

Search with multi-worker firms

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Abstract

We present a generalization of the standard random-search model of unemployment in which firms hire multiple workers and in which the hiring process is time-consuming as well as costly. We follow Stole and Zwiebel (1996) and assume that wages are determined by continuous bargaining between the firm and its employees. The model generates a non-trivial dispersion of firm sizes; when firms' production technologies exhibit decreasing returns to labor, it also generates wage dispersion, even when all firms and all workers are ex ante identical. We characterize the steady-state equilibrium and show that, with a suitably chosen distribution of ex ante heterogeneity across firms, it is consistent with several important stylized facts about the joint distribution of firm size, firm growth, and wages in the U.S. economy. We also conduct a numerical investigation of the out-of-steady state dynamics of our model. We find that the responses of unemployment and of the vacancy to unemployment ratio to a shock to labor productivity can be somewhat more persistent than in the Mortensen-Pissarides benchmark where each firm employs a single worker.

JEL Codes: E24, E32, J31, J64

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1 Introduction

In this paper, we study what is arguably the prototypical model for thinking about multi-worker firms in a frictional labor market. We characterize a steady-state equilibrium and show that, with a suitably chosen distribution of ex ante heterogeneity across firms, it is consistent with many stylized facts about the cross-sectional relationships between firm size, firm growth, wages, and profits in U.S. data.

Our model has three key elements. First, firms operate a production technology which exhibits decreasing returns to labor. Second, hiring workers is costly, and the cost of contacting more workers is a convex function of the contact rate. Third, wages are bargained following the standard generalization of Nash bargaining studied by Stole and Zwiebel (1996a,b).

The combination of these three elements generates several interesting results. First, even when firms all operate the same production technology, the time-consuming hiring process generates endogenous dispersion in the distribution of firm sizes, as new firms are born and cannot grow immediately to their desired number of workers. Because of decreasing returns in production, this generates dispersion in the marginal product of labor across firms, and because of our wage determination assumption, it also generates wage dispersion even in this case when all firms and all workers are ex ante identical. All firms have the same long-run employment target, but the younger the firm, the further it is from that target, the faster it grows, and, through bargaining, the higher the wages that it pays.

The resulting equilibrium of the model with ex ante identical firms can match some important stylized facts about modern labor markets. For example, both in the model and in the data, firms with higher profits per worker pay higher wages (van Reenen, 1996), as do faster-growing firms (Belzil, 2000). However, this simple version of the model is inconsistent with U.S. data in several other respects. Smaller firms pay higher wages in the model, while wages increase with firm size in U.S. data (Davis and Haltiwanger, 1991; Oi and Idson, 1999). The model-implied distribution of firm size has bounded support but an increasing density on that support, whereas its empirical counterpart has a decreasing density but a right tail that approximately satisfies Zipf's law (Axtell, 2001). The model implies a strong negative relationship between firm size and firm growth, contrary to empirical evidence on Gibrat's law, which finds that mean firm growth rates do not vary much with size (Mansfield, 1962; Evans, 1987; Hall, 1987; Sutton, 1997; Amaral et al., 1997). Accordingly, we then allow for firms to differ in a fixed component of productivity, and show that this allows us to make the model consistent with all these features of the data, without affecting the dimensions along which the model was already successful.

Our model also allows for a novel mechanism for generating persistent responses of un-

employment and of labor market tightness to productivity shocks. This is interesting since it is well known that the benchmark Mortensen-Pissarides model lacks an internal propagation mechanism by which a transitory productivity shock could have long-lasting effects on the vacancy-unemployment ratio (Shimer, 2005; Fujita and Ramey, 2007; Hagedorn and Manovskii, 2011). In our model, in response to a positive shock to total factor productivity, new firms enter and are initially small since hiring is time-consuming. Under our bargaining assumption, these firms pay high wages while they remain small. This drives up wages for workers in other firms by driving up the value of their outside option, unemployment. Accordingly, these incumbent firms reduce their recruitment effort, offsetting the boost to labor market tightness caused by new entrants. As the new entrant firms progressively increase their size, their wage payments fall, and so does the value of unemployment and the wages paid by other firms. Market tightness gradually increases and unemployment gradually decreases. Our simulations confirm a modest increase in persistence resulting from our mechanism. In a stylized example intended to match several important features of the U.S. labor market, unemployment undergoes around 53 percent of the transition to its new steady-state value by one quarter after the initial productivity shock. In a comparably parameterized Mortensen-Pissarides model, for example Shimer's (2005) calibration of the Pissarides (1985) model, this number would be 79 percent. We also show that this additional persistence does crucially depend on all of the model's key building blocks.

Our work is related to several literatures. First, we work in the tradition of Mortensen and Pissarides (1994) and Pissarides (2000) in modeling search frictions using undirected search and bargained wages. We differ from this literature in studying large firms with decreasing returns to labor, a topic of increasing interest following the availability of improved establishment-level data on hiring and vacancy creation (Davis et al., 2006a).¹

The model of firm-worker bargaining with diminishing returns at the firm level builds on Stole and Zwiebel (1996a,b), who develop it in the context of a model without labor market frictions (and show it can be thought of as a microfoundation for the well-known Shapley value for multi-player bargaining problems). Smith (1999) was the first to study a related environment with frictions, and focused on the efficiency implications. Stole and Zwiebel's approach to wage bargaining is now widely used, including in work studying contracting and technology adoption (Acemoglu et al., 2007), wage and employment dynamics (Roys, 2012), the interaction of product market regulation and the labor market (Felbermayr and Prat, 2007; Delacroix and Samaniego, 2009; Ebell and Haefke, 2009), trade (Coşar et al., 2010; Helpman

¹Moscarini and Postel-Vinay (2008, 2010) develop a framework based on Burdett and Mortensen's search model to study labor market fluctuations and emphasize the differential behavior of small and large firms (but there are no decreasing returns to labor in their framework).

and Itskhoki, 2010; Helpman et al., 2010; Felbermayr et al., 2011), as well as several others. Hawkins (2013) studies whether the Stole and Zwiebel bargaining assumption can be tested empirically.

Five papers deserve particular note since they are closely related to our work in studying a dynamic Stole and Zwiebel-style bargaining problem between workers and firms in the presence of search frictions and decreasing returns. First among these is the important paper by Bertola and Caballero (1994). Bertola and Caballero assume Nash bargaining over the marginal surplus. Interestingly, their approach, in steady state, coincides with that of Stole and Zwiebel (a fact that is not widely appreciated in the literature). Thus, Bertola and Caballero's steady-state analysis is closely related to ours. Nevertheless, our contribution is more general than theirs in several respects. First, in contrast to their analysis, we do not specialize to a quadratic production function (although we do find this a useful special case). Second, we prove a general existence theorem for the steady-state equilibrium. Third, we allow for rich permanent productivity heterogeneity of firms. Fourth, and most importantly, the focus of the two papers is quite different. We study the life-cycle growth of possibly heterogeneous firms to their long-run target employment level and the implications of this process for unemployment and for the firm size and growth distributions, while Bertola and Caballero study mostly how labor is reallocated across firms in response to mean-reverting idiosyncratic productivity shocks.

A second important and related paper is Wolinsky (2000). Wolinsky was the first to study an environment with bargaining, search frictions, and decreasing returns in general, but his analysis is essentially partial equilibrium since the arrival of new workers to firms is assumed to be exogenous. Consequently, Wolinsky's model does not endogenize the unemployment rate and cannot be used for equilibrium analysis in the labor market.

Equilibrium models of bargaining in such an environment are presented by Cahuc et al. (2008), Elsby and Michaels (2013), and Hawkins (2011). However, in these three papers the possibility of firms being away from their target size is assumed away. That rules out the most distinctive feature of our model, which is the cross-sectional dispersion of firm sizes and of worker productivity and wages arising from time-consuming hiring, and the persistence of the response to shocks that results from this. The assumption that firms are always at their target size is explicit in the case of Cahuc et al. (2008), and is made for the sake of tractability, since their emphasis is on hold-up problems whose study would be intractable if they assumed time-consuming hiring as we do. Elsby and Michaels (2013) and Hawkins (2011) focus explicitly on cross-sectional dispersion and on aggregate fluctuations, but assume that the cost of posting additional vacancies exhibits constant returns to scale. Under this assumption, the optimal vacancy-posting policy of a firm takes a 'bang-bang' form, in the sense that a new entrant firm posts an enormous number of vacancies for a vanishingly short period of time, grows

immediately to its desired size, and then remains there until the arrival of an idiosyncratic or aggregate shock.² By contrast, the slow dynamics that arise when we model firm growth in a more realistic way are at the heart of our paper.

Although the Stole and Zwiebel bargaining assumption we make is the usual one where firms with decreasing returns to labor attempt to hire workers in a setting of random search, it is worth noting there is also a more recent literature using the alternative assumption of directed search to study such environments.³ Hawkins (2006) was the first to study a related environment. Kaas and Kircher (2011) investigate the response of the economy to productivity shocks in such an environment, while Schaal (2012) focuses on the effect of uncertainty shocks.

Finally, our paper also relates to a literature focused on explaining features of the firm size and firm growth distributions through mechanisms other than labor market frictions. Our model is able to match the fat tail of the firm size distribution if firms draw their productivity upon entry from a Pareto distribution, which is the same modeling device that enables several important studies in the trade literature to do the same (Melitz, 2003; Bernard et al., 2003; Helpman et al., 2004; Eaton et al., 2011). This assumption, while standard, is limited in that it is silent as to the source of the underlying Pareto distribution of productivity. Since Gibrat (1931), the leading candidate explanation has been that if a firm's growth rate is independent of its size, a Pareto distribution naturally arises.⁴ Various authors have studied random growth models in which size differences across firms reflect either differences in organization capital (Simon and Bonini, 1958; Sutton, 1997; Luttmer, 2006, 2010b) or productivity (Atkeson and Kehoe, 2005; Luttmer, 2007; Rossi-Hansberg and Wright, 2007). Luttmer (2010a) surveys this literature.

Two features of our paper are noteworthy with respect to this literature. First, our model can be consistent with Gibrat's law even though we assume that a firm's productivity is a fixed characteristic. Because the labor market is frictional in our model, fast-growing firms are those which have not yet hired enough workers to be close to their target employment level, and accordingly have a high marginal benefit of hiring. When there is enough heterogeneity in productivity, conditioning on a firm's current size is uninformative about how far the firm is from its target employment level, and accordingly about its growth rate. Second, the persistence mechanism that we study arises because the firm size distribution is a state variable of the economy, and labor market frictions cause this to adjust only slowly to changes in aggregate conditions. The idea that the firm size distribution can be important in generating a sluggish

²Applications of Stole-Zwiebel bargaining to dynamic settings, for example Felbermayr et al. (2011), also usually make this assumption.

³See Montgomery (1991), Peters (1991), Moen (1997), Shimer (1996), Acemoglu and Shimer (1999a,b) and Burdett et al. (2001) for directed search models.

⁴This assumption is stronger than necessary. See Gabaix (2009) for a survey of relevant work.

response of the economy to aggregate shocks is also found in the random growth literature (see for example Luttmer, 2011, 2012).

The structure of the remainder of the paper is as follows. We first present a general version of the model in [Section 2](#). In [Section 3](#) we introduce our equilibrium concept and characterize the implications of our bargaining solution for wages. In [Section 4](#), we study steady-state equilibrium when firms share a common production technology. We prove existence of equilibrium and characterize the cross-sectional properties of the resulting economy. We then generalize the analysis to allow for ex ante productivity heterogeneity of firms in [Section 5](#). In [Section 6](#) we investigate the adjustment dynamics of the economy in response to a productivity shock. [Section 7](#) concludes. Omitted proofs are in [Appendix A](#), while [Appendix B](#) describes our numerical procedure.

2 Model

There is a unit measure of risk-neutral workers in the economy and a large measure of risk-neutral firms. Time is continuous; workers and firms discount the future at rate $r \geq 0$.

Firms are either inactive or active. At any moment, any inactive firm can elect to become active by paying an entry cost of k units (all production and costs are measured in units of the single good produced in this economy). When a firm becomes active, it draws the level of its idiosyncratic productivity $z > 0$ from a distribution $F(\cdot)$. Idiosyncratic productivity is constant over the life of the firm. A newly active firm begins life with no workers; we describe the hiring technology below.

Active firms have the ability to operate a production technology which uses labor as the only input. The flow output of the final good produced by an active firm with productivity z together with n workers is denoted $y(n; z)$. We assume that $y(n; z)$ is strictly increasing, strictly concave, and continuously differentiable in n , strictly increasing in z , and satisfies $y(0; z) = 0$ for any z .

The labor market is frictional; in order to hire workers, firms must post vacancies. We assume that to post $v \geq 0$ vacancies, firms pay a strictly increasing, strictly convex, continuously differentiable vacancy-posting cost of $c(v)$ which satisfies the Inada conditions

$$\lim_{v \rightarrow 0^+} c'(v) = 0 \quad \text{and} \quad \lim_{v \rightarrow \infty} c'(v) = +\infty. \quad (1)$$

An aggregate matching function $M(u, \bar{v})$ determines the flow of new meetings between firms and workers, as a function of the measure of unemployed workers, u and the total measure of vacancies posted by active firms, \bar{v} . Each firm meets a worker at a Poisson rate proportional to

the number of vacancies it posts. Each worker meets a firm at a Poisson rate that is identical across workers. Which unemployed worker meets which vacancy is randomly determined, and does not depend on any characteristics of worker or firm (except that, as already mentioned, a firm that posts a greater number of vacancies is more likely to meet a worker). We assume that M exhibits constant returns to scale in (u, \bar{v}) , and decreasing returns to scale in u or in \bar{v} separately. Denote labor market tightness, that is, the vacancy-to-unemployment ratio \bar{v}/u , by θ ; then the Poisson rate at which a firm that posts v vacancies meets a firm is $vq(\theta)$, where $q(\theta) \equiv M(u, \bar{v})/\bar{v}$. The rate at which an unemployed worker meets some firm is $\theta q(\theta) = M(u, \bar{v})/u$. In general, $\theta = \theta(t)$ may be time-varying. For notational convenience, we will often omit the dependence of $q(\theta) = q(\theta(t))$ on θ and abuse notation by writing just $q(t)$ (or, in steady state, just q).

Wages paid by firms to workers are determined following Aumann and Shapley (1974) and Stole and Zwiebel (1996a,b) by assuming that firms and workers bargain over the incremental surplus generated by their employment relationship. To formulate this, denote the Hamilton-Jacobi-Bellman (HJB) value of a firm with n employees and productivity z at date t by $J(n, t; z)$, and the value of a worker employed at such a firm by $V(n, t; z)$. Denote the value of an unemployed worker by $V^u(t)$. Then we assume that wages are determined in such a way that⁵

$$\phi J_n(n, t; z) = (1 - \phi) [V(n, t; z) - V^u(t)]. \quad (2)$$

By symmetry, the firm will pay the same wage to all its workers. We denote by $w(n, t; z)$ the wage paid by a firm with n workers at time t if its idiosyncratic productivity is z .

Employment relationships are subject to two types of shocks. At a Poisson rate $\delta > 0$, an active firm is destroyed; in this case, all its workers are returned to unemployment, and the firm is removed from the economy with zero scrap value. At a Poisson rate $s > 0$, each worker employed by the firm is separated from the firm; in this case, the firm continues in existence with all its other incumbent workers. These shocks are independent across active firms and across employed workers.

Unemployed workers generate unemployment income $b > 0$. We assume that the marginal

⁵In an earlier version of this paper, Acemoglu and Hawkins (2007), we solved a version of the model with workers of positive size ε and derived the form of the HJB equation more formally by taking limits as $\varepsilon \rightarrow 0$. The reader is referred to that version of the paper for more detail.

We denote partial derivatives by subscripts (thus, $J_n(n, t; z)$ denotes the second partial derivative of $J(n, t; z)$ with respect to n).

product of labor is eventually below b if the firm is large enough, that is, for any z ,⁶

$$\lim_{n \rightarrow \infty} y'(n; z) < b. \quad (3)$$

This concludes the formal description of the model. Before we commence our analysis of equilibrium, it might be helpful to describe intuitively how firm dynamics work. We will prove the results in the rest of this section more formally below.

First, consider the life of an individual firm. When it has just entered and has very few employees, a firm has a high marginal product of labor, and under our assumption on bargaining, the same is true for the marginal value to the firm of an additional worker. The firm therefore finds it optimal to post many vacancies, and it grows quickly. Nevertheless, because of the convex vacancy-posting cost function, its speed of growth is finite. As the firm grows, the marginal value of an additional hire falls, and so it reduces its intensity of vacancy posting and its growth rate slows. (This slow-down of net firm growth is exacerbated as the flow rate of separation of existing workers is greater when the firm's employment is larger.) In the long run, conditional on its continued survival, the firm reaches a point where its growth due to new hires only just offsets the separation flow of existing workers, and the firm then remains at that 'target' size. How large that size is depends on the firm's production technology (low idiosyncratic productivity z or strongly decreasing returns to labor lead to a smaller target size), on the importance of labor market frictions (the more costly vacancy posting, the smaller the target size), on the separation rate s , and on aggregate variables (market tightness $\theta(t)$, which affects the cost of hiring, and the value of an unemployed worker $V^u(t)$, which affects wages via bargaining).

In equilibrium there is firm size dispersion for two reasons, namely productivity dispersion and time-consuming hiring. If one firm has a higher idiosyncratic productivity z than another, then it will also have a greater target size. In addition, if there is entry of new firms, then not all firms will have grown to their target size, and so there will also be dispersion in firm sizes associated with firms' different ages and therefore different degrees of progress towards that target. In steady state, the second type of dispersion only arises if $\delta > 0$, so that continual entry of new firms occurs to compensate for exit by incumbents. If we shut down entry ($\delta = 0$), then in steady state, all firms would have reached their target size, and the only dispersion would arise from dispersion in productivity, that is, in z . If we shut down productivity dispersion, so that all firms have the same z , and set $\delta > 0$, then all firms would share a common target size, but different firms would have progressed different amounts towards that target size.

⁶Although in general we use subscripts to denote partial derivatives, it is convenient to use the prime notation to denote the derivative of the production function with respect to its first argument. We will never need to differentiate $y(\cdot)$ with respect to productivity, so this will not cause confusion.

3 Equilibrium

3.1 Value Functions and Definitions

We will begin by analyzing our model in a relatively general setting which allows for endogenous variables such as the labor market tightness θ and the value of an unemployed worker $V^u(t)$ to change over time. In this setting, we are able to define equilibrium and to solve for wages. In [Section 4](#) and [Section 5](#) below, we will specialize by studying the steady state of our model; in [Section 6](#) we will study the transitional dynamics following a productivity shock.⁷

We will characterize equilibrium only under the assumption that the firm's value $J(n, t; z)$ is strictly concave and twice continuously differentiable in n . (We will show below that such an equilibrium exists, at least in the steady-state context.)

To begin the analysis, observe that the HJB equation for the value of a firm with n workers at time t takes the form:

$$(r + \delta)J(n, t; z) - J_t(n, t; z) = y(n; z) - nw(n, t; z) - snJ_n(n, t; z) + \max_{v \geq 0} \{-c(v) + q(\theta(t))vJ_n(n, t; z)\}. \quad (4)$$

The intuition for the form of this equation is standard. On the left side, the firm's effective discount rate is $r + \delta$, accounting for both time preference and the destruction rate. On the right side, current flow output is $y(n; z)$, and total flow wage payments $nw(n, t; z)$. In addition, the firm loses a flow of measure sn workers per unit time to the separation shock, with a flow capital loss per unit measure of workers separated of $J_n(n, t; z)$. Finally, the firm chooses its vacancy posting strategy, denoted by v , to maximize the difference between the flow capital gains of $q(\theta(t))vJ_n(n, t; z)$ arising from the hiring flow of qv workers per unit time and the flow cost of vacancy posting, $c(v)$.

Denote by $v(n, t; z)$ the optimal vacancy-posting strategy of the firm, which satisfies

$$v(n, t; z) = \arg \max_{v \geq 0} [-c(v) + q(\theta(t))vJ_n(n, t; z)]. \quad (5)$$

The first-order condition characterizing the optimal choice of vacancy posting is

$$c'(v) = qJ_n(n, t; z); \quad (6)$$

denote the solution by $v(n, t; z)$. The solution is unique conditional on $J(\cdot)$ and does in fact

⁷It would be possible to define equilibrium in greater generality than we do: in particular, we could allow for time-varying aggregate productivity, and we could allow for the firm size distribution to have atoms. We do not pursue this since it would complicate the notation unnecessarily beyond what we need for our analysis.

define the optimal vacancy-posting policy; this follows by the convexity of $c(\cdot)$.

The HJB equation for the value $V(n, t; z)$ of a worker is similar to that of the firm, with two differences. First, separation causes the worker to return to unemployment, whereas it merely reduces the employment level of the firm. Second, the worker takes the firm's vacancy-posting strategy $v(\cdot)$ as given. Thus the HJB equation is

$$rV(n, t; z) - V_t(n, t; z) = w(n, t; z) + (s + \delta) [V^u(t) - V(n, t; z)] + [q(\theta(t))v(n, t; z) - sn]V_n(n, t; z). \quad (7)$$

The final HJB equation is that for an unemployed worker. This takes the form

$$rV^u(t) - V_t^u(t) = b + \theta(t) q(\theta(t)) \mathbf{E} [V(n, t; z) - V^u(t)]. \quad (8)$$

The second term on the right side is the expected capital gain associated with the unemployed worker meeting a firm. This event occurs at a Poisson rate $\theta(t)q(\theta(t))$. Because search is random, the worker meets a firm with n employees and productivity z in proportion to the frequency of these firms in the population multiplied by the number of vacancies such a firm posts; the expectation is taken with respect to this distribution.

To write this expectation more formally, we need to introduce notation for the distribution of firms by number of employees n and by idiosyncratic productivity z . Let $x(t)$ be the total measure of firms in the population. Because firms learn their idiosyncratic productivity z only after entry and because firm death is independent of z , the marginal distribution of productivity across firms always has cdf $F(\cdot)$. Next, let $G(n, t; z)$ be defined so that the fraction of firms with productivity z which have no more than n employees at time t is $G(n, t; z)/x(t)$.⁸ We will only study equilibria in which $G(n, t; z)$ is almost everywhere twice continuously differentiable in n , and we write $g(n, t; z) = G_n(n, t; z)$. Note that the total number of firms in the economy is just

$$x(t) = \int_0^\infty \int_0^\infty g(n, t; z) dn dF(z). \quad (9)$$

Using this notation, we can write the HJB equation for an unemployed worker more fully as

$$rV^u(t) - V_t^u(t) = b + \theta(t) q(\theta(t)) \frac{\int_0^\infty \int_0^\infty [V(\nu, t; z) - V^u(t)] v(\nu, t) g(\nu, t; z) d\nu dF(z)}{\int_0^\infty \int_0^\infty v(\nu, t) g(\nu, t; z) d\nu dF(z)}. \quad (10)$$

It is worth noting that we can eliminate the worker value $V(\cdot)$ from this equation using our

⁸Note that $G(\cdot, t; z)$ is not a probability measure since $\sup_n G(n, t; z) = x(t)$ and not unity. The cumulative distribution function of firm size for firms with productivity z is $G(n, t; z)/x(t)$. It is convenient to write the firm size distribution this way since it simplifies the partial differential equation for its evolution over time, given by (13) below.

bargaining assumption (2) to obtain that

$$rV^u(t) - V_t^u(t) = b + \frac{\phi}{1-\phi} \theta(t) q(\theta(t)) \frac{\int_0^\infty \int_0^\infty J_\nu(\nu, t; z) v(\nu, t) g(\nu, t; z) d\nu dF(z)}{\int_0^\infty \int_0^\infty v(\nu, t) g(\nu, t; z) d\nu dF(z)}. \quad (11)$$

Free entry holds at all dates, so we require that

$$\int_0^\infty J(0, t; z) dF(z) \leq k, \quad (12)$$

with equality whenever there is entry, that is, whenever $e(t) > 0$. (This equation implicitly pins down the level of entry whenever that level is positive. Higher entry today implies higher vacancy-posting and higher market tightness both today and in the future, as new entrant firms grow to their target size. It also implies higher wages. Both these effects drive down the value of entry.)

To complete the description of the environment, we need to specify the evolution of the distribution of firm sizes over time. Suppose that for each z and each t , the distribution of firm sizes admits a continuously differentiable density $g(n, t; z)$. Standard arguments establish that the partial differential equation governing the evolution of this density takes the form

$$g_t(n, t; z) = -\frac{\partial}{\partial n} [(q(\theta(t))v(n, t; z) - sn)g(n, t; z)] - \delta g(n, t; z) + e(t)f(z)j(n) \quad (13)$$

where $e(t)$ is the rate of entry at time t and $j(\cdot)$ is the indicator function taking the value 1 if $n = 0$ and 0 otherwise.⁹

The final equilibrium condition arises from the requirement that $\theta(t)$, taken as given up to now, must actually be consistent with the vacancy-posting strategies of firms. To write this condition, first observe that the measure of unemployed workers, $u(t)$, is given by subtracting

⁹Equation (13) is the Liouville equation for this system; it is the deterministic analog to the Kolmogorov forward equation in Brownian dynamics. To derive it heuristically at $n > 0$, fix $\varepsilon > 0$ small. The measure of firms in the interval $(n - \varepsilon/2, n + \varepsilon/2)$ is approximately $\varepsilon g(n, t; z)$. The net growth rate, taking into account vacancy-posting and exogenous worker separation, of firms with size near $n - \varepsilon/2$ is $q(\theta(t))v(n - \varepsilon/2, t; z) - s[n - \varepsilon/2]$. Therefore in a small time interval $(t, t + dt)$ with $dt = o(\varepsilon)$, the measure of firms which grow from a size smaller than $n - \varepsilon/2$ and enter the size interval $(n - \varepsilon/2, n + \varepsilon/2)$ is approximately $(q(\theta(t))v(n - \varepsilon/2, t; z) - s[n - \varepsilon/2])g(n - \varepsilon/2, t; z)dt$. Similarly, the measure of firms that exit this size interval due to net growth is approximately $(q(\theta(t))v(n + \varepsilon/2, t; z) - s[n + \varepsilon/2])g(n + \varepsilon/2, t; z)dt$. In addition, a measure $\delta \varepsilon g(n, t; z)$ firms exits exogenously. This gives that

$$\begin{aligned} \frac{\partial}{\partial t} [\varepsilon g(n, t; z)] &= (q(\theta(t))v(n - \varepsilon/2, t; z) - s[n - \varepsilon/2])g(n - \varepsilon/2, t; z) \\ &\quad - (q(\theta(t))v(n + \varepsilon/2, t; z) - s[n + \varepsilon/2])g(n + \varepsilon/2, t; z) - \delta \varepsilon g(n, t; z). \end{aligned}$$

Dividing by ε and taking limits as $\varepsilon \rightarrow 0^+$ establishes (13). It is straightforward to formalize this argument or to allow for entry at $n = 0$, recalling that the density of firms with productivity z among entrants is $f(z)$.

the measure of employed workers from the unit labor force

$$u(t) = 1 - \int_0^\infty \int_0^\infty ng(n, t; z) dn dF(z). \quad (14)$$

This is also the unemployment rate since the labor force has measure 1. Similarly, the total measure of vacancies posted by firms is given by

$$\bar{v}(t) = \int_0^\infty \int_0^\infty v(n, t; z)g(n, t; z) dn dF(z). \quad (15)$$

Then we require that $\theta(t) = \bar{v}(t)/u(t)$.

Given these equations, we can define a dynamic equilibrium as follows:

Definition 1. A tuple $\langle \theta(t), V^u(t), G(n, t; z), g(n, t; z), x(t), J(n, t; z), V(n, t; z), v(n, t; z), w(n, t; z) \rangle$ is a *dynamic equilibrium* if for all t ,

- $J(\cdot)$, $V(\cdot)$, and V^u satisfy the HJB equations (4), (7), and (8), as well as the bargaining equation (2);
- vacancy posting is optimal, so that $v(\cdot)$ satisfies (5);
- there is free entry in the sense that equation (12) holds, and with equality if $e(t) > 0$;
- $G(\cdot)$ has density $g(\cdot)$ satisfying (13);
- $x(\cdot)$ satisfies (9); and
- $\theta(t) = \bar{v}(t)/u(t)$ where $u(t)$ and $\bar{v}(t)$ are given by (14) and (15).

We could have incorporated the differential equation for the evolution of the unemployment rate,

$$u_t(t) = (s + \delta)(1 - u(t)) - \theta(t)q(\theta(t))u(t),$$

into the equilibrium definition, but this is not necessary since it is implied by the evolution of the distribution of total employment $1 - u(t) = \int_0^\infty \int_0^\infty ng(n, t; z) dn dF(z)$ along with the fact that all meetings between a worker and a firm are accounted for in (13).

Similarly, the differential equation for the evolution of the measure of active firms,

$$x_t(t) = e(t) - \delta x(t),$$

is implied by the equation for the evolution of the density $g(\cdot)$, that is, (13), together with the accounting equation (9).

3.2 Wages

While a closed-form characterization of equilibrium is in general not available, in this section we use the bargaining equations and value functions to find the wage consistent with the Stole and Zwiebel bargaining equation (2). It turns out that by manipulating the HJB equations for firms and workers (4) and (7), together with the bargaining equation, we can establish that a simple expression for the wage function $w(\cdot)$ holds, at least under the following mild assumption on the production function.

Assumption 1. $\lim_{n \rightarrow 0^+} n^{-\frac{1-\phi}{\phi}} \int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu; z) d\nu = 0$.

Assumption 1 requires that the marginal product of labor should remain bounded, or at least not diverge to infinity too fast as the firm's employment falls to zero (holding constant productivity z). It is a relatively weak condition, satisfied, for example, if the production function is quadratic (for example, $y(n; z) = zn - \frac{1}{2}\sigma n^2$) or Cobb-Douglas ($y(n; z) = zn^\alpha$ for $\alpha \in (0, 1)$).

We can now establish the following characterization of wages.¹⁰

Lemma 1. *The unique solution for wages satisfying $\limsup_{n \rightarrow 0^+} |nw(n, z)| < +\infty$ is*

$$w(n, t; z) = (1 - \phi) [rV^u(t) - V_t^u(t)] + \phi \frac{\int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu; z) d\nu}{\int_0^n \nu^{\frac{1-\phi}{\phi}} d\nu}. \quad (16)$$

We will focus on the unique solution for wages described in **Lemma 1** in the remainder of the paper. Despite the additional general equilibrium interactions, wages in this model take a form essentially identical to those in Stole and Zwiebel (1996a,b) and Wolinsky (2000).¹¹ The wage is a weighted average of two terms. The first term is the contribution of the (flow value of the) outside option of the worker to his wage. The second term is the worker's share of his contribution to the value of the firm, taking into account that if the worker were to quit, this would also influence the wages of other employees of the firm. This term is itself a weighted average of all the inframarginal products, $y'(\nu; z)$, for $\nu \in (0, n)$.

Generically equation (16) cannot be further simplified to give a closed form solution. Two notable cases where this is nonetheless possible are:

¹⁰The Proof of **Lemma 1**, along with all other omitted proofs, can be found in Appendix A.

¹¹This wage equation also takes the same form as in other papers using the Stole-Zwiebel framework, such as Cahuc et al. (2008), and Elsbay and Michaels (2013).

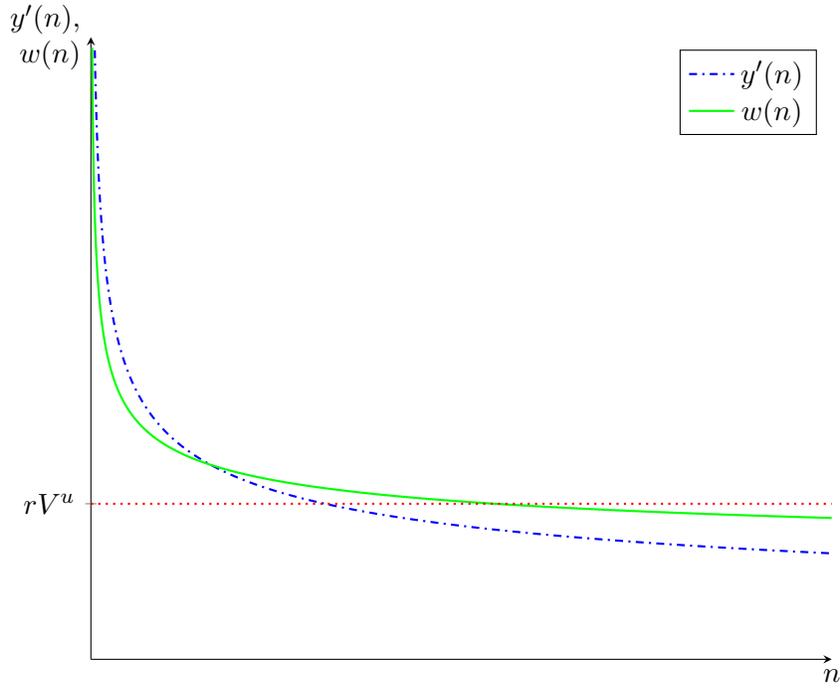


Figure 1: Wages and marginal products

1. if $y(n) = zn - \frac{1}{2}\sigma n^2$ is quadratic, then the wage takes the form¹²

$$w(n) = (1 - \phi) [rV^u(t) - V_t^u(t)] + \phi z - \frac{\phi}{1 + \phi} \sigma n;$$

2. if $y(n) = zn^\alpha$ is Cobb-Douglas, then the wage takes the form

$$w(n) = (1 - \phi) [rV^u(t) - V_t^u(t)] + \frac{\alpha\phi}{1 - \phi + \alpha\phi} zn^{\alpha-1};$$

A graphical representation of the dependence of wages on the number of workers employed at the firm is indicated in [Figure 1](#). (This Figure shows the wage function arising in the example of a quadratic production function, as used in [Section 6](#) below.) Also shown are a horizontal line indicating the flow value of the unemployed, rV^u , and the marginal product function, $y'(n; z)$.

We also find it useful to record that, consistent with what is as depicted in [Figure 1](#), wages and flow profits satisfy convenient boundary conditions.

¹²The quadratic formula given in the text defines a function that is decreasing in n for $n > z/\sigma$. Our results do extend directly to this case. An alternative is to assume that $y(n)$ is given by the quadratic functional form only for $n \leq z/\sigma$, with $y(n) = z^2/(2\sigma)$ for $n > z/\sigma$. [Lemma 3](#) below establishes that the firm would never hire more than z/σ workers in equilibrium in either case, so that the precise assumption on the production function for large n does not matter much.

Lemma 2. *Provided that $rV^u(t) - V_t^u(t) > 0$, wages are strictly positive, strictly decreasing with firm size, and, satisfy*

$$\begin{aligned}\lim_{n \rightarrow 0^+} w(n, t; z) &= (1 - \phi) [rV^u(t) - V_t^u(t)] + \phi \lim_{n \rightarrow 0^+} y'(n; z) \\ \lim_{n \rightarrow \infty} w(n, t; z) &= (1 - \phi) [rV^u(t) - V_t^u(t)] + \phi \lim_{n \rightarrow \infty} y'(n; z).\end{aligned}$$

The flow profit $\pi(n, t; z) \equiv y(n; z) - nw(n, t; z)$ is strictly concave and satisfies

$$\lim_{n \rightarrow 0^+} \pi(n, t; z) = 0.$$

If $\lim_{n \rightarrow \infty} y'(n; z) < rV^u(t) - V_t^u(t)$, then $\lim_{n \rightarrow \infty} \pi(n, t; z) = -\infty$ and the maximizer of $\pi(\cdot, t; z)$ is finite. If in addition $\lim_{n \rightarrow 0^+} y'(n; z) > rV^u(t) - V_t^u(t)$, the maximizer is strictly positive.

The condition that $\lim_{n \rightarrow \infty} y'(n; z) < rV^u(t) - V_t^u(t)$ will always be satisfied in steady state if $\lim_{n \rightarrow \infty} y'(n; z) < b$. If the production function satisfies the Inada condition $\lim_{n \rightarrow 0^+} y'(n; z) = +\infty$, then [Lemma 2](#) implies that $\lim_{n \rightarrow 0^+} w(n, t; z) = +\infty$, and in consequence, the employment level which maximizes a firm's flow profit is strictly positive.

We now turn to study the steady state of our model, where we can give a fuller characterization of equilibrium.

4 Steady-State Equilibrium with Homogeneous Productivity

In this section and the next, we are interested in steady-state equilibria in which all aggregate variables are constant over time and in which wages and the vacancy-posting strategies of firms depend only on the firm's employment level n and idiosyncratic productivity z .

We start our analysis of the steady state in this section by restricting to the special case where all firms have the same idiosyncratic productivity z . In this case, all dispersion in firm size arises from the transitional dynamics of individual firms growing from zero employment at entry towards their target size. We prove existence of a steady-state equilibrium and give a characterization. In [Section 5](#), we introduce dispersion in idiosyncratic productivity z .

We drop the time-dependence from our notation throughout our study of steady-state equilibria. In particular, the key endogenous variables of the model, namely market tightness θ (and the firm's vacancy-filling efficiency q) and the flow value of an unemployed worker, rV^u , are constants. In this section, we also drop the dependence on idiosyncratic productivity z . This simplifies the forms of the HJB and wage equations somewhat.

A greater simplification comes from imposing that the firm size distribution is constant over

time. The partial differential equation for the evolution of the firm size distribution (13) then becomes an ordinary differential equation

$$\frac{g'(n)}{g(n)} = \frac{s - \delta - qv'(n)}{qv(n) - sn}, \quad (17)$$

which can be integrated to obtain that

$$g(n) = D \exp \left(\int_0^n \frac{s - \delta - qv'(\nu)}{qv(\nu) - s\nu} d\nu \right), \quad (18)$$

where the constant of integration D is chosen so that $g(\cdot)$ integrates over the region $[0, n^*]$ to the total number of active firms x .¹³ Equating the flows of firm entry and firm destruction establishes that $x = e/\delta$.

It is also worth noting that under homogeneous firm productivity the free entry condition (12) simplifies in steady state to the form

$$J(0) \leq k \text{ and } \theta \geq 0, \text{ with complementary slackness.} \quad (19)$$

The definition of a steady-state equilibrium is the obvious specialization of [Definition 1](#), that is, a tuple $\langle \theta, V^u, G(n), g(n), x, J(n), V(n), v(n), w(n) \rangle$ with the properties $J(\cdot)$, $V(\cdot)$ and V^u satisfy (4), (7), and (8); $v(\cdot)$ satisfies (5); $g(\cdot)$ satisfies (18); (19) holds so that there is free entry; and $\theta(t) = \bar{v}(t)/u(t)$ where $u(t)$ and $\bar{v}(t)$ are given by (14) and (15).

The unemployment rate u can be calculated directly using the steady-state analog of (14). However, in steady state it can also be calculated more straightforwardly using flow balance. The mass u of unemployed workers will be matched and thus hired at the flow rate $\theta q(\theta)$. On the other side, workers lose their job because of separations at the flow rate s and because of firm shutdowns at the flow rate δ . Consequently, the steady-state unemployment rate is given by equating flows into unemployment, $(1 - u)(s + \delta)$, with flows out of unemployment, $u\theta q(\theta)$, so that

$$u = \frac{s + \delta}{s + \delta + \theta q(\theta)}.$$

It is straightforward to verify that, as in the standard Mortensen-Pissarides model, u is a monotonically decreasing function of θ : steady-state unemployment is lower when the labor market is tighter.

¹³Observe that since the stochastic process for a firm's size satisfies an ergodicity condition, the steady-state distribution $G(\cdot)$ is unique, so that conditional on the assumption that the economy is in steady state, there is no loss of generality in solving only for a distribution in which $g(\cdot)$ is continuously differentiable on $(0, n^*)$. For the ergodicity argument, it is convenient to think of firm death as a shock that changes the size of a firm to 0, rather than causing entry of a new firm; in this case the uniqueness of the invariant distribution is immediate from Theorem 11.9 of Stokey et al. (1989).

We next characterize the behavior of firms in steady-state equilibrium in more detail.

Denote by n^* the smallest value of n such that $qv(n) = sn$.¹⁴ At $n = n^*$, the firm posts just enough vacancies to keep its size constant, by hiring exactly the same number of workers that it loses to separation. A firm with this size remains there until it is exogenously destroyed, earning a constant flow profit of $\pi(n^*)$ and paying a constant flow vacancy-posting cost of $c(v(n^*))$. It follows immediately that the value $J(n^*)$ satisfies

$$(r + \delta)J(n^*) = \pi(n^*) - c(v(n^*)).$$

We now look for an equation characterizing n^* further. To obtain this equation, first evaluate the first-order condition for $v(n)$, equation (6), at $n = n^*$ to see that

$$J'(n^*) = \frac{1}{q}c'(v(n^*)) = \frac{1}{q}c'\left(\frac{sn^*}{q}\right). \quad (20)$$

On the other hand, differentiating the steady-state version of the HJB equation for the firm, (4), with respect to n and substituting from the first-order condition for $v(n)$ and from the definition of n^* to eliminate terms in $qJ'(n^*) - c'(v(n^*))$ and $qv(n^*) - sn^*$ (both of which are equal to zero), we obtain

$$(r + \delta + s)J'(n^*) = \pi'(n^*).$$

Eliminating $J'(n^*)$ from the previous two equations then establishes that

$$\frac{r + \delta + s}{q}c'\left(\frac{sn^*}{q}\right) = \pi'(n^*). \quad (21)$$

According to Lemma 2, $\pi'(0)$ is strictly positive and $\pi(\cdot)$ is strictly concave, so that $\pi'(\cdot)$ is strictly decreasing. $c'(\cdot)$ is strictly increasing by assumption and satisfies the Inada conditions (1), so that (21) has a unique solution for n^* (conditional on the values of the endogenous variables q and rV^u). Having solved for n^* , the HJB equation for the firm (which satisfies the usual Lipschitz condition on $[0, n^*]$) together with the boundary condition (20) now gives an initial value problem which can be solved uniquely for $J(\cdot)$ on $[0, n^*]$.

The characterization of n^* given by (21) can be understood intuitively when expressed in terms of wages. Differentiate the definition of profits, $\pi(n) = y(n) - nw(n)$, substitute from the closed form solution for wages, (16), and rearrange to obtain that for all n , $w(n) = rV^u + \phi\pi'(n)/(1 - \phi)$. This equation holds for all $n > 0$; substitute $n = n^*$ and use (21) to see

¹⁴The existence of such an $n < \infty$ follows from Lemma 2, together with the assumption in (3) that for n sufficiently large, $y'(n) < b$ and the observation from (16) that $rV^u \geq b$.

that

$$w(n^*) = rV^u + (r + \delta + s) \frac{\phi}{1 - \phi} \frac{1}{q} c'(v(n^*)).$$

This equation is intuitive. The firm continues to hire until the wage it pays equals the outside option of the worker, rV^u , plus a term that is proportional to the severity of the labor market friction (measured by the marginal flow cost of the last vacancy required to maintain a workforce of size n^* , which is $c'(v(n^*))/q$). This term arises because the cost of hiring a replacement worker generates a match-specific quasi-rent that allows workers to bargain for higher wages. This wage equation is comparable to the result obtained in a static environment by Stole and Zwiebel (1996a) (see their Corollary 1 on page 396) and generalized to a dynamic setting by Wolinsky (2000).¹⁵ In these previous analyses, since there is no hiring margin (and no frictions), the second term is absent. Consequently, those models always imply “over-hiring” relative to a hypothetical competitive benchmark; firms will hire more than this competitive benchmark in order to reduce the marginal product of workers and thus their bargaining power according to the Shapley bargaining protocol (see Stole and Zwiebel, 1996a). Our analysis shows that this over-hiring result may or may not apply in general equilibrium in a frictional labor market, depending on the importance of hiring frictions and on the worker’s bargaining power.

We summarize the analysis in this section as the following Proposition (proof in the text):

Proposition 1. *Let $q = q(\theta) > 0$ and $rV^u > 0$ be given. Then there is a steady-state equilibrium allocation in which firms contact workers at rate $q(\theta)$ per unit measure of posted vacancies and the value of an unemployed worker is given by rV^u if and only if*

- $J(\cdot)$ is the unique solution to the initial value problem given by the differential equation (4) with boundary condition (20) and in which n^* is the unique solution to (21), ;
- rV^u satisfies (11) where $G(\cdot)$ is given (18) with $v(n)$ given by (6); and
- the free entry condition (19) is satisfied.

We are now in a position to establish the existence of a steady-state equilibrium.

Theorem 1. *A steady-state equilibrium with cutoff hiring strategies exists.*

The proof of the Theorem consists of showing that there exist (q, rV^u) satisfying the hypothesis of Proposition 1. Here, we present a diagrammatic exposition, emphasizing the intuition. The proof of Theorem 1 establishes that an equilibrium with positive activity exists if

$$k < \frac{1}{r + \delta} \max_{n > 0} \left\{ y(n) - n^{-\frac{1-\phi}{\phi}} \int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu) d\nu - n(1 - \phi)b \right\},$$

¹⁵The equation also corresponds to equations obtained by Cahuc et al. (2008) and Elsby and Michaels (2013), who considered cases where firms doing positive amounts of hiring are always at their target hiring level n^* .

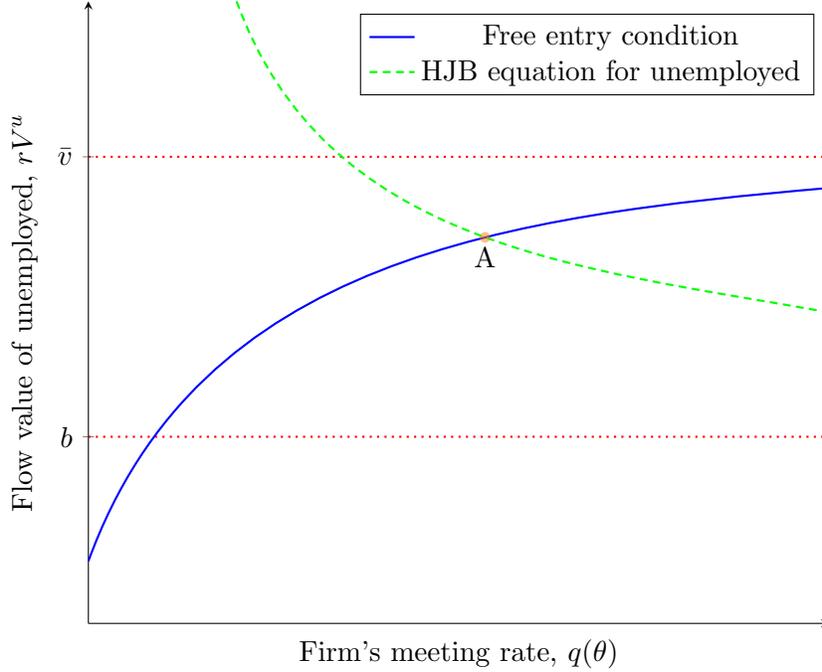


Figure 2: Graphical proof of equilibrium existence. Any equilibrium $(q(\theta), rV^u)$ satisfies the HJB equation for the unemployed, (10), as well as the zero-profit condition for firms, (19). An equilibrium always exists since the two curves always intersect.

where the existence of the maximum on the right side follows as in the proof of Lemma 2.

Figure 2 shows a diagram depicting two curves in (q, rV^u) -space. The upward-sloping curve represents the locus of (q, rV^u) pairs consistent with the free-entry condition of firms, equation (19); intuitively, it is upward-sloping since, all else equal, an increase in rV^u must be compensated by an increase in q . This is because a higher rV^u translates into higher wages, so that the profit margins of firms decline. Starting from a point (q, rV^u) consistent with the free entry condition, if rV^u increases, then to keep the value of $J(0) = k$ unchanged, the firm needs to be able to hire more rapidly. This requires that the cost of hiring must decline, which is achieved by an increase in q . The downward-sloping curve is the HJB equation for unemployed workers, equation (10). Intuitively, this is a downward-sloping locus in (q, rV^u) -space since an increase in rV^u on the left side of (10) corresponds to an increase in wages; to keep the flow value of an unemployed worker satisfying this equation, it must be that hiring is more rapid (that is, q is larger), so that when hired, the worker spends less time earning the high wages paid by smaller firms as firms expand more rapidly. (We are not, however, generally able to show that this second curve is everywhere downward-sloping, although this has been true in all calibrated examples we have investigated.) The proof of Theorem 1 given in Appendix A uses a continuity argument to establish that an intersection of these two curves must exist.

Comparative statics of the response of the endogenous variables $q(\theta)$ and rV^u can be obtained from the diagrammatic representation of the equilibrium. The effect of an increase in the entry cost, k , is particularly clear. If k decreases, the curve corresponding to the free-entry condition, (19), moves upwards in response: holding constant the ease of filling vacancies q , wages paid, and hence rV^u , must increase so as to reduce the value of a new entrant firm and keep the free-entry condition satisfied. Since a change in k does not affect the other curve, it has unambiguous effects on the steady-state equilibrium in the situation presented in Figure 2, in which the worker's Bellman equation is downward-sloping.¹⁶ Also, a decrease in k leads to an increase in rV^u and a decrease in q . This also corresponds to an increase in θ and therefore to a decrease in the steady-state unemployment rate.

The comparative statics with respect to firms' productivity are slightly more complex. Intuitively, the effect of a productivity increase is clear: it should lead to more entry, a tighter labor market, lower unemployment, and higher wages, just as in the benchmark Mortensen-Pissarides model, and this is indeed what we typically find in numerical examples. An example where this is the case is shown in Figure 3. The dashed lines indicate the movement of the curves after a Hicks-neutral increase in productivity.

However, the effect of an increase in steady-state productivity on market tightness and on the value of an unemployed worker are potentially ambiguous. The curve corresponding to the firm's free entry condition, (19), must shift upwards as shown in Figure 3. For a given (q, rV^u) , greater productivity increases flow profits for all firms and so increases the implied value of entry, $J(0)$; to keep the free entry condition satisfied, rV^u must increase for each q . The movement of the curve corresponding to the worker's HJB equation, (10), is ambiguous. First, for a given (q, rV^u) , the wage paid by a firm with any fixed number n of workers, $w(n)$, increases; however, the increase in $q(\theta)$ corresponds to an increase in firms' hiring rates which means that more workers are employed at larger firms, which, all else equal, pay lower wages. In calibrated examples, the first effect dominates, so that the curve moves upwards as one might expect, but we do not have a proof that it always does so. If both curves move upwards, then it is clear from the graph that the effect of a productivity increase is positive for the flow value of the unemployed worker, but the effect on market tightness is ambiguous. Nevertheless, in many calibrated examples, $q(\theta)$ decreases in response to the increase in productivity, so that, as shown in Figure 3, workers' job-finding rate rises and steady-state unemployment falls.

Another interesting feature of this example is that n^* is lower in the higher productivity steady state. This implies that firms are, on average, smaller.¹⁷ Consequently, much of the

¹⁶If this curve is not downward-sloping and there are multiple equilibria, a standard lattice-theoretic argument shows that this comparative static applies to the smallest and largest equilibria.

¹⁷This result, common to many large-firm models with endogenous firm size and a free-entry condition, arises from the assumption that the fixed cost of entry does not change with productivity, together with the assumption

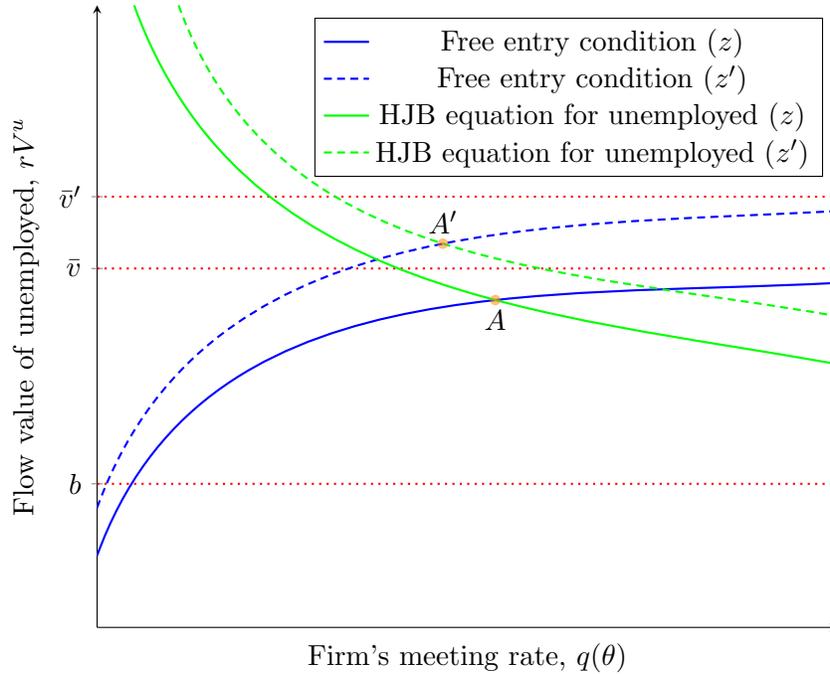


Figure 3: Comparative statics associated with an increase in the productivity of all firms. The equilibrium moves from point A to point A' .

adjustment to the new steady state takes place *at the extensive margin*, that is, by the entry of new firms (while existing firms in fact decline in size). This is a pattern we find consistently in quantitative examples, and underlines the importance of separating the intensive and extensive margins of employment creation.

To summarize, in this section we have shown that our model allows for endogenous dispersion in firm sizes in equilibrium, as firms are born and, because of labor market frictions, take time to grow to their target size. However, because we assumed in this section that all firms operate identical production technologies, they all have the same target size. The only reason for firm size dispersion is that younger firms are far from their target size and older firms nearer. Thus, all firm size heterogeneity in the model without productivity heterogeneity is transitory. Young firms, far from their target size, pay higher wages and post vacancies more intensively, both as a result of their higher marginal product of labor; as they age and grow, the marginal product of labor falls, wages fall, and vacancy-posting falls. Thus, the model is consistent with the stylized fact that fast-growing firms pay higher wages.

However, it should not be surprising that the limited heterogeneity we allow for in this of decreasing returns to labor. When productivity rises, more firms enter, but because employment does not change much, this means firm size decreases. This prediction, probably counterfactual, suggests that it might be interesting to investigate alternative ways of modeling changes in productivity.

section is not adequate for understanding the empirical distribution of firm sizes, wages, and employment in the U.S. economy. For example, although the model endogenously generates dispersion in firm sizes, the steady-state firm size distribution, as indicated by equations (17) and (18), is in several respects inconsistent with U.S. data. First, in the model smaller firms grow faster than larger ones, while empirically Gibrat’s law holds approximately at least for large firms. Because the model gets the distribution of growth rates by firm employment wrong, it also fails to generate an empirically plausible firm size distribution. In the model, the steady-state density of firms with respect to employment is upward sloping for $n < n^*$, but is bounded above by n^* .¹⁸ The empirical distribution of firms by employment, on the other hand, is everywhere decreasing but has a right tail that approximately satisfies Zipf’s law (Axtell, 2001). In addition, the model predicts that large firms pay lower wages, which is inconsistent with U.S. data. Allowing for heterogeneity in firm productivity is required in order to generate a tighter match between the firm size, firm growth, and wage distributions generated by the model and their empirical counterpart.¹⁹ We turn to this in the next section.

5 Steady-State Equilibrium with Heterogeneous Productivity

Up to now, we focused on the version of our model with no ex ante heterogeneity in firms’ production technologies. This allowed us to focus on the most important feature of our model, the time-consuming growth of firms towards their common target size. As we observed in the previous section, however, this simple environment is not adequate for matching the nature of firm heterogeneity observed in the U.S. economy. In this section, we sketch how allowing for ex ante heterogeneity in firm productivity significantly improves the model’s performance.

Recall that we allow for heterogeneity in firm productivity by assuming that after paying the entry cost k , a firm draws a productivity shock z from a fixed distribution $F(\cdot)$. This

¹⁸Because new entrant firms initially have zero employees, the model can also generate a downward-sloping density of firms by employment if firm death occurs at a sufficient rate, but this case is not empirically plausible. A sufficient condition to ensure that the density of firms is upward-sloping on $(0, n^*)$ is that $s > \delta$. In this case (18) would deliver an increasing density $g(\cdot)$ even if $v(n)$ were constant in n , and this occurs all the more since $v(n)$ actually decreases with n . The case $s > \delta$ is the realistic case to consider since the vast majority of separations do not occur at establishment shutdown. For example, Davis et al. (1996, Figure 2.3, page 29) report for U.S. manufacturing plants that in quarterly data, 11.6 percent of job destruction occurs in plant shutdowns, while in annual data, 22.9 percent of job destruction occurs in plant shutdowns. (The reasons for the apparent discrepancy between these two numbers are that closure takes time and that transitory plant-level employment changes are more important in higher frequency data (fn. 9, p. 27).)

¹⁹This is the same solution adopted by other authors in the search-and-matching literature when faced by similar problems. For example, the classic on-the-job search model of Burdett and Mortensen (1998), in the absence of firm heterogeneity, predicts a wage distribution which is qualitatively similar to the one arising in our model, since it has bounded support with most of its mass near the maximum observed wage. Allowing for different firms to have different productivities lets their model generate a more reasonable distribution in their setting, as it does in ours.

productivity level is constant over the life of the firm.

The definition of a steady-state equilibrium in this environment is the obvious specialization of [Definition 1](#) (or the obvious generalization of the definition given in the homogeneous firms case in [Section 4](#)). It is straightforward to extend the arguments given to establish [Theorem 1](#) to show that a steady-state equilibrium exists in this more general environment; for the sake of brevity, we omit the formal proof.

Since our goal in this section is to show that allowing for firm heterogeneity can help the model be consistent with stylized facts about the empirical patterns of firm size, firm growth, and wages, it suffices to specialize henceforth to a specific parametric example. The most tractable assumption for this purpose is that of a quadratic production function and a quadratic vacancy-posting cost function.²⁰ Therefore, for the remainder of the paper, we make the following two assumptions.

Assumption 2. All firms have production function

$$y(n; z) = zn - \frac{1}{2}\sigma n^2,$$

where firms may differ in the coefficient z on the linear term, but all share the same $-\frac{1}{2}\sigma$ coefficient on the quadratic term.

Assumption 3. The vacancy-posting cost function takes the form

$$c(v) = \frac{1}{2}\gamma v^2.$$

This allows us to establish simple closed-form solutions for several key variables.

Lemma 3. *In a steady-state equilibrium, wages $w(n; z)$ and a firm's flow profit $\pi(n; z) \equiv y(n; z) - nw(n; z)$ satisfy*

$$\begin{aligned} w(n; z) &= (1 - \phi)rV^u + \phi z - \frac{\phi}{1 + \phi}\sigma n \\ \pi(n; z) &= (1 - \phi)(z - rV^u)n - \frac{1 - \phi}{2(1 + \phi)}\sigma n^2. \end{aligned}$$

The target size of a firm with productivity is

$$n^*(z) = \frac{z - rV^u}{\frac{\sigma}{1 + \phi} + \frac{\gamma s(r + \delta + s)}{q^2(1 - \phi)}}.$$

²⁰The main results reported in this section, specifically Propositions 3 and 4 and the signs of the correlations reported in Propositions 2 and 5, do not appear to be qualitatively sensitive to the assumption of quadratic production. Numerical work confirms this under Cobb-Douglas production. We do not, however, have a formal proof.

Vacancy-posting satisfies

$$v(n; z) = \frac{sn^*(z)}{q} + \lambda(n^*(z) - n),$$

where λ is the positive solution to

$$q\lambda^2 + (r + \delta + 2s)\lambda - \frac{q}{\gamma} \frac{1 - \phi}{1 + \phi} \sigma = 0.$$

The fraction of firms with productivity equal to z which have employment greater than n is given by

$$\bar{G}(n; z) \equiv 1 - \frac{G(n; z)}{x} = \begin{cases} \left[1 - \frac{n}{n^*(z)}\right]^{\frac{\delta}{s+q\lambda}} & n < n^*(z) \\ 0 & n \geq n^*(z) \end{cases}$$

The assumption of quadratic production alone is sufficient for the first two equations in the Lemma. When we additionally assume quadratic vacancy-posting, we are able to solve for the firm's value function $J(\cdot)$ in closed form; it also turns out to be quadratic. This then implies that vacancies, like wages and per-worker profits, are an affine function of employment n . The simple functional form for vacancies can then be used in (17) to establish the closed-form solution for the firm size distribution.

The expression for the target firm size is intuitive. More productive firms (with higher z) have a larger target size, but the target size is smaller if the value of unemployment (and hence wages) is higher, if separations are more frequent, if the future is discounted more, if vacancy-posting is more costly, or if the production function exhibits stronger decreasing returns. (The latter effect is somewhat offset by the increased incentive for strategic over-hiring caused by greater curvature in the production function, but in the quadratic case, the direct effect dominates.)

We are now ready to show that incorporating heterogeneity in z improves the ability of our model to match the empirical properties of the joint distribution of firm size, growth, and wages.

First, consider the case where in fact there is no heterogeneity, so that all firms share the same z , as in the model of Section 4. (That is, we reconsider the results of Section 4 in this particular parametric example.) In this case, we can show precisely that our model without productivity heterogeneity does a poor job of matching stylized features of the empirical pattern of firm size, firm growth, and wages. This is summarized in the following Proposition.

Proposition 2. *In a steady-state equilibrium with homogeneous firms, firm size is perfectly negatively correlated with firm growth, with the wage a firm pays, and with profits per worker.*

Proof. Immediate since by Lemma 3 a firm's growth rate $\dot{n} = qv(n) - sn$, the wage it pays, and its profit per worker are all affine functions of firm size n . \square

One success of the model, and an immediate corollary of [Proposition 2](#), however, is that fast-growing firms pay higher wages. In fact, under the parametric assumptions considered in this section, wages and firm growth are perfectly positively correlated, since each is perfectly negatively correlated with firm size.

[Figure 4](#) demonstrates the behavior of the model graphically in a particular numerical example.²¹ In this Figure, we represent the cross-sectional distribution of firms by drawing 101 firms evenly spaced in the CDF of firm sizes (thus, one at each percentile from the zeroth, with $n = 0$, to the hundredth, with $n \approx n^*$). For each firm, we calculate the wage, the growth rate in levels ($qv(n) - sn$), flow profit $\pi(n)$, and flow profit per worker $\pi(n)/n$. As can be seen, large firms grow slower than small firms and pay lower wages, consistent with [Proposition 2](#). Profits increase with firm size, but per-worker profits fall with firm size. Thus, more profitable firms (in a per-worker sense) do pay higher wages and do grow faster.

Another dimension in which the model without heterogeneity fails is that for plausible parameters, the firm size distribution implied by [Lemma 3](#) is not a good match to its empirical counterpart. The model-generated distribution has bounded support, with an increasing density on that support, at least if $(s + q\lambda)/\delta$ is larger than one, which is the empirically relevant case.²² Most firms have grown to near their target size in steady state, provided that the rate of firm destruction δ is not too high, the cost of vacancy posting γ is not too high, and the vacancy-filling rate q is not too low.

We now allow for heterogeneity in z . In this case, few results can be proved analytically without specifying the distribution $F(\cdot)$.²³ Accordingly, we proceed by making a distributional assumption on $F(\cdot)$. We assume that z follows a Pareto distribution.²⁴ It is well-known that the empirical distribution of firm sizes is approximately Pareto (Axtell, 2001); the Pareto distributional assumption on z allows us to ensure that this is true at least for large firms in our model also. This is the subject of the following Proposition.

²¹The parameters shown here are $r = 0.0123$, consistent with a 5 percent annual discount rate if the unit of time is a quarter; $\delta = 0.0167$ and $s = 0.0833$ (consistent with a 10 percent quarterly separation rate for workers, one-sixth of which from plant closure), $q = 1.355$ (consistent with the job-finding rate for workers being as in Shimer (2005) if vacancies are measured so that tightness θ is normalized to unity), $z = 3$, $\sigma = 0.1$, and $rV^u = 1$. The vacancy-posting cost parameter is $\gamma = 1$, and the worker bargaining power parameter is $\phi = 0.72$.

²²To see this, note that $(s + q\lambda)/\delta > s/\delta = (\delta/(s + \delta))^{-1} - 1$. $\delta/(s + \delta)$ is the fraction of worker separations due to plant closure; as already noted, Davis et al. (1996) suggest that this number is around one sixth, so that $(s + q\lambda)/\delta > 5$.

²³It can be shown that if $s = 0$, so that all worker separations occur as a result of firm exit, then wages $w(n; z)$ and firm growth in levels $qv(n; z) - sn$ are perfectly correlated, independently of $F(\cdot)$, but this special case is not empirically relevant.

²⁴Accordingly, we assume that there are parameters z_m and κ such that the distribution $F(\cdot)$ is supported on $[z_m, \infty)$ and has density in that range given by $f(z) = \kappa z_m^\kappa z^{-\kappa-1}$. It is worth noting that if we assumed that $z - rV^u$, rather than z , were distributed Pareto, then [Proposition 3](#) and [Proposition 4](#) would hold for all firm sizes and not just asymptotically, but because rV^u , the flow value of an unemployed worker, is an endogenous variable, we cannot make such an assumption.

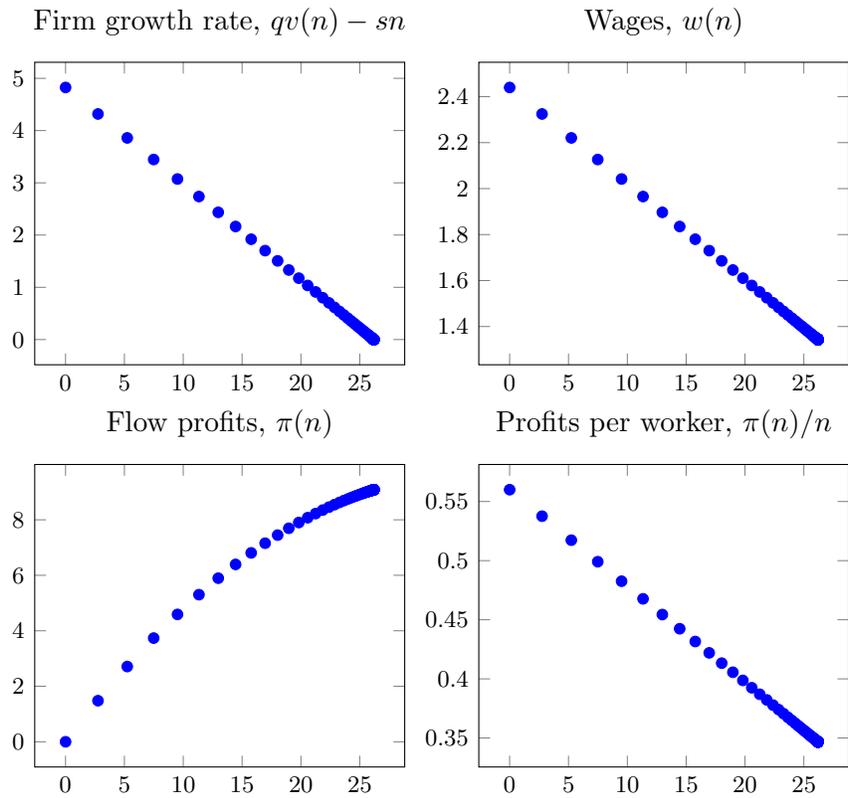


Figure 4: Cross-sectional distributions of firm sizes, growth rates, profits, and wages: homogeneous productivity. In each graph, employment n is on the horizontal axis; the variable of interest is on the vertical axis.

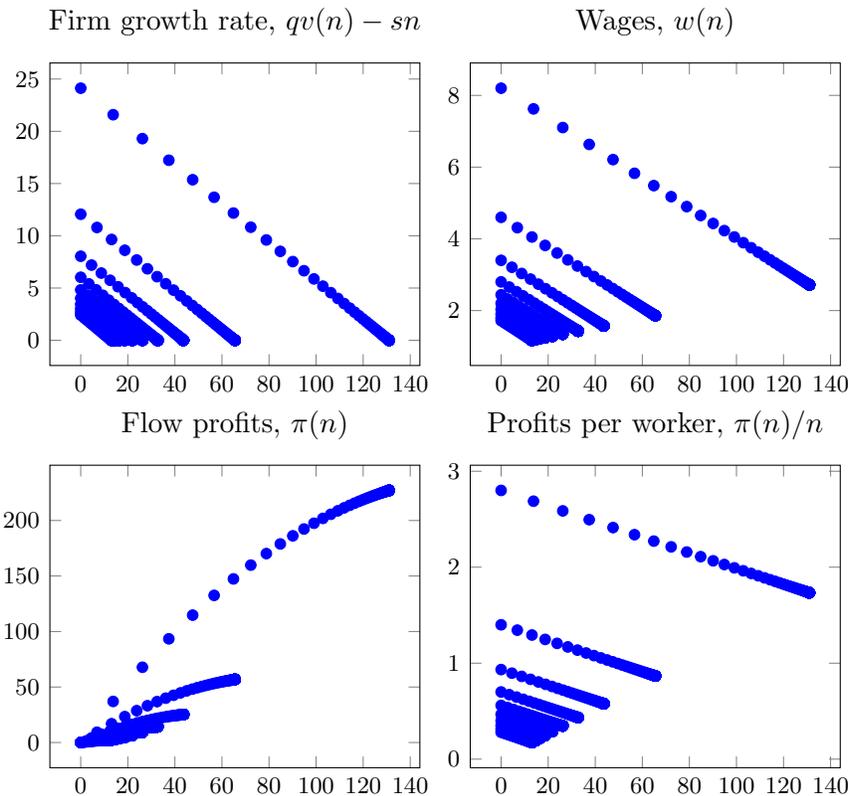


Figure 5: Cross-sectional distributions of firm sizes, growth rates, profits, and wages: heterogeneous productivity. In each graph, employment n is on the horizontal axis; the variable of interest is on the vertical axis.

Proposition 3. *In steady-state equilibrium, if the productivity distribution $F(\cdot)$ is Pareto, then the firm size distribution has a Pareto tail with the same shape parameter.*

It is not particularly surprising that, given enough heterogeneity in z , our model can generate an empirically plausible firm size distribution. However, more interestingly, this assumption also makes the model consistent with Gibrat’s law, that is, the observation that firm size and firm growth are independent.

Proposition 4. *In steady-state equilibrium, if the productivity distribution $F(\cdot)$ is Pareto, then Gibrat’s law holds for large firms.*

The model with productivity heterogeneity also delivers a very different set of correlations between firm size, wages, and growth. In fact, conditional on a firm’s age (that is, on its distance from its target size), we can establish a very sharp result.²⁵

²⁵Note that this result does not rely on the form of the productivity distribution $F(\cdot)$.

	n	\dot{n}	w	π	π/n
n	1	0.197	0.789	0.904	0.949
\dot{n}		1	0.758	0.217	0.496
w			1	0.738	0.942
π				1	0.871
π/n					1

Table 1: Cross-correlations between firm size, firm growth, wages, profits, and profits per worker in a numerical example

Proposition 5. *In steady-state equilibrium, conditional on firm age, firm size is perfectly positively correlated with firm growth, with the wage a firm pays, and with profits per worker.*

To understand the properties of the model with productivity heterogeneity more intuitively, we simulate the model and display the results graphically. To do this, we draw 10 points from the CDF of the productivity parameter z , and for each such z , we repeat the exercise used to obtain Figure 4. We then display all the resulting 1010 data points on Figure 5.²⁶ We also report in Table 1 the cross-sectional correlations between firm size, growth (in levels), wages, and profits (both in levels and per worker).²⁷ The introduction of productivity heterogeneity can allow for the signs of all cross-correlations to be consistent with U.S. data. Growth rates (in levels) are now only weakly correlated with firm size, wages are strongly positively correlated with firm size, and profits per worker are also now strongly positively correlated with firm size.

Comparing Proposition 2 and Proposition 5, or alternatively, comparing Figure 4 with Figure 5, the importance of productivity heterogeneity is transparent. In our model, the *transitory* heterogeneity associated with the growth towards its target size of a firm of fixed productivity generates a negative correlation between firm size and any one of firm growth, wages, or profits per worker. However, the *permanent* heterogeneity associated with permanent differences in z has a very different effect. Holding constant the age of a firm (and accordingly, how far it is from its target size), the greater its productivity z , the larger it is, the higher the wage it pays, and the higher the profits it makes per worker.

The overall pattern of correlations between firm size, firm growth, wages, and per-worker profits in our model therefore depends on how two forces balance each other. If we only have labor market frictions but no productivity heterogeneity, our model predicts successfully that

²⁶The parameters are the same as in the exercise without heterogeneity in z , except that we assume that $z - rV^u = z - 1$ is distributed according to Zipf's law, that is, a Pareto distribution with $k = 1$, and with minimum value $z_m = 1$. It can be shown that if $z - rV^u$, rather than z , is distributed Pareto, then Gibrat's law holds precisely for all firm sizes, and not merely asymptotically. The 10 data points for z shown are the zeroth percentile, the tenth percentile, the twentieth percentile, and so on, up to the ninetieth percentile.

²⁷Table 1 uses the same parameterization as in Figure 5, but simulates a much finer grid of firms (using 4 million points). We truncate the distribution of z above at the 99.95th percentile, since otherwise the expectations we are approximating by simulation would not be well-defined.

fast-growing firms, far from their target size, pay higher wages, and so generates correctly a positive correlation between firm growth and wages, but it would fail in other dimensions, as made clear by [Proposition 2](#). If we only have productivity heterogeneity but no labor market frictions, our model would successfully predict that more productive firms are larger, pay higher wages, and make higher profits per worker, but firms would grow instantaneously to their target size, so that there would be no growth for all but a measure zero of firms, and our model would be unable to account for the positive wage-firm growth correlation. Thus both labor market frictions and productivity heterogeneity are key to allow our model to match a rich set of cross-sectional features found in U.S. data.

6 Propagation of Productivity Shocks

6.1 General Discussion and Intuition

We have so far focused on the steady state and shown that with permanent productivity differences between firms, our model generates steady-state patterns broadly consistent with U.S. data. We now turn to the response of labor market outcomes to shocks (for example, to productivity). The main result is that our model generates somewhat persistent labor market responses to one-off shocks. Our investigation is numerical, although we can show theoretically that the persistence result relies on three key features of our model: decreasing returns to labor in production, convex vacancy-posting costs, and bargained wages that increase with a firm's productivity.

Our persistence result is interesting because in the benchmark Pissarides (1985) or Mortensen and Pissarides (1994) (MP) models of a frictional labor market, the vacancy-unemployment ($v-u$) ratio is a jump variable and so has no persistence beyond what is present in the underlying labor productivity shock process, while empirically the response of this variable to changes in labor productivity is quite persistent (Fujita and Ramey, 2007). Our model is a fairly minimal modification of the MP model, so it is notable that it contains a propagation mechanism the basic model lacks. The key feature of our model that allows for propagation is the slow adjustment of the firm size distribution following a change in aggregate productivity.

The persistence mechanism we identify works through wages. Consider a permanent, unanticipated one-time increase in labor productivity of all firms. In the new steady state, the economy will have more active firms than before.²⁸ How does the entry of these firms influence the transition path to the new steady state? Following the productivity shock, a positive mass of new firms enters. Because hiring is time-consuming in our model, these new entrant firms

²⁸This statement is true in the quantitative examples we have studied, but we do not have a general proof that it is always true.

remain small relative to their target size for some time. Because there are decreasing returns to labor in production, this means these firms have a high marginal product. Under the bargaining assumption we use, their employees also earn high wages. The possibility of matching with such a firm causes the flow value of unemployment to jump above its new steady-state value on impact. According to (16), this then causes wages to increase for all firms, including incumbents. This reduces vacancy-posting by incumbents.

What is the resulting effect on the dynamics of the v-u ratio? One intuition suggests that it should be ambiguous. One set of firms (new entrants) temporarily have a high marginal benefit of hiring, so they post vacancies more intensively than in steady state. Another set of firms (incumbents) reduce their vacancy posting, because hiring is more expensive during the transition to the new steady state. Why, then, is the v-u ratio persistent, that is, why is the total effect that the v-u ratio is lower during the transition than in the new steady state?

To understand the reason why our model generates persistence, think of the dynamics of vacancy-posting and entry that would arise if (counterfactually) the flow value of an unemployed worker, $rV^u(t) - V_t^u(t)$, jumped immediately to its new steady-state value and stayed there, as would occur in the absence of persistence. According to our wage equation (16), wages (as a function of firm size) would also jump immediately to their new steady-state value. It follows that the v-u ratio θ should jump immediately to its new steady-state value in order that the free entry condition be satisfied along the transition. (If both wages and market tightness were to jump to their new steady-state values, then a firm's optimization problem would be exactly the same as the one it faces in the steady state. Thus, the free-entry condition would always be satisfied with equality at all points along the transition path.) The transitional dynamics of the unemployment rate would then be straightforward to determine: the unemployment rate would evolve according to the differential equation

$$u_t(t) = (s + \delta)(1 - u(t)) - \theta qu(t). \quad (22)$$

Because the job-finding rate θq would be at its new steady-state value, there would be no more persistence in unemployment than in the standard MP model, where market tightness is also a jump variable.

Of course, in the true equilibrium, neither wages nor the value of an unemployed worker remain constant in the transition path. As already explained, after a positive shock to aggregate productivity, the economy has a disproportionate number of new entrant firms that are far from their target size. Because these firms pay high wages, in fact the flow value of an unemployed worker is temporarily higher than in the steady state. This raises the wages paid by all firms (conditional on size), and so reduces the intensity of vacancy-posting by all

firms in the transition, conditional on the firm’s current size and relative to the new steady state. This applies equally to incumbent firms and new entrant firms: all firms reduce their vacancy-posting relative to the new steady state.²⁹

The persistence mechanism is reinforced because the fact that wages are temporarily higher than they will be in the long run renders it unprofitable for all the firms needed to accomplish the full transition to the new steady state to enter at once. The entry process of new firms therefore now continues for some time after the initial shock. For the same reason, these later entrants then pay high wages when they do enter, and this keeps wages higher for longer, further slowing the transition to the new steady state.

The analysis in the rest of this section proceeds as follows. To illustrate the persistence mechanism, we present a numerical simulation of a specific parameterized version of our model. We then argue that four key features of our model are all needed to generate the persistence result. If there were constant returns to scale in production, constant returns to scale in vacancy-posting, or if wages were not bargained in such a way that a higher marginal product firm pays higher wages, then the v-u ratio in the resulting variant of our model would be a jump variable just as in the benchmark MP model. If we did not allow for entry of new establishments, but retained the other three assumptions, there is still some persistence in the response of the v-u ratio, but numerically we can show that it is much attenuated with respect to the benchmark version of the model. Understanding these four variants of the model makes clear the source of the persistence mechanism in our basic model.

6.2 Numerical Example

We first illustrate the persistence mechanism in a numerical example. Our goal in this section is to demonstrate the existence of our mechanism. We therefore focus for simplicity on the simple version of our model studied in [Section 4](#) which lacks the productivity heterogeneity required to match, for example, the empirical distributions of firm size and firm growth. The numerical results should therefore be taken as illustrative, even though we do choose several parameters to be consistent with some cross-sectional features of the U.S. economy.

We exhibit the dynamics of our model following an unanticipated, one-time, permanent increase in productivity. As in the previous section, we impose [Assumptions 2](#) and [3](#) so that production and the cost of posting vacancies take a quadratic form.³⁰ Our parameterization

²⁹It is true that entrant firms are hiring more aggressively than they will in steady state while incumbent firms are reducing their hiring, but this is not relevant. To generate persistence, what is important is not that entrant firms hire more aggressively than they will in steady state (when they will be larger) or that incumbent firms hire less than they did in the old steady state, but that all firms post fewer vacancies, conditional on their size, than they would if rV^u jumped immediately to its new steady-state value.

³⁰Note that both of these assumptions serve to reduce the quantitative importance of our propagation mech-

strategy is to match as far as possible the calibration targets used in Shimer (2005), for the sake of comparability to the literature which uses that paper as a reference point. We somewhat arbitrarily choose $\sigma = 0.05$, and then choose z and γ so that average employment per firm is 23.8 as in Davis et al. (2006b) and so as to normalize the flow value of unemployment rV^u to unity.³¹

In order to solve for the transitional dynamics following an unanticipated permanent productivity shock, we take a discrete-time approximation to the model, as well as approximating the value function on a discrete state space for n (and using cubic splines to interpolate for other n). This is necessary since numerically differentiating an incorrect guess for $J(\cdot)$ induces errors which are intractable numerically. The numerical solution procedure we use is described in Appendix B.

Figure 6 shows the impulse responses of key endogenous variables to an unanticipated, permanent 1 percent increase in productivity for all firms (that is, we multiply z and σ by 1.01). The left panel in the Figure shows that on impact, wages jump up as new firms enter and temporarily pay high wages. In the long run, average wages rise slightly more than 1 percent, but on the impact of the shock, wages for workers hired in the current quarter (new hires) jump by around 2.5 percent as the share of hiring done by small firms with a high marginal product of labor increases. Because of the increase in wages together with the increase in vacancy posting, the flow value of an unemployed worker increases by nearly 3 percent. The increase in wages paid to new hires, however, is transitory, and over time, the average wages earned by new hires declines. This reduction in the price of labor allows an increase in vacancy posting to be profitable, and so labor market tightness gradually rises over the transition. On impact, labor market tightness initially rises only slightly as vacancy posting by incumbent firms falls. Entry is large enough that the number of active firms actually slightly overshoots its steady-state

animum. Because the marginal product of labor for very small firms is not too high, wages do not rise much when new firms enter following the positive aggregate shock. Moreover, because vacancy posting costs are not too convex, these firms do not remain small for too long.

³¹The remainder of the parameterization is as follows. The values of z and γ required to hit the targets described in the text are $z = 1.776$ and $\gamma = 0.1131$. The matching function is Cobb-Douglas, $M(u, \bar{v}) = Zu^\eta \bar{v}^{1-\eta}$, with $Z = 1.355$ and $\eta = 0.72$, following Shimer (2005) and consistent with an unemployment rate of 6.87 percent. We set the unemployment income b so that $b/rV^u = 0.4$. We normalize the steady-state value of θ to 1; this is a normalization since multiplying Z by $\xi^{\eta-1}$ and γ by $\xi^{-(1+p)\eta}$ changes the equilibrium only by multiplying θ by ξ , q by $\xi^{-\eta}$, and $v(\cdot)$ by ξ^η ; the matching rates for firms and workers, $qv(\cdot)$ and θq , then remain unchanged. We set $\phi = 0.72$ for comparability with Shimer. We set $s + \delta = 0.10$ to match Shimer, and follow the evidence of Davis et al. (1996) in assuming that one-sixth of job destruction is attributable to firm shut-down. Finally, we choose k so that the free entry condition is satisfied in steady state; this requires $k = 87.57$. To put these derived parameters in context, note that at the mean level of vacancy-posting in the benchmark calibration, the flow cost incurred by a firm in posting vacancies, $c(v)$, is around 0.204, or around 35 percent of the marginal product of a worker of a firm at the mean firm size. It takes in expectation $(qv)^{-1} \approx 0.42$ quarters for a worker to be hired. That is, generating a new hire costs around 14.7 percent of one quarter's production by a single (marginal) worker. The free-entry cost corresponds to around 149.5 quarters of the flow output of the marginal worker.

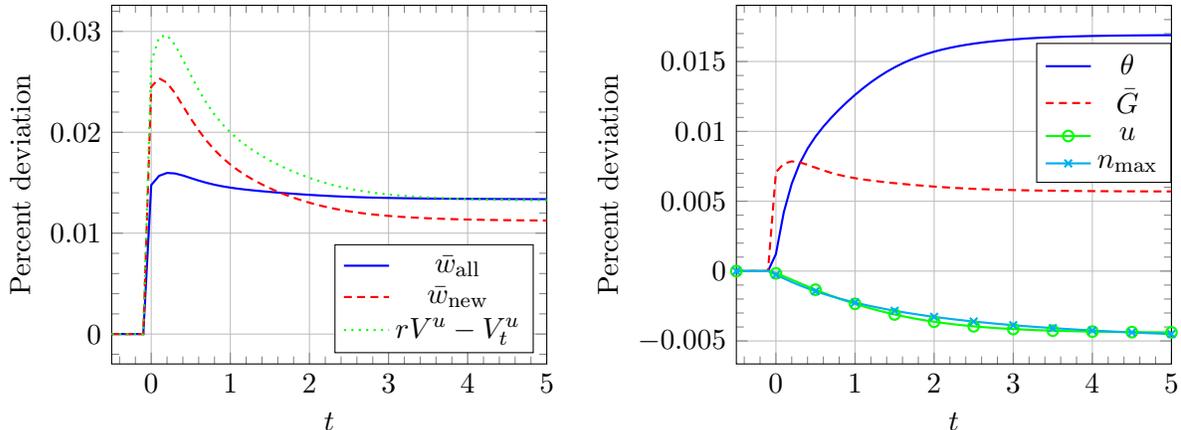


Figure 6: Impulse response to productivity shock: quadratic production, free entry. The left panel shows the paths of the average wage of all employed workers ($\bar{w}_{all}(t)$), the average wage of workers hired in the current period (\bar{w}_{new}), and the flow value of an unemployed worker, $rV^u - V_t^u$. The right panel shows the paths of market tightness θ , the number of active firms \bar{G} , unemployment u , and the size of the largest active firms, n_{max} . All variables are shown as log deviations from the initial steady state. Time is measured in quarters.

value. The process of adjustment in these two variables is then modestly persistent. After one quarter, market tightness has only increased by 75 percent of the long-run response and unemployment has undergone 53 percent of its eventual change. The size of the largest firms declines monotonically and slowly to its new steady state, and is also persistent.³²

In summary, a novel qualitative result of our model is the additional persistence in key labor market variables in response to shocks. Recall that these variables are much less, or not at all, persistent in the benchmark Mortensen-Pissarides model. For example, in that benchmark, market tightness has no persistence and unemployment completes 79 percent of its total adjustment after one quarter, as compared to 53 percent in our model.

6.3 Understanding the Persistence Result

In order to understand the source of the persistence in the benchmark model, we now briefly investigate four variants of the model. The contrast between these four variants, in three of which there is no persistence in the v-u ratio and in one of which there is only limited persistence, makes clear the source of propagation in our (benchmark) model.

³²The precise values reported in this paragraph are, of course, sensitive to the exact specification we investigate. In earlier versions of this paper, we also investigated the persistence properties of the model under Cobb-Douglas production. Because of the Inada condition at zero employment, under Cobb-Douglas production new entrant firms pay very high wages, which generates greater persistence than the current quadratic specification. Another way to generate more persistence is to increase the degree of convexity of the vacancy posting cost function, which slows down the growth of new firms.

6.3.1 Constant Returns to Scale in Production

Consider a model that is identical to our benchmark, except that the production function at establishment level takes the form

$$y(n) = zn.$$

Under this assumption, the wage equation (16) now takes the simpler form

$$w(n, t) \equiv w(t) = (1 - \phi) [rV^u(t) - V_t^u(t)] + \phi z,$$

independent of n . That is, the size of the firm is now irrelevant for the wage it pays. It is straightforward to establish that the value function of a firm is now linear in employment, and therefore that vacancy-posting is constant across firms.³³ The model still generates non-trivial firm size dispersion, because although all firms make the same number of hires at any time (given by qv), separations are greater for larger firms (since they are proportional to employment). Thus firm size dynamics are similar to those in the model with decreasing returns to scale in production—firms eventually converge to a ‘target’ size at which hiring just offsets separations—but unlike in the benchmark, vacancy-posting does not fall as the firm grows.

Because wages are independent of firm size, so too is the value of an employed worker, which can simply be denoted $V^e(t)$. This in turn means that the HJB equation for an unemployed worker takes the simple form

$$rV^u(t) = b + \theta q [V^e(t) - V^u(t)].$$

It follows that after a positive productivity shock, the evolution of the firm size distribution in the future is irrelevant for the equilibrium that obtains in the economy today.³⁴ The transitional dynamics following such a shock are therefore simple to characterize. On arrival of the unanticipated productivity shock, a positive measure of firms enters instantaneously, and the labor market tightness θ jumps immediately to its new steady-state level, as does the vacancy-posting policy v of each active firm, the wage w (which is independent of firm size), and the values of a firm with n workers, $J(n, t)$, of an unemployed worker V^u , and of an employed worker V^e (the latter is also independent of firm size).³⁵

Of course, some variables do exhibit transitional dynamics, notably the unemployment

³³To prove this, guess and verify a linear solution to the HJB equation.

³⁴That is, the model now has the same ‘block recursivity’ property as either the Pissarides (1985) benchmark or the large-firm model of Pissarides (2000).

³⁵This would not be correct if we were considering an unanticipated decrease in productivity, following which the number of entrant firms would not suddenly decrease to the new steady-state value – instead, it would decline slowly as firms exit due to exogenous destruction – so that it is important here that we are considering a positive shock to productivity.

rate and the size distribution of firms, but these variables do not enter in the equations that characterize the set of variables mentioned in the previous paragraph. The differential equation indicating the evolution of unemployment over time is as in (22); its solution is

$$u(t) = u^1 + (u^0 - u^1) \exp(-(s + \delta + \theta q)t),$$

where u^0 and u^1 are respectively the pre- and post-shock steady-state values of unemployment. As in the standard MP model, the transitional dynamics of unemployment are very rapid for plausible calibrations. In the benchmark model with decreasing returns to scale, the unemployment rate underwent only 54 percent of its transition to the new steady state in the first two quarters after the arrival of the shock. In the model presented in this section, if the values of s , δ , and θq are the same, then the corresponding statistic is greater than 79 percent. For some variables the contrast is more striking: in the linear model, the value of an unemployed worker jumps immediately to its new steady-state value, while in the model with decreasing returns, this value overshoot the final steady-state value significantly on the impact of the shock, and after a quarters was still 2.0 percent above the original steady-state value, significantly above the new steady-state value (1.3 percent above the former value).

6.3.2 Constant Returns to Scale in Vacancy Posting

A second key assumption of our model is that the vacancy-posting cost function is convex. That is, a firm that wishes to hire more rapidly pays more than a proportional cost in order to do so. If we instead assumed that the cost of hiring was proportional to the intensity of hiring, the dynamics both of growth for an individual firm and for the time path of the adjustment of the economy to a change in aggregate productivity would be quite different. This case is studied more formally in Hawkins (2011); here we give an intuitive discussion of the results in that paper.

First, for an individual firm, growth dynamics are ‘bang-bang.’ The firm still has a target employment level, determined by the point at which the benefits from hiring another worker (the marginal product of labor plus the effect of the increased firm-level employment in driving down wages for inframarginal workers) just balance the required vacancy-posting cost, which is now independent of the amount of vacancy-posting the firm is doing. Because it is profitable to hire workers up to this point, the firm does so as quickly as possible. Thus, a new entrant firm posts a very large measure of vacancies for a vanishingly small instant of time, arrives at its target size immediately, and after this, only posts vacancies just sufficient so that new hires just offset the flow of exogenous separations. If the firm’s target size increases in response to an aggregate or idiosyncratic shock, it immediately hires enough workers to reach its new

target; if it falls, the firm allows exogenous separations to return it to its target over time.³⁶ An important implication is that all new hire workers share the same value of employment $V(n, t; z)$, because all firms that post any vacancies are at their hiring target, and this implies that the current values of aggregate productivity and of market tightness are sufficient statistics for any firm’s hiring target.³⁷

For the aggregate economy, the adjustment dynamics in response to a shock share the property of the benchmark Pissarides (1985) model that the v-u ratio is a jump variable that responds instantaneously to the productivity shock: that is, the model with constant returns to scale in vacancy posting lacks any propagation mechanism for market tightness. To see why, consider the response, for example, to a small change in aggregate productivity. Suppose that the v-u ratio jumps to its new steady-state value, which is lower than before. In response, each incumbent firm’s target size changes. If the target size is higher, then because the v-u ratio jumps, it is optimal for the firm immediately to post vacancies sufficient to jump to its new target employment level; thus, there is a burst of hiring by incumbent firms. If the target size is lower, then firms allow their employment to fall gradually by attrition. The amount of entry adjusts so as to keep the correct measure of vacancies posted, given the value of the unemployment rate (which is a state variable).

In summary, when the vacancy-posting technology is linear, then a change in aggregate productivity does not lead to the presence of firms that are below their hiring target which pay temporarily higher wages. This means that the propagation mechanism present in the benchmark model is absent.

6.3.3 No Worker Bargaining Power

A third possible modification to our benchmark model is the assumption that workers have no bargaining power, that is, $\phi = 0$. In this case, according to (16), wages are constant and equal to b for all employed workers, so that the value of an unemployed worker is also $rV^u = b$. (This is the world of the Diamond paradox: all workers earn a wage equal to their outside option.) What do the transitional dynamics look like in this environment? The answer is that again there is no persistence: the v-u ratio jumps on the arrival of a change in aggregate productivity. The reason for this result is straightforward in view of the intuitive explanation of the persistence mechanism already given in the introductory part of Section 6, which relied on a comparison with the case when rV^u was constant and equal to its new steady-state value

³⁶Very large decreases in the target size, for example, associated with large falls in productivity z , may also be associated with firing; this would be unlikely in response to (small) aggregate shocks.

³⁷If we allowed for differences in idiosyncratic productivity across firms, a productive firm would have a larger target size than a less productive counterpart, but the two firms would have the same marginal value of hiring.

during the whole transition. When workers lack bargaining power, this is not a counterfactual, but exactly correct. Along the transition $rV^u \equiv b$ is constant, so the only dynamics for the v-u ratio consistent with optimal vacancy-posting by incumbent firms and free entry of new firms are the jump dynamics just described, which do not generate any persistence. This result emphasizes that our wage determination assumption is key for our persistence result.

6.3.4 No Free Entry

Finally, let us return to the benchmark model with the modification that the number of firms in the economy is fixed. Thus, in responses to changes in aggregate productivity, there will not be entry. In order to ensure our model has a non-trivial steady-state equilibrium under this assumption, we set $\delta = 0$, so that there is also no firm exit.

In this environment, unlike in the previous three subsections, the v-u ratio exhibits some dynamics in response to changes in aggregate productivity, but those dynamics are only mildly persistent. In response to an increase in aggregate productivity, all increased vacancy posting and job creation must be done by incumbents, which grow slightly. As these firms increase vacancy-posting immediately after the shock, the probability of job-finding for an unemployed worker rises above the value it will take in the new steady state. However, a greater fraction of vacancies than in the new steady state are in this case posted by relatively large firms, who offer relatively low wages.

Figure 7 shows the effect of a 1 percent increase in aggregate productivity z in the numerical example of quadratic production from Section 6.2 when there is no entry.³⁸ In this numerical example, the two offsetting effects mentioned in the previous paragraph almost exactly cancel, so that rV^u nearly jumps to its new steady-state value. Market tightness continues to increase slightly in transition as unemployment falls; without free entry on the extensive margin and with existing firms unwilling to increase vacancy posting too much due to the increasing marginal vacancy-posting costs, firms are unwilling to post enough vacancies at the time of impact for market tightness to increase all the way to its new steady-state values. However, this effect is quantitatively insignificant, and unemployment dynamics are very close to those in the MP benchmark.

To summarize, the effects present in our benchmark model would not be present or would be much muted without all three contributing factors, time-consuming hiring, bargained wages, and decreasing returns to scale production functions. This highlights that the persistence result

³⁸We make only two modifications to the previous example. First, we set $\delta = 0$ and $s = 0.1$, thus holding constant the total worker employment-unemployment transition rate $s + \delta$. Second, while we continue to assume that the measure of active firms is such that the free-entry condition does hold in the initial steady state, we hold this measure constant thereafter and do not impose the free-entry condition after the change in productivity.

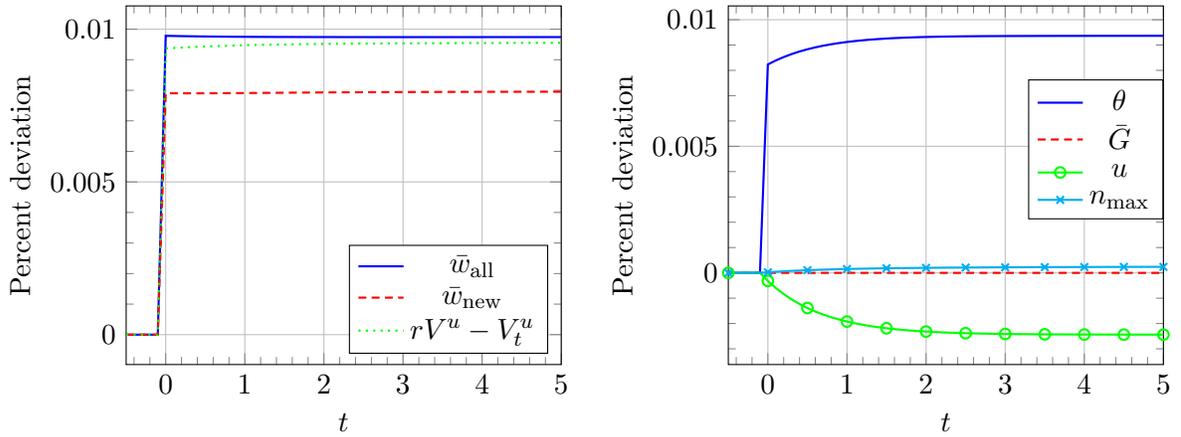


Figure 7: Impulse response to productivity shock: quadratic production, no entry. Notation is as in Figure 6.

relies on all of the key building blocks of our model.

7 Conclusion

This paper considered a generalization of the canonical Diamond-Mortensen-Pissarides search model to an environment in which firms employ multiple workers (with decreasing returns to labor) and hiring is explicitly time-consuming (because there is a convex cost of posting vacancies). We followed Stole and Zwiebel (1996a,b) and assumed that wages are determined by continuous bargaining between the firm and its employees. Our model introduces a meaningful distinction between the intensive and extensive margins of hiring – entry of new firms and hiring by existing firms, respectively. There is also an endogenous dispersion of firm sizes and wages (because small firms have higher marginal product of labor and pay higher wages). We proved the existence of a steady-state equilibrium, characterized several properties of steady-state equilibria, and showed that the steady-state equilibrium was consistent in many important respects with U.S. data, at least if we allow for enough productivity heterogeneity across firms. We then numerically studied the dynamics out of steady state, and found that the simple ingredients of our model are by themselves enough to induce a quantitatively relevant channel for propagating aggregate productivity shocks which is lacking in the constant returns benchmark.

Our paper suggests several interesting directions for further research. Our assumption that a firm’s productivity is fixed is clearly an unrealistic simplification. It would be interesting to generalize our model to allow for transitory productivity shocks. This would allow marrying the insights of Bertola and Caballero (1994) on how labor market frictions slow down reallocation of labor from unproductive to productive uses with our focus on the growth of firms. It would

also improve the realism of large firm bargaining models such as Elsby and Michaels (2013), Fujita and Nakajima (2009), Hawkins (2011), and Schaal (2012) which often unrealistically assume ‘bang-bang’ hiring dynamics.

It would also be interesting to investigate the business cycle dynamics of a full-fledged quantitative version of the model. Our model suggests that the temporarily higher wages that firms below their steady-state employment play an important role in business cycle dynamics. This is potentially consistent with recent evidence provided by Moscarini and Postel-Vinay (2008) on the important role played by small firms in the early stages of expansions.

A Appendix: Omitted Proofs

Proof of Lemma 1. Rearrange the worker HJB equation (7) to observe that

$$(r + \delta + s) [V(n, t; z) - V^u(t)] - V_t(n, t; z) = w(n, t; z) - rV^u(t) + [qv(n, t; z) - sn] V_n(n, t; z).$$

Use the bargaining equations (2), together with its derivatives with respect to n and t , to replace terms involving the worker’s value $V(\cdot)$ with those involving derivatives of the firm’s value $J(\cdot)$. Comparing the results with the derivative with respect to n of the firm’s HJB equation (4) establishes immediately that

$$\phi [y'(n; z) - w(n, t; z) - nw_n(n, t; z)] = (1 - \phi) [w(n, t; z) - (rV^u(t) - V_t^u(t))],$$

or equivalently

$$w(n, t; z) + \phi nw_n(n, t; z) = \phi y'(n; z) + (1 - \phi) [rV^u(t) - V_t^u(t)].$$

Integrating this equation with respect to n implies that

$$w(n, t; z) = (1 - \phi) [rV^u(t) - V_t^u(t)] + n^{-\frac{1}{\phi}} \left[c + \int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu; z) d\nu \right],$$

where c is a constant of integration; this is well-defined because of [Assumption 1](#). The assumption that $nw(n, t; z)$ remains finite as $n \rightarrow 0^+$ implies that $c = 0$. The expression in the statement of the Lemma now follows by writing $n^{-1/\phi} = \phi \left(\int_0^n \nu^{-(1-\phi)/\phi} d\nu \right)^{-1}$. \square

Proof of Lemma 2. Define

$$\psi(n; z) = \frac{\int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu; z) d\nu}{\int_0^n \nu^{\frac{1-\phi}{\phi}} d\nu} = \frac{1}{\phi} n^{-\frac{1}{\phi}} \int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu; z) d\nu.$$

Because $\phi\psi(n; z) = w(n; z) - (1 - \phi)[rV^u(t) - V_t^u(t)]$, it suffices to show that $\psi(n; z) > 0$, $\psi'(n; z) < 0$, $\lim_{n \rightarrow \infty} \psi(n; z) = \lim_{n \rightarrow \infty} y'(n; z)$, and $\lim_{n \rightarrow 0^+} \psi(n; z) = \lim_{n \rightarrow 0^+} y'(n; z)$ in order to establish the claims about wages. The first of these properties is obvious. The second follows from writing

$$\phi\psi'(n; z) = -\frac{1}{\phi}n^{-\frac{1+\phi}{\phi}} \int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu; z) d\nu + n^{-1}y'(n) = \frac{1}{\phi}n^{-\frac{1+\phi}{\phi}} \int_0^n \nu^{\frac{1-\phi}{\phi}} [y'(n; z) - y'(\nu; z)] d\nu$$

which is strictly negative because of the strict concavity of $y(\cdot; z)$. To prove the third property, observe that

$$y'(n; z) < \psi(n; z) = n^{-1}y(n; z) - \frac{1-\phi}{\phi}n^{-\frac{1}{\phi}} \int_0^n \nu^{\frac{1-2\phi}{\phi}} y(\nu; z) d\nu < n^{-1}y(n; z),$$

where the first inequality follows because $y'(\cdot; z)$ is a decreasing function and $\psi(n; z)$ is a weighted average of terms of the form $y(\nu; z)$ for $\nu \in [0, n]$, while the second follows because $y(\cdot; z)$ is strictly positive. Since $\lim_{n \rightarrow \infty} n^{-1}y(n; z) = \lim_{n \rightarrow \infty} y'(n; z)$, the result follows by the squeeze principle. Finally, the last result is again obvious from the fact that $\psi(n; z)$ is a weighted average of the values of $y'(\nu; z)$ on the interval $\nu \in (0, n)$.

To obtain the results concerning the profit function, note that using our characterization of wages, by definition

$$\pi(n, t; z) = y(n; z) - n^{-\frac{1-\phi}{\phi}} \int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu; z) d\nu - n(1 - \phi)[rV^u(t) - V_t^u(t)],$$

so that $\pi_n(n, t; z) = (1 - \phi)(\psi(n; z) - [rV^u(t) - V_t^u(t)])$. Thus $\pi(\cdot, t; z)$ is strictly concave because $\psi(\cdot; z)$ is strictly decreasing. Hence the maximizer of $\pi(\cdot, t; z)$ on $[0, \infty)$ is unique. Since $\lim_{n \rightarrow 0^+} \psi(n, t; z) = \lim_{n \rightarrow 0^+} y'(n; z)$, the maximizer is strictly positive provided that $\lim_{n \rightarrow 0^+} y'(n; z) > rV^u(t) - V_t^u(t)$. To establish that $\pi(n, t; z) \rightarrow 0$ as $n \rightarrow 0^+$, note that $y(0; z) = 0$ by assumption, while it is straightforward to verify from (16), together with **Assumption 1**, that $nw(n; z) \rightarrow 0$ as $n \rightarrow 0$. To establish that $\pi(n, t; z) \rightarrow -\infty$ as $n \rightarrow \infty$, note that under the condition in the statement of the Lemma,

$$\frac{1}{1-\phi} \lim_{n \rightarrow \infty} \pi_n(n, t; z) = -[rV^u(t) - V_t^u(t)] + \lim_{n \rightarrow \infty} \psi(n; z) = -[rV^u(t) - V_t^u(t)] + \lim_{n \rightarrow \infty} y'(n; z),$$

so that $\pi_n(n, t; z)$ is bounded away from zero uniformly in n for n sufficiently large. \square

Proof of Theorem 1. We define two functions $\chi, \omega : \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R}$. Let $\chi(q, rV^u)$ be the value of $J(0) - k$, where $J(\cdot)$ is the unique solution to the ordinary differential equation given by the steady-state version of (4) where, on the right side of (4), the wage function $w(\cdot)$ is given

by (16), and where the boundary condition for the initial value problem is $(n^*, J(n^*))$ with n^* defined as the unique solution to (21) and the value of $J(n^*)$ given by (20). Let

$$\omega(q, rV^u) = rV^u - \left[b + \frac{\phi}{1-\phi} \theta q \frac{\int_0^{n^*} v(n) J'(n) g(n) dn}{\int_0^{n^*} v(n) g(n) dn} \right],$$

where the density $g(\cdot)$ is given by (18) and where $J(\cdot)$ is as just defined. That is, $\chi(q, rV^u)$ is the net surplus over the entry cost for a potential entrant firm that takes q and rV^u as given, expects to pay wages $w(\cdot)$ as given by (16), and chooses its vacancy-posting optimally; $rV^u - \omega(q, rV^u)$ is the value of an unemployed worker in an economy populated by such firms.

As observed in the discussion preceding Proposition 1, (q, rV^u) is part of an equilibrium allocation iff

$$\chi(q, rV^u) = \omega(q, rV^u) = 0.$$

We will prove that such an intersection exists as follows. We first show that the set S of tuples (q, rV^u) satisfying $k = \chi(q, rV^u)$ is a connected subset of $\mathbb{R}^+ \times \mathbb{R}$. Next, we observe that ω restricted to this set defines a continuous function which takes both positive and negative values. The result then follows from the intermediate value theorem.

$\chi(\cdot)$ is a continuous function by the theorem of the maximum, so to prove that the set of points such that $\chi(q, rV^u) = 0$ is a continuous 1-manifold, it suffices to show that $\chi(\cdot)$ is nondecreasing in its first argument and nonincreasing in its second. (In fact, both relationships are strict if there is positive activity, that is, if it is optimal for a firm with zero workers to hire.) To see why, recall that $\chi(q, rV^u)$ is the maximized value of the problem for the firm as described. If rV^u decreases, then for any q , $\chi(q, rV^u)$ must increase, since if the firm keeps the same hiring strategy as before, then it would increase the value of its program as $w(n)$ decreases for each n ; re-optimizing the hiring strategy can only increase this effect. The increase in value is strict provided the firm ever hires a positive number of workers, and this is guaranteed by Lemma 2 provided that $y'(0) > rV^u$ (since in that case $\pi(n) > 0$ for $n > 0$ small).

Second, if q increases to $q' > q$, then the firm could increase the value of its program by replacing its former vacancy-posting strategy $v(\cdot)$ by $qv(\cdot)/q'$. This would lead to the same dynamics of its size and cost strictly less (again, the inequality is strict provided that the firm ever posts any vacancies).

Next, define \bar{v} to solve

$$k = \frac{1}{r + \delta} \max_{n > 0} \left[y(n) - n^{-\frac{1-\phi}{\phi}} \int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu) d\nu - n(1-\phi)\bar{v} \right]. \quad (\text{A.1})$$

A firm that pays wages given by

$$w(n) = (1 - \phi)\bar{v} + n^{-\frac{1}{\phi}} \int_0^n \nu^{\frac{1-\phi}{\phi}} y'(\nu) d\nu$$

to each of its employees will just break even if and only if it can reach that employment level n^* that maximizes the right side of (A.1) instantaneously on entry and at zero cost.

If $\bar{v} - b \leq 0$ then it is clear that there is an equilibrium in which no firm ever enters. Otherwise, in the case $\bar{v} > b$, we show that there are points $(q_1, v_1), (q_2, v_2) \in S$ such that $\omega(q_1, v_1)$ and $\omega(q_2, v_2)$ differ in sign.

To find a point at which $\omega(\cdot)$ takes a positive value, observe that $\lim_{q \rightarrow \infty} \chi(q, \bar{v}) = 0$. Also, for $v > \bar{v}$, $\chi(q, v) < k$ by construction. If $q \rightarrow \infty$, then any firm will instantaneously hire n^* ; thus in the limit, $J(n) = J(n^*)$ for all $n \in [0, n^*]$. From the definition of $\omega(\cdot)$, it follows that for q sufficiently large, equivalently, for $\theta q(\theta)$ sufficiently small, $\omega(q, \bar{v}) = \bar{v} - b > 0$.

To find a point at which $\omega(\cdot)$ takes a negative value, let $\hat{q} > 0$ satisfy $\chi(\hat{q}, b) = 0$. Such a \hat{q} will exist since $\bar{v} > b$. By definition

$$\omega(\hat{q}, b) = -\frac{\phi}{1 - \phi} \theta \hat{q} \frac{\int_0^{n^*} v(n) J'(n) g(n) dn}{\int_0^{n^*} v(n) g(n) dn},$$

which is strictly negative because $J'(n)$ is strictly positive for any $n < n^*$, because $\theta \hat{q} > 0$ (otherwise $q = +\infty$, which is impossible since $\bar{v} \neq b$). Thus if $\bar{v} - b > 0$, then S contains points at which ω takes values of opposite signs, which completes the proof of the existence of an equilibrium via the intermediate value theorem (more formally, applied to $\omega(\cdot)$ restricted to a continuous path connecting two such points). \square

Proof of Lemma 3. The equations for wages and profits are immediate from specializing (16) using the quadratic production function.

Next, substitute for wages into the firm's HJB equation (4) and impose steady state to specialize this equation to the form

$$(r + \delta)J(n; z) = \pi(n; z) - snJ_n(n; z) + \frac{1}{2\gamma}q^2J_n(n; z).$$

This can be solved in closed form. We guess and verify that there is a quadratic solution for $J(\cdot)$, of the form

$$J(n; z) = A(z) + B(z)n - \frac{1}{2}Cn^2,$$

and solve for the unknown coefficients. (Note that C turns out not to depend on z .)

$$\begin{aligned}
C &= \frac{\gamma}{2q^2} \left(-(r + \delta + 2s) + \sqrt{(r + \delta + 2s)^2 + \frac{4q^2(1-\phi)}{\gamma(1+\phi)}\sigma} \right) \\
B(z) &= \frac{2(1-\phi)(z - rV^u)}{r + \delta + \sqrt{(r + \delta + 2s)^2 + \frac{4q^2(1-\phi)}{\gamma(1+\phi)}\sigma}} \\
A(z) &= \frac{q^2 B(z)^2}{2\gamma(r + \delta)}
\end{aligned} \tag{A.2}$$

generates the unique concave solution.

Under quadratic vacancy-posting, the first-order condition for optimal vacancy posting requires that $c'(v(n; z)) = qJ_n(n; z)$. Substituting for $J_n(n; z)$ and simplifying gives that the functional form for vacancies is

$$v(n; z) = \frac{q}{\gamma}(B(z) - Cn). \tag{A.3}$$

The expression for $n^*(z)$ follows by using this expression in the equation that characterizes $n^*(z)$, that is, $qv(n^*(z)) = sn^*(z)$. Substituting back into the vacancy equation just derived gives the claimed expression for vacancies.

To solve for the firm size distribution, substitute the functional form for vacancies into (17) and integrate. After some algebra, we obtain that among firms with productivity equal to z , the density of firms with employment equal to n is

$$\frac{g(n)}{x} = \delta \left(\frac{\gamma}{q^2 B(z)} \right)^{\frac{\delta}{s+q^2 C/\gamma}} \left(\frac{q^2 B(z)}{\gamma} - \left(s + \frac{q^2 C}{\gamma} \right) n \right)^{-1 + \frac{\delta}{s+q^2 C/\gamma}}, \tag{A.4}$$

and a second integration step establishes the claimed expression for $\bar{G}(n; z)$. \square

Proof of Proposition 3. Using the expressions for $n^*(z)$ and $\bar{G}(n; z)$ in Lemma 3, the fraction of firms with productivity z which have size greater than n can be written

$$\bar{G}(n; z) = \begin{cases} \left(1 - \frac{c_1 n}{z - rV^u} \right)^{c_2} & n < n^*(z); \\ 0 & n \geq n^*(z). \end{cases}$$

Here the constants c_1 and c_2 do not depend on either n or z . For any $n > 0$, the maximum value of z such that $\bar{G}(n; z) = 0$ is given by solving $n^*(z) = n$ for z , that is, by $z = rV^u + c_1 n$.

It follows that the fraction of firms with size greater than n is given by

$$\bar{G}(n) = \int_{rV^u + c_1 n}^{\infty} \bar{G}(n; z) f(z) dz.$$

Using the change of variable $\hat{z} = z - rV^u$, this can be written explicitly as

$$\bar{G}(n) = \int_{c_1 n}^{\infty} \left(1 - \frac{c_1 n}{\hat{z}}\right)^{c_2} \frac{kz_m^k}{(\hat{z} + rV^u)^{k+1}} d\hat{z}.$$

Now observe that if $\lambda \geq 1$ is a constant, then

$$\begin{aligned} \bar{G}(\lambda n) &= \int_{c_1 \lambda n}^{\infty} \left(1 - \frac{c_1 \lambda n}{\hat{z}}\right)^{c_2} \frac{kz_m^k}{(\hat{z} + rV^u)^{k+1}} d\hat{z} \\ &= \int_{c_1 n}^{\infty} \left(1 - \frac{c_1 \lambda n}{\lambda \tilde{z}}\right)^{c_2} \frac{kz_m^k}{(\lambda \tilde{z} + rV^u)^{k+1}} \lambda d\tilde{z} \\ &= \lambda^{-k} \int_{c_1 n}^{\infty} \left(1 - \frac{c_1 n}{\tilde{z}}\right)^{c_2} \frac{kz_m^k}{(\tilde{z} + rV^u)^{k+1}} \left(\frac{1 + \frac{rV^u}{\tilde{z}}}{1 + \frac{rV^u}{\lambda \tilde{z}}}\right)^{k+1} d\tilde{z} \end{aligned}$$

where in the second line we use the change of variables $\hat{z} = \lambda \tilde{z}$. Because rV^u is a fixed constant in equilibrium, the third factor inside the integral on the right side converges uniformly to 1 as n becomes large, so that for n large, $\bar{G}(\lambda n) / (\lambda^{-k} \bar{G}(n))$ is also arbitrarily close to 1 as n becomes large. \square

Proof of Proposition 4. To establish Gibrat's law, it suffices to prove that the joint density of firms over firm size n and firm growth $\zeta = \dot{n}/n$ can be factored as the product of a function of n and a function of ζ .

First observe that using (A.3), a firm's growth rate can be written as

$$\zeta = \frac{\dot{n}}{n} = \frac{1}{n} (qv(n) - sn) = \frac{1}{n} \left(\frac{q^2}{\gamma} (B(z) - Cn) - sn \right) = \frac{q^2}{\gamma} \frac{B(z)}{n} - \left(s + \frac{q^2 C}{\gamma} \right),$$

so that

$$B(z) = \left(\zeta + s + \frac{q^2 C}{\gamma} \right) \frac{\gamma}{q^2} n. \quad (\text{A.5})$$

Use this to eliminate $B(z)$ from the expression for the firm size distribution (A.4), obtaining that

$$\frac{g(n)}{x} = \delta n^{-1} \zeta^{-1 + \frac{\delta}{s + q^2 C / \gamma}} \left(\zeta + s + \frac{q^2 C}{\gamma} \right)^{-\frac{\delta}{s + q^2 C / \gamma}}. \quad (\text{A.6})$$

Next, the value of productivity z consistent with firm size n and growth rate ζ can be found

by combining (A.2) and (A.5) to observe that

$$z = rV^u + c_1 \left(\zeta + s + \frac{q^2 C}{\gamma} \right) n \quad (\text{A.7})$$

where

$$c_1 = \frac{r + \delta + \sqrt{(r + \delta + 2s)^2 + \frac{4q^2(1-\phi)}{\gamma(1+\phi)} \sigma} \gamma}{2(1-\phi) q^2}$$

is a constant parameter independent of z , n , and ζ . (A.7) has two implications. First, holding n constant, we have that the value of z consistent with growth rate ζ satisfies

$$\frac{\partial z}{\partial \zeta} = c_1 n. \quad (\text{A.8})$$

Second, the density of z can be written under the Pareto distributional assumption as

$$\begin{aligned} f(z) &= \frac{\kappa z_m^\kappa}{\left(rV^u + c_1 \left(\zeta + s + \frac{q^2 C}{\gamma} \right) n \right)^{\kappa+1}} \\ &= \frac{\kappa z_m^\kappa}{c_1^{\kappa+1}} \left(\zeta + s + \frac{q^2 C}{\gamma} \right)^{-(\kappa+1)} n^{-(\kappa+1)} \left(1 + \frac{rV^u}{c_1 \left(\zeta + s + \frac{q^2 C}{\gamma} \right) n} \right)^{-(\kappa+1)}. \end{aligned} \quad (\text{A.9})$$

Combining (A.6), (A.8), and (A.9), we obtain that the density of firms with size n and growth rate ζ can be written

$$\begin{aligned} &\frac{g(n)}{x} \cdot f(z) \cdot \left(\frac{\partial z}{\partial \zeta} \right)^{-1} \\ &= \frac{\delta \kappa z_m^\kappa}{c_1^{\kappa+2}} n^{-(\kappa+3)} \zeta^{-1 + \frac{\delta}{s + q^2 C / \gamma}} \left(\zeta + s + \frac{q^2 C}{\gamma} \right)^{-(\kappa+1) - \frac{\delta}{s + q^2 C / \gamma}} \left(1 + \frac{rV^u}{c_1 \left(\zeta + s + \frac{q^2 C}{\gamma} \right) n} \right)^{-(\kappa+1)}. \end{aligned}$$

The right side of this equation is the product of a function of n , a function of ζ , and an expression that converges to 1 uniformly in ζ as $n \rightarrow +\infty$. It follows that Gibrat's law holds asymptotically. \square

Proof of Proposition 5. Because a firm's death rate is independent of productivity z and because firms grow monotonically towards their target size, a firm's age is a monotone function of its rank in the firm size distribution, with older firms being larger. It follows in turn from the expression for the firm size distribution in Lemma 3 that any two firms which have the same age have the same value of the expression $c(n; z) \equiv n/n^*(z)$. Note that $c(n; z) \in [0, 1)$, with older firms having higher values of $c(n, z)$. ($c(n, z) = 1$ would correspond to a firm that has

reached its target size, but this does not occur in finite time.)

Now, suppose that two firms have the same age, and therefore the same value of $c(n, z)$. Denote this common value by \bar{c} . Then for such firms, we can write

$$z - rV^u = \left[\frac{\sigma}{1 + \phi} + \frac{\gamma s(r + \delta + s)}{q^2(1 - \phi)} \right] \frac{1}{\bar{c}} n. \quad (\text{A.10})$$

Using this equation, it is straightforward to complete the proof. A firm's growth rate (in levels) is

$$\dot{n}(n; z) = qv(n; z) - sn = \frac{q^2}{\gamma} \left[B(z) - \left(\frac{s\gamma}{q^2} + C \right) n \right];$$

where the second expression follows from (A.3). After some algebra, this can be rewritten as

$$\dot{n}(n; z) = \frac{q^2}{\gamma} \left[\frac{s\gamma}{q^2} + C \right] \left[\frac{1}{\bar{c}} - 1 \right] n.$$

Second, write the wage a firm pays as

$$w(n; z) = rV^u + \phi \left(z - rV^u - \frac{\sigma}{1 + \phi} n \right) = rV^u + \phi \left[\frac{\sigma}{1 + \phi} \left[\frac{1}{\bar{c}} - 1 \right] + \frac{\gamma s(r + \delta + s)}{q^2(1 - \phi)\bar{c}} \right] n.$$

Finally, use the expression for profits per worker in Lemma 3 to write

$$\frac{\pi(n; z)}{n} = (1 - \phi) \left[z - rV^u - \frac{\sigma n}{2(1 + \phi)} \right] = (1 - \phi) \left[\frac{\sigma}{1 + \phi} \left[\frac{1}{\bar{c}} - \frac{1}{2} \right] + \frac{\gamma s(r + \delta + s)}{q^2(1 - \phi)\bar{c}} \right] n.$$

In each of the last three equations, the coefficient on n on the right side is strictly positive since $\bar{c} \in (0, 1)$. This establishes the result.³⁹ \square

B Appendix: Numerical Approximate Solution Method

We use the following approximate solution procedure for the transitional dynamics of the model in Section 6.2.

- Select a time T by which the transition will be largely complete, and impose that from time T onwards, the economy will be in the steady state corresponding to the new, higher productivity level.
- Guess time paths for $\{\theta(t)\}_{t=0}^{T-1}$ and $\{rV^u(t)\}_{t=0}^{T-1}$. If entry is allowed, guess also a time path for firm entry, $\{e(t)\}_{t=0}^{T-1}$.

³⁹The observant reader will notice that this argument does not apply to the case of $n = 0$, which corresponds to $\bar{c} = 0$. It is straightforward to include this case also by rewriting (A.10) to express n in terms of \bar{c} and $z - rV^u$.

- Solve for the initial steady-state firm size distribution, $G(\cdot, 0)$.
- Solve for the final steady-state value function, $J(\cdot, T)$.
- Solve recursively for the functions $\{J(\cdot, t)\}_{t=0}^{T-1}$, iterating backwards in time, and using the assumed time paths for $\theta(t)$ and $rV^u(t)$. In this process, calculate the optimal vacancy posting policies of firms, $v(n, t)$ for $t \in 0, 1, 2, \dots, T - 1$.
- Using the guessed time paths of $\theta(\cdot)$ and $e(\cdot)$ and the calculated vacancy posting policies $v(\cdot, \cdot)$, simulate the evolution of the firm size distribution $G(\cdot, t)$ and of the unemployment rate $u(t)$.
- Use these, together with the value function equation for the unemployed worker (8), to calculate the resulting time paths of $rV^u(t)$ and $\theta(t)$; denote these time paths by $r\hat{V}^u(t)$ and $\hat{\theta}(t)$. If the calculated time paths are sufficiently close to the guesses, stop. If not, update the guesses by selecting new guesses

$$V_{\text{new}}^u(t) = (1 - \lambda_V)V^u(t) + \lambda_V\hat{V}^u(t) \quad \text{and} \quad \theta_{\text{new}}(t) = (1 - \lambda_\theta)\theta(t) + \lambda_\theta\hat{\theta}(t),$$

where λ_V and λ_θ are constants chosen small enough that the procedure converges. (In the case of free entry, check also whether the free entry condition holds for all $t = 0, 1, \dots, T - 1$, and if not, reduce (respectively, increase) entry slightly at times when the calculated value of entry, $J(0, t)$, is less than (respectively, greater than) k .)

- Verify that the distribution of firm sizes at time T is sufficiently close to the steady-state distribution. If not, choose a larger T and repeat the whole algorithm.

The solution generated has minimal error. At the reported allocation, relative error in $|\hat{V}^u(t)/V^u(t) - 1|$ and $|\hat{\theta}(t)/\theta(t) - 1|$ are all less than 10^{-5} .

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