^k$, then i is still of order k at G', so $x_i = s_i \leq \lambda'^k \wedge s_i = x'_i$; if $s_i > \lambda^k$, then $x_i = \lambda^k$ and i is of order no less than k at G', so $x'_i = \lambda'^{k'} \wedge s_i \geq x_i$.

The argument is identical for demanders. □

Note that the addition of a link ij may well hurt agents other than i, j. In Figure 5, we show an example with short demand in which our rule picks the allocation $x_1 = 3$ and $x_2 = 1$. Adding the link between supplier 2 and demander 1 gives $x'_1 = x'_2 = 2$.

In the rest of this section, we discuss properties for which the graph G is fixed, so we write a rule simply as $\psi(R)$ for $R \in \mathcal{R}^{S \cup D}$. The next incentive property is the familiar *strategy-proofness*. It is useful to decompose it into a monotonicity and an invariance condition.

PEAK MONOTONICITY. An agent's net transfer is weakly increasing in her reported peak: for all $R \in \mathcal{R}^{S \cup D}$, $i \in S$, $j \in D$, and $R'_i, R'_j \in \mathcal{R}$,

$$\begin{aligned} s[R'_i] \leq s[R_i] \quad &\Rightarrow \quad \psi_i(R'_i, R_{-i}) \leq \psi_i(R) \\ d[R'_j] \leq d[R_j] \quad &\Rightarrow \quad \psi_j(R'_j, R_{-j}) \leq \psi_j(R). \end{aligned}$$

INVARIANCE. For all $R \in \mathcal{R}^{S \cup D}$, $i \in S$, and $R'_i \in \mathcal{R}$,

$$\begin{aligned} &\{s[R_i] < \psi_i(R) \text{ and } s[R'_i] \leq \psi_i(R)\} \quad \text{or} \quad \{s[R_i] > \psi_i(R) \text{ and } s[R'_i] \geq \psi_i(R)\} \\ &\Rightarrow \quad \psi_i(R'_i, R_{-i}) = \psi_i(R) \end{aligned} \tag{3}$$

and similarly $\psi_j(R'_j, R_{-j}) = \psi_j(R)$ when agent $j \in D$ such that $\psi_j(R) \neq d[R_j]$ reports $R'_j \in \mathcal{R}$ with peak $d[R'_j]$ on the same side of $\psi_j(R)$ as $d[R_j]$.

STRATEGY-PROOFNESS. For all $R \in \mathcal{R}^{S\cup D}$, $i \in S$, $j \in D$, and $R'_i, R'_j \in \mathcal{R}$,

$$\psi_i(R)\, R_i\, \psi_i(R'_i, R_{-i}) \quad \text{and} \quad \psi_j(R)\, R_j\, \psi_j(R'_j, R_{-j}).$$

Each one of Peak Monotonicity or Invariance implies *own-peak-only:* my net transfer depends only on the peak of my preferences, and not on the way I compare allocations across my peak.

The next lemma connects these three properties and Pareto optimality.

LEMMA 3. *(i) If a rule is peak monotonic and invariant, it is strategy-proof.*

(ii) An efficient and strategy-proof rule is peak monotonic and invariant.

PROOF. We omit the easy argument and prove statement (i) just as in the Sprumont model.

(ii) We prove (peak) monotonicity for a given supplier i (and omit the entirely similar argument for a demander). Fix a Pareto optimal and strategy-proof rule ψ, a preference profile $R \in \mathcal{R}^{S\cup D}$, a supplier $i \in S$, and an alternative preference $R'_i \in \mathcal{R}$. Notation: $s_i = s[R_i]$, $s'_i = s[R'_i]$, $R' = (R'_i, R_{-i})$, and (s, d), (s', d) are the profiles of peaks at R and R', respectively. Finally, $x_i = \psi_i(R)$, $x'_i = \psi_i(R')$.

We assume $s'_i \leq s_i$ and show $x'_i \leq x_i$. Fix some partitions $S_{+,-}, D_{+,-}$ as in Lemma 2 for the problem (s, d) and consider two cases.

Case 1: $i \in S_-$. First assume $s'_i > x_i$. Then $S_{+,-}, D_{+,-}$ are valid partitions at (s', d), because inequalities (1) still hold: the left-hand one is clear; for the right-hand one, feasibility of ψ implies $d_{D'} \leq x_{g(D')\cap S_-}$, while efficiency (and Proposition 1) gives $x \leq s$ on S_-, so that $d_{D'} \leq s'_{g(D')\cap S_-}$. By Proposition 1, again $x'_i \leq s'_i$. Assume $x_i < x'_i$. Then we have $x_i < x'_i \leq s'_i \leq s_i$ and we get a contradiction of strategy-proofness (SP) for agent i at profile R.

Assume next $s'_i \leq x_i$. Then $x_i < x'_i$ gives $s'_i \leq x_i < x'_i$, contradicting SP for agent i at R'.

Case 2: $i \in S_+$. Then efficiency gives $s_i \leq x_i$, so $x_i < x'_i$ implies $s'_i \leq s_i \leq x_i < x'_i$, a violation of SP for agent i at R'.

We show invariance next, again in the case of a supplier i and with the same notation. Under the premises of property (3) inside the left bracket, if $x'_i > x_i$, we have $s'_i \leq x_i < x'_i$, hence a violation of SP for agent i at R'. If $x'_i < x_i$, we can find a preference R^*_i with peak $s^*_i = s_i$ such that $x'_i\, P^*_i\, x_i$. By own-peak-only (a consequence of Monotonicity), $\psi_i(R^*_i, R_{-i}) = x_i$, so agent i with preferences R^*_i benefits by reporting s'_i. The proof under the premises of (3) inside the right bracket is identical. □

PROPOSITION 3. *The egalitarian transfer rule is peak monotonic and invariant, hence strategy-proof as well.*

PROOF. Because the egalitarian transfer rule is peak-only, it is enough to speak of the profiles of peaks, instead of the full-fledged preferences.

Peak Monotonicity. We fix a benchmark profile (s, d) with the corresponding egalitarian transfers (x, y) and consider a change of peak by a supplier $i \in S$ to $s'_i > s_i$. Write (s', d) for the new profile of peaks. Let (x', y') be the egalitarian allocation for the profile (s', d). Compare the algorithms that define our solution at (s, d) and (s', d). We use the same notation as in the proof of Proposition 2.

Let the decomposition at (s, d) be $S_{-,+}$ and $D_{-,+}$, and the decomposition at (s', d) be $S'_{-,+}$ and $D'_{-,+}$.

Assume that at (s, d), $i \in S_-$ and x_i is of order k. Hence at (s', d), $i \in S'_-$ as well. Clearly the first $k-1$ steps of the algorithm are unchanged at (s', d); in particular, $\lambda^t = \lambda'^t$ for $t = 1, \dots, k-1$. Moreover, $\lambda^k \geq \lambda'^k$ because the left-hand term in (2) increases weakly while the right-hand term stays put. However, $\lambda'^k \geq s_i$ is guaranteed because up to $\lambda = s_i$, (2) is the same at (s, d) and at (s', d). Distinguish two cases: if $s_i \leq \lambda^k$, then i is of order no less than k at (s', d), so $x'_i \geq \lambda'^k \wedge s'_i \geq s_i = x_i$; if $\lambda^k < s_i$, then $i \in X^k$ at G, but also at G', so $x'_i = x_i$.

Now assume that at (s, d), $i \in S_+$. Then the egalitarian transfer rule gives $x_i = s_i$. If at (s', d), $i \in S'_+$, the egalitarian rule gives $x'_i = s'_i > s_i$. So suppose that at (s', d), $i \in S'_-$. Then clearly $S_- \subsetneq S'_-$. Let $T = S'_- \setminus S_-$. Because $T \subseteq S_+$, inequalities (1) imply

$$\sum_{k \in T} s_k \leq \sum_{j \in f(T) \cap D_-} d_j. \tag{4}$$

Observe that at (s', d), all demanders in $f(T) \cap D_-$ receive transfers only from T, because $G(S_-, f(T) \cap D_-) = \varnothing$; moreover, they are in D'_+ (again by (1)). This shows

$$\sum_{k \in T} x'_k \geq \sum_{j \in f(T) \cap D_-} d_j. \tag{5}$$

By Pareto optimality for all $k \in T$ such that $k \neq i$, $x'_k \leq s_k$, which together with (4) and (5) implies $x'_i \geq s_i = x_i$.

As usual, we omit the entirely similar argument for a change of peak by a demander j.

Invariance. In the premises of (3), the case $s[R_i] < \psi_i(R)$ never happens with the egalitarian solution. Now we fix as above $i \in S$, and two profiles of peaks (s, d) and (s', d) that differ only in the i-coordinate. As in the premises of (3), we also assume $s_i > x_i$, $s'_i \geq x_i$. Hence $i \in S_-$ necessarily, implying $i \in S'_-$ as well.

Assume that at (s, d), x_i is of order k. Again the first $k-1$ steps are unchanged at (s', d). Now $s_i > x_i$ implies $\lambda^k < s_i$ and $i \in X^k$. Therefore, the algorithm at (s', d) proceeds exactly as at (s, d) and $x' = x$ (for all suppliers). □

Our next property resembles the type of cross-monotonicity property that appeared first in Shapley and Shubik's (1972) bilateral assignment games.

CROSS-MONOTONICITY. Increasing the peak of a supplier (resp. demander) weakly benefits agents on the other side and weakly hurts those on the same side: for all $R \in \mathcal{R}^{S \cup D}$,

$j^* \in D$, and $R'_{j^*} \in \mathcal{R}$, we have

$$d[R_{j^*}] \leq d[R'_{j^*}] \quad \Rightarrow \quad \{\psi_i(R'_{j^*}, R_{-j^*})\, R_i\, \psi_i(R) \quad \text{for all } i \in S,$$
$$\text{and} \quad \psi_j(R)\, R_j\, \psi_j(R'_{j^*}, R_{-j^*}) \quad \text{for all } j \in D \setminus \{j^*\}\},$$

and a similar statement where we exchange the role of demanders and suppliers.

PROPOSITION 4. *The egalitarian transfer rule is cross-monotonic.*

We now turn to equity properties. The familiar equity test of no envy must be adapted to our model because of the feasibility constraints. If supplier 1 envies the net transfer x_2 of supplier 2, it might not be possible to give him x_2 because the demanders connected to agent 1 have insufficient demands. Even if we can exchange the net transfers of 1 and 2, this may require construction of a new flow and alteration of some of the other agents' allocations. In either case, we submit that supplier 1 has no legitimate claim against the allocation x. An envy argument by agent 1 against agent 2 is legitimate only if it is feasible to improve upon agent 1's allocation without altering the allocation of anyone other than agent 2.

NO ENVY. For any preference profile $R \in \mathcal{R}^{S\cup D}$ and any $i_1, i_2 \in S$ such that $\psi_{i_2}(R)\, P_{i_1}\, \psi_{i_1}(R)$, there exists no $(x, y) \in \mathcal{A}(G)$ such that

$$\psi_i(R) = x_i \quad \text{for all } i \in S \setminus \{i_1, i_2\},$$
$$\psi_j(R) = y_j \quad \text{for all } j \in D, \quad \text{and} \quad x_{i_1}\, P_{i_1}\, \psi_{i_1}(R),$$

and a similar statement where we exchange the role of demanders and suppliers.

Note that if i_1, i_2 have identical connections, $i_1 j \in G \Leftrightarrow i_2 j \in G$, then we can exchange their allocations without altering any other net transfer. Therefore, No Envy implies $\psi_{i_1}(R)\, I_{i_1}\, \psi_{i_2}(R)$.

The familiar horizontal equity property must be similarly adapted to account for the bilateral constraints on transfers.

EQUAL TREATMENT OF EQUALS (ETE). For any preference profile $R \in \mathcal{R}^{S\cup D}$ and any i_1, $i_2 \in S$ such that $R_{i_1} = R_{i_2}$, there exists no $(x, y) \in \mathcal{A}(G)$ such that

$$\psi_i(R) = x_i \quad \text{for all } i \in S \setminus \{i_1, i_2\}, \qquad \psi_j(R) = y_j \quad \text{for all } j \in D$$
$$|x_{i_1} - x_{i_2}| < |\psi_{i_1}(R) - \psi_{i_2}(R)|, \tag{6}$$

and a similar statement where we exchange the role of demanders and suppliers.

Again, if i_1, i_2 have identical connections, ETE implies $\psi_{i_1}(R) = \psi_{i_2}(R)$. In general, ETE requires the rule to equalize as much as possible the allocations of two agents with identical preferences.

PROPOSITION 5. *(i) No Envy plus Pareto optimality imply Equal Treatment of Equals.*

(ii) The egalitarian transfer rule ψ^e satisfies No Envy.

PROOF. (i) Suppose the rule ψ violates ETE, and check that it violates No Envy and/or Pareto optimality (PO). Fix a profile $R \in \mathcal{R}^{S\cup D}$ and two suppliers 1 and 2 such that $s_1[R_1] = s_2[R_2] = s^*$, and there exists (x, y) satisfying (6). Note that $x_1 + x_2 = \psi_1(R) + \psi_2(R)$ because x and $\psi(R)$ coincide on $S \cup D \setminus \{1, 2\}$ and by conservation of flows. Assume without loss of generality $\psi_1(R) < \psi_2(R)$. Then only two cases are possible: $\psi_1(R) < x_1 \le x_2 < \psi_2(R)$ or $\psi_1(R) < x_2 \le x_1 < \psi_2(R)$.

Assume the first case. If $s^* \ge \psi_2(R)$, supplier 1 envies 2 via (x, y); similarly $s^* \le \psi_1(R)$ implies a violation of No Envy. If $x_1 \le s^* \le x_2$, the profile of transfers (x, y) is Pareto superior to $\psi(R)$ (for both agents). If $x_2 < s^* < \psi_2(R)$, the profile (x', y), $x'_2 = s^*$, $x'_1 = x_1 + x_2 - s^*$, $x'_k = x_k$ else, is a convex combination of (x, y) and $\psi(R)$, so it is feasible ($\mathcal{A}(G)$ is convex) and Pareto superior to $\psi(R)$ (for both agents). The case $\psi_1(R) < s^* < x_1$ leads to a similar violation of PO.

In the second case, observe that the profile (x', y), $x'_1 = x'_2 = \frac{1}{2}(x_1 + x_2)$, $x'_k = x_k$ otherwise, is a convex combination of (x, y) and $\psi(R)$, so it is feasible and we are back to the first case.

(ii) Let R be a profile at which supplier 1 envies supplier 2 via (x, y). We have $\psi^e_1(R) < s_1$, because 1 is not envious if $\psi^e_1(R) = s_1$. Single-peakedness of R_1, and the fact that 1 prefers both x_1 and $\psi^e_2(R)$ to $\psi^e_1(R)$, imply $\psi^e_1(R) < x_1, \psi^e_2(R)$. As above, conservation of flows implies $x_1 + x_2 = \psi^e_1(R) + \psi^e_2(R)$. Therefore, $x_2 < \psi^e_2(R)$. By Proposition 1, we see that for ε small enough, the allocation $\varepsilon x + (1 - \varepsilon)\psi^e(R)$ is in $\mathcal{PO}^*(G, s, d)$. It is a Pigou–Dalton transfer from 2 to 1 in this set, contradicting the Lorenz dominance of $\psi^e(R)$ (Theorem 1). □

8. CHARACTERIZATION RESULT

Our last incentive property states that each agent is entitled to keep her endowment of the commodity and refuse to trade. It is weaker than Link Monotonicity.

VOLUNTARY TRADE. For all $R \in \mathcal{R}^{S\cup D}$, $i \in S \cup D$, we have $\psi_i(R)\, R_i\, 0$.

THEOREM 2. *The egalitarian transfer rule ψ^e is characterized by Pareto optimality, Strategy-proofness, Voluntary Trade, and Equal Treatment of Equals.*

9. CONCLUDING COMMENTS

Summary

Our model generalizes the two-sided version of the fair division model with single-peaked preferences (Klaus et al. 1998) by adding bipartite feasibility constraints. The market is divided into two independent submarkets: one with excess supply and one with excess demand. The relevant Pareto optimal allocations are described in each submarket as the core of a submodular game. Our solution is such an allocation, that is

Lorenz dominant, corresponding to Dutta and Ray's (1989) egalitarian solution in each one of these cooperative games.

The corresponding rule is characterized by the combination of efficiency, strategy-proofness and a version of equal treatment of equals where equalizing transfers are restricted to those that do not affect the shares of agents not involved in the transfer.

We conjecture that the egalitarian transfer rule is also group-strategy-proof, i.e., robust against coordinated misreport of preferences by subgroups of agents.

Extensions

First, following Sasaki (2001) and Ehlers and Klaus (2003) for the division model under single-peaked preferences, we can think of a "discrete" variant where indivisible units have to be traded between sellers and demanders. Both papers above offer a characterization of the randomized uniform rule, and it is likely that their result can be adapted to our model with bilateral constraints. Second, we have considered here only rules that treat agents with identical preferences as equally as possible given the bilateral constraints. Dropping the strong assumption of Equal Treatment of Equals, we would like to understand what rules meet the other three properties in Theorem 2. That question is already difficult in the standard rationing model (Barberà et al. 1997, Moulin 1999).

Appendix: Remaining proofs

A.1 *Proof of Proposition 1*

Step 0. The *if* statement in (ii) is easy. If (x', y') Pareto dominates (x, y), we have, by single-peakedness,

$$x \geq x' \quad \text{on } S_+, \qquad y' \geq y \quad \text{on } D_-.$$

As $x_{S_+} = y_{D_-}$ and D_- can receive only from S_+, these inequalities are all equalities. A similar argument shows $x = x'$ on S_- and $y' = y$ on D_+.

For the rest of the proof, fix an economy (G, R, s, d), the partitions $S_{-,+}, D_{-,+}$ corresponding to (s, d), and a Pareto optimal allocation (x, y) implemented by the flow φ. Consider the following color coding scheme with respect to the flow φ: supplier nodes i with $x_i < s_i$ are colored green and those with $x_i > s_i$ are colored red; demander nodes j with $y_j < d_j$ are colored red and those with $y_j > d_j$ are colored green. Let every other node be colored black. Observe that green suppliers prefer to send more flow and red suppliers prefer to send less, whereas green demanders prefer to receive less flow and red demanders prefer to receive more. Consider the following directed graph G^φ associated with the given flow φ: Orient all edges of G from the supplier to the demander. Moreover, if $\varphi_{ij} > 0$, introduce a directed edge from the demander node j back to supplier node i. These new edges are called *backward* edges. This graph is called the residual network with respect to φ. It captures all possible ways in which the current flow can be modified.

Because (x, y) is Pareto optimal, there is no path from a green node to a red node in G^φ. Indeed, if there is such a path, we can increase flow along that path (keeping in

mind that whenever we use backward edges, we are actually *reducing* flow on that edge in the original network, as would be the case on the first edge of a path from a green demander to a red supplier), so the flow φ is clearly not Pareto optimal. Define X to be the set of all green nodes and all the nodes that one can reach from any green node in the residual network G^φ. Let Y be the set of all red nodes and all the nodes from which one can reach a red node in G^φ. Notice that X and Y are disjoint: if they had any member in common, then we would find a path from a green node to a red node in G^φ. Thus every node in X is green or black; every node in Y is red or black. Let Z be the set of the remaining nodes, so that X, Y, and Z partition the nodes of G; clearly every node in Z is black.

Step 1. If (x, y) is Pareto optimal, we have $x \le s$ on S_- and $y \ge d$ on D_+. For this step, we consider only the suppliers in S_- and the demanders in D_+. To that end, let $X_- \equiv X \cap S_-$ and $X_+ \equiv X \cap D_+$. Likewise for the sets Y and Z. By the definition of X, Y, and Z, there cannot be any edge in the original graph G between X_- and Y_+, between X_- and Z_+, or between Z_- and Y_+. Also, there is no positive flow between Y_- and X_+ or Y_- and Z_+ or Z_- and X_+.

We want to show that there are no red nodes in S_- and D_+. Note that the red nodes, if any, are all in Y_- and Y_+. Because every node in Y_- and Y_+ is red or black, we have

$$y_{Y_+} \le d_{Y_+} \quad \text{and} \quad s_{Y_-} \le x_{Y_-}.$$

By Lemma 2 ((1)) and the fact that the only nodes in S_- that a node in Y_+ is connected to are in Y_-,

$$d_{Y_+} \le s_{g(Y_+)\cap S_-} \le s_{Y_-}.$$

Because the only positive flow from Y_- is to Y_+, we finally have $x_{Y_-} \le y_{Y_+}$ and we conclude that all inequalities above are equalities. But $y_{Y_+} = d_{Y_+}$ means that there is no red node in Y_+; in view of $x_{Y_-} = s_{Y_-}$, Y_- contains no red nodes either. This establishes Step 1. Moreover, this also establishes that Y_+ cannot receive any flow from any supplier in S_+.

We now turn to Z_+ and Z_-. Because the only positive flow from $Y_- \cup Z_-$ is to $Y_+ \cup Z_+$, we have $x_{Y_-\cup Z_-} \le y_{Y_+\cup Z_+}$. Because all the nodes in Z are black, $y_{Z_+} = d_{Z_+}$ and $s_{Z_-} = x_{Z_-}$; we already know that $y_{Y_+} = d_{Y_+} = x_{Y_-} = s_{Y_-}$. Therefore,

$$s_{Y_-\cup Z_-} = x_{Y_-\cup Z_-} \le y_{Y_+\cup Z_+} = d_{Y_+\cup Z_+}.$$

By (1) and the fact that the only nodes in S_- that a node in $Z_+ \cup Y_+$ is connected to are in $Z_- \cup Y_-$,

$$d_{Y_+\cup Z_+} \le s_{g(Y_+\cup Z_+)\cap S_-} \le s_{Y_-\cup Z_-}.$$

Thus we conclude that the inequalities above are all equalities. In particular, the nodes in Z_+ cannot receive any flow from any supplier in S_+.

Step 2. If (x, y) is Pareto optimal, we have $x \ge s$ on S_+ and $y \le d$ on D_-. We omit the entirely similar proof. As in the proof of Step 1, the proof here also yields the additional conclusion that the suppliers in $S_+ \cap (X \cup Z)$ cannot send any flow to D_+.

Step 3. If (x, y) is Pareto optimal, then φ is null between S_- and D_- or between S_+ and D_+. The first statement follows from Lemma 2(i). To prove the second, suppose $\varphi_{ij} > 0$ for some $i \in S_+$ and $j \in D_+$. From the proofs of Steps 1 and 2, we know that i must be in Y and j must be in X. By the definition of X and Y, G^φ must contain a path from a green node to j and a path from i to a red node; this along with arc (j, i) in the residual network implies the existence of a path from a green node to a red node in G^φ, a contradiction. □

A.2 *Proving Theorem 1*

We first give an alternative characterization of the Pareto* set that is critical to the analysis of the egalitarian solution. Define two cooperative games (S, v) and (D, w), of which the players are, respectively, the suppliers and the demanders:

$$v(T) = \min_{T' \subseteq T} \{s_{T'} + d_{f(T \setminus T')}\} \quad \text{for all } T \subseteq S$$

$$w(E) = \min_{E' \subseteq E} \{d_{E'} + s_{g(E \setminus E')}\} \quad \text{for all } E \subseteq D.$$

LEMMA 4. *The games (S, v) and (D, w) are submodular. Moreover,*

$$v(S) = w(D) = s_{S_+} + d_{D_+}; \qquad v(S_-) = d_{D_+}; \qquad w(D_-) = s_{S_+}. \tag{7}$$

PROOF. A set function $h(\cdot)$ is *submodular* if for all sets X and X',

$$h(X \cup X') + h(X \cap X') \leq h(X) + h(X').$$

It is *modular* or *additive* if the inequality above is satisfied as an equality (for all sets X and X'). Given a modular function $l(\cdot)$ and a submodular function $h(\cdot)$, define

$$k(X) = \min_{U \subseteq X} \{h(U) + l(X \setminus U)\}.$$

We omit the straightforward argument showing that $k(\cdot)$ is submodular as well. This implies that v and w are submodular, because $T \to s_T$ and $E \to d_E$ are modular, while $T \to d_{f(T)}$ and $E \to s_{g(E)}$ are submodular.

We check (7). For each $S' \subseteq S$, the set $\{\sigma\} \cup S \setminus S' \cup f(S \setminus S')$ is a cut of $\Gamma(G, s, d)$ with capacity $s_{S'} + d_{f(S \setminus S')}$, and any other cut has infinite capacity. Therefore, $v(S)$ is a min-cut of $\Gamma(G, s, d)$; hence Lemma 2 gives $v(S) = s_{S_+} + d_{D_+}$. Next $v(S_-) = d_{D_+}$ easily follows from the fact that the transfer d is feasible in $G(S_-, D_+)$ under the capacity constraint s. Similar arguments give the rest of (7). □

The core of the game (S, v), denoted $\text{Core}(S, v)$, is the set of allocations $x \in \mathbb{R}_+^S$ such that $x_T \leq v(T)$ for all $T \subset S$ and $x_S = v(S)$. Similarly the core of the game (D, w) is the set of allocations $y \in \mathbb{R}_+^D$ such that $y_E \leq w(E)$ for all $E \subset D$ and $y_D = w(D)$. Notice that $v(T) \leq s_T$ for all $T \subset S$. Therefore, $x \in \text{Core}(S, v)$ implies $x \leq s$; similarly, $y \in \text{Core}(D, w) \Rightarrow y \leq d$.

LEMMA 5. *Fix the problem (G, s, d), and two partitions S_+, S_- and D_+, D_- as in Lemma 2. Then the allocation (x, y) is in $\mathcal{PO}^*(G, s, d)$ if and only if it satisfies one of the following equivalent properties:*

(i) $x \in \text{Core}(S, v)$ *and* $y \in \text{Core}(D, w)$.

(ii) $\{x = s$ *on* S_+*, and on* S_-*,* $x \in \text{Core}(S_-, v)\}$ *and* $\{y = d$ *on* D_+*, and on* D_-*,* $y \in \text{Core}(D_-, w)\}$.

PROOF. Recall that $\mathcal{PO}^*(G, s, d)$ is the set of net transfers (x, y) implemented by a maximal flow of $\Gamma(G, s, d)$. Thus $x_S = v(S)$ and $y_D = w(D)$ (all equal to the max-flow). Next fix $T' \subseteq T \subset S$ and observe that $s_{T'} + d_{f(T \setminus T')}$ is an upper bound on the transfers to T under the capacity constraints s, d. This gives $x_T \leq v(T)$ and $x \in \text{Core}(S, v)$. The rest of statement (i) is checked similarly.

Conversely, fix $x \in \text{Core}(S, v)$ and $y \in \text{Core}(D, w)$. From $v(i) \leq s_i$ and $v(S_-) = d_{D_+}$, we get

$$x \leq s \quad \text{on } S_+ \quad \text{and} \quad x_{S_-} \leq d_{D_+},$$

and the sum of these inequalities is an equality, so they all are equalities. Finally for any $T \subset S_-$, the core property gives $x_T \leq v(T) \leq d_{f(T)}$, so by Lemma 1, the transfers (x, d) are feasible in $G(S_-, D_+)$. Combining this with $x = s$ on S_+, and, symmetrically, $y = d$ on D_+, and (s, y) feasible in $G(S_+, D_-)$ implies that (x, y) is feasible in $\Gamma(G, s, d)$ and maximizes the flow. □

PROOF OF THEOREM 1. For $z, w \in \mathbb{R}^N$, we say that z *lexicographically dominates* w if the first coordinate a in which z^* and w^* are not equal is such that $z^{*a} > w^{*a}$. We show that the egalitarian solution lexicographically dominates any other solution. Recall that in an arbitrary submodular cooperative game, the egalitarian core selection introduced in Dutta and Ray (1989) *Lorenz dominates* every other core allocation. As the set $\mathcal{PO}^*(G, s, d)$ is the intersection of the cores of two submodular games (Lemma 5), it has a unique Lorenz dominant element, which must also be lexicographically optimal. As the lexicographically optimal element is always unique, it must also be Lorenz dominant.

We prove the result for the suppliers by induction on the number of agents. An analogous argument for the demanders, omitted as usual, completes the proof. The result is clearly true when there is a single supplier and when the max-flow function (defined earlier) $\alpha(\lambda)$ does not have any type-2 breakpoints. In the latter case, every supplier will be allocated his peak, which clearly Lorenz dominates every other allocation. Let λ^* be the first type-2 breakpoint of the max-flow function $\alpha(\lambda)$ and let X^* be the corresponding largest bottleneck set of suppliers (2). The following facts about the egalitarian allocation are clear:

- Each supplier $i \in X^*$ will send s_i or λ^*, whichever is smaller.
- Each supplier $i \notin X^*$ with $s_i \leq \lambda^*$ will send s_i.
- Each supplier $i \notin X^*$ with $s_i > \lambda^*$ will send a flow that is *strictly above* λ^*.

(The last statement is valid because $\lambda^{**} > \lambda^*$.) Therefore, all the suppliers with a peak at or below λ^* transfer their peak values; every other supplier sends at least λ^* and those in X^* send exactly λ^*. Let W be the set of suppliers (both in X^* and outside) with peak at or below λ^*. Clearly, the allocations of the suppliers in W cannot be improved. It is also clear that in any other allocation, at least one of the suppliers in $X^* \setminus W$ who is not sending his peak must send at most λ^*. This is because, in the egalitarian allocation, they equally split the $d_{f(X^*)} - s_{X^* \cap W}$ units of flow they collectively send. In any other allocation, they send *at most* these many units of flow, so the smallest allocation of a supplier in $X^* \setminus W$ is at most λ^*. And if this smallest allocation is exactly λ^*, the allocation coincides with the egalitarian allocation on $X^* \cup W$. Thus the egalitarian allocation lex-dominates any allocation that does not agree with it on the allocations of the suppliers in $W \cup X^*$. We can, therefore, fix the allocations of the suppliers in $W \cup X^*$ to their egalitarian allocation for the purposes of proving lex-dominance. Let $\mathcal{W}$ be the subset of Pareto optimal allocations that gives each supplier in $W \cup X^*$ their egalitarian allocation. Note that in every allocation in $\mathcal{W}$, each demander $j \in f(X^*)$ receives his peak demand, all of which flows from the suppliers in X^*. Thus, none of these demanders receives additional flow from the suppliers in $S \setminus X^*$ in any allocation in $\mathcal{W}$. By construction, no supplier in X^* has links to a demander in $D \setminus f(X^*)$. Thus, proving lex-dominance of the egalitarian allocation for the original problem is equivalent to proving the following statement: when restricted to the suppliers in $S \setminus X^*$, the egalitarian allocation lexicographically dominates all the allocations in $\mathcal{W}$. The restriction of the egalitarian allocation to the suppliers in $S \setminus X^*$ is identical to the egalitarian allocation of the subproblem $(S \setminus X^*, D \setminus f(X^*))$. This, however, is a smaller problem, so, by the induction hypothesis, the egalitarian allocation of this subproblem lexicographically dominates any other Pareto optimal allocation, and, in particular, those in $\mathcal{W}$. □

The above proof implies the following corollary.

Corollary 1. *For any problem (G, s, d), the allocation x^e (resp. y^e) is the egalitarian selection in* Core(S, v) *(resp.* Core(D, w)*).*

A.3 *Proof of Proposition 4*

Step 1: A demander's peak increases ⇒ all suppliers shares increase weakly. The initial problem is (S, D, G, s, d). The new demand profile is $\widetilde{d}$, $d \leq \widetilde{d}$. We want to show that all suppliers are weakly better off. Let the successive bottlenecks for the suppliers' algorithm be X^k, $1 \leq k \leq K$, at d, with corresponding values λ^k, then $\widetilde{X}^l$, $1 \leq l \leq L$, and $\widetilde{\lambda}^l$ at $\widetilde{d}$; the corresponding shares are $x_i = \lambda^k \wedge s_i$ and $\widetilde{x}_i = \widetilde{\lambda}^l \wedge s_i$. We use the notation $Y^k = \bigcup_{t=1}^k X^k$, and similarly for $\widetilde{Y}^l$. Note that $X^K = S_-$ and $\widetilde{Y}^L = \widetilde{S}_-$ (i.e., they give us one of the decompositions). Finally we write $\lambda \wedge s_T = \sum_{i \in T} \min\{\lambda, s_i\}$.

We need to show $x_i \leq \widetilde{x}_i$ for $i \in \widetilde{Y}^L$ (there is nothing to prove for $i \in \widetilde{S}_+$). Note that with the convention $X^{K+1} = S_+$ (i.e., $S \setminus S_-$), and $\lambda^{K+1} = \infty$, for every l, $1 \leq l \leq L$, there is a unique k, $1 \leq k \leq K$, such that

$$\widetilde{X}^l \cap X^{k+1} \neq \varnothing; \quad \widetilde{X}^l \subseteq Y^{k+1}. \tag{8}$$

We prove by induction on l, $1 \le l \le L$, the statement $\mathcal{P}_l$,

$$[\lambda^k < \widetilde{\lambda}^l] \quad \text{and} \quad [\{\lambda^{k+1} \le \widetilde{\lambda}^l\} \text{ and/or } \{s_i \le \widetilde{\lambda}^l \text{ for all } i \in \widetilde{X}^l \cap X^{k+1}\}],$$

where k is defined by (8). Note that $\mathcal{P}_l$ implies $x_i = \lambda^k \wedge s_i \le \widetilde{x}_i = \widetilde{\lambda}^l \wedge s_i$ for $i \in \widetilde{X}^l \cap Y^k$; for $i \in \widetilde{X}^l \cap X^{k+1}$, we have $\widetilde{x}_i = s_i \ge x_i$ by Pareto optimality. Thus all we need to prove is $\mathcal{P}_l$.

Check $\mathcal{P}_1$. By definition of X^k, λ^k, we have

$$\lambda^k \wedge s_{\widetilde{X}^1 \cap X^{k+1}} < d_{f(\widetilde{X}^1) \setminus f(Y^k)} \tag{9}$$

and by definition of X^{k+1}, λ^{k+1} (including the case $k = K$),

$$\lambda^{k+1} \wedge s_{\widetilde{X}^1 \cap X^{k+1}} \le d_{f(\widetilde{X}^1) \setminus f(Y^k)}. \tag{10}$$

Next the definition of $\widetilde{X}^1$, $\widetilde{\lambda}^1$ gives

$$\begin{aligned} \widetilde{d}_{f(\widetilde{X}^1)} = \widetilde{\lambda}^1 \wedge s_{\widetilde{X}^1} &= \widetilde{\lambda}^1 \wedge s_{\widetilde{X}^1 \cap X^{k+1}} + \widetilde{\lambda}^1 \wedge s_{\widetilde{X}^1 \cap Y^k} \\ \Rightarrow \quad \widetilde{d}_{f(\widetilde{X}^1)} &\le \widetilde{\lambda}^1 \wedge s_{\widetilde{X}^1 \cap X^{k+1}} + \widetilde{d}_{f(\widetilde{X}^1 \cap Y^k)}. \end{aligned} \tag{11}$$

Finally we check the inequality

$$d_{f(\widetilde{X}^1) \setminus f(Y^k)} + \widetilde{d}_{f(\widetilde{X}^1 \cap Y^k)} \le \widetilde{d}_{f(\widetilde{X}^1)}. \tag{12}$$

If some demander j whose demand increased is in $f(\widetilde{X}^1 \cap Y^k)$, she is in $f(\widetilde{X}^1)$ as well, so (12) follows from the corresponding inequality with d instead of $\widetilde{d}$. Combining the latter with (9), (10), and (11) gives

$$\lambda^k \wedge s_{\widetilde{X}^1 \cap X^{k+1}} < \widetilde{\lambda}^1 \wedge s_{\widetilde{X}^1 \cap X^{k+1}} \quad \text{and} \quad \lambda^{k+1} \wedge s_{\widetilde{X}^1 \cap X^{k+1}} \le \widetilde{\lambda}^1 \wedge s_{\widetilde{X}^1 \cap X^{k+1}}$$

from which $\mathcal{P}_1$ follows at once.

Assume $\mathcal{P}_1, \dots, \mathcal{P}_l$ and check $\mathcal{P}_{l+1}$. Let k_0 be the *largest* k associated by (8) with some l', $1 \le l' \le l$, and let l_0 be the *smallest* l' achieving k_0. By assumption $\lambda^{k_0} < \widetilde{\lambda}^{l_0} < \widetilde{\lambda}^{l+1}$. We distinguish three cases.

Case 1. $\widetilde{X}^{l+1} \subseteq Y^{k_0}$.

Case 2. $\widetilde{X}^{l+1} \subseteq Y^{k_0+1}$, $\widetilde{X}^{l+1} \cap X^{k_0+1} \ne \varnothing$.

Case 3. $\widetilde{X}^{l+1} \subseteq Y^{k+1}$, $\widetilde{X}^{l+1} \cap X^{k+1} \ne \varnothing$ for some $k > k_0$.

In Case 1, the integer k associated with $l+1$ by (8) is strictly smaller than k_0, so $\lambda^{k+1} \le \lambda^{k_0} < \widetilde{\lambda}^l < \widetilde{\lambda}^{l+1}$ and we are done. In Case 2, the integer defined by (8) for $l+1$ is k_0; we already have $\lambda^{k_0} < \widetilde{\lambda}^{l_0} < \widetilde{\lambda}^{l+1}$, so we only need to prove the second part of $\mathcal{P}_l$.

By definition of X^{k_0+1}, λ^{k_0+1} we have

$$\lambda^{k_0+1} \wedge s_{\widetilde{X}^{l_0,\dots,l+1} \cap X^{k_0+1}} \le d_{f(\widetilde{X}^{l_0,\dots,l+1}) \setminus f(Y^{k_0})},$$

where we use the notation $\widetilde{X}^{l_0,\dots,l+1} = \bigcup_{t=l_0}^{l+1} X^t$.

Next, the definitions of $\widetilde{X}^{l_0}, \widetilde{X}^{l_0+1}, \ldots, \widetilde{X}^{l+1}$ give

$$\widetilde{d}_{f(\widetilde{X}^{l_0},\ldots,l+1)\setminus f(\widetilde{Y}^{l_0-1})} = \sum_{t=l_0}^{l+1} \widetilde{\lambda}^t \wedge s_{\widetilde{X}^t} = \sum_{t=l_0}^{l+1} \widetilde{\lambda}^t \wedge s_{\widetilde{X}^t \cap X^{k_0+1}} + \sum_{t=l_0}^{l+1} \widetilde{\lambda}^t \wedge s_{\widetilde{X}^t \cap Y^{k_0}}.$$

Then we compute

$$\sum_{t=l_0}^{l+1} \widetilde{\lambda}^t \wedge s_{\widetilde{X}^t \cap X^{k_0+1}} \leq \sum_{t=l_0}^{l+1} \widetilde{\lambda}^{l+1} \wedge s_{\widetilde{X}^t \cap X^{k_0+1}} = \widetilde{\lambda}^{l+1} \wedge s_{\widetilde{X}^{l_0},\ldots,l+1 \cap X^{k_0+1}}$$

$$\sum_{t=l_0}^{l+1} \widetilde{\lambda}^t \wedge s_{\widetilde{X}^t \cap Y^{k_0}} \leq \sum_{t=l_0}^{l+1} \widetilde{d}_{f(\widetilde{X}^t \cap Y^{k_0})\setminus f(\widetilde{Y}^{t-1})} \leq \sum_{t=l_0}^{l+1} \widetilde{d}_{f(\widetilde{X}^t \cap Y^{k_0})\setminus f(\widetilde{Y}^{l_0-1})}$$

$$= \widetilde{d}_{f(\widetilde{X}^{l_0},\ldots,l+1 \cap Y^{k_0})\setminus f(\widetilde{Y}^{l_0-1})}.$$

Finally, we check the inequality

$$d_{f(\widetilde{X}^{l_0},\ldots,l+1)\setminus f(Y^{k_0})} + \widetilde{d}_{f(\widetilde{X}^{l_0},\ldots,l+1 \cap Y^{k_0})\setminus f(\widetilde{Y}^{l_0-1})} \leq \widetilde{d}_{f(\widetilde{X}^{l_0},\ldots,l+1)\setminus f(\widetilde{Y}^{l_0-1})}.$$

Indeed any demander whose demand goes up, appearing in the $\widetilde{d}$ term on the right, is also present in the left term, so we can replace $\widetilde{d}$ by d. Then the two sets $f(\widetilde{X}^{l_0},\ldots,l+1) \setminus f(Y^{k_0})$ and $f(\widetilde{X}^{l_0},\ldots,l+1 \cap Y^{k_0}) \setminus f(\widetilde{Y}^{l_0-1})$ are disjoint; by definition of l_0, $\widetilde{Y}^{l_0-1} \subseteq Y^{k_0}$; therefore, our two sets are both contained in $f(\widetilde{X}^{l_0},\ldots,l+1) \setminus f(\widetilde{Y}^{l_0-1})$.

Combining the four inequalities and one equality above gives

$$\lambda^{k_0+1} \wedge s_{\widetilde{X}^{l_0},\ldots,l+1 \cap X^{k_0+1}} \leq \widetilde{\lambda}^{l+1} \wedge s_{\widetilde{X}^{l_0},\ldots,l+1 \cap X^{k_0+1}}$$

and we are done.

In Case 3, we have $\widetilde{Y}^l \subseteq Y^k$ and we proceed as in the proof of $\mathcal{P}_1$, with the arguments

$$\lambda^k \wedge s_{\widetilde{X}^{l+1} \cap X^{k+1}} < d_{f(\widetilde{X}^{l+1})\setminus f(Y^k)}; \qquad \lambda^{k+1} \wedge s_{\widetilde{X}^{l+1} \cap X^{k+1}} \leq d_{f(\widetilde{X}^{l+1})\setminus f(Y^k)}$$

(by definition of X^k, λ^k, X^{k+1}, λ^{k+1}). Use next the definition of $\widetilde{X}^{l+1}$, $\widetilde{\lambda}^{l+1}$:

$$\widetilde{d}_{f(\widetilde{X}^{l+1})\setminus f(\widetilde{Y}^l)} = \widetilde{\lambda}^{l+1} \wedge s_{\widetilde{X}^{l+1}} = \widetilde{\lambda}^{l+1} \wedge s_{\widetilde{X}^{l+1} \cap X^{k+1}} + \widetilde{\lambda}^{l+1} \wedge s_{\widetilde{X}^{l+1} \cap Y^k}$$

$$\Rightarrow \quad \widetilde{d}_{f(\widetilde{X}^{l+1})\setminus f(\widetilde{Y}^l)} \leq \widetilde{\lambda}^{l+1} \wedge s_{\widetilde{X}^{l+1} \cap X^{k+1}} + \widetilde{d}_{f(\widetilde{X}^{l+1} \cap Y^k)\setminus f(\widetilde{Y}^l)}.$$

Then the inequality

$$d_{f(\widetilde{X}^{l+1})\setminus f(Y^k)} + \widetilde{d}_{f(\widetilde{X}^{l+1} \cap Y^k)\setminus f(\widetilde{Y}^l)} \leq \widetilde{d}_{f(\widetilde{X}^{l+1})\setminus f(\widetilde{Y}^l)}$$

follows from the usual argument to replace $\widetilde{d}$ by d and the inclusion $\widetilde{Y}^l \subseteq Y^k$, showing that the two disjoint sets on the left are included in the set on the right.

Combining these inequalities and equality, we now have

$$\lambda^k \wedge s_{\widetilde{X}^{l+1} \cap X^{k+1}} < \widetilde{\lambda}^{l+1} \wedge s_{\widetilde{X}^{l+1} \cap X^{k+1}} \quad \text{and} \quad \lambda^{k+1} \wedge s_{\widetilde{X}^{l+1} \cap X^{k+1}} \leq \widetilde{\lambda}^{l+1} \wedge s_{\widetilde{X}^{l+1} \cap X^{k+1}}$$

and the proof of Step 1 is complete.

Step 2: *A supplier's peak increases ⇒ other suppliers' shares decrease weakly.* We fix a supplier i_0 whose peak increases, $s_{i_0} < \widetilde{s}_{i_0}$, and use the same notation, so $\widetilde{s}$ is the new profile of suppliers' peaks, etc. We need to show that all agents in Y^K are weakly worse off.

For every k, $1 \le k \le K$, there is a unique l, $1 \le l \le L$, such that

$$X^k \cap \widetilde{X}^{l+1} \neq \varnothing; \quad X^k \subseteq \widetilde{Y}^{l+1}. \tag{13}$$

We prove by induction on k, $1 \le k \le K$, the statement $\mathcal{Q}_k$,

$$[\widetilde{\lambda}^l < \lambda^k] \quad \text{and} \quad [\{\widetilde{\lambda}^{l+1} \le \lambda^k\} \text{ and/or } \{s_i \le \lambda^k \text{ for all } i \in X^k \cap \widetilde{X}^{l+1}\}],$$

where l is defined by (13). Note that $\mathcal{Q}_k$ implies $x_i = \lambda^k \wedge s_i \ge \widetilde{\lambda}^l \wedge s_i \ge \widetilde{x}_i$ for $i \in X^k \cap \widetilde{Y}^l \setminus \{i_0\}$, and for $i \in X^k \cap \widetilde{X}^{l+1} \setminus \{i_0\}$, we have $x_i = \lambda^k \wedge s_i \ge \widetilde{\lambda}^{l+1} \wedge s_i = \widetilde{x}_i$ and/or $x_i = s_i \ge \widetilde{x}_i$. So proving $\mathcal{Q}_k$ will be enough.

We prove $\mathcal{Q}_1$. We have

$$\widetilde{\lambda}^l \wedge \widetilde{s}_{X^1 \cap \widetilde{X}^{l+1}} < d_{f(X^1)\setminus f(\widetilde{Y}^l)}; \quad \text{and} \quad \widetilde{\lambda}^{l+1} \wedge \widetilde{s}_{X^1 \cap \widetilde{X}^{l+1}} \le d_{f(X^1)\setminus f(\widetilde{Y}^l)}$$

$$d_{f(X^1)} = \lambda^1 \wedge s_{X^1} = \lambda^1 \wedge s_{X^1 \cap \widetilde{X}^{l+1}} + \lambda^1 \wedge s_{X^1 \cap \widetilde{Y}^l}$$

$$\Rightarrow \quad d_{f(X^1)} \le \lambda^1 \wedge s_{X^1 \cap \widetilde{X}^{l+1}} + d_{f(X^1 \cap \widetilde{Y}^l)} \quad \Leftrightarrow \quad d_{f(X^1)\setminus f(\widetilde{Y}^l)} \le \lambda^1 \wedge s_{X^1 \cap \widetilde{X}^{l+1}}$$

so that

$$\widetilde{\lambda}^l \wedge s_{X^1 \cap \widetilde{X}^{l+1}} \le \widetilde{\lambda}^l \wedge \widetilde{s}_{X^1 \cap \widetilde{X}^{l+1}} < \lambda^1 \wedge s_{X^1 \cap \widetilde{X}^{l+1}} \quad \Rightarrow \quad \widetilde{\lambda}^l < \lambda^1$$

and a similar argument gives $\widetilde{\lambda}^{l+1} \wedge s_{X^1 \cap \widetilde{X}^{l+1}} \le \lambda^1 \wedge s_{X^1 \cap \widetilde{X}^{l+1}}$. If $\widetilde{\lambda}^{l+1} > \lambda^1$, this last inequality gives $\widetilde{\lambda}^{l+1} \wedge s_i = \lambda^1 \wedge s_i$ for $i \in X^1 \cap \widetilde{X}^{l+1}$; hence $\lambda^{k+1} \ge s_i$ as desired.

The induction step from $\mathcal{Q}_1, \ldots, \mathcal{Q}_k$ to $\mathcal{Q}_k$, $\mathcal{Q}_{k+1}$ distinguishes three cases as above. We let l_0 be the largest l associated by (13) to some $k' \le k$ and let k_0 be the smallest k achieving l_0.

Case 1. $X^{k+1} \subseteq \widetilde{Y}^{l_0}$.

Case 2. $X^{k+1} \subseteq \widetilde{Y}^{l_0+1}$, $X^{k+1} \cap \widetilde{X}^{l_0+1} \neq \varnothing$.

Case 3. $X^{k+1} \subseteq \widetilde{Y}^{l+1}$, $X^{k+1} \cap \widetilde{X}^{l+1} \neq \varnothing$ for some $l > k_0$.

In Case 1, l associated to $k+1$ is below l_0; therefore, $\widetilde{\lambda}^{l+1} \le \widetilde{\lambda}^{l_0} < \lambda^{k_0} < \lambda^{k+1}$ and we are done. In Case 2, the l associated to $k+1$ by (13) is l_0; we already have $\widetilde{\lambda}^{l_0} < \lambda^{k_0} < \lambda^{k_0+1}$, so we only need to prove the second part of $\mathcal{Q}_{k+1}$. We have three successive inequalities:

$$\widetilde{\lambda}^{l_0+1} \wedge \widetilde{s}_{X^{k_0,\ldots,k+1} \cap \widetilde{X}^{l_0+1}} \le d_{f(X^{k_0,\ldots,k+1})\setminus f(\widetilde{Y}^{l_0})}$$

$$d_{f(X^{k_0,\ldots,k+1})\setminus f(Y^{k_0-1})} = \sum_{t=k_0}^{k+1} \lambda^t \wedge s_{X^t}$$

$$= \sum_{t=k_0}^{k+1} \lambda^t \wedge s_{X^t \cap \widetilde{X}^{l_0+1}} + \sum_{t=k_0}^{k+1} \lambda^t \wedge s_{X^t \cap \widetilde{Y}^{l_0}}$$

$$\leq \lambda^{k+1} \wedge s_{X^{k_0,\dots,k+1} \cap \widetilde{X}^{l_0+1}} + \sum_{t=k_0}^{k+1} d_{f(X^t \cap \widetilde{Y}^{l_0}) \setminus f(Y^{t-1})}$$

$$\leq \lambda^{k+1} \wedge s_{X^{k_0,\dots,k+1} \cap \widetilde{X}^{l_0+1}} + d_{f(X^{k_0,\dots,k+1} \cap \widetilde{Y}^{l_0}) \setminus f(Y^{k_0-1})}$$

$$d_{f(X^{k_0,\dots,k+1}) \setminus f(\widetilde{Y}^{l_0})} + d_{f(X^{k_0,\dots,k+1} \cap \widetilde{Y}^{l_0}) \setminus f(Y^{k_0-1})} \leq d_{f(X^{k_0,\dots,k+1}) \setminus f(Y^{k_0-1})}.$$

The last inequality is because $f(X^{k_0,\dots,k+1}) \setminus f(\widetilde{Y}^{l_0})$ and $f(X^{k_0,\dots,k+1} \cap \widetilde{Y}^{l_0}) \setminus f(Y^{k_0-1})$ are disjoint; moreover, $Y^{k_0-1} \subseteq \widetilde{Y}^{l_0}$ by definition of l_0, thus both subsets are contained in $f(X^{k_0,\dots,k+1}) \setminus f(Y^{k_0-1})$. Combining the three inequalities gives

$$\widetilde{\lambda}^{l_0+1} \wedge s_{X^{k_0,\dots,k+1} \cap \widetilde{X}^{l_0+1}} \leq \widetilde{\lambda}^{l_0+1} \wedge \widetilde{s}_{X^{k_0,\dots,k+1} \cap \widetilde{X}^{l_0+1}} \leq \lambda^{k+1} \wedge s_{X^{k_0,\dots,k+1} \cap \widetilde{X}^{l_0+1}}.$$

If $\widetilde{\lambda}^{l_0+1} > \lambda^{k+1}$, this implies $\widetilde{\lambda}^{l_0+1} \wedge s_i = \lambda^{k+1} \wedge s_i$ for $i \in X^{k_0,\dots,k+1} \cap \widetilde{X}^{l_0+1}$; hence $\lambda^{k+1} \geq s_i$ as desired.

The proof of $\mathcal{Q}_{k+1}$ in Case 3 is entirely similar to that of $\mathcal{Q}_1$. We omit it for brevity. □

A.4 *Proof of Theorem 2*

The egalitarian transfer rule selects by construction $\psi^e(R)$ in $\mathcal{PO}^*(G,s,d)$ for all R with peaks (s,d); thus $\psi_i^e(R) \leq s_i$, $\psi_j^e(R) \leq d_j$ ensures Voluntary Trade. The other properties are proven in Propositions 3 and 5. Conversely, we fix a rule ψ that meets the four properties in the statement of the theorem.

Step 1. If the rule ψ satisfies Pareto optimality, Strategy-proofness, and Voluntary Trade, then $\psi(R) \in \mathcal{PO}^*(G,s,d)$ for all $R \in \mathcal{R}^{S \cup D}$ with peaks (s,d).

By Proposition 1, this amounts to showing that $\psi_i(R) > s_i$ is impossible for $i \in S_+(s,d)$ and $\psi_j(R) > d_j$ is impossible for $j \in D_+(s,d)$. Say $\psi_i(R) > s_i$ and choose $R_i' \in \mathcal{R}$ such that $s[R_i'] = s_i$ and $0\, P_i'\, \psi_i(R)$. Recall from Lemma 4 and the comments immediately before that ψ is *own-peak-only*; in particular, $\psi_i(R) = \psi_i(R_i', R_{-i})$. Now $0\, P_i'\, \psi_i(R_i', R_{-i})$ contradicts Voluntary Trade. As usual, we omit the similar proof of the other statement.

Step 2. It remains to prove that for all R with peaks (s,d), and any corresponding partitions $S_{+,-}, D_{+,-}$, the projections of $\psi(R)$ on S_- and D_- coincide with that of ψ^e. We focus on S_-, omitting the similar argument for D_-. By Lemma 5(ii), the projection of $\psi(R)$ on S is in Core(S_-, v), and this inclusion can be written as the system

$$x \leq s \quad \text{on } S_- \quad \text{and} \quad x_T \leq d_{f(T)} \quad \text{for all } T \subset S_- \tag{14}$$

$$x_{S_-} = d_{D_+}. \tag{15}$$

Step 2.1. In this step we assume that in the profile R, all suppliers have identical preferences: $R_i = R_{i'}$ for all $i, i' \in S$ (there are no constraints on the preferences of demanders). We use ETE to show that x is precisely the Lorenz dominant allocation $\overline{x}$ among those satisfying system (14) and (15).

CLAIM 1. Pick an agent 1 in S_- such that $x_1 = x^{*p}$ (so $x_1 = \max_{S_-} x_i$). Then

$$x_1 = \overline{x}_1 = x^{*p} = \overline{x}^{*p}. \tag{16}$$

As $\overline{x}$ is Lorenz dominant, we have $x^{*p} \geq \overline{x}^{*p}$. If $x_i = x^{*p}$ for all $i \in S_-$, then $x = \overline{x}$ (because $x_{S_-} = \overline{x}_{S_-}$) and we are done. Suppose next there is at least one $i \in S_-$ such that $x_i < x^{*p}$. We show that if $x_i < s_i$, there exists a coalition $S(i) \subset S_-$ containing i but not 1, such that $x_{S(i)} = d_{f(S(i))}$. Suppose, to the contrary, $x_T < d_{f(T)}$ for all $T \subset S_-$ containing i but not 1. A Pigou–Dalton transfer from x_1 to x_i transforms x into x' such that $x'_1 = x_1 - \varepsilon$, $x'_i = x_i + \varepsilon$, and $x_j = x'_j$ elsewhere. If ε is small enough, x' satisfies (14) and (15), in contradiction to ETE.

We set $S^* = \bigcup_{i:x_i<x^{*p},s_i} S(i)$. By submodularity of $T \to d_{f(T)}$, we have $x_{S^*} = d_{f(S^*)}$. By construction, for all $i \in N \setminus S^*$, x_i is x^{*p} or s_i, hence $x_i \geq \overline{x}_i$; moreover, $N \setminus S^*$ contains 1. Alternatively, we have

$$\overline{x}_{S^*} \leq d_{f(S^*)} = x_{S^*} \quad \Rightarrow \quad \overline{x}_{N\setminus S^*} \geq x_{N\setminus S^*}. \tag{17}$$

Combining this with $x_i \geq \overline{x}_i$ on $N \setminus S^*$ gives (16).

CLAIM 2. Pick agent 2 in S_-, $2 \neq 1$, such that $x_2 = x^{*(p-1)}$. Then

$$x_2 = \overline{x}_2 = x^{*(p-1)} = \overline{x}^{*(p-1)}. \tag{18}$$

As $\overline{x}$ Lorenz dominates x, we have $x^{*p} + x^{*(p-1)} \geq \overline{x}^{*p} + \overline{x}^{*(p-1)} \Rightarrow x^{*(p-1)} \geq \overline{x}^{*(p-1)}$. If $x_i = x^{*(p-1)}$ for all $i \in S_- \setminus \{1\}$, then $x = \overline{x}$ and we are done. Suppose now there is at least one $i \in S_- \setminus \{1\}$ such that $x_i < x^{*(p-1)}$. If $x_i < s_i$ there exists a coalition $S(i) \subset S_-$ that contains i but not 2, such that $x_{S(i)} = d_{f(S(i))}$; otherwise, we can construct as above a Pigou–Dalton transfer from 2 to i. Set $S^* = \bigcup_{i:x_i<x^{*(p-1)},s_i} S(i)$. Then $x_{S^*} = d_{f(S^*)}$ by submodularity of $T \to d_{f(T)}$. Moreover, for all i in $N \setminus (S^* \cup \{1\})$, x_i is $x^{*(p-1)}$ or s_i; in particular, $x_i \geq \overline{x}_i$. Combining this with $x_1 = \overline{x}_1$ and $\overline{x}_{N\setminus S^*} \geq x_{N\setminus S^*}$ ((17)), we see that x and $\overline{x}$ coincide in $N \setminus S^*$, which contains 2. Property (18) follows.

The inductive argument that establishes $x = \overline{x}$ is now clear.

Step 2.2. We just proved that ψ and ψ^e coincide on S when all suppliers have the same preferences. We use another induction argument, introduced in Ching (1994), to establish this equality for an arbitrary profile R. We use the following notation: for $R, \widetilde{R} \in \mathcal{R}^{S\cup D}$ and $T \subset S$, $(R_{[T]}, \widetilde{R}_{[(S\setminus T)\cup D]})$ is the profile equal to R in T and to $\widetilde{R}$ elsewhere.

Fix a profile $\widetilde{R}$ where all suppliers have identical preferences, an integer n, $0 \leq n \leq |S| - 1$, and consider the subset of preference profiles

$$R \in \mathcal{B}(\widetilde{R}, n) \quad \overset{\text{def}}{\Leftrightarrow} \quad \text{for some } T \subset S, |T| \leq n\text{:}$$
$$R_{[(S\setminus T)\cup D]} = \widetilde{R}_{[(S\setminus T)\cup D]} \text{ and } s[\widetilde{R}_i] \geq s[R_i] \text{ if } i \in S.$$

We prove by induction on n the following property $\mathcal{H}^+(n)$: for all $R \in \mathcal{R}^{S\cup D}$ and all $T \subset S$,

$$R \in \mathcal{B}(\widetilde{R}, n) \quad \Rightarrow \quad \psi_i(R) = \psi_i^e(R) \quad \text{for all } i \in S.$$

Step 2.1 establishes $\mathcal{H}^+(0)$. Assume now $\mathcal{H}^+(n-1)$ is true, and fix $R \in \mathcal{B}(\widetilde{R}, n)$ with $R[(S \setminus T) \cup D] = \widetilde{R}[(S \setminus T) \cup D]$ and $|T| = n$.

We prove first $\psi_i(R) = \psi_i^e(R)$ for $i \in T$. Pick such an agent and set $R' = (R_{[T\setminus i]}, \widetilde{R}_{[(S\setminus T)\cup\{i\}\cup D]}) \in \mathcal{B}(\widetilde{R}, n-1)$. By the inductive assumption, $\psi_i(R') = \psi_i^e(R') = x_i'$; by Pareto optimality* (i.e., $\psi(R), \psi^e(R) \in \mathcal{PO}^*(G, s, d)$) and the definition of $\mathcal{B}(\widetilde{R}, n)$,

$$s[\widetilde{R}_i] \geq s[R_i] \geq \psi_i(R), \psi_i^e(R).$$

If $s[R_i] \geq \psi_i^e(R) > \psi_i(R)$, Monotonicity implies $\psi_i^e(R') \geq \psi_i^e(R)$ and Invariance gives $\psi_i(R') = \psi_i(R)$, hence a contradiction. If $s[R_i] \geq \psi_i(R) > \psi_i^e(R)$, we have a similar contradiction from $\psi_i(R') \geq \psi_i(R)$ (Monotonicity) and $\psi_i^e(R') = \psi_i^e(R)$ (Invariance).

It remains to check $\psi_i(R) = \psi_i^e(R)$ for $i \in S \setminus T$. This is clear in $S \setminus S_-(R)$, so we check it for $S_-(R) \setminus T$. Write $S_- = S_-(R)$ and $x = \psi_{[S_-]}(R)$, $\overline{x} = \psi^e_{[S_-]}(R)$ as in Step 2.1. Consider the set

$$\mathcal{C}(R) = \{z \in \mathbb{R}_+^{S_- \setminus T} \mid (z, \overline{x}_{[S_- \cap T]}) \text{ satisfies system (14), (15)}\}.$$

Clearly $\overline{x}_{[S_-\setminus T]}$ is still Lorenz dominant in $\mathcal{C}(R)$, hence we can mimic the proof of Step 2.1 to show that ETE and Pareto optimality* imply $x = \overline{x}$ in $S_- \setminus T$. Indeed, $\widetilde{R}_{[S_-\setminus T]}$ consists of identical preferences; therefore, we can apply ETE to any pair of agents in $S_- \setminus T$. Moreover, $\mathcal{C}(R)$ is defined by the system

$$z \leq s \quad \text{on } S_- \setminus T \quad \text{and} \quad z_{T'} \leq \widetilde{v}(T') = d_{f(T' \cup [T \cap S_-])} - \overline{x}_{[T \cap S_-]} \quad \text{for all } T' \subset S_- \setminus T$$

$$z_{S_- \setminus T} = \overline{x}_{S_- \setminus T}.$$

Then the proof proceeds exactly as in Step 2.1. We omit the details.

We have proved that $\mathcal{H}^+(|S|-1)$ for any choice of $\widetilde{R}$. Now consider an arbitrary profile R and choose i in S and such that $s[R_i] \geq s[R_{i'}]$ for all $i' \in S$. Choosing for $\widetilde{R}$ the profile of preferences $\widetilde{R}_{i'} = R_i$ for all $i' \in S$ and $\widetilde{R}_j = R_j$ for all $j \in D$, we have $R \in \mathcal{B}(\widetilde{R}, |S|-1)$ and the proof is complete. □

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