Strategy-proof voting for multiple public goods

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In a voting model where the set of feasible alternatives is a subset of a product set $A = A_1 \times \cdots \times A_m$ of m finite categories, we characterize the set of all strategy-proof social choice functions for three different types of preference domains over A, namely for the domains of additive, completely separable, and weakly separable preferences over A.

KEYWORDS. Multiple public goods, strategy-proofness, voting under constraints, additive preferences, separable preferences.

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1. INTRODUCTION

In this paper, we consider the issue of strategy-proofness in a voting model where agents vote for alternatives from several categories simultaneously. To be precise, the voters face m finite categories A_k of alternatives ($k = 1, \ldots, m$), and the purpose of the voting is to choose one element from the product set $A = A_1 \times \cdots \times A_m$. If these m choices are entirely independent, one can of course consider the choice from each category A_k separately, and the strategy-proof voting procedures for such "one-dimensional" social choice problems are well known: when A_k contains three or more eligible alternatives, then it follows from the Gibbard–Satterthwaite theorem (Gibbard 1973, Satterthwaite 1975) that only the dictatorial voting procedures are strategy-proof, and when A_k contains exactly two eligible alternatives, then a voting procedure is strategy-proof if and only if it is "voting by committees" (see Barberà et al. 1991). In this study, however, we consider the simultaneous choice from m categories because we want to allow explicitly for the possibility that these m choices cannot be made completely independently, but that they must be coordinated to some extent.

Such a need for coordination can come up for different reasons, as illustrated by the following examples. First, one of the most common reasons is that the overall choice

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from the product set must be balanced in some sense. This can be due to exogenous constraints such as budget restrictions; for example, if the categories A_k represent levels of spending on different parts of the public sector such as infrastructure, public education, health care, etc., then budget limitations usually exclude the simultaneous choice of expensive alternatives from several categories. Alternatively, the balanced choice can be the consequence of fundamental values among the voters that are incorporated in the voting procedure; for example, if a committee has to decide how m open positions at a department (which are represented by the m categories A_k) are to be filled and if an even distribution of sex or age of the staff is considered important, then the committee will probably want to use a voting procedure that avoids unbalanced outcomes. Second, another important reason for coordination may be that certain categories are complementary to some extent. For example, if a laboratory must decide between different research projects (which correspond to the alternatives in A_1), employ a research manager (A_2), choose among different kinds of equipment (A_3), etc., then each alternative in A_1 typically is meaningful only with certain choices from A_2 and A_3, but not with others. Finally, a third important reason for coordination we want to mention here is that, in many societies, basic constitutional principles put restrictions on laws and necessitate coordination. This is the case, for example, when a parliament has to decide what civil rights to assign to different groups in the society that are distinguished by characteristics such as age, gender, nationality, etc. In this case, A_k consists of different possible bundles of rights for group k (e.g., no rights, basic level of rights, full rights including franchise, membership in social security and pension system, etc.). According to many constitutions, the assignment of civil rights must inter alia be such that citizens get at least as many rights as residents without citizenship, and if two groups differ only in gender, then these two groups must receive the same bundle of rights.

The need for coordination can thus be of a very different nature, but from a general perspective, coordination across different categories always means that certain combinations of alternatives are excluded from being elected. Formally, this means that the set of all feasible outcomes of the voting is, in general, not the entire product set $A = A_1 \times \cdots \times A_m$, but rather a proper subset of it, and in our voting model, the range of the voting can be any arbitrary nonempty subset of A.

If the voters can rank the possible outcomes of such a constrained multiple good voting in any conceivable way, then it follows, again by the Gibbard–Satterthwaite theorem, that only the dictatorial voting procedures are strategy-proof, at least if there are more than two possible outcomes. But when eligible alternatives are multiple public goods, as in the model in this paper, it is common to assume that voters' preferences over $A = A_1 \times \cdots \times A_m$, in addition to being rational, also exhibit some kind of separability structure. In the literature on multiple public good voting, one finds mainly three different types of separability assumptions on voters' preferences, namely weak separability, complete separability, and additivity. We consider our voting model for all of these three preference structures.

The discussion so far indicates that two factors are central to our considerations: on the one hand, the structure of the set of feasible outcomes of a voting, and on the other

hand, the structure of individual preferences. In our analysis it turns out that the extent to which it is possible to construct strategy-proof voting procedures for constrained multiple public good voting depends crucially on these two factors and their interplay. In the literature closest to this paper, the role of these two factors is analyzed in the papers Barberà et al. (2005) and Svensson and Torstensson (2008), both of which consider basically the same voting model as in our study, but under additional assumptions on the cardinality of the categories A_k. Barberà et al. (2005) characterize all strategy-proof social choice functions (SCFs) for constrained multiple public good voting for both additive and weakly separable preferences under the assumption that every category A_k contains exactly two alternatives. Svensson and Torstensson (2008) assume instead that every category A_k contains at least three alternatives, and they provide a complete characterization of all strategy-proof SCFs for completely separable preferences. There is, of course, an obvious theoretical interest in finding corresponding characterizations for the general case (i.e., without any restrictions on the number of alternatives in the categories A_k); additionally, the examples above indicate that the cardinality restrictions in these previous results cannot generally be expected to be satisfied. The purpose of this paper is to remove these cardinality restrictions and to characterize all strategy-proof SCFs for constrained multiple public good voting for weakly separable, completely separable, and additive preferences without imposing any additional assumptions.

Our characterization of the strategy-proof SCFs is based on a description of the range of the SCFs. Thereby, we make extensive use of the following result from Svensson and Torstensson (2008), which allows us to express the functional structure of the strategy-proof SCFs in an easily understandable way: If f is an SCF whose range $\mathcal{R}_f$ is a subset of the product set $A = A_1 \times \cdots \times A_m$, then there is a unique maximal decomposition $\mathcal{R}_f = B_1 \times \cdots \times B_q$ of the range into $q \leq m$ components. To construct nondictatorial strategy-proof SCFs for multiple public good voting, it is natural to try to divide the choice from the entire product set A into a number of independent choices from subcollections of categories A_k, and the maximal range decomposition $B_1 \times \cdots \times B_q$ provides the upper bound for how far the choice can be decomposed.

When preferences are additive or completely separable, voters consider the elements in each component B_j independently of the other components. For these preference structures, we can show (Theorem 1 and Theorem 2) that an SCF f is strategy-proof if and only if f can be decomposed into q independent choices from the q components B_j in the maximal range decomposition of f, and the choice from every component B_j is made in a strategy-proof way. To be precise, f is strategy-proof if and only if f is dictatorial on every component B_j with three or more elements (where different such components can have different dictators), and voting by committees on every component B_j with two elements. In particular, this means that the purely dictatorial result from the Gibbard–Satterthwaite theorem can be avoided when the maximal range decomposition of f contains at least two components. Our characterization of the strategy-proof SCFs for additive and completely separable preferences may appear intuitive, but it cannot be obtained by applying the standard results for strategy-proof voting procedures to each of the components B_j in the maximal range decomposition, because if such a component covers two or more of the categories in the product set A, then the domain

of (marginal) preferences on this component is not unrestricted and, therefore, it is not a priori obvious which SCFs are strategy-proof on B_j. Instead, we present a proof technique that allows us to simplify the structure of the domain of marginal preferences on every component B_j considerably, and we hope that this proof strategy turns out to be fruitful in connection with other problems as well.

When preferences are weakly separable, we require only that voters rank the alternatives in every category A_k independently of the other categories, but the same need no longer be true for the elements in a component B_j. But if the voters' rankings of the elements in a component B_j depend on which elements are chosen from the other components, then it is intuitively reasonable that every nondictatorial decomposed SCF provides incentives for strategic voting. This is indeed the case, and we show that for all nontrivial range restrictions, only the dictatorial SCFs are strategy-proof (Theorem 3), which means that we get back to the conclusion of the Gibbard–Satterthwaite theorem when preferences are weakly separable.

This paper is organized as follows: Section 2 introduces the social choice model. Section 3 provides our characterizations of strategy-proof SCFs for constrained multiple good voting for the three cases when voters' preferences are either additive, completely separable, or weakly separable. Section 4, finally, contains a discussion of the related literature and some concluding comments. All proofs are collected in the Appendix.

2. The social choice model and some important results from the literature

2.1 *The basic model*

The basic formal framework of this study is as follows. Let $N = \{1, \ldots, n\}$ be a finite society of *n individuals*, who consider a finite set A of *social alternatives*. The individuals have complete, transitive, and asymmetric[1] preferences over the alternatives in A, and the preference of individual i is denoted by P_i. The set of all admissible preferences over A is denoted by $\mathcal{P}_A$, and $\mathcal{P}_A$ is referred to as the *preference domain*; normally, $\mathcal{P}_A$ is a proper subset of the set of all complete, transitive, and asymmetric preferences. If $P \in \mathcal{P}_A$ and $B \subset A$ is a subset of A, then $\tau_B(P)$ denotes the (unique) *maximal element* of P in B, i.e., $\tau_B(P) \in B$ and $\tau_B(P) \, P \, b$ for all $b \in B \setminus \{\tau_B(P)\}$; in particular, $\tau_A(P)$ is the unrestricted maximal element of P in A. The preferences of the n individuals are collected in a *(preference) profile* $(P_1, \ldots, P_n) \in \mathcal{P}_A^n$, and so as to focus on individual i's preference, the profile $(P_1, \ldots, P_n)$ is sometimes rewritten as (P_i, P_{-i}), where $P_{-i} \in \mathcal{P}_A^{n-1}$ thus denotes the profile of all individuals except individual i.

A *social choice function* (SCF) is a mapping $f : \mathcal{P}_A^n \to A$ that assigns to every profile $(P_1, \ldots, P_n) \in \mathcal{P}_A^n$ a unique social choice $a \in A$. For a given SCF f, the set of all alternatives that can be attained by f is denoted by $\mathcal{R}_f$, i.e., $\mathcal{R}_f = \{a \in A : a = f(P_1, \ldots, P_n)$ for some $(P_1, \ldots, P_n) \in \mathcal{P}_A^n\}$, and $\mathcal{R}_f$ is called the *range* of f. An SCF $f : \mathcal{P}_A^n \to A$ is *manipulable* at the profile $(P_1, \ldots, P_n) \in \mathcal{P}_A^n$ if there is some $i \in N$ and some $P_i' \in \mathcal{P}_A$ such

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[1]A preference P on A is *complete* if for all $a, a' \in A$, $a \neq a'$, either $a \, P \, a'$ or $a' \, P \, a$; P is *transitive* if $a \, P \, a'$ and $a' \, P \, a''$ implies $a \, P \, a''$ for all $a, a', a'' \in A$. Finally, P is *asymmetric* if, for all $a, a' \in A$, $a \, P \, a'$ implies that $a' \, P \, a$ does not hold.

that $f(P_i', P_{-i}) \, P_i \, f(P_i, P_{-i})$. If f cannot be manipulated at any admissible profile, then f is *strategy-proof*. Finally, $f : \mathcal{P}_A^n \to A$ is *dictatorial* if there is some $i \in N$ such that $f(P_i, P_{-i}) = \tau_{\mathcal{R}_f}(P_i)$ for all $(P_i, P_{-i}) \in \mathcal{P}_A^n$.

In this formal framework, the theoretical starting point for our study is the well known Gibbard–Satterthwaite theorem (Gibbard 1973, Satterthwaite 1975).

THE GIBBARD–SATTERTHWAITE THEOREM. *Let A be a finite set of alternatives, and suppose that the preference domain $\mathcal{P}_A$ consists of all complete, transitive, and asymmetric preferences over A. Then $f : \mathcal{P}_A^n \to A$ with $\#\mathcal{R}_f \geq 3$ is strategy-proof if and only if f is dictatorial.*

2.2 *The model with multiple public goods*

The basic model is now extended by imposing a product set structure on the set of alternatives. To be precise, there are $m < \infty$ finite and nonempty *categories* A_k ($1 \leq k \leq m$), indexed by the set $M = \{1, \dots, m\}$, and the set of alternatives is $A = \prod_{k \in M} A_k$. Thus, an alternative is now an m-tuple $a = (a_1, \dots, a_m)$ of m public goods, and if $a \in A$, then $a_k \in A_k$ denotes the public good in a that is chosen from category k. If $S \subset M$ is a nonempty subset of the set of coordinate indices, we denote by A_S the product set of the corresponding categories, i.e., $A_S = \prod_{k \in S} A_k$. The *projection* from A onto A_S is denoted by $\pi_{A_S} : A \to A_S$, so for $a \in A$, $\pi_{A_S}(a)$ is the suballocation of a that belongs to the categories corresponding to S; for the special case when $S = \{k\}$ for some $k \in M$, we write π_{A_k} instead of $\pi_{A_{\{k\}}}$. If $a \in A$ and $S \subset M$, we can decompose a as $a = (a_S, a_{-S})$, where $a_S = \pi_{A_S}(a)$ and $a_{-S} = \pi_{A_{M \setminus S}}(a)$; in the special case when $S = \{k\}$, we write $a = (a_k, a_{-k})$.

Consider now the preference domain $\mathcal{P}_A$. When A is a product set, it is often reasonable to assume that not all conceivable preferences over A belong to $\mathcal{P}_A$, but that $\mathcal{P}_A$ instead exhibits a structure that takes the product set structure of A into account. In the literature, there are mainly three types of preference domains that are commonly considered.

1. *Additive preferences.* A complete, transitive, and asymmetric preference P on $A = \prod_{k \in M} A_k$ is *additive* if there exist m functions $u_k : A_k \to \mathbb{R}$, $k \in M$, such that the utility function $u : A \to \mathbb{R}$ defined by $u(a) = \sum_{k \in M} u_k(a_k)$ represents P. The set of all additive preferences over A is denoted by $\mathcal{P}_A^{\text{add}}$.

2. *Completely separable preferences.* Given a nonempty collection $S \subset M$ of categories, a complete, transitive, and asymmetric preference P on $A = \prod_{k \in M} A_k$ is *separable with respect to S* if the ranking of P of the elements in A_S is independent of the categories $M \setminus S$. Formally, P is separable with respect to S if for all $a, a' \in A$, we have $(a_S, a_{-S}) P(a_S', a_{-S})$ if and only if $(a_S, a_{-S}') P(a_S', a_{-S}')$. If P is separable with respect to S, then P induces a well defined preference on A_S, called the *marginal preference* of P on A_S and denoted by $\pi_{A_S}(P)$, which satisfies $a_S \, \pi_{A_S}(P) a_S'$ for $a_S, a_S' \in A_S$ if and only if $(a_S, a_{-S}) P(a_S', a_{-S})$ for some $a_{-S} \in A_{M \setminus S}$. A preference P on A is *completely separable* if P is separable with respect to every nonempty subset $S \subset M$, and the set of all completely separable preferences on A is denoted by $\mathcal{P}_A^{\text{cs}}$.

3. *Weakly separable preferences.* A complete, transitive, and asymmetric preference
   P on $A = \prod_{k \in M} A_k$ is *weakly separable* if the ranking of the alternatives in every
   category A_k is independent of the categories $M \setminus \{k\}$, i.e., if P is separable with
   respect to $\{k\}$ for every $k \in M$. The set of all weakly separable preferences on A is
   denoted by $\mathcal{P}_A^{\mathrm{ws}}$.

From these definitions, it follows immediately that $\mathcal{P}_A^{\mathrm{add}} \subset \mathcal{P}_A^{\mathrm{cs}} \subset \mathcal{P}_A^{\mathrm{ws}}$. However, the
converse inclusions do not hold in general: Bradley et al. (2005) show that the inclu-
sion $\mathcal{P}_A^{\mathrm{add}} \subset \mathcal{P}_A^{\mathrm{cs}}$ is strict if A contains $m \geq 5$ categories with $\# A_k \geq 2$,[2] and the following
example illustrates why the inclusion $\mathcal{P}_A^{\mathrm{cs}} \subset \mathcal{P}_A^{\mathrm{ws}}$ is strict whenever $m \geq 3$.

EXAMPLE 1. Let $A = \{0, 1\} \times \{0, 1\} \times \{0, 1\}$. If the preference P on A is such that

$$(1, 1, 1) \, P \, (1, 1, 0) \, P \, (1, 0, 1) \, P \, (1, 0, 0)$$
$$P \, (0, 1, 1) \, P \, (0, 1, 0) \, P \, (0, 0, 1) \, P \, (0, 0, 0), \tag{1}$$

then it is easily checked that the utility function $u(a) = 4a_1 + 2a_2 + a_3$ represents P, so P
is additive, and hence also completely and weakly separable.

Alternatively, if P is defined by

$$(1, 1, 1) \, P \, (0, 1, 1) \, P \, (1, 0, 1) \, P \, (0, 0, 1)$$
$$P \, (1, 1, 0) \, P \, (1, 0, 0) \, P \, (0, 1, 0) \, P \, (0, 0, 0), \tag{2}$$

then in every category, 1 is preferred to 0 for every fixed pair of alternatives from the
other two categories, and hence P is weakly separable. But the ranking of the pairs $(0, 1)$
and $(1, 0)$ in the first two categories depends on whether the third coordinate is 0 or 1,
and, therefore, P is neither completely separable nor additive.   ◇

Example 1 can also be used to explain the differences between the three domains
$\mathcal{P}_A^{\mathrm{ws}}$, $\mathcal{P}_A^{\mathrm{cs}}$, and $\mathcal{P}_A^{\mathrm{add}}$ in a more intuitive way. Suppose, for example, that the three cat-
egories in $A = \{0, 1\}^3$ correspond to three candidates under consideration for employ-
ment, and that $a_k = 1$ indicates that candidate k is employed, while $a_k = 0$ represents
the opposite. In this context, weak separability means that each candidate is evaluated
independently of whether the other candidates are employed; this condition is met in
both (1) and (2), where all three candidates are considered appropriate for employment.
Complete separability requires here that, in addition, the relative ranking of any two
candidates does not depend on whether the third candidate is employed; for example,
in (1), candidate 1 is unambiguously preferred to candidate 2 in the sense that both
$(1, 0, 0) \, P \, (0, 1, 0)$ and $(1, 0, 1) \, P \, (0, 1, 1)$. This requirement may be violated if there is
some interplay between the candidates; for example, if candidate 1 is considered more
skilled than candidate 2, but candidate 2 is able to cooperate much more efficiently with
candidate 3, then it is possible that $(0, 1, 1) \, P \, (1, 0, 1)$ despite $(1, 0, 0) \, P \, (0, 1, 0)$, as in (2)

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[2]In a classical paper, Debreu (1960) shows however, that in the "continuous" case, for example, when
$A = \mathbb{R}_+^m$ and preferences are continuous, the identity $\mathcal{P}_A^{\mathrm{add}} = \mathcal{P}_A^{\mathrm{cs}}$ holds.

where the preference is only weakly, but not completely separable. Finally, if preferences are completely separable, it is often convenient to impose also the structure of additivity as a technical assumption to obtain a tractable representation of preferences.

2.3 *The maximal range decomposition and some special SCFs*

Consider now an SCF $f : \mathcal{P}_A^n \to A$. It is, of course, possible that the range of f equals the entire product set A, but for reasons illustrated in the Introduction, it is also of particular interest to consider the case when $\mathcal{R}_f$ is a proper subset of A. Therefore, we now describe the structure of subsets of A. If $B \subset A$ is a proper subset of the product set A, then B can also be considered as a product set, simply because the entire set B can be seen as a product set with only one factor, but in many cases there exists a product set decomposition $B = B_1 \times \cdots \times B_q$ of B into q *components* B_j $(1 \le j \le q)$ that is finer than the one-factor product set B, but coarser than the complete decomposition into m factors. Formally, that $B_1 \times \cdots \times B_q$ is a decomposition of B means that for every B_j $(j = 1, \ldots, q)$, there exists a corresponding set of coordinates $C(B_j) \subset M$ "covered" by B_j such that (i) $B_j \subset \prod_{k \in C(B_j)} A_k$, (ii) $C(B_j) \cap C(B_{j'}) = \varnothing$ for all $j \ne j'$, (iii) $\bigcup_{1 \le j \le q} C(B_j) = M$, and (iv) B and $B_1 \times \cdots \times B_q$ are equal as sets. A decomposition $B = B_1 \times \cdots \times B_q$ is *maximal* if there exists no decomposition $B = B'_1 \times \cdots \times B'_{q'}$ with $q' > q$ components. It turns out that every subset of A has a unique maximal decomposition.

SVENSSON AND TORSTENSSON's (2008) PROPOSITION 1. *Let $B \subset \prod_{k \in M} A_k$. Then there is a decomposition $B = B_1 \times \cdots \times B_{\bar{q}}$ such that the number $\bar{q}$ is maximal. For the maximal $\bar{q}$, the various components B_j $(j = 1, \ldots, \bar{q})$ are unique. If $B = B'_1 \times \cdots \times B'_{q'}$ is another decomposition of B, then for all j and j', $C(B_j) \subset C(B'_{j'})$ if $C(B_j) \cap C(B'_{j'}) \ne \varnothing$.*[3]

In some cases, B is its own maximal decomposition (i.e., $\bar{q} = 1$), and then B is called *indecomposable*; for example, $B = \{(0,0), (0,1), (1,0)\}$ is an indecomposable subset of $A = \{0,1\} \times \{0,1\}$. At the other extreme, the maximal decomposition of a subset can have at most $q = m$ components, which, of course, is the case when $B = A$, but this can also happen for proper subsets of A: for example, $B = \{0,1\} \times \{0\}$ is a proper subset of $A = \{0,1\} \times \{0,1\}$ with $q = m = 2$. The following example illustrates the maximal decomposition of a subset in a nontrivial case.

EXAMPLE 2. Let $A = \{0,1\} \times \{0,1\} \times \{0,1\}$ and set $B = A \setminus \{(1,1,0), (1,1,1)\}$. This means that B is obtained by excluding exactly those two alternatives from A whose first two coordinates are $(1,1)$. Thus, we have the decomposition $B = B_1 \times B_2$, where $B_1 = \{(0,0), (0,1), (1,0)\}$ and $B_2 = \{0,1\}$, with $C(B_1) = \{1,2\}$ and $C(B_2) = \{3\}$. Since it is not possible to decompose B further, $B_1 \times B_2$ is the unique maximal decomposition of B. ◇

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[3]This result can be considered as a set-theoretical counterpart to the *fundamental theorem of arithmetic*, which states that every integer greater than 1 has a unique prime factor decomposition.

In connection with product set decompositions, we also use the following notation: if $B_1 \times \cdots \times B_q$ is a decomposition of a subset $B \subset A$, then we denote by π_{B_j} the projection of A onto the categories covered by B_j, i.e., $\pi_{B_j} = \pi_{A_{C(B_j)}}$, and the complementary projection is denoted by π_{-B_j}, i.e., $\pi_{-B_j} = \pi_{A_{M \setminus C(B_j)}}$.

In addition to the types of separable preferences defined above, we also use the following notion to indicate when the separability of a preference is compatible with a decomposition of a subset B of A: a preference P on A is *componentwise separable* with respect to the decomposition $B_1 \times \cdots \times B_q$ of $B \subset A$ if P is separable with respect to $C(B_j)$ for every $j = 1, \ldots, q$.

We now define some special functional forms of SCFs for multiple public good voting. If the set of feasible alternatives is a proper subset of $A = \prod_{k \in M} A_k$, then it is, in general, not possible to decompose the voting completely in the sense that the individuals in the society make independent choices from the m categories, because this may lead to infeasible alternatives. But if the set of feasible alternatives is decomposed as $B_1 \times \cdots \times B_q$, then society can choose without restrictions from every component B_j, and the choices from the different components can be combined in any conceivable way. Consider now $f : \mathcal{P}_A^n \to A$ and let $\mathcal{R}_f = B_1 \times \cdots \times B_q$ be some decomposition of $\mathcal{R}_f$, which is not necessarily maximal. We say that f is *dictatorial on a component B_j* if there is some individual i, called the *dictator on B_j*, such that $\pi_{B_j}(f(P_i, P_{-i})) = \pi_{B_j}(\tau_{\mathcal{R}_f}(P_i))$ for all profiles $(P_i, P_{-i}) \in \mathcal{P}_A^n$. It is possible that f is dictatorial on some components B_j but not on others, and two components B_j and $B_{j'}$ can have different individuals as dictators. If the same individual i is a dictator on every component B_j ($j = 1, \ldots, q$), then f is, of course, dictatorial in the usual sense.

For components with exactly two elements, one can define the following special form of voting rule: If $\#B_j = 2$ for some B_j in $\mathcal{R}_f = B_1 \times \cdots \times B_q$, i.e., $B_j = \{b_j^1, b_j^2\}$, then $f : \mathcal{P}_A^n \to A$ is *voting by committees* on B_j if there exist two nonempty families $\mathcal{W}_{b_j^1}$ and $\mathcal{W}_{b_j^2}$ of nonempty subsets of N, called the *winning coalitions* for b_j^1 and b_j^2, such that for $s = 1, 2$ and all $(P_1, \ldots, P_n) \in \mathcal{P}_A^n$, we have

$$\pi_{B_j}(f(P_1, \ldots, P_n)) = b_j^s \quad \Leftrightarrow \quad \{i \in N : \pi_{B_j}(\tau_{\mathcal{R}_f}(P_i)) = b_j^s\} \in \mathcal{W}_{b_j^s},$$

and, in addition, $\mathcal{W}_{b_j^s}$ satisfies *coalition monotonicity* in the sense that if $I \in \mathcal{W}_{b_j^s}$ and $I \subset J \subset N$, then also $J \in \mathcal{W}_{b_j^s}$. Intuitively, coalition monotonicity means that increasing the support for a certain alternative must not worsen that alternative's chances to be elected. A common example of voting by committees is *voting by quota*, where one of the alternatives, say b_j^1, needs to get the support of a quota $Q \in [0, 1]$ from the individuals to be elected, i.e., $I \in \mathcal{W}_{b_j^1}$ if and only if $\#I/n \geq Q$, and the special case $Q = 1/2$ corresponds of course to the majority rule. Another example of voting by committees is obtained when f is dictatorial on B_j with individual i being the dictator, and in this case $I \in \mathcal{W}_{b_j^s}$ if and only if $\{i\} \subset I$ ($s = 1, 2$). For the case when $\#A_k = 2$ for all $k \in M$, the domain $\mathcal{P}_A$ is either the domain of additive or weakly separable preferences, and $f : \mathcal{P}_A^n \to A$ is onto, Barberà et al. (1991) show that f is strategy-proof if and only if f is voting by committees on every A_k.

We conclude this section with a result from the literature that characterizes all strategy-proof SCFs on the domain of additive preferences in the presence of arbitrary range restrictions, but under the additional requirement that every category contains exactly two alternatives; this result is used in the following analysis.

Barberà et al.'s (2005) Theorem 1.[4] *Let $A = \prod_{k \in M} A_k$ be a product set of alternatives with $\#A_k = 2$ for all $k \in M$. An SCF $f : (\mathcal{P}_A^{\mathrm{add}})^n \to A$ is strategy-proof if and only if f has the following properties with respect to its maximal range decomposition $\mathcal{R}_f = B_1 \times \cdots \times B_q$.*

(i) *If $\#B_j = 2$, then f is voting by committees on B_j ($1 \le j \le q$).*

(ii) *If $\#B_j \ge 3$, then f is dictatorial on B_j ($1 \le j \le q$).*

3. Strategy-proof voting for multiple public goods under range restrictions

We now provide complete characterizations of the strategy-proof SCFs for multiple public good votings with arbitrary range restrictions for the three cases when preferences are additive, componentwise separable, or weakly separable.

3.1 *Additive preferences*

For the domain of additive preferences, Theorem 1 in Barberà et al. (2005) provides, under the additional assumption that every category contains exactly two alternatives, the complete characterization of the strategy-proof SCFs for constrained multiple public good voting cited in the previous section. It turns out that the same characterization also holds in the absence of the restriction $\#A_k = 2$ ($k \in M$), and we have the following result.

Theorem 1. *Let $A = \prod_{k \in M} A_k$ be a product set of m finite categories A_k. An SCF $f : (\mathcal{P}_A^{\mathrm{add}})^n \to A$ is strategy-proof if and only if f has the following properties with respect to its maximal range decomposition $\mathcal{R}_f = B_1 \times \cdots \times B_q$.*

(i) *If $\#B_j = 2$, then f is voting by committees on B_j ($1 \le j \le q$).*

(ii) *If $\#B_j \ge 3$, then f is dictatorial on B_j ($1 \le j \le q$).*

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[4]We have reformulated this result in the language of our model so as to apply and generalize this result appropriately. In the original voting model in Barberà et al. (2005), society must choose a subset of a set of m different objects under the constraint that not all possible subsets are feasible. Identifying each subset of the m objects with a binary m-dimensional characteristic vector, this voting problem of course turns out to be equivalent to choosing one element from the product set $A = \{0, 1\}^m$. Furthermore, the terminology in Barberà et al. (2005) translates as follows: *sections* and *active components* in Barberà et al. (2005) are called *components B_j* and *elements* in B_j, respectively, in our model, and for the range of the SCF, Barberà et al. (2005) obtain a *minimal Cartesian decomposition* (in the sense that no section can be decomposed further), which corresponds to our *maximal range decomposition* (where "maximal" refers to the number of components). Moreover, in Barberà et al. (2005), the SCF is defined by coordinatewise voting with coordination conditions on the winning coalitions so as to meet the range restrictions, while we define the SCF componentwise, which makes the coordination conditions redundant.

Theorem 1 states that a strategy-proof SCF $f : (\mathcal{P}_A^{\mathrm{add}})^n \to A$ with maximal range decomposition $\mathcal{R}_f = B_1 \times \cdots \times B_q$ must be *componentwise decomposable* in the sense that the choice from every component depends only on the individuals' marginal preferences over that component, which of course are well defined when preferences are additive. Moreover, to obtain strategy-proofness for the entire choice from $\mathcal{R}_f$, it is necessary and sufficient that the choice on every component is strategy-proof according to the results for "one-dimensional" voting, which means that f must be voting by committees on components with two elements and dictatorial on components with three or more elements. Note, however, that the second part of this result does *not* follow directly from the Gibbard–Satterthwaite theorem because if $\#C(B_j) \geq 2$, then additivity of preferences implies that the domain of marginal preferences on B_j is not unrestricted, and hence the Gibbard–Satterthwaite theorem cannot be applied to components with three or more elements. Instead, Theorem 1 can be derived from the corresponding result in Barberà et al. (2005) when the choice problem in Theorem 1 is reformulated in a convenient way, which is explained in the following example. In the formal proof of Theorem 1, which is provided in the Appendix, the arguments from Example 3 are made rigorous and general.

EXAMPLE 3. Consider the product set $A = A_1 \times A_2 \times A_3$, where $A_k = \{a_{k1}, a_{k2}, a_{k3}\}$ for $k = 1, 2, 3$. To rewrite the social choice problem of choosing one alternative from A into a multiple binary choice problem, start by identifying the alternatives in each category A_k with binary unit vectors of length three as follows:

$$a_{k1} \leftrightarrow (1, 0, 0), \qquad a_{k2} \leftrightarrow (0, 1, 0), \qquad a_{k3} \leftrightarrow (0, 0, 1), \quad k = 1, 2, 3.$$

Then every $a \in A$ can be identified with a binary vector of length 9 by joining the binary unit vectors corresponding to its three coordinates; for example, $a = (a_{12}, a_{23}, a_{31})$ corresponds to $(0, 1, 0, 0, 0, 1, 1, 0, 0)$. In this way, A can be identified with a proper subset of the binary product set $\bar{A} = \{0, 1\}^9$. Next, if $\bar{P}$ is a preference on $\bar{A}$, we associate $\bar{P}$ with that preference P on A that ranks the alternatives in A in the same way as $\bar{P}$ ranks the corresponding alternatives in $\bar{A}$.[5] Thereby, the additive preferences on $\bar{A}$ are associated with additive preferences on A: If $\bar{P} \in \mathcal{P}_{\bar{A}}^{\mathrm{add}}$ is represented by the utility function $\bar{u} = \sum_{l=1}^{9} \bar{u}_l$, define the utility function $u : A \to \mathbb{R}$ by letting $u(a)$ be equal to the utility assigned by $\bar{u}$ to the alternative in $\bar{A}$ that corresponds to $a \in A$; for example, if $a = (a_{12}, a_{23}, a_{31})$, set

$$u(a) = \bar{u}_1(0) + \bar{u}_2(1) + \bar{u}_3(0) + \bar{u}_4(0) + \bar{u}_5(0) + \bar{u}_6(1) + \bar{u}_7(1) + \bar{u}_8(0) + \bar{u}_9(0).$$

It is easily checked that u is an additive representation of the preference P on A that corresponds to $\bar{P}$. For any given SCF $f : (\mathcal{P}_A^{\mathrm{add}})^n \to A$, we can now define a corresponding SCF $\bar{f} : (\mathcal{P}_{\bar{A}}^{\mathrm{add}})^n \to \bar{A}$ as follows: If $(\bar{P}_1, \ldots, \bar{P}_n) \in (\mathcal{P}_{\bar{A}}^{\mathrm{add}})^n$ and $(P_1, \ldots, P_n) \in (\mathcal{P}_A^{\mathrm{add}})^n$ is a preference profile that corresponds to $(\bar{P}_1, \ldots, \bar{P}_n)$, then we let $\bar{f}(\bar{P}_1, \ldots, \bar{P}_n)$ be that alternative in $\bar{A}$ that corresponds to $f(P_1, \ldots, P_n) \in A$.

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[5]Note that since A corresponds only to a proper subset of $\bar{A}$, there exist for every P on A several $\bar{P}$ on $\bar{A}$ that rank the alternatives in $\bar{A}$, which correspond to alternatives in A, in the same way as P.

Suppose now, as a concrete example, that $f:(\mathcal{P}_A^{\mathrm{add}})^n \to A$ is a strategy-proof SCF with range $\mathcal{R}_f = ((A_1 \times A_2) \setminus \{(a_{11}, a_{21})\}) \times (A_3 \setminus \{a_{33}\})$. Then the maximal range decomposition of f is, of course, $B_1 \times B_2$, where $B_1 = (A_1 \times A_2) \setminus \{(a_{11}, a_{21})\}$ and $B_2 = A_3 \setminus \{a_{33}\}$. The SCF $\bar{f}:(\mathcal{P}_{\bar{A}}^{\mathrm{add}})^n \to \bar{A}$, which corresponds to f, is also strategy-proof and has the maximal range decomposition $\mathcal{R}_{\bar{f}} = \bar{B}_1 \times \bar{B}_2$, where, using the abbreviation $\hat{A} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$,

$$\bar{B}_1 = (\hat{A} \times \hat{A}) \setminus \{(1, 0, 0, 1, 0, 0)\} \quad \text{and} \quad \bar{B}_2 = \hat{A} \setminus \{(0, 0, 1)\}.$$

By Theorem 1 in Barberà et al. (2005), $\bar{f}$ is dictatorial on $\bar{B}_1$ and voting by committees on $\bar{B}_2$, and translating this functional structure to f gives that f must be dictatorial on B_1 and voting by committees on B_2.   ◇

REMARK 1. In Theorem 1, the functional form of f is formulated in such a way that the individuals vote for the elements in the components B_j. Alternatively, it would be possible to express the functional form of f in terms of how the individuals vote for the m categories A_k in A, but this would require introducing coordination conditions that ensure the feasibility of the overall outcome. Such coordination conditions are not uncommon in the literature on constrained multiple public good votings, but unfortunately they tend to be rather complicated. If the choice of f, as here, is described in terms of the components B_j, feasibility of the overall outcome is automatically guaranteed, and the characterization of f becomes more transparent and understandable.

3.2 *Componentwise separable preferences*

Intuitively, the characterization of strategy-proof SCFs in Theorem 1 as component-wise decomposable rules should not depend ultimately on the additivity of preferences, but rather on the fact that marginal preferences are well defined for every component B_j. The next theorem states that componentwise separability of preferences is indeed a sufficient condition to obtain the same decomposability characterization of strategy-proof SCFs as in Theorem 1.

THEOREM 2. *Let $A = \prod_{k \in M} A_k$ be a product set of m finite categories A_k. Let $\mathcal{P}_A$ be a domain of complete, transitive, and asymmetric preferences over A such that $\mathcal{P}_A^{\mathrm{add}} \subset \mathcal{P}_A$. Let further $f : \mathcal{P}_A^n \to A$ be an SCF with maximal range decomposition $\mathcal{R}_f = B_1 \times \cdots \times B_q$, and assume that every $P \in \mathcal{P}_A$ is componentwise separable with respect to $B_1 \times \cdots \times B_q$. Then f is strategy-proof if and only if f satisfies the following two conditions.*

  (i) *If $\#B_j = 2$, then f is voting by committees on B_j ($1 \le j \le q$).*

  (ii) *If $\#B_j \ge 3$, then f is dictatorial on B_j ($1 \le j \le q$).*

The assumption $\mathcal{P}_A^{\text{add}} \subset \mathcal{P}_A$ is a richness condition that allows us to use Theorem 1 in the proof of Theorem 2. Without this condition, the class of strategy-proof SCFs may be larger than specified by Theorem 2.[6]

For the domain $\mathcal{P}_A^{\text{cs}}$ of all completely separable preferences, the assumptions of Theorem 2 are, of course, always satisfied, regardless of the structure of the maximal range decomposition $B_1 \times \cdots \times B_q$, which leads to the following corollary.

COROLLARY 1. *Let $A = \prod_{k \in M} A_k$ be a product set of m finite categories A_k. An SCF $f : (\mathcal{P}_A^{\text{cs}})^n \to A$ is strategy-proof if and only if f has the following properties with respect to its maximal range decomposition $\mathcal{R}_f = B_1 \times \cdots \times B_q$.*

(i) *If $\#B_j = 2$, then f is voting by committees on B_j ($1 \leq j \leq q$).*

(ii) *If $\#B_j \geq 3$, then f is dictatorial on B_j ($1 \leq j \leq q$).*

REMARK 2. Corollary 1 confirms a conjecture in the concluding discussion in Svensson and Torstensson (2008): The main result in that paper provides a characterization of all strategy-proof SCFs $f : (\mathcal{P}_A^{\text{cs}})^n \to A$ under the additional assumptions that $\#A_k \geq 3$ for every $k \in M$ and that f is *weakly onto* in the sense that every alternative in every category is elected at least at some profile (formally, this means $\{\pi_{A_k}(a) : a \in \mathcal{R}_f\} = A_k$ for every $k \in M$). With these assumptions, every component in the maximal range decomposition of f satisfies $\#B_j \geq 3$, and Svensson and Torstensson (2008) obtain a componentwise dictatorial result that corresponds to case (ii) in Corollary 1. However, they conjecture that in the absence of their additional assumptions, Corollary 1 should hold.

3.3 *Weakly separable preferences*

For weakly separable preferences, Theorem 2 in Barberà et al. (2005) characterizes completely the strategy-proof SCFs for constrained multiple public good voting under the restriction that $\#A_k = 2$ for every $k \in M$. The following result provides the corresponding characterization for the general case, i.e., without making any assumptions on the cardinality of A_k except finiteness.

THEOREM 3. *Let $A = \prod_{k \in M} A_k$ be a product set of m finite categories A_k. An SCF $f : (\mathcal{P}_A^{\text{ws}})^n \to A$ is strategy-proof if and only if f has the following properties with respect to its maximal range decomposition $\mathcal{R}_f = B_1 \times \cdots \times B_q$.*

(i) *If $q = m$, then f is voting by committees on every B_j with $\#B_j = 2$ and dictatorial on every B_j with $\#B_j \geq 3$ ($1 \leq j \leq q$).*

(ii) *If $q < m$ and $\#\mathcal{R}_f \geq 3$, then f is dictatorial.*

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[6]If $\mathcal{P}_A$ is a domain of componentwise separable preferences, it is straightforward to check that any SCF $f : \mathcal{P}_A^n \to A$, which is voting by committees on components with two elements and dictatorial on components with at least three elements, is strategy-proof, even if $\mathcal{P}_A^{\text{add}} \not\subset \mathcal{P}_A$. But if $\mathcal{P}_A^{\text{add}} \not\subset \mathcal{P}_A$, there may be additional strategy-proof SCFs; for example, if $\mathcal{P}_A$ contains only componentwise separable preferences that also are multidimensional single-peaked as defined in Barberà et al. (1997), then it follows from Barberà et al. (1997) that there also exist strategy-proof SCFs $f : \mathcal{P}_A^n \to A$ that are generalized median voter schemes.

Part (i) of Theorem 3 follows, of course, as a special case from Theorem 2. Part (ii) states that if preferences are weakly separable, then in the case of a nontrivial range restriction (i.e., $q < m$), it is no longer possible to construct a strategy-proof SCF by composing it via componentwise strategy-proof rules, but instead there must be a single dictator on the entire range of the SCF. The intuition for this purely dictatorial result is explained in the following example.

EXAMPLE 4. Let $A = \{0, 1\} \times \{0, 1\} \times \{0, 1\}$ and suppose that $f : (\mathcal{P}_A^{ws})^2 \to A$ is a strategy-proof two-person SCF with range $\mathcal{R}_f = A \setminus \{(1, 1, 0), (1, 1, 1)\}$. By Example 2, the maximal decomposition of $\mathcal{R}_f$ is $B_1 \times B_2$, where $B_1 = \{(0, 0), (0, 1), (1, 0)\}$ and $B_2 = \{0, 1\}$. Since $\mathcal{P}_A^{cs} \subset \mathcal{P}_A^{ws}$, we can consider the restriction of f to profiles of completely separable preferences; denote this restriction by $\bar{f}$. By Corollary 1, $\bar{f}$ is dictatorial on B_1 and voting by committees on B_2, but suppose here for simplicity that $\bar{f}$ is also dictatorial on B_2. If B_1 and B_2 have the same individual as dictator, then $\bar{f}$ is dictatorial, and from standard extension arguments it follows then that f is also dictatorial. This is, however, the only possible scenario here, because if B_1 and B_2 have different dictators under $\bar{f}$, then it is possible to construct a profile at which f is manipulable, which is explained in the following discussion. Assume, therefore, that individual 1 is the dictator associated with B_1, while individual 2 is the dictator on B_2. Now allow individual 1 to extend his preference to $\mathcal{P}_A^{ws}$, while individual 2's preference still belongs to $\mathcal{P}_A^{cs}$; in particular, this means that the maximal element of P_1 in B_1 may now depend on whether 0 or 1 is chosen from B_2. For such profiles $(P_1, P_2) \in \mathcal{P}_A^{ws} \times \mathcal{P}_A^{cs}$, the choice from B_2 must still be individual 2's maximal element in B_2: If there were some $(\hat{P}_1, \hat{P}_2) \in \mathcal{P}_A^{ws} \times \mathcal{P}_A^{cs}$ such that $\pi_{B_2}(f(\hat{P}_1, \hat{P}_2)) \neq \pi_{B_2}(\tau_{\mathcal{R}_f}(\hat{P}_2))$, then individual 1 would be able to manipulate f at every profile $(\hat{P}_1', \hat{P}_2) \in (\mathcal{P}_A^{cs})^2$ where $\tau_{\mathcal{R}_f}(\hat{P}_1') = f(\hat{P}_1, \hat{P}_2)$ because $f(\hat{P}_1, \hat{P}_2) = \tau_{\mathcal{R}_f}(\hat{P}_1')$, while $f(\hat{P}_1', \hat{P}_2) \neq \tau_{\mathcal{R}_f}(\hat{P}_1')$ since $\pi_{B_2}(f(\hat{P}_1', \hat{P}_2)) = \pi_{B_2}(\tau_{\mathcal{R}_f}(\hat{P}_2)) \neq \pi_{B_2}(\tau_{\mathcal{R}_f}(\hat{P}_1'))$. For the choice from B_1, this has the following implication: If $(P_1, P_2) \in \mathcal{P}_A^{ws} \times \mathcal{P}_A^{cs}$, then $\pi_{B_1}(f(P_1, P_2))$ must be individual 1's maximal element in B_1 *conditional* on the choice made by individual 2 in B_2, because, otherwise, individual 1 could manipulate f by representing some $P_1' \in \mathcal{P}_A^{cs}$, which ranks this conditional maximal element at the (unconditional) top.

Consider now the profile $(\bar{P}_1, \bar{P}_2)$, where $\bar{P}_1$ and $\bar{P}_2$ are defined as

$$\cancel{(1, 1, 1)}\,\bar{P}_1\,(0, 1, 1)\,\bar{P}_1\,(1, 0, 1)\,\bar{P}_1\,(0, 0, 1)$$

$$\bar{P}_1\,\cancel{(1, 1, 0)}\,\bar{P}_1\,(1, 0, 0)\,\bar{P}_1\,(0, 1, 0)\,\bar{P}_1\,(0, 0, 0)$$

$$\cancel{(1, 1, 1)}\,\bar{P}_2\,\cancel{(1, 1, 0)}\,\bar{P}_2\,(1, 0, 1)\,\bar{P}_2\,(1, 0, 0)$$

$$\bar{P}_2\,(0, 1, 1)\,\bar{P}_2\,(0, 1, 0)\,\bar{P}_2\,(0, 0, 1)\,\bar{P}_2\,(0, 0, 0),$$

where alternatives that do not belong to $\mathcal{R}_f$ are scored out. By Example 1, $\bar{P}_1 \in \mathcal{P}_A^{ws} \setminus \mathcal{P}_A^{cs}$ and $\bar{P}_2 \in \mathcal{P}_A^{cs}$. From the preceding paragraph, we get first $\pi_{B_2}(f(\bar{P}_1, \bar{P}_2)) =$

$\pi_{B_2}(\tau_{\mathcal{R}_f}(\bar{P}_2)) = 1$, and given that 1 is chosen from B_2, individual 1's most preferred element in B_1 is $(0, 1)$, so $\pi_{B_1}(f(\bar{P}_1, \bar{P}_2)) = (0, 1)$. Thus, $f(\bar{P}_1, \bar{P}_2) = (0, 1, 1)$. But now individual 2 can manipulate f: If $\bar{P}_2' \in \mathcal{P}_A^{\mathrm{cs}}$ is some completely separable preference with $\pi_{B_2}(\tau_{\mathcal{R}_f}(\bar{P}_2')) = 0$, then $\pi_{B_2}(f(\bar{P}_1, \bar{P}_2')) = 0$ and hence $\pi_{B_1}(f(\bar{P}_1, \bar{P}_2')) = (1, 0)$, so $f(\bar{P}_1, \bar{P}_2') = (1, 0, 0)$; since $(1, 0, 0)\bar{P}_2(0, 1, 1)$, this means that individual 2 can manipulate f at $(\bar{P}_1, \bar{P}_2)$ via $\bar{P}_2'$. Thus, f can only be strategy-proof if the restriction of f to completely separable preferences has the same individual as dictator on B_1 and B_2, and it is then straightforward to show that this dictatorship extends to f itself.

The reason for individual 2's manipulation opportunity at $(\bar{P}_1, \bar{P}_2)$ should now be clear: f can only be strategy-proof if the choice in B_1 is the most preferred alternative of individual 1 given the choice of individual 2 in B_2, but if individual 1's most preferred alternative in B_1 depends on the choice in B_2, which is possible here because B_1 covers more than one category and preferences need only be weakly separable, then this dependency makes it possible for individual 2 to influence the choice in B_1 and hence also to manipulate f. Note that the inversion of individual 1's most preferred element in B_1, which is crucial for the argument above, is only possible when individual 1's unrestricted top element from the first two categories (in this example, the pair $(1, 1)$) is excluded from the range of f; in particular, this means that if there are no range restrictions (i.e., $\mathcal{R}_f = A$), then the manipulation opportunity indicated above does not exist.   ◇

REMARK 3. Theorem 3 *cannot* be obtained from the corresponding result in Barberà et al. (2005, Theorem 2), where $\#A_k = 2$, using the binary transformation that allowed us to derive our result for additive preferences from the corresponding result in Barberà et al. (2005) as illustrated in Example 3 for the following reason: In Example 3, we started by identifying the set of alternatives $A = A_1 \times A_2 \times A_3$ with a subset of the binary product set $\bar{A} = \{0, 1\}^9$. If $\bar{P}$ is now a weakly separable preference on $\bar{A}$, then $\bar{P}$ ranks each of the nine coordinates in $\bar{A}$ independently of the other coordinates, but the same is not necessarily true, for example, for the collection of the first three coordinates in $\bar{A}$, which correspond to the category A_1, and hence $\bar{P}$ does not necessarily correspond to a weakly separable preference on A. Therefore, the set of all preferences on A, which correspond to weakly separable preferences on $\bar{A}$, is a strict superdomain of $\mathcal{P}_A^{\mathrm{ws}}$, and the dictatorial result for the former domain, which can be obtained from Barberà et al. (2005, Theorem 2), cannot be used to infer the dictatorial result for the smaller domain $\mathcal{P}_A^{\mathrm{ws}}$. Instead, the proof of Theorem 3 is based on the ideas outlined in Example 4.

We conclude this section with two examples that illustrate how our results can be applied.

EXAMPLE 5. Suppose that $A = \prod_{k \in M} A_k$ is a product set and let $B = A \setminus \{a\}$ for some $a \in A$. This special form of range restriction is considered in Aswal et al. (2003), who show, using their criterion of *linked domains*, that if $A_k = \{0, 1\}$ for every $k \in M$ (in their paper, every category A_k corresponds to one of m candidates, which can be either elected or not elected) and preferences are weakly separable, then every strategy-proof SCF $f : (\mathcal{P}_A^{\mathrm{ws}})^n \to A$ whose range equals B must be dictatorial. However, B is obviously

indecomposable, and, therefore, it follows directly from our results that under the assumptions in Aswal et al. (2003), there is no nondictatorial strategy-proof SCF with range B, and, moreover, this dictatorial result also holds when $\#A_k \geq 2$ ($k \in M$) and also when preferences are completely separable or additive.　　　　　　　　　　　$\diamond$

EXAMPLE 6. Aswal et al. (2003) also consider the following slightly more complicated range restriction: There are again m candidates (i.e., $A_k = \{0, 1\}$ for $k \in M$), but the range is now $B = \{a \in A = \prod_{k \in M} A_k, K_1 \leq \sum_{k \in M} a_k \leq K_2\}$, where K_1 and K_2 are integers such that $0 < K_1 \leq K_2 < m$, which means that the number of elected candidates must be between K_1 and K_2. This kind of range restriction occurs, for example, when an understaffed department wants to employ at least K_1 new researchers and, due to budget restrictions, at most K_2 researchers can be employed. Using the criterion of linked domains, Aswal et al. (2003) show that there is no nondictatorial strategy-proof SCF $f : (\mathcal{P}_A^{\text{ws}})^n \to A$ with range B.

With our results, this dictatorial result can be established in a more general setting: Let $A = \prod_{k \in M} A_k$ be a product set of integer intervals $A_k = [\underline{\alpha}_k, \overline{\alpha}_k] = \{\underline{\alpha}_k, \underline{\alpha}_k + 1, \ldots, \overline{\alpha}_k\}$ and let $B = \{a \in A : K_1 \leq \sum_{k \in M} a_k \leq K_2\}$, where $\sum_{k \in M} \underline{\alpha}_k \leq K_1 \leq K_2 \leq \sum_{k \in M} \overline{\alpha}_k$ with at least two of the inequalities being strict. Then B is indecomposable,[7] and it follows, hence, from our results that if $\mathcal{P}_A$ is the domain of weakly separable, completely separable, or additive preferences over A, then there is no nondictatorial strategy-proof SCF $f : \mathcal{P}_A^n \to A$ with range B.　　　　　　　$\diamond$

4. RELATED LITERATURE AND CONCLUDING COMMENTS

The issue of strategy-proofness in multiple public good voting is studied from different perspectives in a number of papers in the literature. The most fundamental papers in this area, among which Border and Jordan (1983), Barberà et al. (1991), Le Breton and Sen (1999), Le Breton and Weymark (1999), Weymark (1999), and Nehring and Puppe (2007) should be mentioned, consider voting models where preferences are separable and the range is unrestricted (i.e., $\mathcal{R}_f = A$), and the papers differ mainly in their assumptions on the form of separability and on the cardinality of the categories A_k. The main common finding in these papers is that strategy-proof SCFs with full range must be decomposable across the categories A_k and be strategy-proof on every category.

The basic model for multiple public good voting has been modified in different ways, and our study contributes to a variant of the model where preferences are still separable, but the range is now allowed to be restricted. A first contribution in this direction is by Aswal et al. (2003), who consider the two special cases of range restrictions that were

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[7]The indecomposability of B can be shown as follows: We have $\sum_{k \in M} \underline{\alpha}_k < K_j < \sum_{k \in M} \overline{\alpha}_k$ either for $j = 1$ or $j = 2$, and suppose that this condition holds here for $j = 2$ (the case $j = 1$ is symmetric). Suppose that $B = B_1 \times B_2$ and let $a = (b_1, b_2) \in B_1 \times B_2$ be such that $\sum_{k \in M} a_k = K_2$. Then $\sum_{k \in M} \underline{\alpha}_k < \sum_{k \in M} a_k < \sum_{k \in M} \overline{\alpha}_k$, and hence either $\sum_{k \in C(B_1)} a_k > \sum_{k \in C(B_1)} \underline{\alpha}_k$ and $\sum_{k \in C(B_2)} a_k < \sum_{k \in C(B_2)} \overline{\alpha}_k$ or $\sum_{k \in C(B_1)} a_k < \sum_{k \in C(B_1)} \overline{\alpha}_k$ and $\sum_{k \in C(B_2)} a_k > \sum_{k \in C(B_2)} \underline{\alpha}_k$. Assume that the latter two inequalities hold. Now construct $a' = (b'_1, b'_2)$ by increasing one of the coordinates in b_1 by 1 and decreasing one of the coordinates in b_2 by 1. Then $\sum_{k \in M} a'_k = \sum_{k \in M} a_k = K_2$, so $a' \in B$. But $(b'_1, b_2) \notin B$ because $\sum_{k \in C(B_1)} a'_k + \sum_{k \in C(B_2)} a_k = K_2 + 1$. Thus, B cannot be decomposed as a product set.

explained and generalized in Examples 5 and 6. Barberà et al. (2005) and Svensson and Torstensson (2008) are so far the most general papers on multiple public good voting under constraints, in the sense that all possible kinds of range restrictions are allowed, but both papers impose additional assumptions on the cardinality of the categories A_k. In the literature, these two papers are closest to our study, and from the discussion in the Introduction and in the preceding section, it should be clear how our results are related to and generalize the results in these two papers.

Another interesting way to modify the basic model for multiple public good voting is to keep the full range assumption, but to impose a structure other than separability on preferences. In this context, special attention is given to multidimensional single-peaked preferences, and here we find the important contributions by Barberà et al. (1993) and Chichilnisky and Heal (1997). In Barberà et al. (1993), the set A of alternatives is a discrete multidimensional grid, and single-peaked preferences P are defined by the condition that if $a, b \in A$ are such that a belongs to the minimal box spanned by $\tau_A(P)$ and b, then $a \, P \, b$. In this framework, an SCF is strategy-proof if and only if it is categorywise decomposable and is a generalized median voter scheme on every category. Chichilnisky and Heal (1997) consider instead a continuous m-dimensional set of alternatives on which single-peaked preferences are symmetric in the sense that indifference curves are ellipsoids; strategy-proof SCFs are coordinatewise decomposable and they are characterized as locally constant or dictatorial rules.

Finally, it is, of course, possible to combine multidimensional single-peaked preferences with range restrictions, which is done in Serizawa (1996) and Barberà et al. (1997, 1998). Serizawa (1996) considers componentwise single-peaked preferences and a special form of range restrictions, which occur as a consequence of limited production resources in the economy, and strategy-proof SCFs are found to be generalized schemes of voting by committees. Barberà et al. (1997) start with the same assumptions as in Barberà et al. (1993), but now all kinds of range restrictions are allowed, and strategy-proof SCFs are characterized as generalized median voter schemes on every dimension together with a condition, the *intersection property*, that ensures the feasibility of elected alternatives. Barberà et al. (1998) establish a similar characterization in a continuous setting.

We conclude with a remark on how the voting problem studied in this paper can be considered further. Our results provide characterizations of all strategy-proof SCFs for constrained multiple public good voting for the three important domains of additive, completely separable, and weakly separable preferences. Even though these domains of separable preferences are considered frequently in the literature, they are only particular cases of domains of separable preferences. More generally, one can define a domain $\mathcal{P}_A^{\mathcal{S}}$ of separable preferences over A by specifying a set $\mathcal{S} \subset 2^M$ of subsets of M and letting $\mathcal{P}_A^{\mathcal{S}}$ be the set of all preferences over A that are separable with respect to every $S \in \mathcal{S}$.[8] It would then be interesting to find characterizations of all strategy-proof SCFs

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[8]Not all $\mathcal{S} \subset 2^M$ can be used to define a domain $\mathcal{P}_A^{\mathcal{S}}$ of separable preferences because separability with respect to certain groups of categories may also imply separability with respect to other groups of categories. For example, when every A_k is finite, then any preference that is separable with respect to $S, T \in \mathcal{S}$

$f : (\mathcal{P}_A^{\mathcal{S}})^n \to A$ for an arbitrary separability structure $\mathcal{S}$.[9] Obviously, such a characterization must depend on the interplay between $\mathcal{S}$ and the maximal range decomposition $B_1 \times \cdots \times B_q$, and generalizing the arguments in the proofs of Theorems 2 and 3, it can be shown that if $q \geq 2$, then there exist nondictatorial strategy-proof SCFs $f : (\mathcal{P}_A^{\mathcal{S}})^n \to A$, which are voting by committees on every B_j with $\#B_j = 2$ and componentwise dictatorial on every B_j with $\#B_j \geq 3$, if and only if every $P \in \mathcal{P}_A^{\mathcal{S}}$ has a well defined top element in every component B_j (i.e., independently of which elements are chosen from the other components). This condition is, of course, met when preferences are component-wise separable, which is the content of Theorem 2, but $C(B_j) \in \mathcal{S}$ for all j is only a sufficient, but not necessary condition to avoid a purely dictatorial result, which is illustrated in the following example.

EXAMPLE 7. Let $A = \{0,1\}^4$ and define $\mathcal{P}_A^{\mathcal{S}}$ by

$$\mathcal{S} = \{\{1\}, \{2\}, \{3\}, \{4\}, \{1,2\}, \{1,3\}, \{2,3\}\}. \tag{3}$$

Furthermore, let $B = B_1 \times B_2$, where $B_1 = \{(1,0,0), (0,1,0), (0,0,1)\}$ and $B_2 = \{0,1\}$. Then every $P \in \mathcal{P}_A^{\mathcal{S}}$ ranks $(1,0,0) \in B_1$ and $(0,1,0) \in B_1$ independently of whether 0 or 1 is chosen from B_2, because $\{1,2\} \in \mathcal{S}$ implies that $(1,0,0,0) \, P \, (0,1,0,0)$ if and only if $(1,0,0,1) \, P \, (0,1,0,1)$. Similarly, since $\{1,3\} \in \mathcal{S}$ and $\{2,3\} \in \mathcal{S}$, also $(1,0,0)$ and $(0,0,1)$ as well as $(0,1,0)$ and $(0,0,1)$ are ranked independently of the choice from B_2. From this it follows that the top of P in B_1 does not depend on the choice from B_2. Further, since $\{4\} \in \mathcal{S}$, also the top in B_2 is independent of the choice from B_1. Therefore, it is possible to construct nondictatorial (componentwise) strategy-proof SCFs $f : (\mathcal{P}_A^{\mathcal{S}})^n \to A$, whose range equals B.

Note, however, that this nondictatorial result holds even though there exist preferences $P \in \mathcal{P}_A^{\mathcal{S}}$ that are not separable with respect to $C(B_1) = \{1,2,3\}$. For example, it is straightforward to check that P defined by

$$(1,1,1,1) \, P \, (1,1,0,1) \, P \, (1,0,1,1) \, P \, (1,0,0,1) \, P \, (0,1,1,1) \, P \, (0,1,0,1)$$

$$P \, (0,0,1,1) \, P \, (0,0,0,1) \, P \, (1,1,1,0) \, P \, (1,1,0,0) \, P \, (1,0,1,0) \, P \, (0,1,1,0) \tag{4}$$

$$P \, (1,0,0,0) \, P \, (0,1,0,0) \, P \, (0,0,1,0) \, P \, (0,0,0,0)$$

belongs to $\mathcal{P}_A^{\mathcal{S}}$, but since $(0,1,1,0) \, P \, (1,0,0,0)$ while $(1,0,0,1) \, P \, (0,1,1,1)$, it follows that $C(B_1) \notin \mathcal{S}$.[10] $\diamond$

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is also separable with respect to $S \cap T$, i.e., $\mathcal{S}$ must be closed under intersections (Bradley et al. 2005, Proposition 4.2). In the "continuous" case, $\mathcal{S}$ must, in addition, also be closed under unions, differences, and symmetric differences (Gorman 1968).

[9]A similar problem is studied in Le Breton and Sen (1999), but from the following different perspective: In a model where $\#A_k \geq 3$ for all $k \in M$ and $\mathcal{R}_f = A$ (i.e., range restrictions are not permitted), they consider the special case when $\mathcal{S}$ is an arbitrary *partition* of M, which moreover can be different for different individuals, and they find that strategy-proof SCFs must be categorywise dictatorial and, in addition, satisfy a condition that ensures consistency with the individual separability structures.

[10]It might seem obvious that $C(B_1) \notin \mathcal{S}$ because $\mathcal{S}$ in (3) does not contain the set $\{1,2,3\}$. Hypothetically, it could have been the case that separability with respect to the $S \in \mathcal{S}$ specified in (3) also implies separability with respect to $\{1,2,3\}$, but P in (4) shows that this is not the case.

Example 7 shows that componentwise separability is not necessary for the existence of a nondictatorial strategy-proof SCF. However, we do not think that there exists a simple condition to decide whether a given combination of a separability structure $\mathcal{S}$ and a maximal range decomposition $B_1 \times \cdots \times B_q$ admits nondictatorial decomposable strategy-proof SCFs.

Appendix: The proofs

This appendix provides the proofs of Theorem 1, Theorem 2, and Theorem 3.

PROOF OF THEOREM 1. If $P \in \mathcal{P}_A^{\mathrm{add}}$, then for every j ($1 \leq j \leq q$), $\pi_{B_j}(\tau_{\mathcal{R}_f}(P))$ is the well defined most preferred element of P in B_j, independently of which element in B_{-j} it is combined with. Hence, if $f : (\mathcal{P}_A^{\mathrm{add}})^n \to A$ is voting by committees on every B_j with $\#B_j = 2$ and dictatorial on every B_j with $\#B_j \geq 3$, then f is strategy-proof.

For the *only if* part, assume now that f is strategy-proof. In the following proof, we employ Theorem 1 in Barberà et al. (2005) as indicated in Example 3 and we divide the proof into six steps.

Step 1: It suffices to prove the only if part for SCFs that are weakly onto.[11] Assume that the *only if* part has already been proved for all weakly onto SCFs and consider some SCF $f : (\mathcal{P}_A^{\mathrm{add}})^n \to A$ that is not weakly onto. This means that there is some category $A_{\hat{k}}$ and some $\hat{a}_{\hat{k}} \in A_{\hat{k}}$ such that $\hat{a}_{\hat{k}} \notin \pi_{A_{\hat{k}}}(\mathcal{R}_f)$.[12] Every profile $(P_1, \ldots, P_n) \in (\mathcal{P}_A^{\mathrm{add}})^n$ of additive preferences can be identified with a profile $(\sum_{k \in M} u_{1k}, \ldots, \sum_{k \in M} u_{nk})$ of corresponding additive utility functions, and f can hence be considered as a function of the numbers $\{u_{ik}(a_{l_k})\}_{i \in N, k \in M, 1 \leq l_k \leq \#A_k}$. However, since $\hat{a}_{\hat{k}} \notin \pi_{A_{\hat{k}}}(\mathcal{R}_f)$, f must be independent of the numbers $\{u_{i\hat{k}}(\hat{a}_{\hat{k}})\}_{i \in N}$, because if there existed $P_i, P_i' \in \mathcal{P}_A^{\mathrm{add}}$ and $P_{-i} \in (\mathcal{P}_A^{\mathrm{add}})^{n-1}$ such that P_i and P_i' rank all $a \in \mathcal{R}_f$ identically but $f(P_i, P_{-i}) \neq f(P_i', P_{-i})$, then individual i would be able to manipulate f either at (P_i, P_{-i}) or at (P_i', P_{-i}). Set $\hat{A}_{\hat{k}} = A_{\hat{k}} \setminus \{\hat{a}_{\hat{k}}\}$ and $\hat{A} = A_1 \times \cdots \times \hat{A}_{\hat{k}} \times \cdots \times A_m$, and let $\hat{f} : (\mathcal{P}_{\hat{A}}^{\mathrm{add}})^n \to \hat{A}$ be the restriction of f to $(\{u_{ik}(a_{l_k})\}_{i \in N, k \in M, 1 \leq l_k \leq \#A_k}) \setminus (\{u_{i\hat{k}}(\hat{a}_{\hat{k}})\}_{i \in N})$. Then $\hat{f}$ is strategy-proof because f is strategy-proof, and, moreover, since $\mathcal{R}_{\hat{f}} = \mathcal{R}_f$, $\hat{f}$ and f have the same maximal range decomposition. If $\hat{f}$ is weakly onto, $\hat{f}$ is voting by committees on every B_j with $\#B_j = 2$ and dictatorial on every B_j with $\#B_j \geq 3$, and f must then have the same properties. If $\hat{f}$ is not weakly onto, continue eliminating alternatives that cannot be elected until a weakly onto SCF is obtained. Hence, it suffices to prove the *only if* part for weakly onto SCFs.

Step 2: Define a binary transformation T_A of A. For $k \in M$, set $\alpha_k \equiv \#A_k$ and let $A_k = \{a_{l_k}^k\}_{l_k=1}^{\alpha_k}$ be indexations of the elements in A_k. Define $T_k : A_k \to \{0, 1\}^{\alpha_k}$ by $T_k(a_{l_k}^k) = e_{l_k}$, where e_{l_k} denotes the l_kth unit vector of length α_k. Set $\bar{A} = \prod_{k=1}^m \{0, 1\}^{\alpha_k}$ and define $T_A : A \to \bar{A}$ by $T_A(a_1, \ldots, a_m) = (T_1(a_1), \ldots, T_m(a_m))$. Note that T_A is injective and for all $a \in A$, $T_A(a)$ contains exactly m ones, the other entries being zero.

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[11]An SCF $f : \mathcal{P}_A^n \to A$ is *weakly onto* if $\{\pi_{A_k}(a) : a \in \mathcal{R}_f\} = A_k$ for every $k \in M$.

[12]Here, the following convention is used: When $\varphi : X \to Y$ is a mapping and $Z \subset X$, then $\varphi(Z)$ denotes the set $\varphi(Z) = \{\varphi(x) : x \in Z\}$.

Step 3: Define a surjective mapping $Q: \mathcal{P}_{\bar{A}}^{\mathrm{add}} \to \mathcal{P}_A^{\mathrm{add}}$ *that preserves the ranking of alternatives in* $\bar{A}$ *and* A *that correspond to each other under* T_A. For $\bar{P} \in \mathcal{P}_{\bar{A}}^{\mathrm{add}}$, define the preference $Q(\bar{P})$ on A by setting $a \, Q(\bar{P}) \, a'$ for $a, a' \in A$ if and only if $T_A(a) \, \bar{P} \, T_A(a')$. To show that $Q(\bar{P}) \in \mathcal{P}_A^{\mathrm{add}}$, suppose that $\bar{P}$ is represented by the utility function $\bar{u} = \sum_{k=1}^m \sum_{l_k=1}^{\alpha_k} \bar{u}_{l_k}^k$, where $\bar{u}_{l_k}^k : \{0, 1\} \to \mathbb{R}$, and after a linear transformation of $\bar{u}$ if necessary, we can assume that $\bar{u}_{l_k}^k(0) = 0$ for $k \in M$ and $1 \le l_k \le \alpha_k$. Introduce utility functions $u_k : A_k \to \mathbb{R}$ by setting $u_k(a_{l_k}) = \bar{u}_{l_k}^k(1)$ for $k \in M$ and $1 \le l_k \le \alpha_k$, and define $u : A \to \mathbb{R}$ by $u(a) = \sum_{k=1}^m u_k(a_k)$ for $a = (a_1, \dots, a_m) \in A$. Then $u(a) = \bar{u}(T_A(a))$, so u is an additive representation of $Q(\bar{P})$. Furthermore, since $\bar{P}$ is asymmetric, $Q(\bar{P})$ is also asymmetric. Hence, $Q(\bar{P}) \in \mathcal{P}_A^{\mathrm{add}}$.

It remains to show that Q is surjective. Consider some arbitrary but fixed $P \in \mathcal{P}_A^{\mathrm{add}}$ and let P be represented by $u = \sum_{k=1}^m u_k$, where $u_k : A_k \to \mathbb{R}$ for $k \in M$. Since every A_k is finite, we can, after small adjustments, assume that every u_k takes values only in $\mathbb{Q}$, and after multiplying u by a convenient factor, every u_k can, in fact, be assumed to be integer-valued. Define now, for every $k \in M$ and $1 \le l_k \le \alpha_k$, utility functions $\bar{u}_{l_k}^k : \{0, 1\} \to \mathbb{R}$ by setting $\bar{u}_{l_k}^k(0) = 0$ and[13]

$$\bar{u}_{l_k}^k(1) = u_k(a_{l_k}) + 2^{-(l_k + \sum_{k'=1}^{k-1} \alpha_{k'})}, \tag{5}$$

and let $\bar{P}$ be that preference on $\bar{A}$ that is represented by the utility function $\bar{u} = \sum_{k=1}^m \sum_{l_k}^{\alpha_k} \bar{u}_{l_k}^k$. The terms $2^{-(l_k + \sum_{k'=1}^{k-1} \alpha_{k'})}$ in (5) work as a tie-breaking device and ensure that $\bar{P}$ is asymmetric. Hence, $\bar{P} \in \mathcal{P}_{\bar{A}}^{\mathrm{add}}$. Since $\sum_{z=1}^Z 2^{-z} < 1$ for every $Z < \infty$, we have $u(a) > u(a')$ if and only if $\bar{u}(T_A(a)) > \bar{u}(T_A(a'))$ for all $a, a' \in A$. Therefore, $Q(\bar{P}) = P$ and Q is surjective.

Step 4: Associate to $f : (\mathcal{P}_A^{\mathrm{add}})^n \to A$ *a strategy-proof SCF* $\bar{f} : (\mathcal{P}_{\bar{A}}^{\mathrm{add}})^n \to \bar{A}$. Define the SCF $\bar{f} : (\mathcal{P}_{\bar{A}}^{\mathrm{add}})^n \to \bar{A}$ by $\bar{f}(\bar{P}_1, \dots, \bar{P}_n) = T_A(f(Q(\bar{P}_1), \dots, Q(\bar{P}_n)))$ for all $(\bar{P}_1, \dots, \bar{P}_n) \in (\mathcal{P}_{\bar{A}}^{\mathrm{add}})^n$. Then for all $(\bar{P}_i, \bar{P}_{-i}) \in (\mathcal{P}_{\bar{A}}^{\mathrm{add}})^n$ and $\bar{P}_i' \in \mathcal{P}_{\bar{A}}^{\mathrm{add}}$, we have[14]

$$\bar{f}(\bar{P}_i', \bar{P}_{-i}) \, \bar{P}_i \, \bar{f}(\bar{P}_i, \bar{P}_{-i}) \quad \Leftrightarrow \quad f(Q(\bar{P}_i'), Q(\bar{P}_{-i})) \, Q(\bar{P}_i) \, f(Q(\bar{P}_i), Q(\bar{P}_{-i})),$$

and since f is strategy-proof, it follows that $\bar{f}$ is strategy-proof.

Step 5: Let $\mathcal{R}_f = B_1 \times \cdots \times B_q$ *be the unique maximal decomposition of* $\mathcal{R}_f$ *according to* *Svensson and Torstensson (2008, Proposition 1). If the binary transformation corresponding to the categories covered by* B_j *($1 \le j \le q$) are denoted by* $T_{B_j} : \prod_{k \in C(B_j)} A_k \to \{0, 1\}^{\sum_{k \in C(B_j)} \alpha_k}$, *then* $\mathcal{R}_{\bar{f}} = T_{B_1}(B_1) \times \cdots \times T_{B_q}(B_q)$ *is the unique maximal decomposition of* $\mathcal{R}_{\bar{f}}$. From the definition of $\bar{f}$, it follows that $\mathcal{R}_{\bar{f}} = T_A(\mathcal{R}_f) = T_{B_1}(B_1) \times \cdots \times T_{B_q}(B_q)$. It remains to show that $T_{B_1}(B_1) \times \cdots \times T_{B_q}(B_q)$ cannot be decomposed further. Suppose, therefore, so as to obtain a contradiction, that for some j ($1 \le j \le q$), there exists a decomposition $T_{B_j}(B_j) = \bar{B}_j^1 \times \bar{B}_j^2$. Consider some $k \in C(B_j)$. For every $\bar{b}_j \in T_{B_j}(B_j)$,

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[13]For $k = 1$, we use in (5) the convention $\sum_{k'=1}^0 \alpha_{k'} = 0$.

[14]Making a slight abuse of notation, $Q(\bar{P}_{-i})$ denotes here the profile obtained when Q is applied to every preference in $\bar{P}_{-i}$.

exactly one of the coordinates in $\bar{b}_j$ corresponding to A_k equals 1, and, conversely, since f is weakly onto, for every coordinate in $T_{B_j}(B_j)$ corresponding to A_k there is some $\bar{b}_j \in T_{B_j}(B_j)$ for which this coordinate is equal to 1. Therefore, the coordinates in $T_{B_j}(B_j)$ that correspond to A_k must either belong completely to $\bar{B}_j^1$ or completely to $\bar{B}_j^2$. This means that T_{B_j} can be written as $(T_{B_j}^1, T_{B_j}^2)$, where $T_{B_j}^s$ is the binary transformation of the categories A_k that belong to $\bar{B}_j^s$ ($s = 1, 2$). But then $B_j = (T_{B_j}^1)^{-1}(\bar{B}_j^1) \times (T_{B_j}^2)^{-1}(\bar{B}_j^2)$, which contradicts the assumption that $\mathcal{R}_f = B_1 \times \cdots \times B_q$ is the maximal decomposition of $\mathcal{R}_f$. Thus, $\mathcal{R}_{\bar{f}} = T_{B_1}(B_1) \times \cdots \times T_{B_q}(B_q)$ must be the unique maximal decomposition of $\mathcal{R}_{\bar{f}}$.

Step 6: Apply Theorem 1 in Barberà et al. (2005) to $\bar{f}$, and translate the structure of $\bar{f}$ to f. By Theorem 1 in Barberà et al. (2005), $\bar{f}$ is voting by committees on $T_{B_j}(B_j)$ with $\#T_{B_j}(B_j) = 2$ and dictatorial on $T_{B_j}(B_j)$ with $\#T_{B_j}(B_j) \geq 3$. To show that f has a corresponding structure, consider first some component B_j with $\#B_j = 2$. Then $\bar{f}$ is voting by committees on $T_{B_j}(B_j)$. Because $\pi_{B_j} = T_{B_j}^{-1} \circ \pi_{T_{B_j}(B_j)} \circ T$ and $\tau_{\mathcal{R}_f} \circ Q = T^{-1} \circ \tau_{\mathcal{R}_{\bar{f}}}$, we have

$$\left\{ i \in N : \pi_{B_j}(\tau_{\mathcal{R}_f}(Q(\bar{P}_i))) = b_j \right\} = \left\{ i \in N : \pi_{T_{B_j}(B_j)}(\tau_{\mathcal{R}_{\bar{f}}}(\bar{P}_i)) = T_{B_j}(b_j) \right\}$$

for all $(\bar{P}_1, \ldots, \bar{P}_n) \in (\mathcal{P}_{\bar{A}}^{\mathrm{add}})^n$ and $b_j \in B_j$; since, furthermore, $\pi_{B_j}(f(Q(\bar{P}_1), \ldots, Q(\bar{P}_n))) = b_j$ if and only if $\pi_{T_{B_j}(B_j)}(\bar{f}(\bar{P}_1, \ldots, \bar{P}_n)) = T_{B_j}(b_j)$ and Q is surjective, the winning coalitions for $T_{B_j}(b_j)$ under $\bar{f}$ must be exactly the same as the winning coalitions for b_j under f. Thus, f is voting by committees on B_j. By a similar argument, it follows that if $\#B_j \geq 3$, then f is dictatorial on B_j.   □

In the proofs of Theorem 2 and Theorem 3, we use the following monotonicity property of strategy-proof SCFs, which is frequently used in the literature and goes back to Muller and Satterthwaite (1977).

LEMMA 1 (Monotonicity). *Let $\mathcal{P}_A$ be a domain of complete, transitive, and asymmetric preferences over a set A, and suppose that $f : \mathcal{P}_A^n \to A$ is a strategy-proof SCF. If $f(P_1, \ldots, P_n) = a$ for some profile $(P_1, \ldots, P_n) \in \mathcal{P}_A^n$, and $(P_1', \ldots, P_n') \in \mathcal{P}_A^n$ is such that $a P_i a'$ implies $a P_i' a'$ for all $i \in N$ and $a' \in A$, then also $f(P_1', \ldots, P_n') = a$.*

PROOF OF THEOREM 2. If f is voting by committees on every B_j with $\#B_j = 2$ and dictatorial on every B_j with $\#B_j \geq 3$, then it follows by the same argument as in the beginning of the proof of Theorem 1 that f is strategy-proof.

For the *only if* part, assume now that f is strategy-proof. Let $\hat{f} : (\mathcal{P}_A^{\mathrm{add}})^n \to A$ be the restriction of f to profiles of additive preferences. Since f is strategy-proof, $\hat{f}$ is also strategy-proof. Moreover, $\mathcal{R}_{\hat{f}} = \mathcal{R}_f$, because if $a \in \mathcal{R}_f$ so that $f(P_1, \ldots, P_n) = a$ for some $(P_1, \ldots, P_n) \in \mathcal{P}_A^n$, we can pick some $P_a \in \mathcal{P}_A^{\mathrm{add}}$ with $\tau_A(P_a) = a$ and get, by Lemma 1, that $\hat{f}(P_a, \ldots, P_a) = a$. In particular, this means that $\hat{f}$ has the same maximal range decomposition $B_1 \times \cdots \times B_q$ as f. By Theorem 1, $\hat{f}$ is voting by committees on every B_j with

$\#B_j = 2$ and dictatorial on every B_j with $\#B_j \geq 3$, and we show now that this functional structure extends to f.

Consider first some component B_j with $\#B_j \geq 3$, and let individual i be the dictator associated with B_j by $\hat{f}$, that is, $\pi_{B_j}(\hat{f}(P_i, P_{-i})) = \pi_{B_j}(\tau_{\mathcal{R}_{\hat{f}}}(P_i))$ for all $(P_i, P_{-i}) \in (\mathcal{P}_A^{\mathrm{add}})^n$. Suppose, to obtain a contradiction, that individual i is not a dictator on B_j under f, i.e., there is some profile $(\bar{P}_i, \bar{P}_{-i}) \in \mathcal{P}_A^n$ such that $f(\bar{P}_i, \bar{P}_{-i}) = \bar{a}$, but $\pi_{B_j}(\bar{a}) \neq \pi_{B_j}(\tau_{\mathcal{R}_f}(\bar{P}_i))$. Take some $\bar{P}' \in \mathcal{P}_A^{\mathrm{add}}$ with $\tau_A(\bar{P}') = \bar{a}$, and set $\bar{P}'_{-i} = (\bar{P}', \ldots, \bar{P}') \in (\mathcal{P}_A^{\mathrm{add}})^{n-1}$. By Lemma 1, $f(\bar{P}_i, \bar{P}'_{-i}) = \bar{a}$. Let $\bar{a}' \in \mathcal{R}_f$ be the alternative that satisfies $\pi_{B_j}(\bar{a}') = \pi_{B_j}(\tau_{\mathcal{R}_f}(\bar{P}_i))$ and $\pi_{-B_j}(\bar{a}') = \pi_{-B_j}(\bar{a})$. Since $\bar{P}_i$ is separable with respect to $C(B_j)$, we have $\bar{a}' \bar{P}_i \bar{a}$. Now if $\bar{P}'_i \in \mathcal{P}_A^{\mathrm{add}}$ is such that $\tau_A(\bar{P}'_i) = \bar{a}'$, then $f(\bar{P}'_i, \bar{P}'_{-i}) = \hat{f}(\bar{P}'_i, \bar{P}'_{-i}) = \bar{a}'$, which contradicts strategy-proofness of f at $(\bar{P}_i, \bar{P}'_{-i})$. Thus, individual i must be a dictator on B_j under f.

Now consider some component B_j with $\#B_j = 2$ and let $B_j = \{b_j^1, b_j^2\}$. Let $\mathcal{W}_{b_j^s}$ be the winning coalitions for b_j^s under $\hat{f}$ ($s = 1, 2$). Using arguments similar to those in the preceding paragraph, we show that the same coalition structure applies for f on B_j. Suppose, therefore, that there exists some profile $(\bar{P}_1, \ldots, \bar{P}_n) \in \mathcal{P}_A^n$ such that the coalition $\bar{W} = \{i \in N : \pi_{B_j}(\tau_{\mathcal{R}_f}(\bar{P}_i)) = b_j^1\}$ belongs to $\mathcal{W}_{b_j^1}$, but $f(\bar{P}_1, \ldots, \bar{P}_n) = \bar{a}$ and $\pi_{B_j}(\bar{a}) = b_j^2$. Let $\bar{P}' \in \mathcal{P}_A^{\mathrm{add}}$ be such that $\tau_A(\bar{P}') = \bar{a}$, and set $\bar{P}'_{-\bar{W}} = (\bar{P}', \ldots, \bar{P}') \in (\mathcal{P}_A^{\mathrm{add}})^{n-\#\bar{W}}$. Then, by Lemma 1, $f(\bar{P}_{\bar{W}}, \bar{P}'_{-\bar{W}}) = \bar{a}$. Furthermore, set $\bar{a}' = (b_j^1, \bar{a}_{-j})$, and note that $\bar{a}' \in \mathcal{R}_f$ and $\bar{a}' \bar{P}_i \bar{a}$ for every $i \in \bar{W}$. Let $\bar{P}'' \in \mathcal{P}_A^{\mathrm{add}}$ be such that $\tau_A(\bar{P}'') = \bar{a}'$ and set $\bar{P}''_{\bar{W}} = (\bar{P}'', \ldots, \bar{P}'') \in (\mathcal{P}_A^{\mathrm{add}})^{\#\bar{W}}$. Then $f(\bar{P}''_{\bar{W}}, \bar{P}'_{-\bar{W}}) = \hat{f}(\bar{P}''_{\bar{W}}, \bar{P}'_{-\bar{W}}) = \bar{a}'$. But this means that if we replace the preferences from $(\bar{P}_{\bar{W}}, \bar{P}'_{-\bar{W}})$ to $(\bar{P}''_{\bar{W}}, \bar{P}'_{-\bar{W}})$ successively one at a time, there must be some $i \in \bar{W}$ that can manipulate f. Hence, every $W \in \mathcal{W}_{b_j^1}$ is a winning coalition for b_j^1 under f and, by symmetry, a corresponding statement must be true for every $W \in \mathcal{W}_{b_j^2}$. Thus, f is voting by committees on B_j. □

PROOF OF THEOREM 3. (i) The case $q = m$ is a direct consequence of Theorem 2.

(ii) For the case $q < m$, note first that a dictatorial SCF obviously is strategy-proof. For the converse implication, assume now that $f : (\mathcal{P}_A^{\mathrm{ws}})^n \to A$ is strategy-proof. To begin with, we consider only the case when $n = 2$. Let $\bar{f} : (\mathcal{P}_A^{\mathrm{cs}})^2 \to A$ be the restriction of $f : (\mathcal{P}_A^{\mathrm{ws}})^2 \to A$ to completely separable preferences. Since f is strategy-proof, it follows immediately that $\bar{f}$ is also strategy-proof. Moreover, $\mathcal{R}_{\bar{f}} = \mathcal{R}_f$, because if $a \in \mathcal{R}_f$ so that $f(P_1, P_2) = a$ for some $(P_1, P_2) \in (\mathcal{P}_A^{\mathrm{ws}})^2$, we can pick some $P_a \in \mathcal{P}_A^{\mathrm{cs}}$ with $\tau_A(P_a) = a$ and get, by Lemma 1, that $\bar{f}(P_a, P_a) = a$. In particular, this means that $\bar{f}$ has the same maximal range decomposition $B_1 \times \cdots \times B_q$ as f. By Theorem 2, $\bar{f}$ is voting by committees on every B_j with $\#B_j = 2$ and dictatorial on every B_j with $\#B_j \geq 3$. Since $q < m$, the maximal range decomposition $B_1 \times \cdots \times B_q$ must contain some component, say B_1, such that $\#C(B_1) \geq 2$. We now consider the two cases $\#B_1 \geq 3$ and $\#B_1 = 2$ separately.

Case 1: When $\#B_1 \geq 3$. If $\#B_1 \geq 3$, then $\bar{f}$ is dictatorial on B_1, and assume without loss of generality that $i = 1$ is the dictator associated with B_1. Introduce first some simplifying notational conventions: Assume that the categories in $A = A_1 \times \cdots \times A_m$

are ordered in such a way that all categories belonging to B_1 are collected leftmost in A, then all categories belonging to B_2 follow, and so on; formally, this means that if $k \in C(B_j)$ and $k' \in C(B_{j'})$, then $j < j'$ implies $k < k'$. To define preferences on A in a convenient way below, identify each A_k with the integer interval $\{0, 1, \ldots, \#A_k - 1\}$. Further assume without loss of generality that $b_j^\circ \equiv (0, \ldots, 0) \in B_j$ for all $j \geq 2$. Note also that since $\#C(B_1) \geq 2$, there must be some $b_1^\circ \in \prod_{k \in C(B_1)} \pi_{A_k}(B_1)$ such that $b_1^\circ \notin B_1$, because otherwise B_1 could be decomposed as $B_1 = \prod_{k \in C(B_1)} \pi_{A_k}(B_1)$, and assume that $b_1^\circ \equiv (0, \ldots, 0)$. Suppose further that the categories A_k belonging to B_1 are ordered in such a way that, among the elements in B_1 with a maximal number of zeros, there is at least one element $\hat{b}_1$ such that all zeros are collected "rightmost" in $\hat{b}_1$, and denote the set of all $\hat{b}_1$ with the precisely described property by $\hat{B}_1$. Formally, $\hat{b}_1 \in \hat{B}_1$ if (i) $\hat{b}_1$ maximizes $\#\{k \in C(B_1) : \pi_{A_k}(b_1) = 0\}$ among all $b_1 \in B_1$, and (ii) if $\pi_{A_k}(\hat{b}_1) = 0$ and $k + 1 \in C(B_1)$, then $\pi_{A_{k+1}}(\hat{b}_1) = 0$.

In the following part of the proof, we show that $i = 1$ must be a dictator for $\bar{f}$; from this it then follows almost directly that $i = 1$ is also a dictator for f. To obtain a contradiction, assume now that there is some component B_j for which $i = 1$ is not a dictator under $\bar{f}$. Now we prove in five steps that it is then possible to construct a profile at which f is manipulable.

Step 1: Define preferences $\bar{P}_1 \in \mathcal{P}_A^{\text{ws}} \setminus \mathcal{P}_A^{\text{cs}}$ and $\bar{P}_2, \bar{P}_2' \in \mathcal{P}_A^{\text{cs}}$. Let $\alpha \equiv \max_{k \in M}\{\#A_k\}$ and define for $i = 1$, the preference $\bar{P}_1 \in \mathcal{P}_A^{\text{ws}} \setminus \mathcal{P}_A^{\text{cs}}$ by the utility function

$$
\bar{u}_1(a) = \begin{cases} -\displaystyle\sum_{k \in M} a_k \alpha^k & \text{if } a_k = 0 \text{ for all } k \in M \setminus C(B_1) \\[2ex] -\displaystyle\sum_{k \in C(B_1)} a_k \alpha^{\#C(B_1)-k+1} - \displaystyle\sum_{k \in M \setminus C(B_1)} a_k \alpha^k & \text{if } a_k \neq 0 \text{ for some } k \in M \setminus C(B_1). \end{cases}
$$

Intuitively, $\bar{P}_1$ is lexicographic on $B_2 \times \cdots \times B_q$ with priority decreasing from the right to the left. Moreover, the component B_1 has least priority and the priority order within B_1 depends on whether $b_{-1} = (0, \ldots, 0)$. Let $\pi_{B_1}(\bar{P}_1 | b_{-1})$ be the marginal preference of $\bar{P}_1$ on B_1 conditional on $b_{-1} \in B_2 \times \cdots \times B_q$ being fixed. For $b_{-1}^\circ = (0, \ldots, 0)$, define $\bar{b}_1 \equiv \tau_{B_1}(\pi_{B_1}(\bar{P}_1 | b_{-1}^\circ))$ and note that $\bar{b}_1 \in \hat{B}_1$. Define further $\bar{b}_1' \equiv \tau_{B_1}(\pi_{B_1}(\bar{P}_1 | b_{-1}))$ for $b_{-1} \neq (0, \ldots, 0)$ and note that we must have $\pi_{A_1}(\bar{b}_1') = 0$ and thus $\bar{b}_1' \neq \bar{b}_1$ because $\pi_{A_1}(\bar{b}_1) = 0$.

Next, let $\bar{P}_2 \in \mathcal{P}_A^{\text{cs}}$ be the lexicographic preference on A given by the utility function $\bar{u}_2(a) = -\sum_{k=1}^m a_k \alpha^{m-k}$, whose priority thus is decreasing from the left to the right.

Finally, to define $\bar{P}_2'$, consider each component B_j in $B_1 \times \cdots \times B_q$ on which $i = 1$ is not a dictator under $\bar{f}$. If $\#B_j \geq 3$, then $i = 2$ must be a dictator on B_j under $\bar{f}$; in this case, let $\bar{b}_j' \in B_j$ be some arbitrary element in B_j with $\bar{b}_j' \neq (0, \ldots, 0)$. If $\#B_j = 2$, then there must be at least one $\bar{b}_j' \in B_j$ for which $\{2\}$ is a winning coalition under $\bar{f}$, and assume without loss of generality that $\bar{b}_j' \neq (0, \ldots, 0)$. In both cases, we get that if $P_2 \in \mathcal{P}_A^{\text{cs}}$ is such that $\pi_{B_j}(\tau_{\mathcal{R}_f}(P_2)) = \bar{b}_j'$, then $\pi_{B_j}(\bar{f}(P_1, P_2)) = \bar{b}_j'$ for all $P_1 \in \mathcal{P}_A^{\text{cs}}$. Now let $\bar{P}_2' \in \mathcal{P}_A^{\text{cs}}$ be such that $\pi_{B_j}(\tau_{\mathcal{R}_f}(\bar{P}_2')) = \bar{b}_j'$ on every B_j on which $i = 1$ is not a dictator.

Step 2: We have $f(\bar{P}_1, \bar{P}_2) = \tau_{\mathcal{R}_f}(\bar{P}_1)$ *and, in particular,* $\pi_{B_1}(f(\bar{P}_1, \bar{P}_2)) = \bar{b}_1$. If $P_1 \in \mathcal{P}_A^{cs}$ is such that $\tau_{\mathcal{R}_f}(P_1) = \tau_{\mathcal{R}_f}(\bar{P}_1)$, then $f(P_1, \bar{P}_2) = \bar{f}(P_1, \bar{P}_2) = \tau_{\mathcal{R}_f}(P_1)$, because on B_1, $i = 1$ is a dictator, and on B_j with $j \geq 2$, we have the unanimous vote $\pi_{B_j}(\tau_{\mathcal{R}_f}(P_1)) = \pi_{B_j}(\tau_{\mathcal{R}_f}(\bar{P}_2)) = (0, \dots, 0)$. Hence, by Lemma 1, $f(\bar{P}_1, \bar{P}_2) = \tau_{\mathcal{R}_f}(\bar{P}_1)$.

Step 3: For $j \geq 2$, $\pi_{B_j}(f(\bar{P}_1, \bar{P}_2')) = \pi_{B_j}(\bar{f}(P_1, \bar{P}_2'))$ *for all* $P_1 \in \mathcal{P}_A^{cs}$ *with* $\tau_{\mathcal{R}_f}(P_1) = \tau_{\mathcal{R}_f}(\bar{P}_1)$.[15] Consider first a component B_j on which $i = 1$ is not a dictator under $\bar{f}$. Then $\pi_{B_j}(f(P_1, \bar{P}_2')) = \bar{b}_j'$ for all $P_1 \in \mathcal{P}_A^{cs}$. If now $\pi_{B_j}(f(\bar{P}_1, \bar{P}_2')) = \hat{b}_j \neq \bar{b}_j'$, then $i = 1$ is able to manipulate f at a profile $(P_1', \bar{P}_2')$, where $P_1' \in \mathcal{P}_A^{cs}$ is some lexicographic preference that gives highest priority to the categories belonging to B_j and that satisfies $\pi_{B_j}(\tau_A(P_1')) = \hat{b}_j$. Thus, $\pi_{B_j}(f(\bar{P}_1, \bar{P}_2')) = \pi_{B_j}(\bar{f}(P_1, \bar{P}_2'))$ for all B_j on which $i = 1$ is not a dictator.

Consider now components B_j on which $i = 1$ is a dictator under $\bar{f}$. Let B_{j*} be the first component from the right in $B_1 \times \cdots \times B_q$ such that $i = 1$ is a dictator on B_{j*} under $\bar{f}$, but $\pi_{B_{j*}}(f(\bar{P}_1, \bar{P}_2')) \neq \pi_{B_{j*}}(\tau_{\mathcal{R}_f}(\bar{P}_1)) = (0, \dots, 0)$. If $P_1' \in \mathcal{P}_A^{cs}$ is such that $\tau_A(P_1') = \tau_{\mathcal{R}_f}(\bar{P}_1)$, then $\pi_{B_{j'}}(f(P_1', \bar{P}_2')) = \pi_{B_{j'}}(f(\bar{P}_1, \bar{P}_2'))$ for all $j' > j^*$ by the preceding paragraph, but $\pi_{B_{j*}}(f(P_1', \bar{P}_2')) = (0, \dots, 0)$, and thus $i = 1$ is able to manipulate f at $(\bar{P}_1, \bar{P}_2')$. Thus, there is no such B_{j*}, and $\pi_{B_j}(f(\bar{P}_1, \bar{P}_2')) = \pi_{B_j}(\bar{f}(P_1, \bar{P}_2'))$ also holds for all B_j on which $i = 1$ is a dictator.

Step 4: We have $\pi_{B_1}(f(\bar{P}_1, \bar{P}_2')) = \bar{b}_1'$. From the assumption that $i = 1$ is not a dictator on all B_j with $j \geq 2$ and Step 3, it follows that $\pi_{B_{-1}}(f(\bar{P}_1, \bar{P}_2')) \neq (0, \dots, 0)$, and hence the top alternative of $\bar{P}_1$ in B_1 conditional on $\pi_{B_{-1}}(f(\bar{P}_1, \bar{P}_2'))$ is $\bar{b}_1'$. Therefore, $\pi_{B_1}(f(\bar{P}_1, \bar{P}_2')) = \bar{b}_1'$, because otherwise $i = 1$ is able to manipulate f by representing some $P_1 \in \mathcal{P}_A^{cs}$ whose top satisfies $\pi_{B_1}(\tau_A(P_1)) = \bar{b}_1'$ and $\pi_{B_{-1}}(\tau_A(P_1)) = \pi_{B_{-1}}(f(\bar{P}_1, \bar{P}_2'))$.

Step 5: We have $f(\bar{P}_1, \bar{P}_2') \, \bar{P}_2 \, f(\bar{P}_1, \bar{P}_2)$. The preceding three steps imply that $\pi_{B_1}(f(\bar{P}_1, \bar{P}_2)) = \bar{b}_1$ and $\pi_{B_1}(f(\bar{P}_1, \bar{P}_2')) = \bar{b}_1'$. Since $\bar{P}_2$ is lexicographic with highest priority given to A_1 and since $\pi_{A_1}(\bar{b}_1') = 0$ while $\pi_{A_1}(\bar{b}_1) \neq 0$, we have $f(\bar{P}_1, \bar{P}_2') \, \bar{P}_2 \, f(\bar{P}_1, \bar{P}_2)$.

Since Step 5 contradicts the strategy-proofness of f, the assumption that $i = 1$ is not a dictator for $\bar{f}$ must be wrong, and we conclude that $\bar{f}(P_1, P_2) = \tau_{\mathcal{R}_f}(P_1)$ for all $(P_1, P_2) \in (\mathcal{P}_A^{cs})^2$.

The final step is to show that $i = 1$ is also a dictator for f. Suppose, therefore, there is some profile $(P_1, P_2) \in (\mathcal{P}_A^{ws})^2$ such that $f(P_1, P_2) = a \neq \tau_{\mathcal{R}_f}(P_1)$. If $P_2' \in \mathcal{P}_A^{cs}$ is such that $\tau_A(P_2') = a$, then $f(P_1, P_2') = a$ by Lemma 1. But then $i = 1$ can manipulate f at (P_1, P_2') by representing some $P_1' \in \mathcal{P}_A^{cs}$ with $\tau_A(P_1') = \tau_{\mathcal{R}_f}(P_1)$ because $f(P_1', P_2') = \bar{f}(P_1', P_2') = \tau_{\mathcal{R}_f}(P_1')$. Thus, $i = 1$ must be a dictator for f.

Case 2: When $\#B_1 = 2$. Consider now the case when $\#B_1 = 2$ and let $B_1 = \{b_1^1, b_1^2\}$. If $\bar{f} : (\mathcal{P}_A^{cs})^2 \to A$ is, as above, the restriction of f to completely separable preferences, then $\bar{f}$ is by Theorem 2 voting by committees on B_1, and there are two cases to consider.

(i) If $\bar{f}$ is dictatorial on B_1, it follows by exactly the same arguments as in Case 1 that $\bar{f}$ and hence also f must be dictatorial.

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[15]Intuitively, Step 3 shows that on all B_j with $j \geq 2$, the outcome $\pi_{B_j}(f(\bar{P}_1, \bar{P}_{-1}'))$ is the same as if $i = 1$ reported some completely separable preference with the same top as $\bar{P}_1$.

(ii) If $\bar{f}$ is not dictatorial on B_1, then one of the two alternatives in B_1 is chosen if and only if both individuals vote for it. Suppose without loss of generality that $\pi_{B_1}(\bar{f}(P_1, P_2)) = b_1^2$ if and only if $\pi_{B_1}(\tau_{\mathcal{R}_{\bar{f}}}(P_1)) = \pi_{B_1}(\tau_{\mathcal{R}_{\bar{f}}}(P_2)) = b_1^2$, and otherwise, $\pi_{B_1}(\bar{f}(P_1, P_2)) = b_1^1$. To replicate the arguments above, set $\bar{b}_1 \equiv b_1^1$ and $\bar{b}_1' \equiv b_1^2$. Let A_k with $k \in C(B_1)$ be identified with integer intervals $\{0, 1, \ldots, \#A_k - 1\}$ in such a way that if $\bar{P}_2 \in \mathcal{P}_A^{cs}$ is defined as in Step 1 above, then $\pi_{B_1}(\tau_{\mathcal{R}_{\bar{f}}}(\bar{P}_2)) = \bar{b}_1'$. Further, choose $\bar{P}_2' \in \mathcal{P}_A^{cs}$ as in Step 1 above with the additional requirement that $\pi_{B_1}(\tau_{\mathcal{R}_{\bar{f}}}(\bar{P}_2')) = \bar{b}_1'$. Then both $\pi_{B_1}(\bar{f}(P_1, \bar{P}_2))$ and $\pi_{B_1}(\bar{f}(P_1, \bar{P}_2'))$ must be equal to $\pi_{B_1}(\tau_{\mathcal{R}_{\bar{f}}}(P_1))$ for all $P_1 \in \mathcal{P}_A^{cs}$, exactly as if $i = 1$ were a dictator for $\bar{f}$ on B_1, and we can, therefore, apply the same arguments as in the five steps above to obtain the contradiction that $i = 2$ can manipulate f. Thus, also when $\#B_1 = 2$, $f : (\mathcal{P}_A^{ws})^2 \to A$ must be dictatorial.

We now use an induction argument to prove the theorem for general n. Assume thus that $n \geq 3$, and suppose that the theorem has already been proved for all $n' < n$. Let $f : (\mathcal{P}_A^{ws})^n \to A$ be a strategy-proof SCF with $\#\mathcal{R}_f \geq 3$. Define $\hat{f} : (\mathcal{P}_A^{ws})^2 \to A$ by $\hat{f}(P_1, P_2) = f(P_1, \ldots, P_1, P_2)$, where P_1 thus is replicated $n - 1$ times in the argument of f. Note that $\mathcal{R}_{\hat{f}} = \mathcal{R}_f$, because if $a \in \mathcal{R}_f$ and $P_a \in \mathcal{P}_A^{ws}$ is such that $\tau_A(P_a) = a$, then $\hat{f}(P_a, P_a) = a$ as a consequence of Lemma 1, and hence $a \in \mathcal{R}_{\hat{f}}$; in particular, $\#\mathcal{R}_{\hat{f}} \geq 3$. Note further that $\hat{f}$ is strategy-proof: To see that $\hat{f}$ is strategy-proof in its first argument, let $P_1, P_1', P_2 \in \mathcal{P}_A^{ws}$ be arbitrary but fixed. For $\eta = 0, 1, \ldots, n - 1$, define the coalitions $W_\eta' = \{1, \ldots, \eta\} \subset N$ and $W_\eta = \{\eta + 1, \ldots, n - 1\} \subset N$ with the convention $W_0' = W_{n-1} = \varnothing$, and set $P_{W_\eta'} = (P_1', \ldots, P_1') \in (\mathcal{P}_A^{ws})^\eta$ and $P_{W_\eta} = (P_1, \ldots, P_1) \in (\mathcal{P}_A^{ws})^{n-\eta-1}$. Further, set $a_\eta = f(P_{W_\eta'}, P_{W_\eta}, P_2)$ for $\eta = 0, 1, \ldots, n - 1$, and note that $a_0 = \hat{f}(P_1, P_2)$ and $a_{n-1} = \hat{f}(P_1', P_2)$. Since f is strategy-proof, we must have either $a_{\eta-1} P_1 a_\eta$ or $a_{\eta-1} = a_\eta$ for $\eta = 1, \ldots, n - 1$. By transitivity it follows that either $\hat{f}(P_1, P_2) P_1 \hat{f}(P_1', P_2)$ or $\hat{f}(P_1, P_2) = \hat{f}(P_1', P_2)$, which means that $\hat{f}$ is strategy-proof in its first argument. It further follows directly from the strategy-proofness of f that $\hat{f}$ is also strategy-proof in its second argument. By the induction hypothesis, $\hat{f}$ is thus dictatorial, and we consider the two cases when the dictator of $\hat{f}$ is $\hat{i} = 1$ and $\hat{i} = 2$.

Suppose first that $\hat{i} = 1$ is the dictator for $\hat{f}$. Take some arbitrary but fixed $\bar{P}_n \in \mathcal{P}_A^{ws}$ and define $\bar{f}_{\bar{P}_n} : (\mathcal{P}_A^{ws})^{n-1} \to A$ by $\bar{f}_{\bar{P}_n}(P_1, \ldots, P_{n-1}) = f(P_1, \ldots, P_{n-1}, \bar{P}_n)$. Since f is also strategy-proof, it follows immediately that $\bar{f}_{\bar{P}_n}$ is also strategy-proof. Furthermore, $\mathcal{R}_{\bar{f}_{\bar{P}_n}} = \mathcal{R}_f$, because if $a \in \mathcal{R}_f$ and $P_a \in \mathcal{P}_A^{ws}$ is such that $\tau_A(P_a) = a$, then $\bar{f}_{\bar{P}_n}(P_a, \ldots, P_a) = \hat{f}(P_a, \bar{P}_n) = a$. By the induction hypothesis, $\bar{f}_{\bar{P}_n}$ is dictatorial, and it remains to show that the identity of the dictator of $\bar{f}_{\bar{P}_n}$ does not depend on the choice of $\bar{P}_n \in \mathcal{P}_A^{ws}$. Suppose, therefore, that there exist $\bar{P}_n, \bar{P}_n' \in \mathcal{P}_A^{ws}$ such that individual i is the dictator for $\bar{f}_{\bar{P}_n}$, while individual i' ($i' \neq i$) is the dictator for $\bar{f}_{\bar{P}_n'}$. If $P_i, P_{i'} \in \mathcal{P}_A^{ws}$ are now chosen such that $\tau_{\mathcal{R}_f}(P_i) \neq \tau_{\mathcal{R}_f}(\bar{P}_n)$ while $\tau_{\mathcal{R}_f}(P_{i'}) = \tau_{\mathcal{R}_f}(\bar{P}_n)$, then this would imply that individual n is able to manipulate f at $(P_1, \ldots, P_i, \ldots, P_{i'}, \ldots, \bar{P}_n)$ by representing $\bar{P}_n'$ instead of $\bar{P}_n$. Thus, the identity of the dictator of $\bar{f}_{\bar{P}_n}$ must be independent of $\bar{P}_n$, and it follows that f is dictatorial.

Consider finally the case when $\hat{i} = 2$ is the dictator for $\hat{f}$. In this case, individual n must be a dictator for f, which follows from a contradiction argument: If there were some $(P_n, P_{-n}) \in (\mathcal{P}_A^{\text{ws}})^n$ such that $f(P_n, P_{-n}) = a$ but $a \neq \tau_{\mathcal{R}_f}(P_n)$, we could take some $P_a \in \mathcal{P}_A^{\text{ws}}$ with $\tau_A(P_a) = a$ and set $P'_{-n} = (P_a, \ldots, P_a)$. Then $f(P_n, P'_{-n}) = a$ by Lemma 1, but since $\hat{i} = 2$ is a dictator for $\hat{f}$, we also have $f(P_n, P'_{-n}) = \hat{f}(P_a, P_n) = \tau_{\mathcal{R}_f}(P_n)$, which is a contradiction. □

References

Aswal, Navin, Shurojit Chatterji, and Arunava Sen (2003), "Dictatorial domains." *Economic Theory*, 22, 45–62. [676, 677]

Barberà, Salvador, Faruk Gul, and Ennio Stacchetti (1993), "Generalized median voter schemes and committees." *Journal of Economic Theory*, 61, 262–289. [678]

Barberà, Salvador, Jordi Massó, and Alejandro Neme (1997), "Voting under constraints." *Journal of Economic Theory*, 76, 298–321. [674, 678]

Barberà, Salvador, Jordi Massó, and Alejandro Neme (2005), "Voting by committees under constraints." *Journal of Economic Theory*, 122, 185–205. [665, 671, 672, 673, 674, 676, 678, 680, 682]

Barberà, Salvador, Jordi Massó, and Shigehiro Serizawa (1998), "Strategy-proof voting on compact ranges." *Games and Economic Behavior*, 25, 272–291. [678]

Barberà, Salvador, Hugo Sonnenschein, and Lin Zhou (1991), "Voting by committees." *Econometrica*, 59, 595–609. [663, 670, 677]

Border, Kim C. and James S. Jordan (1983), "Straightforward elections, unanimity and phantom voters." *Review of Economic Studies*, 50, 153–170. [677]

Bradley, W. James, Jonathan K. Hodge, and D. Marc Kilgour (2005), "Separable discrete preferences." *Mathematical Social Sciences*, 49, 335–353. [668, 679]

Chichilnisky, Graciela and Geoffrey M. Heal (1997), "The geometry of implementation: A necessary and sufficient condition for straightforward games." *Social Choice and Welfare*, 14, 259–294. [678]

Debreu, Gerard (1960), "Topological methods in cardinal utility theory." In *Mathematical Methods in the Social Sciences* (Kenneth J. Arrow, Samuel Karlin, and Patrick Suppes, eds.), 16–26, Stanford University Press, Stanford. [668]

Gibbard, Allan (1973), "Manipulation of voting schemes: A general result." *Econometrica*, 41, 587–601. [663, 667]

Gorman, William M. (1968), "The structure of utility functions." *Review of Economic Studies*, 35, 369–390. [679]

Le Breton, Michel and Arunava Sen (1999), "Separable preferences, strategyproofness, and decomposability." *Econometrica*, 67, 605–628. [677, 679]

Le Breton, Michel and John A. Weymark (1999), "Strategy-proof social choice with continuous separable preferences." *Journal of Mathematical Economics*, 32, 47–85. [677]

Muller, Eitan and Mark A. Satterthwaite (1977), "The equivalence of strong positive association and strategy-proofness." *Journal of Economic Theory*, 14, 412–418. [682]

Nehring, Klaus and Clemens Puppe (2007), "The structure of strategy-proof social choice—Part I: General characterization and possibility results on median spaces." *Journal of Economic Theory*, 135, 269–305. [677]

Satterthwaite, Mark A. (1975), "Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions." *Journal of Economic Theory*, 10, 187–217. [663, 667]

Serizawa, Shigehiro (1996), "An impossibility theorem in pure public goods economies with feasibility constraints: Voting by committees in non-rectangular feasible sets." *Japanese Economic Review*, 47, 368–383. [678]

Svensson, Lars-Gunnar and Pär Torstensson (2008), "Strategy-proof allocation of multiple public goods." *Social Choice and Welfare*, 30, 181–196. [665, 669, 674, 678, 681]

Weymark, John A. (1999), "Decomposable strategy-proof social choice functions." *Japanese Economic Review*, 50, 343–355. [677]