Serial Dictatorship: the Unique Optimal Allocation Rule when Information is Endogenous.

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Abstract

The study of matching problems typically assumes that agents precisely know their preferences over the goods to be assigned. Within applied contexts, this assumption stands out as particularly counterfactual. Parents typically do invest a large amount of time and resources to find the best school for their children; doctors run costly tests to establish the best kidney for a given patient. In this paper I introduce the assumption of endogenous information acquisition into otherwise standard house allocation problems. I find that there is a unique ex-ante Pareto-optimal, strategy-proof and non-bossy allocation mechanism: serial dictatorship. This stands in sharp contrast to the very large set of such mechanisms for house allocation problems without endogenous information acquisition.

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1 Introduction

Many allocation problems of indivisible goods have to be solved without explicit markets. For some such goods, be it school slots or kidneys, the use of markets to determine allocations is perceived as immoral or repugnant. In many cases markets are explicitly forbidden. A - prospering - subfield of mechanism design asks the question how to best allocate such objects to recipients; many mechanisms that are optimal according to a host of different criteria have been found. These mechanisms have usually been designed for the case of agents precisely knowing their preferences over the goods to be assigned. However, this assumption seems counterfactual in many of the areas in which such mechanisms are used. Parents typically invest a significant amount of time on school choice; doctors need to run costly tests on kidneys to figure out which would be best for a given patient.

This paper sets out to study the allocative properties of mechanisms in conjunction with their impact on the agents' incentives to acquire information. To this end I modify the standard model of house allocation problems in which a set of agents needs to be matched to a set of equally many objects, henceforth called houses, allowing for costly information acquisition on these houses. The goal is to characterize the set of strategy-proof, non-bossy and Pareto-optimal mechanisms in this environment.

Over the years, various classes of such mechanisms have been identified for the standard case of known preferences. Pycia and Unver [9] characterize the – very large – set of all such mechanisms. Lots of room remains to impose additional requirements to select among these mechanisms. The case of housing problems with endogenous information acquisition differs sharply. In that case there is a unique strategy-proof, non-bossy and ex-ante-Pareto-optimal mechanism: serial dictatorship. The following example illustrates the outstanding role of serial dictatorship.

Example 1 Two agents called 1 and 2 start out owning two houses, k and g, respectively; this initial allocation only changes if both agents agree to exchange houses.¹ In an environment without endogenous learning, this

¹This mechanism is Gale's top trading cycles mechanism for two agents and two houses; a formal definition can be found in Section 2.2.

mechanism is strategy-proof, non-bossy and Pareto-optimal. To see that this mechanism can be Pareto dominated when agents have a choice to learn consider the following setup. Neither agent knows whether he values house k at 8 or at 0: the agents' valuations of this house are independent draws from a distribution according to which the two possible values are equally likely. Both agents value house g at 2. Agent 1 has to pay .8 to learn his value of house k, learning is free for agent 2. Both agents need to announce simultaneously whether or not they would like to swap.

Now let's consider agent 1's decision problem. If he does not learn the value of house k, he prefers to keep it (expected value of 4 vs. 2, the known value of g). If he learns the value, he prefers to swap houses if and only if he values house k at 0. Agent 2, in turn, is willing to swap with a probability of $\frac{1}{2}$. If agent 1 learns his value of house k, he obtains an expected utility of $\frac{1}{2} \times 8 + \frac{1}{2}(\frac{1}{2} \times 2 + \frac{1}{2} \times 0) - .8 = 3.7$, with the last term reflecting agent 1's cost of learning. So agent 1 prefers to keep house k without learning, implying that in equilibrium agent 2 is stuck with house g, yielding an ex-ante utility profile of (4,2).

The serial dictatorship with agent 1 as the first dictator ex-ante Pareto-dominates the given mechanism. For agent 1 as the first dictator it is worth-while to learn the value of house k and to choose it if and only if he finds it of high value (expected utility: $\frac{1}{2} \times 8 + \frac{1}{2} \times 2 - .8 = 4.2$). So agent 2 is matched to his ex-ante preferred house with a probability of $\frac{1}{2}$. The profile of ex-ante utilities is (4.2,3).

This example shows that some of the bedrock of matching theory starts to crumble if one allows for endogenous information acquisition. Both mechanisms described in the example, the top trading cycles mechanism and serial dictatorship, are Pareto-optimal, strategy-proof and non-bossy in an environment without endogenous information acquisition. Either one of these mechanisms traces out the full set of Pareto optimal matchings when one allows for all possible orderings of dictators or for all possible initial allocations respectively. With endogenous information acquisition, something very different happens. In that case serial dictatorship may ex-ante Pareto domi-

²Since learning is costless agent 2 will be willing to exchange with agent 1 if and only if he values house k at 8, which happens with probability $\frac{1}{2}$.

nate Gale's top trading cycles mechanism as shown in Example 1. The main result of the paper significantly generalizes this observation. I show that for any non-bossy and strategy-proof mechanism that is not a serial dictatorship one can find a housing problem and a (path-dependent) serial dictatorship, such that the serial dictatorship strictly ex-ante Pareto-dominates the named mechanism in the given housing problem. Conversely, serial dictatorships are never dominated in this way.

The essential difference between the two mechanisms in Example 1 is that the strong incentives for learning under serial dictatorship are dampened under top trading cycles. While agent 1's knowledge of the value of house k is always useful under serial dictatorship, the same knowledge is irrelevant in half of all cases under the alternative mechanism. Serial dictatorship stands out as the only mechanism which always combines optimal learning incentives with optimal allocation incentives. It is well known that serial dictatorship sets the "right" incentives for allocations: it belongs to the set of strategy-proof, non-bossy and Pareto-optimal mechanisms. What distinguishes serial dictatorship is that it is the only mechanism in this set which also sets the "right" incentives for information acquisition: given that any agent knows his exact choice-set when he decides to learn, no information is ever wastefully acquired. The paper gives two variants of the uniqueness statement on serial dictatorship pertaining to the case of sequential and simultaneous learning, Theorems 1 and 2.

The set of information structures considered in this article is constrained in two ways: first, the agents' preferences are independent draws, implying that agents never wish to delegate their choices to better informed agents. Second, in line with the literature on standard housing problems, a no-indifference condition ensures that at least some mechanisms work optimally. With these two constraints in place we can be sure that the the sub-optimality of mechanisms other than serial dictatorship is due to the agents' ability to acquire information. Serial dictatorship might outperform other mechanisms in matching problems with indifferences or correlated preferences. Still, the domination of serial dictatorship in the present article cannot be attributed to such arguments as these classical "trouble-makers" have been ruled out. The uniqueness result of the article is indeed driven by the assumption of

endogenous information acquisition.

The compromise between the quality of information acquisition and of allocations is one of the main themes of the growing literature on mechanism design with endogenous information acquisition. Mechanisms are often characterized in terms of an optimal trade-off between informative and allocative efficiency. Gerardi and Yariv [5] as well as Bergemann and Valimaki [2] respectively illustrate this trade-off in voting and auctions environments. This trade-off is relevant in the present paper: simple serial dictatorship is the one mechanism under which allocative and informative efficiency coexist. For any other mechanism we have to face the trade off between the two kinds of efficiency. The optimality of sequential learning is another theme of the literature on mechanisms with endogenous learning that is echoed in the present paper. Gershkov and Szentes [6] as well as Smorodinsky and Tennenholtz [12] present voting models in which the voters' optimal acquisition of information is sequential. Similarly, for auctions, Compte and Jehiel [3] find that ascending price auctions can dominate sealed bid auctions in terms of expected welfare. In this vein the present paper shows the unique optimality of sequential simple serial dictatorship when allowing for any sequence of information elicitation.

In Section 2 I provide formal definitions of the housing problems and mechanisms under study. There I define Example 2 to argue that sequential elicitation procedures might outperform simultaneous ones in the present context. With all the relevant terminology in hand, I state the two main results of the article Theorems 1 and 2 in Section 3. The proof of these two theorems revolves around three examples: Example 2, which is already presented in Section 2, the introductory Example 1, which is revisited in Section 4, and Example 5 which is presented in in Section 4. To extend the arguments gleaned from these examples to the case of large housing problems, I rely on Pycia and Unver's [9] "trading cycles" mechanisms (Section 6). The proof of the two results is contained in Section 7. The presentation of Pycia and Unver's representation and the proof are preceded by Section 5 which sheds light on possible extensions and limits of the unique ex-ante Pareto optimality of serial dictatorship.

2 The Model

2.1 Agents, Houses, Values

Fix two sets of agents $I = \{1, \dots, n\}$ and houses H with equally many elements (|H| = n) and generic elements $i, j \in N$ and $h, d, g, k \in H$. There is a finite state space Ω which consists of profiles of values $\omega := (\omega_h^i)_{h \in H, i \in I}$, where ω_h^i is the value that agent i assigns to house h and $\omega^i := (\omega_h^i)_{h \in H}$ is the vector of agent i's valuations. Denote the set of all partitions of Ω by \mathcal{P} . The state $\omega \in \Omega$ is drawn from the probability distribution π , with $\pi(\omega) > 0$ for all $\omega \in \Omega$. The prior π is common knowledge among the designer and all agents.

A vector $c := (c^i)_{i \in N}$ of cost functions $c^i : \mathcal{P} \to \mathbb{R}_0^+ \cup \{\infty\}$ describes the agents' learning technologies where $c^i(P)$ is agent i's non-negative (and possibly infinite) cost to learn P. Staying ignorant is free in the sense that $c^i(\{\Omega,\emptyset\}) = 0$ is assumed to hold for all i. Adopt the understanding that $\hat{\omega}_h^i$ not only denotes agent i's value of house h in state $\hat{\omega}$ but also the event $\{\omega \mid \omega_h^i = \hat{\omega}_h^i\}$ that agent i values house h at $\hat{\omega}_h^i$. Define ζ^i as the algebra on Ω which is generated by all events $\hat{\omega}_h^i$. It is assumed that no agent can learn anything about any other agent's preferences: formally $c^i(P) = \infty$ holds for all $P \not\subset \zeta^i$. An agent who has acquired the partition P knows the event $P(\omega)$ at state ω . The partition according to which agent i knows his value for each of the houses is called \overline{P}^i .

To ensure the comparability of the present model to standard housing models, I impose two further assumptions: First, the agents' preferences are drawn independently, formally $\pi(E^i \cap E^j) = \pi(E^i)\pi(E^j)$ holds for all $E^i \in \zeta^i$ and $E^j \in \zeta^j$. The assumption implies that agent i' posterior value of a house does not change if he finds out what some other agent knows. To see this define $\overline{\omega}_h^i(E)$ as agent i's expected value of house h when he knows event E. Observe that $\overline{\omega}_h^i(E) = \overline{\omega}_h^i(E \cap G)$ holds when $E \in \zeta^i$ and $G = \bigcap_{j \neq i} G^j$ with $G^j \in \zeta^j$, since

$$\overline{\omega}_h^i(E \cap G) = \frac{\sum_{\omega_h^i \subset E} \pi(\omega_h^i \cap G) \omega_h^i}{\pi(E \cap G)} = \frac{\sum_{\omega_h^i \subset E} \pi(\omega_h^i) \pi(G) \omega_h^i}{\pi(E) \pi(G)} = \overline{\omega}_h^i(E),$$

³So \overline{P}^i is the finest partition P with the feature $P \subset \zeta^i$, $\overline{P}^i(\omega) = \omega^i$ holds for all $\omega \in \Omega$.

where the crucial equality follows from the independence of any any event ω_h^i and G^4 . Without the assumption of independence some agents might find it beneficial to delegate their decision, there would also be scope for signalling.

Second, to avoid the difficulties that arise in housing problems with indifferences, I assume that any agent i who is faced with the non-strategic problem of choosing a house from some subset $S \subset H$ has a unique optimal plan of action which consists of a partition P together with a choice function $C:\Omega\to S$ which prescribes a unique choice from S for every cell of P, so $C(\omega)=C(\omega')$ for $\omega\in P(\omega')$. The condition is satisfied if $\overline{\omega}_h^i(E)\neq\overline{\omega}_g^i(E)$ holds for any $h\neq g$ and any E that is an element of a partition P with $c^i(P)<\infty$ and if for every $S\subset H$ there is a unique P that maximizes $\sum_{E\in P}\max_{h\in S}\overline{\omega}_h^i(E)-c^i(P)^5$. The two assumptions of independence and no-indifference imply that the following results are indeed driven by the novel assumption of endogenous information acquisition. The outstanding role of serial dictatorship cannot be attributed to an appearance of weak or correlated preferences under the guise of endogenous information acquisition. The two assumptions are discussed at length in Section 5.

The vector $\mathcal{H} = (I, H, \Omega, \pi, c)$ of sets of agents and houses I and H, a state space Ω , a probability distribution π on Ω and cost functions c which all satisfy the assumptions discussed above constitutes a **housing problem** (with endogenous information acquisition).

A matching is a bijection $\mu: I \to H$. The set of all matchings is denoted by \mathcal{M} . A submatching $\sigma: I_{\sigma} \to H_{\sigma}$ is a bijection with $I_{\sigma} \subset I$ and $H_{\sigma} \subset H$. The set of all all submatchings that are not matchings is denoted by $\overline{\mathcal{M}}$. For any particular submatching $\sigma \in \overline{\mathcal{M}}$ the sets of unmatched agents and houses are denoted by \overline{I}_{σ} and \overline{H}_{σ} . The house assigned to agent i under the submatching σ is $\sigma(i)$. Submatchings σ are also interpreted as sets, where a pair (i,h) belongs to the set σ if and only if $\sigma(i) = h$ under the interpretation of σ as a function.

An **outcome function** $f: \Omega \to \mathcal{M} \times \mathcal{P}^n$ maps any state ω to a matching $\mu[\omega] \in \mathcal{M}$ and profile of information partitions $(P^i[\omega])_{i \in I}$. Outcome functions

⁴Note that any $E \in \zeta^i$ can be represented as the the union of events $\omega_h^i \subset E$.

⁵Since $P \not\subset \zeta^i$ implies $c^i(P) = \infty$ agent *i*'s unique utility maximizing partition P must be ζ^i -measurable

describe the different matchings achieved and the learning undertaken at all states ω . At ω agent i knows the event $P^i\omega$ when he acquires the partition $P^i[\omega]$ as prescribed by the outcome function f. The ex-ante utility U^i of agent i associated with a given outcome function $f: \Omega \to \mathcal{M} \times \mathcal{P}^n$ is defined as

$$U^{i}(f) = \sum_{\omega \in \Omega} \pi(\omega) \Big(\omega_{\mu[\omega](i)}^{i} - c^{i}(P^{i}[\omega]) \Big).$$

One outcome function f is said to (ex-ante Pareto-)dominate another outcome function f' if $U^i(f) \geq U^i(f')$ holds for all $i \in I$ and if $U^j(f) > U^j(f')$ holds for some $j \in I$.

2.2 Standard Housing Problems

A housing problem $\mathcal{H} = (I, H, \Omega, \pi, c)$ is a standard housing problem if Ω is a singleton. Dropping I and H and omitting π , c which are irrelevant when Ω is a singleton, I denote a standard housing problem by ω , the profile of preferences (that is known to occur). In a standard housing problem an agent i has a unique optimal plan of action for every choice set $S \subset H$ if and only if his preference over any two different houses is strict $(\omega_h^i \neq \omega_g^i)$ for all $h \neq g$ and i). So in the subset of standard housing problems the no-indifference condition of the present article is equivalent to the standard condition of strict preferences. The condition of independently drawn preferences is trivially satisfied in standard housing problems. The set of all standard housing problems is denoted by Θ : = $\{\omega \mid \omega_h^i \neq \omega_q^i \text{ for all } h \neq g \text{ and } i\}$. An outcome function $f:\Omega\to\mathcal{M}\times\mathcal{P}^n$ for a standard housing problem maps the only state $\omega \in \Omega$ to a matching $\mu[\omega] \in \mathcal{M}$ and the trivial partition $\{\emptyset, \Omega\}$ for every agent. Within the set of standard housing problems Θ , any outcome for a particular problem ω can consequently be identified with the matching $\mu|\omega|$.

A (direct) mechanism is a function $\varphi : \Theta \to \mathcal{M}$ mapping profiles of preferences $\omega \in \Theta$ to matchings $\varphi(\omega) \in \mathcal{M}$. Such a mechanism is considered strategy-proof if the truthful revelation of preferences is a weakly dominant strategy. A mechanism is considered non-bossy (as defined by

Satterthwaite and Sonnenschein [10]) if an agent can only change the allocation of some other agent if he also changes his own allocation. This implies that any misreport of preferences that does not change the agent's own assignment does not change anyone else's assignment. The mechanism φ is considered Pareto-optimal if $\varphi(\omega)$ is Pareto-optimal for any ω . Pycia and Unver [9]'s characterization of the set of all strategy-proof, non-bossy, and Pareto-optimal mechanisms, which I present in Section 6, crucially simplifies my proof. First, though, let me define three canonical matching mechanisms which belong to this set.

According to a **simple serial dictatorship**, one agent, the first dictator, is matched to the best house out of H according to his stated preferences.⁶ Next, another agent, the second dictator, is matched to his most preferred house out of the remainder, and so forth, until all houses are matched. I denote a simple serial dictatorship as a direct mechanism by $\delta:\Theta\to\mathcal{M}$. The simple serial dictatorship in which agent i is the ith dictator is called δ^* . The reason for the qualifier "simple" arises since **path-dependent serial dictatorships**, denoted by $\gamma:\Theta\to\mathcal{M}$, also play a role in the present paper. This type of serial dictatorship generalizes simple serial dictatorships insofar as that the identity of any current dictator is allowed to depend on all preceding dictators choices.⁷

Gale's top trading cycles, the third canonical mechanism⁸ starts out with a matching μ called the initial endowment. Each agent points to the agent who has been endowed with the house he likes best according to his stated preferences. At least one cycle forms. All agents in such cycles are assigned the houses that they point to. The procedure is repeated with the remaining houses and agents until all houses are assigned.⁹

⁶Simple serial dictatorship has been characterized by Svensson [13].

⁷Path-dependent serial dictatorships where introduced by Papai [8] under the name of sequential dictatorship.

 $^{^8}$ This mechanism was first defined by Shapley and Scarf [11], who attribute it to David Gale.

⁹These three mechanisms are well defined when any agent has a unique most preferred house in any set of houses, as is the case for any $\omega \in \Theta$. If we allow for indifferences the mechanisms cede to be well-defined.

2.3 Dynamic Direct Revelation Mechanisms

In this section I define the grand set of mechanisms considered in the present article together with a list of canonical examples. Let me first argue that the sequence of preference announcements matters in mechanisms with endogenous information acquisition.

Example 2 Two different dynamic versions of the serial dictatorship δ^* stand out: the designer might either simultaneously elicit the preferences of all agents; alternatively the designer might elicit the preferences of all agents in order of their index i – and thereby allow each agent to tailor his information acquisition to his actual choice set. To see that this difference matters, let $\mathcal{H}^b = (I, H, \Omega, \pi, c)$ with $H = \{d, g, k\}$ and three a priori identical agents. Each agent assigns value 8 or 0 (with probability $\frac{1}{2}$) to house d. The values of houses g and k are known to be 5 and 2, respectively. Assume that it costs each agent c = .1 to learn his type. If the designer simultaneously elicits preferences, it is worthwhile for agents 1 and 2 to learn their type. However, if the designer elicits preferences sequentially, then agent 2 will only learn his type if agent 1 did not choose house d. The sequential mechanism ex-ante Pareto-dominates the mechanism of simultaneous elicitation, as the second dictator will not spend the cost c = .1 when learning is of no consequence to his decision.

In a **dynamic (direct) mechanism** the designer can fix any order of the agents' announcements. A rooted tree t, called a **c-tree**, describes the agents' communication to the designer. The initial node of a c-tree is labeled with the first agent to declare a preference. The next agent to declare a preference is allowed to depend on the declaration of the prior agent(s); branches terminate when all agents have declared their type. The designer can freely choose the sequencing of announcements as well as the information sets on the c-tree t. An agent's information set on a c-tree determines what he knows about the preceding announcements, when it is his turn to reveal his type to the designer. The dynamic mechanism induced by the c-tree t, and the direct mechanism φ , is denoted as $\langle \varphi, t \rangle$.

Applying the dynamic mechanism $\langle \varphi, t \rangle$ to a housing problem \mathcal{H} one obtains the extensive form game $\langle \varphi, t \rangle(\mathcal{H})$. This game starts with a chance

node in which nature draws the state ω from π . Agents get to declare their preferences in the order determined by the c-tree t. Any agent gets to choose an information partition right before the node in which he declares his preference. The information sets in the extensive form game reflect the privacy of learning as well as the revelations implied by the c-tree t. Agent i's utility in an end node is calculated as the difference between his value of the house he is assigned and the learning cost he incurred on the path to the node.

Of course, in many contexts, sequential learning might be impractical. This is the case when learning takes up much time or when there is a large number of I therefore also study the class of mechanisms in which the designer simultaneously elicits all preferences. Formally a mechanism $\langle \varphi, t^s \rangle$ is defined as a **simultaneous (direct) mechanism** where t^s is the c-tree according to which no agent knows anything about the other agents' announcements when he announces his own preferences.

A sequential simple serial dictatorship or 3S dictatorship is defined as the dynamic direct revelation mechanism $\langle \delta, t^{\delta} \rangle$, where t^{δ} is the c-tree, according to which any dictator knows the preference-announcement of all preceding dictators when it is his turn to announce his preferences. When considering the simple serial dictatorship δ^* (where agent i is the ith dictator) I let $t^{\delta^*} = t^*$. Analogously a dynamic direct revelation mechanism is a sequential path-dependent serial dictatorship $\langle \gamma, t^{\gamma} \rangle$ where t^{γ} is such that agents publicly announce their preferences in the sequence in which they become dictators.

2.4 Equilibria and Implementation

A (mixed) strategy profile in $\langle \varphi, t \rangle(\mathcal{H})$ is considered an **equilibrium** if it is a perfect Bayesian equilibrium and if agents truthfully announce their types in the sense that any agent i reveals his (true) ex post preferences $\overline{\omega}^i$ to the designer. In the standard case there exists at most one equilibrium. In that case each agent knows his ranking ω^i , the question is just whether telling it is a best reply. My next example demonstrates, that matching mechanisms with endogenous information acquisition might have multiple equilibria.

Example 3 Consider a housing problem $\mathcal{H} = (I, H, \Omega, \pi, c)$ as follows: n =

2, $H = \{k, g\}$ and Ω has 4 equiprobable states. Agent 1's valuation of house k might either be 8 or 0; he is sure to value house g at 3. Conversely, agent 2's valuation of house g might either be 8 or 0; he is sure to value house g at 3. It costs each agent .1 to find out his preference. Let g be Gale's top trading cycles mechanism where agent 1 starts out owning house g. The game g and g (in which both agents need to announce their rankings simultaneously) has two equilibria. According to the first, neither agent learns anything and always points to the house he was endowed with. According to the other, both agents learn their true values and point to the house they find to be of higher value. Note that in either one of these equilibria the agents tell the truth.

Every strategy profile in the game $\langle \varphi, t \rangle(\mathcal{H})$ is associated with an outcome function $f: \Omega \to \mathcal{M} \times \mathcal{P}^n$ in the sense that the matching $\mu[\omega]$ and the set of partitions $(P^i[\omega])_{i \in I})$ (so $f(\omega) = (\mu[\omega], (P^i[\omega])_{i \in I})$) obtain at state ω when agents follow the strategy profile. A mechanism $\langle \varphi, t \rangle$ is said to **implement** a vector of ex-ante utilities $(U^1(f); \dots; U^n(f))$ in the housing problem \mathcal{H} if $\langle \varphi, t \rangle(\mathcal{H})$ has an equilibrium strategy profile that is associated with the outcome function f. If all utility vectors implemented by $\langle \varphi, t \rangle(\mathcal{H})$ dominate all utility vectors implemented by a different dynamic direct revelation mechanism $\langle \varphi', t' \rangle$ in \mathcal{H} , then $\langle \varphi, t \rangle$ is said to (ex-ante Pareto-)dominate $\langle \varphi', t' \rangle$ at \mathcal{H} , which is denoted by $\langle \varphi, t \rangle(\mathcal{H}) \succ^* \langle \varphi', t' \rangle(\mathcal{H})$. I say that a mechanism $\langle \varphi, t \rangle$ is (ex-ante) Pareto-optimal in a set of mechanisms, if this set contains no alternative mechanism $\langle \varphi', t' \rangle$ such that $\langle \varphi', t' \rangle(\mathcal{H}) \succ^* \langle \varphi, t \rangle(\mathcal{H})$ holds for some housing problems \mathcal{H} .

Note that the set of Pareto optimal mechanisms might be empty. This is the case if for every $\langle \varphi, t \rangle$ there exists an alternative mechanism $\langle \varphi', t' \rangle$ and a housing problem \mathcal{H} such that $\langle \varphi', t' \rangle(\mathcal{H}) \succ^* \langle \varphi, t \rangle(\mathcal{H})$. Restricted to the set of standard housing problems Θ the present notion of a Pareto optimal mechanism coincides with the standard notion given in Section 2.2. To see this consider a direct mechanism $\varphi : \Theta \to \mathcal{M}$ which is Pareto optimal according to the notion just defined.¹⁰ This implies that there is not alternative mechanism φ' and no housing problem ω such that the matching $\varphi'(\omega)$ Pareto

 $^{^{10}}$ I omit t here, which does not matter given that no agent can learn in a standard problem.

dominates the matching $\varphi(\omega)$. But φ' might be any mechanism, including a constant one that maps all ω to the same matching μ . So φ is Pareto optimal according to the notion defined here if and only if there exists no profile of preferences ω such that $\varphi(\omega)$ is dominated by some matching μ . In sum φ satisfies the standard definition of a Pareto optimal mechanism.

3 The Uniqueness of Serial Dictatorship

It is the goal of this article to characterize all strategy-proof and non-bossy mechanisms φ with c-trees t such that the dynamic direct revelation mechanism $\langle \varphi, t \rangle$ is ex-ante Pareto-optimal. The next two theorems show that simple serial dictatorship is the only such mechanism - whether one allows for all dynamic direct revelation mechanisms or only for the simultaneous ones.

Theorem 1 A mechanism $\langle \varphi^{\circ}, t^{\circ} \rangle$ is Pareto optimal in the set of all mechanisms $\langle \varphi, t \rangle$ with φ non-bossy and strategy proof if and only if $\langle \varphi^{\circ}, t^{\circ} \rangle$ is a 3S dictatorship.

A very similar observation holds when one restricts attention to the set of simultaneous matching mechanisms.

Theorem 2 A simultaneous mechanism $\langle \varphi^{\circ}, t^{s} \rangle$ is Pareto optimal in the set of all simultaneous mechanisms $\langle \varphi, t^{s} \rangle$ with φ non-bossy and strategy proof if and only if φ° is a simple serial dictatorship.

Serial dictatorships have another outstanding welfare property: any mechanism that is not a simple serial dictatorship is dominated by a path-dependent serial dictatorship in some housing problem. Since this observation holds for dynamic as well as simultaneous mechanisms, I state only one remark covering both cases.

Remark 1 Fix any mechanism in the set of dynamic [simultaneous], strategy proof, non-bossy direct revelation mechanisms which is not a 3S [simultaneous simple serial] dictatorship. There exists a housing problem in which

this mechanism is dominated by a dynamic [simultaneous] path-dependent serial dictatorship.

So for any strategy proof and non-bossy φ which is not a simple serial dictatorship and any c-tree t there exist a path-dependent serial dictatorships γ, γ' and housing problems $\mathcal{H}, \mathcal{H}'$ such that $\langle \gamma, t^{\gamma} \rangle$ dominates $\langle \varphi, t \rangle$ at \mathcal{H} and $\langle \gamma', t^s \rangle$ dominates $\langle \varphi, t^s \rangle$ at \mathcal{H}' . In the Appendix I show that Remark 1 cannot be strengthened by replacing path-dependent with simple serial dictatorships. The next section contains two examples with $n \leq 3$ which do not just illustrate the two theorems, they serve as the backbone of the proof.

4 Housing Problems With At Most 3 Agents

With just two agents there is only one strategy-proof and Pareto-optimal mechanism other than serial dictatorship: Gale's top trading cycles mechanism. Let us revisit Example 1 to see that the "only if" part of Theorems 1 and 2 as well as Remark 1 hold for n = 2:

Example 4 Let $n=2, H=\{g,k\}$. Fix $\langle \varphi,t\rangle$ with φ Gale's top trading cycles mechanism in which agents 1 and 2 start out owning house k and g, respectively. According to t, at least one agent, say agent 1, has to declare his preferences before knowing the preference of the other. Reconsider the housing problem defined in Example 1, which can now be defined succinctly as $\mathcal{H}^a=(I,H,\Omega,\pi,c)$ with $n=2, H=\{k,g\}, \pi(\omega_k^i=8)=\pi(\omega_k^i=0)=\frac{1}{2}$ and $\pi(\omega_g^i=2)=1$ for $i=1,2, c^1(\overline{P}^1)=.8$ and $c^2(\overline{P}^2)=0$. As argued in the introduction, agent 1's costs outweigh his benefit of learning; since agent 1 ex-ante prefers house k there is no exchange in equilibrium, yielding the examte utility profile (4,2). Conversely, if agent 1 is the first dictator, learning is worthwhile for him; in this case, agent 2 has a chance to obtain his ex-ante preferred house g, the ex-ante utility profile implemented by $\langle \delta^*, t^* \rangle(\mathcal{H})$ is (4.2;3).

There was only one reference to the sequence of announcements: according to t, (at least) one agent has to announce his preferences before knowing the preferences of the other. As this holds for simultaneous and sequential

versions of the mechanism and as the outcome of serial dictatorship depends on only one announcement when there are just two agents, the above shows the "only if" part of Theorems 1 and 2 for n = 2. Since any simple serial dictatorship is a path-dependent one, Example 4 also proves Remark 1 for the case that n = 2.

To see that the "if" part of Theorem 1 holds when n=2, I fix an arbitrary housing problem \mathcal{H} with $H=\{g,k\}$. I first show that agent 1 (weakly) prefers any equilibrium of $\langle \delta^*, t^* \rangle(\mathcal{H})$ to any equilibrium of any other mechanism $\langle \varphi, t \rangle(\mathcal{H})$. As the first dictator agent 1's obtains the utility

$$U^* \colon = \max_{P \subset \zeta^1} \left(\sum_{E \in P} \pi(E) \max_{h \in \{g,k\}} \overline{\omega}_h^1(E) - c^1(P) \right) \ge \max\{\overline{\omega}_g^1(\Omega), \overline{\omega}_k^1(\Omega)\}$$

where the lower bound represents the utility of not learning and then choosing the ex-ante preferred house.

Now fix an arbitrary strategy for agent 2 in $\langle \varphi, t \rangle(\mathcal{H})$. Observe that this strategy might determine whether agent 1 gets to choose from $\{g, k\}$ or whether he is matched to g or k. Abusing notation, I denote the event that agent 1 gets to choose from S given agent 2's fixed equilibrium strategy by S as well. Any such event must be an element of ζ^2 , given that these are the only events on which agent 2 can condition his strategy.¹¹ Agent 2's strategy implies a distribution over agent 1's choice sets $S \in \{\{g, k\}, \{g\}, \{k\}\}\}$ which is denoted by ρ .

Since the agents' preferences are independently drawn the expected value that agent 1 assigns to any house does not vary with his knowledge of any event in ζ^2 . Under $\langle \varphi, t \rangle$ agent 1 might have to declare (and therefore learn) his preferences before he knows whether he has any choice $(S = \{g, k\})$ or not $(S = \{g\})$ or $\{k\}$. Since agent 1's utility can only increase when he may condition his choice to learn on the event S, the expression $\rho(\{g, k\})U^* + \rho(\{g\})\overline{\omega}_g^1(\Omega) + \rho(\{k\})\overline{\omega}_k^1(\Omega) \leq U^*$ yields an upper bound on agent 1's expected utility in $\langle \varphi, t \rangle(\mathcal{H})$ for the fixed strategy of agent 2.

Consequently, for $\langle \varphi, t \rangle(\mathcal{H})$ to have an equilibrium that Pareto dominates the equilibria of $\langle \delta^*, t^* \rangle(\mathcal{H})$, agent 1 must obtain the utility U^* under the equilibrium of $\langle \varphi, t \rangle(\mathcal{H})$. However the no-indifference condition implies that

¹¹It was assumed that $c^2(P) = \infty$ for $P \not\subset \zeta^2$.

 $\max_{P\subset\zeta^1}\sum_{E\in P}\pi(E)\max_{h\in\{g,k\}}\overline{\omega}_h^1(E)-c^1(P)$ is uniquely maximized by a partition P^* and a function $\mu[\cdot](1):\Omega\to\{g,k\}$ which maps every state ω to a house $\mu[\omega](1)$ for agent 1. This uniqueness implies that agent 1's matches must also be described by $\mu[\cdot](1)$ for agent 1 to obtain U^* under the equilibrium of $\langle \varphi, t \rangle(\mathcal{H})$. Since there are only two agents there is no leeway with respect to agent 2's matches. For agent 1 to obtain utility U^* in $\langle \varphi, t \rangle(\mathcal{H})$ agent 2 must - in every state ω - be matched with the house $\mu[\omega](2) \in \{g,k\}$ that is not $\mu[\omega](1)$. Since the function $\mu[\cdot](2)$ also describes agent 2's matches under the equilibrium of $\langle \delta^*, t^* \rangle(\mathcal{H})$, we can conclude that $\langle \varphi, t \rangle(\mathcal{H})$ cannot have an equilibrium that Pareto dominates the unique equilibrium of $\langle \delta^*, t^* \rangle(\mathcal{H})$.

The "if" part of Theorem 2 follows form the same arguments as the announcement of a single agent (the first dictator) determines the outcome of a serial dictatorship with just two agents, so $\langle \delta^*, t^* \rangle$ and $\langle \delta^*, t^s \rangle$ are identical for n=2. The treatment of the case that n=2 already contains most core arguments of the proof for any n. However, when n=2 all serial dictatorships are simple. Therefore some example with n>2 is in order to preview the arguments pertaining to path-dependent serial dictatorships. In the next example I show that with three agents path-dependent serial dictatorships can be dominated by other path-dependent serial dictatorships.

Example 5 Take n=3 and $H=\{g,k,d\}$. Consider the sequential path-dependent serial dictatorship $\langle \gamma, t^{\gamma} \rangle$ with agent 1 as the first dictator. If he chooses g, then agent 2 gets to choose from $\{k,d\}$; otherwise, agent 3 becomes the next dictator. Define the housing problem \mathcal{H}^c , such that agent 1's utility vector for the three houses is either (2,1,0) or (0,2,1) - each with probability $\frac{1}{2}$. Agent 1 faces a cost of .1 to learn his type. The utility vectors of agent 2 and 3 are known to be $\omega^2=(10,2,0)$ and $\omega^3=(2,10,0)$, respectively. The unique equilibrium of $\langle \gamma, t^{\gamma} \rangle (\mathcal{H}^c)$ yields the vector (1.9;1;1) of expected utilities to agents 1,2 and 3.

Now consider the alternative sequential path-dependent serial dictatorship $\langle \gamma', t^{\gamma'} \rangle$ which also starts with agent 1 as the first dictator, but then

¹²Note that the a correlation of values of houses does not conflict with the independence assumption which only requires that the preferences of different agents are drawn independently.

continues with 3 as the next dictator if agent 1 chooses g, and agent 2 otherwise. The vector of ex-ante utilities implemented by $\langle \gamma', t^{\gamma'} \rangle (\mathcal{H}^c)$ is (1.9; 5; 5). The crucial difference between $\langle \gamma, t^{\gamma} \rangle (\mathcal{H}^c)$ and $\langle \gamma', t^{\gamma'} \rangle (\mathcal{H}^c)$ is that under the latter agents 2 and 3 get to choose from sets where they face a utility differential of 10. Conversely under $\langle \gamma, t^{\gamma} \rangle (\mathcal{H}^c)$ agents 2 and 3 get to choose only in situations of relatively minor relevance: when either one is called to choose he faces a utility differential of just two.

Since there is only one agent in \mathcal{H}^c who has any information to acquire, the timing of announcements does not matter in Example 5; the equilibrium sets of the games $\langle \varphi, t \rangle(\mathcal{H}^c)$ and $\langle \varphi, t^s \rangle(\mathcal{H}^c)$ are identical for any t. Any path-dependent serial dictatorship with just three agents that is not a simple serial dictatorship is – up to renaming – identical to γ . In sum, the example shows the "only if" part of Theorems 1 and 2 as well as Remark 1 are true when one only considers the case of path-dependent serial dictatorships and n = 3.

5 Limitations and Extensions

The "if" part of Theorem 1 states that 3S-dictatorship is Pareto optimal. The "only if" part together with Remark 1 state that for each mechanism $\langle \varphi, t \rangle$ that is not a 3S-dictatorship there exists a housing problem \mathcal{H} and a path-dependent serial dictatorship, such that the latter dominates $\langle \varphi, t \rangle$ at the housing problem \mathcal{H} . So the "if" part could be strengthened by showing that it holds for a yet larger domain of housing problems than the one defined in Section 2. Conversely the "only if" part (together with Remark 1) could be strengthened by showing that it holds on a sub-domain. Here I show that one can not enlarge the domain by much and have the "if" part continue to hold. On the other hand the "only if" part also holds on a much smaller domain.

The domain of housing problems could be enlarged by dropping the condition of no-indifference or of independently drawn preferences or by allowing for there to be differently many houses and agents. Dropping the no-indifference condition is problematic since serial dictatorships cede to be well-defined when we allow for indifferences: the truthful revelation of preferences need not imply unique choices from sets of houses. To circumvent

this problem I modify the notion of (path-dependent and simple) serial dictatorship in housing problems with indifferences. Letting m be the number of some dictator's most preferred houses in the set of houses from which he is entitled to choose, I require that this dictator faces a probability of $\frac{1}{m}$ to be matched to any one of these m houses. Without indifferences this notion of serial dictatorship reduces to the standard notion. The following example 13 shows that 3S dictatorship can be dominated when we drop the no-indifference condition.

Example 6 Consider a housing problem $\mathcal{H} = (I, H, \Omega, \pi, c)$ with n = 2 and $H = \{k, g\}$. Let Ω have 4 equiprobable states where each agent's utility schedule might either be (1, 2) or (2, 1). Agent 1's cost of learning his own preference is 3, agent 2's is 0. As the first dictator agent 1 optimally stays ignorant, assigns a value of 1.5 to each house, and consequently obtains either one of the two houses with probability $\frac{1}{2}$. Facing a probability $\frac{1}{2}$ to be matched to either house, agent 2 also obtains a utility of 1.5. On the other hand agent 2 as the first dictator chooses the house which gives him a utility of 2, implying a utility vector of (1.5; 2). So the serial dictatorship with agent 2 moving first dominates the serial dictatorship with agent 1 moving first.

Example 6 shows that serial dictatorship may be dominated by another mechanism in housing problems in which agents are indifferent between choices. Agent 1 does not have a unique optimal plan of action for the choice set H; not learning and choosing k, not learning and choosing g, as well as any mixture thereof are all optimal. To see that very similar issues arise in the standard model with indifferences consider a variation of Example 6, in which agent 1 and 2's utility schedules are known to be (1.5, 1.5) and (1, 2), respectively. The serial dictatorship with agent 2 as the first dictator yields a utility vector of (1.5; 2) and therefore Pareto dominates the other serial dictatorship which yields a utility vector of (1.5; 1.5), given that agent 1 as the first dictator is equally likely to be matched to either one of the two-indifferent - houses.

Ehlers [4] showed that the set of Pareto optimal, strategy proof and non-bossy mechanisms is empty when the domain of preference profiles in-

 $^{^{13}\}mathrm{I}$ would like to thank one of the referees for this example.

cludes indifferences. So we certainly need to impose *some* condition of noindifference on housing problems with endogenous information acquisition for such mechanisms to exist. Applied to standard housing problems this no-indifference condition has to imply that preferences over different houses are strict.

The no-indifference condition defined in Section 5 is not the only one that satisfies the requirement just mentioned: the strictness of preferences in standard housing problems is also ensured if no agent is indifferent between any two houses according to any ω in the support of π . However, Example 6 shows that this alternative condition does not suffice for the existence of a exante Pareto optimal, strategy proof and non-bossy mechanism. 14 The same example furthermore shows that the alternative condition does not imply the no-indifference condition I impose. To see that the converse implication does not hold either, modify the housing problem defined in Example 4 such that agent 1 is with a small probability indifferent between the two houses, with the complementary probability his preferences are determined as in the description of Example 4. Assume furthermore that the acquisition of any partition according to which agent 1 learns whether he is indifferent or not has infinite cost. Keep all other aspects of Example 4 fixed. The no-indifference condition defined in Section 5 holds since agents 1 and 2 each have unique optimal plans when faced with the choice between house q and k.

Finally observe that, in parallel to the standard case, the no-indifference condition holds generically. It is violated if an agent who learned some partition at a cost below infinity is expost indifferent between two houses or if an agent is indifferent between learning two different partitions when he faces the problem to choose a house from some set S. Mathematically if

¹⁴By Theorem 1 we know that for any mechanism $\langle \varphi, t \rangle$ that is not a 3S dictatorship we can find a housing problem and a mechanism $\langle \varphi', t' \rangle$ such that $\langle \varphi, t \rangle$ is dominated by $\langle \varphi', t' \rangle$ at the given housing problem. Now fix any 3S dictatorship $\langle \delta, t^{\delta} \rangle$ and allow for indifferences. Following Example 6 we can then construct a housing problem \mathcal{H} and an alternative 3S dictatorship $\langle \delta', t^{\delta'} \rangle$ that dominates $\langle \delta, t^{\delta} \rangle$ at \mathcal{H} .

we consider \mathbb{R}^{ν} to be the parameter space¹⁵, then the parameters at which indifferences obtain are defined by a finite set of linear equalities, and thus form a set of Lebesgue measure 0.

To see that the Pareto optimality of serial dictatorship also fails when we allow for correlated preferences consider the following example¹⁶:

Example 7 Consider a housing problem $\mathcal{H} = (I, H, \Omega, \pi, c)$ with n = 2, $H = \{g, k\}$, Ω consisting of two equiprobable states $\tilde{\omega}$ and $\hat{\omega}$ with $\tilde{\omega}^1 = (2,0) = \hat{\omega}^2$, $\hat{\omega}^1 = (0,4) = \tilde{\omega}^2$ and $c^1(\overline{P}^1) = 0.1$ and $c^2(\overline{P}^2) = \infty$. So agent 1 prefers house g whenever agent 2 prefers house k and vice versa, moreover while learning is cheap for agent 1 it is prohibitive for agent 2. Agent 2 therefore always likes to cede the choice to agent 1. At the given housing problem with correlated preferences the serial dictatorship with 2 as the first dictator is Pareto dominated by the serial dictatorship with agent 1 as the first dictator (the respective vectors of ex-ante utilities are (2; 2) for the first kind of serial dictatorship and (2.9; 3) for the second).

There is yet a further problem in environments with correlated preferences: even simple serial dictatorships need not have truthtelling equilibria for all c-trees t.¹⁷ In sum we can say that the "if" part of Theorem 1 (and 2) fails if we drop the conditions of no-indifference and/or of independence. While it might be possible to relax these conditions somewhat the search for the maximal domain of housing problems for which 3S-dictatorship is ex-ante Pareto optimal goes beyond the scope of this paper. Let me just say that the case of $|I| \neq |H|$ can easily be accommodated. If there are less agents than houses the proof of the optimality of 3S dictatorship (and the analogous simultaneous case) goes through unchanged. If there are more agents than houses we can only allow for some trivial changes: any optimal mechanism

¹⁵For a fixed number of states in the space Ω the following parameters need to be determined to describe an housing problem: a value $\omega_h^i \in \mathbb{R}$ for each agent, each house and each state, a probability vector $\pi \in \Delta^{|\Omega|}$ on the state space, a finite set of cost values $c^i(P^i) \in \mathbb{R} \cup \{\infty\}$, assigning a cost for every partition $P^i \subset \zeta^i$ on Ω for every agent i.

¹⁶I would like to thank one of the referees for this example.

¹⁷An example is available on request.

has to start out as a serial dictatorship. Once all (real) houses have been assigned and only dummy-houses remain, we can use any mechanism.

Without stating a result on the minimal domain on which the "only if" part of Theorems 1 and 2 and Remark 1 are valid, let me argue that these results also hold on a much smaller domain of housing problems. To this end restrict the domain of housing problems with the following two additional conditions. Firstly, any house has at most two different values for each agent, so $\omega_h^i \in \{\overline{\omega}_h^i, \underline{\omega}_h^i\}$ holds for all i, h and all $\omega \in \Omega$. Secondly, for each agent i there is at most one nontrivial partition $\widetilde{P}^i \neq \{\emptyset, \Omega\}$ such that $c^i(\widetilde{P}^i) < \infty$. So there does not need to be much uncertainty in the housing problem: it suffices for each agent i to know that house h is either of some high value $\overline{\omega}_h^i$ or some low value $\underline{\omega}_h^i$. Moreover just one informational choice per agent, either learn \widetilde{P}^i or remain ignorant $(\{\Omega,\emptyset\})$, is sufficient. Since these two additional restrictions are satisfied by all examples used to prove the "only if" part of Theorem 1 and 2 as well as Remark 1, these results are also valid for the restricted domain.

6 The Trading Cycles Mechanism

The set of all strategy-proof, non-bossy, and ex-ante Pareto-optimal direct revelation mechanisms φ has been characterized by Pycia and Unver [9] as the set of trading cycles mechanisms. In trading cycles mechanisms, just like in Gale's top trading cycles mechanism, there is an initial allocation of all houses to the agents, and assignments are then determined through trade in cycles. Trading cycles mechanisms generalize Gale's top trading cycles mechanism in two ways: First of all, agents can own more than one house before they leave with their assignment. Once an owner of multiple houses leaves the mechanism his as of yet unmatched houses are passed on to the remaining agents via a fixed inheritance rule. Secondly, there are two types of control rights in trading cycles mechanisms. In addition to ownership which is defined as in Gale's top trading cycles mechanism, there is a new form of control called "brokerage". A broker can exchange the house

¹⁸This first difference between Gale's top trading cycles and the trading cycles mechanism already appears in Papai [7].

he controls for a different house; however, he may not himself appropriate the house.

Formally, any **trading cycles mechanism** $\psi^{c,b}$ following Pycia and Unver [9] is defined through a **control rights structure** (c,b). Such a structure assigns control rights for any submatching: $(c,b) = \{(c_{\sigma},b_{\sigma}) \mid \overline{H}_{\sigma} \rightarrow \overline{I}_{\sigma} \times (ow,br)\}_{\sigma \in \overline{\mathcal{M}}}$ with $c_{\sigma}(h)$ the agent controlling house h and $b_{\sigma}(h)$ the type of control at submatching σ (br for brokerage, and ow for ownership). A control rights structure (c,b) is considered **consistent** if it satisfies the following requirements R1-R6, which I take word by word from Pycia and Unver [9]:

Within-round requirements. Consider any $\sigma \in \overline{\mathcal{M}}$:

- (R1) There is at most one brokered house at σ .
- (R2) If i is the only unmatched agent at σ , then i owns all unmatched houses at σ .
- (R3) If agent i brokers a house at σ , then i does not own any houses at σ .

Across-round requirements. Consider any submatchings σ , σ' , such that $|\sigma'| = |\sigma| + 1$ and $\sigma \subset \sigma' \in \overline{\mathcal{M}}$, and any agent $i \in \overline{I}_{\sigma'}$ and any house $h \in \overline{H}_{\sigma'}$:

- (R4) If i owns h at σ then i owns h at σ' .
- (R5) Assume that at least two agents from $\overline{I}_{\sigma'}$ own houses at σ . If i brokers house h at σ then i brokers h at σ' .
- (R6) Assume that at σ agent i controls h and agent $i' \in \overline{I}_{\sigma}$ controls $h' \in \overline{H}_{\sigma}$. Then, i' owns h at $\sigma \cup \{(i, h')\}$, and if, in addition, i' brokers h' at σ but not at σ' and $i' \in \overline{I}_{\sigma'}$, then i owns h' at σ' .

The following algorithm establishes the outcome $\psi^{c,b}(\omega)$ of the trading cycles mechanism $\psi^{c,b}$ at ω , via a finite sequence of rounds $r=1,2,\cdots$. The submatching of agents and houses matched before round r is denoted by σ^{r-1} , with $\sigma^0=\emptyset$. In round r each house $h\in \overline{H}_{\sigma^{r-1}}$ points to the agent who controls it at σ^{r-1} . If there exists a broker at σ^{r-1} , then he points to his most preferred house among the ones owned at σ^{r-1} . All other agents point to their most preferred house in $\overline{H}_{\sigma^{r-1}}$. There exists at least one cycle of agents and houses pointing to each other. Each agent in each such trading cycle is matched with the house he is pointing to. The union of σ^{r-1} and

the newly matched agent-house pairs defines σ^r . If σ^r is a matching, the mechanism terminates.

A submatching σ is **reached** under $[\psi^{c,b},\omega]$ if there exists a round r of the trading cycles process such that $\sigma = \sigma^r$; a submatching is **reachable** under $\psi^{c,b}$ if it is reached under $[\psi^{c,b}(\omega)]$ for some ω .¹⁹ Any submatching σ that is reached under $[\psi^{c,b},\omega]$ must satisfy $\sigma \subset \psi^{c,b}(\omega)$.

Pycia and Unver [9] show that a direct revelation mechanism φ is strategy-proof, non-bossy and Pareto-optimal if and only if there exists a consistent control rights structure (c, b) such that $\varphi = \psi^{c,b}$. A discussion of the trading cycles mechanism goes beyond of the scope of the present paper and can be found in their paper. The canonical mechanisms introduced at the end of Section 2.2 can now be represented as special cases of trading cycles mechanisms. Path-dependent serial dictatorships require that for each reachable σ there exists an i_{σ} such that $c_{\sigma}(h) = i_{\sigma}$ for all $h \in \overline{H}_{\sigma}$, simple serial dictatorships are special cases of path-dependent serial dictatorships with $i_{\sigma} = i_{\sigma'}$ if $|\sigma| = |\sigma'|$; a control rights structure (c, b) defines Gale's top trading cycles mechanism $\psi^{c,b}$ if there exists a matching μ such that $(c_{\emptyset}, b_{\emptyset})(h) = (\mu^{-1}(h), ow)$ for all $h \in H$.

7 Proofs

To prove that 3S dictatorship cannot be ex-ante Pareto-dominated ("if" part of Theorem 1) suppose the 3S dictatorhip $\langle \delta^*, t^* \rangle$ was dominated by some $\langle \varphi, t \rangle$ at some housing problem \mathcal{H} . Under $\langle \delta^*, t^* \rangle(\mathcal{H})$ agent 1 obtains the ex-ante utility²⁰

$$\max_{P \subset \zeta^1} \Big(\sum_{E \subseteq P} \pi(E) \max_{h \in H} \overline{\omega}_h^i(E) - c^1(P) \Big).$$

Pick an equilibrium of $\langle \varphi, t \rangle(\mathcal{H})$ and fix the strategies of agents $\{2, \dots, n\}$

¹⁹Consider the serial dictatorship δ^* as an example. Any submatching (1,h) is reachable under δ^* since agent 1 as the first dictator is free to appropriate any house. No submatching (2,h) is reachable since 2 can only be matched under δ^* once agent 1 has been matched. Next $(1,h^*)$ is reached under $\delta^*(\omega)$ if any only $\omega_{h^*}^1 > \omega_h^1$ for all $h \in H \setminus \{h^*\}$.

²⁰Note that agent 1 here maximizes over all partitions $P \in \mathcal{P}$ that are subsets of ζ^1 . This is without loss of generality since $c^1(P) = \infty$ holds for any partition P with $P \not\subset \zeta^1$.

to the ones prescribed by that equilibrium. Agent 1's problem then consists in announcing preferences which determine choices from a set $\{H^1, \dots, H^L\}$ of choice sets that are possible according to all other agents' strategies. The strategies of agents $2, \dots, n$ imply a distribution ρ on $\{H^1, \dots, H^L\}$. Let H^l not only denote a choice set for agent 1 but also the event that agent 1 gets to choose from H^l . Since the strategies of agents 2 through n determine the set H^l that agent 1 gets to choose from and since any agent i can only condition his strategy on events in ζ^i , any event H^l can be represented as $\bigcap_{i=2}^n E^l_i$ for some events $E^l_i \in \zeta^i$ for all $i=2,\dots,n$.

Agent 1 may have to announce (and learn) his preferences before he knows which choice set he is facing. Since agent 1's utility can only increase as he gets to choose a separate information partition for every choice set H^l agent 1's ex-ante utility in the fixed equilibrium of $\langle \varphi, t \rangle(\mathcal{H})$ cannot be higher than

$$\begin{split} \sum_{l=1}^{L} \rho(H^l) \max_{P \subset \zeta^1} \Big(\sum_{E \in P} \pi(E \mid H^l) \max_{h \in H^l} \overline{\omega}_h^1(E \cap H^l) - c^1(P) \Big) &= \\ \sum_{l=1}^{L} \rho(H^l) \max_{P \subset \zeta^1} \Big(\sum_{E \in P} \pi(E) \max_{h \in H^l} \overline{\omega}_h^1(E) - c^1(P) \Big) &\leq \\ \max_{P \subset \zeta^1} \Big(\sum_{E \in P} \pi(E) \max_{h \in H} \overline{\omega}_h^1(E) - c^1(P) \Big). \end{split}$$

The equality holds since all agents' preferences are drawn independently which in turn implies that $\overline{\omega}_h^1(E) = \overline{\omega}_h^1(E \cap H^l)$ holds for all $h \in H$ and all $1 \leq l \leq L$. The inequality holds since the maximum in some set $S \subset \mathbb{R}$ cannot be smaller the maximum in any subset of S. So we can conclude that agent 1's utility in the fixed equilibrium of $\langle \varphi, t \rangle(\mathcal{H})$ is no higher than his utility as the first dictator.

Due to the no-indifference condition agent 1's optimal choices as the first dictator imply a unique match for him for every state ω . These matches can be described by the function $\mu[\cdot](1):\Omega\to H$. For agent 1 to be at least as well off under the equilibrium of $\langle \varphi,t\rangle(\mathcal{H})$ as under $\langle \delta^*,t^*\rangle(\mathcal{H})$, the house that agent 1 is matched with under the equilibrium of $\langle \varphi,t\rangle(\mathcal{H})$ must also be described by $\mu[\cdot](1)$.

Fixing a house h^* that agent 1 chooses for some state ω as the first dictator, compare agent 2's utility in the event $E := \{\omega \mid \mu[\omega](1) = h^*\}$

under serial dictatorship and under the equilibrium of $\langle \varphi, t \rangle(\mathcal{H})$. Since agent 1 can only base his decision on a partition $P \subset \zeta^1$, the event E that agent 1 picks h^* is an element of ζ^1 . The independence assumption implies that knowing this event E has no impact on the assessment of any event that is relevant for the other agents' decisions, formally $\pi(\omega_h^i \mid E) = \pi(\omega_h^i)$ holds for all $i \in \{2, \dots, n\}$ and all $h \in H$. Under serial dictatorship agent 2 gets to choose from the set $H \setminus \{h^*\}$. He (weakly) prefers this choice to any other mechanism that matches the houses $H \setminus \{h^*\}$ to the agents $\{2, \dots, n\}$. This preference follows the same arguments given for agent 1's preference to be the first dictator. Since h^* was chosen arbitrarily these observations hold for any possible choice by agent 1 as the first dictator.

We can conclude that conditioning on agent 1 being at least as well off as the first dictator in $\langle \varphi, t \rangle(\mathcal{H})$ agent 2 cannot be made any better off under $\langle \varphi, t \rangle(\mathcal{H})$ than under $\langle \delta^*, t^* \rangle(\mathcal{H})$. The claim then follows by an inductive application of these arguments to all consecutive dictators. The "if" part of Theorem 2 can be shown using a minor modification of the above arguments. I postpone this proof to the Appendix.

The proof of the "only if" part of Theorem 1 together with Remark 1 starts with the observation that for any mechanisms $\langle \varphi^{\circ}, t^{\circ} \rangle$ to be Pareto optimal in the sets of dynamic mechanisms, the direct revelation mechanism φ° must itself be Pareto optimal. Otherwise $\langle \varphi^{\circ}, t^{\circ} \rangle$ is dominated by some constant mechanism at some standard housing problem ω . Pycia and Unver's [9] characterization then implies that φ° can be represented as a trading cycles mechanism $\psi^{c,b}$. I subdivide the set of dynamic direct mechanisms $\langle \psi^{c,b}, t \rangle$ that are not 3S dictatorships into three categories: I) $\psi^{c,b}$ is not a path-dependent serial dictatorship without being a simple serial dictatorship, III) $\psi^{c,b}$ is a simple serial dictatorship δ , but t is not equal to t^{δ} . Lemmas 1, 3, and 4 then show that the "only if" part of Theorem 1 holds restricted to mechanisms belonging to category I), II), and III) respectively. All proofs can be found in the Appendix.

Lemma 1 Fix any trading cycles mechanism $\psi^{c,b}$ that is not a path-dependent serial dictatorship and any c-tree t. There exists a housing problem \mathcal{H}^A and a simple serial dictatorship δ such that $\langle \psi^{c,b}, t \rangle$ is dominated by $\langle \delta, t^{\delta} \rangle$ at \mathcal{H}^A .

The proof of this Lemma for the case of n=2 is contained in Example 1, that demonstrates the domination of Gale's top trading cycles by a serial dictatorship in some housing problems with two agents. To extend this idea of proof to Lemma 1 (for any n), situations similar to Gale's top trading cycles with just two agents need to be identified in any trading cycles mechanism $\psi^{c,b}$, that is not a path dependent serial dictatorship. Lemma 2 shows that any such $\psi^{c,b}$ has at least two owners at some reachable submatching.

Lemma 2 In any trading cycles mechanism $\psi^{c,b}$ that is not a path-dependent serial dictatorship exists a reachable submatching σ^* and two houses g, k such that $c_{\sigma^*}(g) \neq c_{\sigma^*}(k)$ and $b_{\sigma^*}(g) = b_{\sigma^*}(k) = ow$.

The preceding Lemma allows me to prove Lemma 1 by embedding Example 1 into a housing problem with n > 3 agents. Fix any $\psi^{c,b}$ that is not a path dependent serial dictatorship, by Lemma 2 $\psi^{c,b}$ must have a reachable submatching σ^* with two owners, say 1 and 2 who respectively own two houses, say g and k. Restricted to agents 1 and 2 and houses g and k, let \mathcal{H}^A be identical to the problem defined in Example 1. The preferences of all other agents are known and σ^* is reached in all equilibria of $\langle \psi^{c,b}, t \rangle (\mathcal{H}^A)$. Once σ^* is reached we face a housing problem that is strategically identical to Example 1; consequently the mechanism $\langle \psi^{c,b}, t \rangle$ is dominated by a 3S dictatorship at \mathcal{H}^A . The next two Lemmas state the "only if" part of Theorem 1 and Remark 1 for the categories II) and III).

Lemma 3 Fix any path-dependent serial dictatorship γ that is not a serial dictatorship and any c-tree t. There exists a housing problem \mathcal{H}^C and a path-dependent serial dictatorship γ' such that $\langle \gamma, t \rangle$ is dominated by $\langle \gamma', t^{\gamma'} \rangle$ at \mathcal{H}^C .

Lemma 4 Fix a serial dictatorship δ together with a c-tree $t \neq t^{\delta}$. There exists a housing problem \mathcal{H}^B such that the sequential serial dictatorship $\langle \delta, t \rangle$ is dominated by the 3S dictatorship $\langle \delta, t^{\delta} \rangle$ at \mathcal{H}^B .

Lemmas 3 and 4 were proven by Examples 5 and 2, respectively, for the case of n = 3. The proof of these Lemmas for larger n consists in embedding these examples into housing problems with n houses and agents. Jointly

Lemmas 2 through 4 constitute the proof of the "only if" part of Theorem 1 and the relevant portion of Remark 1: they show that for any conceivable deviation from 3S dictatorship there exists a housing problem such that some path-dependent serial dictatorship dominates that mechanism at this housing problem.

Some tweaking of the preceding proof suffices to show the "only if" part of Theorem 2 (and the relevant portion of Remark 1). The reason is that the sequentiality of announcements neither matters for the arguments brought forward with respect to Examples 4 and 5, nor for their embedding in larger housing problems. The housing problems \mathcal{H} chosen to prove Lemmas 1 and 3 are defined such that at most one agent learns in any of the equilibria that are relevant for the proofs. But any equilibrium of a game $\langle \varphi, t \rangle(\mathcal{H})$ in which at most one agent learns is an equilibrium of the game $\langle \varphi, t \rangle(\mathcal{H})$, implying that some minor translation work is needed to make the proof of Theorem 1 applicable to Theorem 2; the details can be found in the Appendix.

8 Conclusion

If one allows for endogenous information acquisition in housing problems, simple serial dictatorships stand out from the large set of strategy-proof, non-bossy and Pareto-optimal mechanisms. Whether one looks at mechanisms that dynamically elicit preferences or only at the subset of mechanisms in which preferences are elicited simultaneously: simple serial dictatorships are the only ex-ante Pareto-optimal mechanisms.

Within the set of strategy-proof and non-bossy mechanisms, serial dictatorships are unique in the sense that they always provide optimal learning incentives. In a 3S dictatorship each agent knows his choice set when it is his turn to learn and choose. Agents can therefore perfectly tailor their learning to fit the questions at stake. Example 1 shows that this is not the case for Gale's top trading cycles mechanism with just two agents: in this case one agent, say agent 1, needs to decide what to learn when he only knows the distribution over his possible choice sets. That example was constructed such that this agent avoids learning and therefore refuses any exchange. In addition, agent 2 would rather award agent 1 dictator rights, to get agent

1 to learn, than to keep the house he is initially endowed with. The main argument of the proof was that any strategy-proof, non-bossy and Pareto-optimal direct choice mechanism that is not a serial dictatorship in a sense embeds Gale's top trading cycles mechanism with just two agents.

Abdulkadiroglu and Sonmez [1] show that serial dictatorship with the order of dictators randomly drawn from a uniform distribution is equivalent to Gale's top trading cycles mechanism with the endowment drawn from a uniform distribution. To see that this equivalence result does not hold for the case of endogenous information acquisition, reconsider the housing problem \mathcal{H}^a defined and discussed in Example 1. Suppose each agent is assigned each of the roles (first or second dictator, and owner of g or k, respectively) with probability $\frac{1}{2}$. The agents get to know the role they are assigned before they decide wether to acquire information about the houses. The expected utility profiles for the two different serial dictatorships are (3,5) and (4.2,3), whereas for Gale's top trading cycles mechanism they are (3,5) and (4,2). So, the vector of expected utilities for serial dictatorship with a random order of dictators and for Gale's top trading cycles mechanism with random endowments are $\frac{1}{2}(3;5) + \frac{1}{2}(4.2;3) = (3.6;4)$ and $\frac{1}{2}(3;5) + \frac{1}{2}(4;2) = (3.5;3.5)$. We can conclude that randomization of serial dictatorship Pareto dominates the randomization of Gale's top trading cycles in the present example.

There is another question relating to random matching mechanisms: could sequential serial dictatorships be replaced by random matching mechanisms in Remark 1? Example 5 suggests that this is not possible. Recall the arguments brought forward to show that agent 1 would have to be the first dictator in any mechanism that dominates the path-dependent serial dictatorship of Example 5. Applying the same arguments to the present case we obtain that agent 1 would have to obtain the same matches as he does as the first dictator in any dominating random matching mechanism. So the randomization can only concern agents 2 and 3. But it is preferable that agents 2 and 3 are not randomly assigned to choose from the set that remains after agent 1 appropriated a house. For each agent there is one set in which his choice matters much too him (high utility differential between the remaining houses) and another set in which his choice matters less (low utility differential between the remaining houses).

It could also be interesting to study ex-ante Pareto optimality without the assumption of endogenous learning. This study could be couched in a version of the model considered here with $c^i(P) = 0$ for any partition P containing only elements $E \in \zeta^i$, meaning that agents face no cost of learning their own types. Observe that the cost of learning in Example 5 played no role, so the argument that any path-dependent serial dictatorship is dominated by another path-dependent serial dictatorship for some housing problem \mathcal{H} also applies to this special case.

Example 5 can be re-interpreted as an illustration of conflict between bossiness and ex-ante Pareto optimality. To see this, change the mechanism γ defined in that example to a very similar type of bossy serial dictatorship in which the identity of the second dictator does not depend on whether agent 1 chooses house g or k, but rather on whether he ranks house d at the bottom or not. Say agent 2 becomes the second dictator if and only if agent 1 ranks house d at the bottom. This is a bossy mechanism, since agent 1's assignment does not change when announcing either (2,1,0) or (2,0,1). However, the assignments to the following two dictators will vary with agent 1's announcement if their preferences are aligned. Now observe that for \mathcal{H}^c the housing problem given in Example 5 this bossy mechanism is essentially identical to the path-dependent serial dictatorship defined there: \mathcal{H}^c is defined such that agent 1 chooses house q if and only if he ranks house d lowest. Consequently, for \mathcal{H}^c the given form of bossy serial dictatorship is dominated by the alternative path-dependent serial dictatorship γ' defined in the same example. This is but one example, it is not known whether ex-ante Pareto optimality generally conflicts with bossiness.

Finally let me say that a relaxation of the restrictions I imposed on the domain of housing problems might lead to a wealth of interesting results on matching with endogenous information acquisition. One stylized fact about matching mechanisms used in practice is that they often do not allow the participants to submit complete rankings, instead they only permit short lists of a few top choices. Maybe such mechanisms fare better than classical matching mechanisms in housing problems in which agents are indifferent over all "ex-ante unknown" objects. So a theory of matching with endogenous information acquisition and ex-ante indifference could explain why such

mechanisms are so common in practical applications. Alternatively, a theory of matching problems under endogenous information acquisition with correlated values could serve to analyze (and possibly optimize) the institutions through which agents share information before submitting their preferences to matching mechanisms.

9 Appendix

Some further definitions and preliminary observations on trading cycles mechanisms are needed for the upcoming proofs. For any fixed a trading cycles mechanism $\psi^{c,b}$ and a submatching σ^* the mechanism $\psi^{c',b'}$ defined through $c'_{\sigma} = c_{\sigma^* \cup \sigma}$ and $b'_{\sigma} = b_{\sigma^* \cup \sigma}$ for all submatchings σ with $I_{\sigma} \subset \overline{I}_{\sigma^*}$ and $H_{\sigma} \subset \overline{H}_{\sigma^*}$ is called the **submechanism** of $\psi^{c,b}$ at σ^* . This mechanism matches the agents in \overline{I}_{σ^*} to the houses in \overline{H}_{σ^*} . It is routine to check that a submechanism of a trading cycles mechanism is itself a trading cycles mechanism.

Pycia and Unver's [9] theorem states that any Pareto optimal, strategy proof and non-bossy mechanism φ can be represented as a trading cycles mechanism $\psi^{c,b}$. This representation is not unique: the requirement that any (c,b) specifies control rights for all submatchings $\overline{\mathcal{M}}$ whereas only the reachable submatchings matter for the determination of matchings $\psi^{c,b}(\omega)$. In that vein, any (c,b) with $(c_{\sigma},b_{\sigma})(h)=(i,ow)$ for all σ with $I_{\sigma}=\{1,2,\cdots,i-1\}$ defines the serial dictatorship $\psi^{c,b}=\delta^*$, no matter how we specify (c_{σ},b_{σ}) for any other σ (say for example $\sigma=(2,h)$). There are yet more types of multiple representations as shown in the following Lemma:

Lemma 5 Take any control rights structure (c,b) and define an alternative control rights structure (c^*,b^*) as follows. If there exists a house $g \in \overline{H}_{\sigma}$ and an agent $i^* \in \overline{H}_{\sigma}$ such that $b_{\sigma}(g) = br$ and $(c_{\sigma},b_{\sigma})(h) = (i^*,ow)$ for all $h \in \overline{H}_{\sigma} \setminus \{g\}$ then let $(c^*_{\sigma},b^*_{\sigma})(h) = (i^*,ow)$ for all $h \in \overline{H}_{\sigma}$, otherwise let $(c_{\sigma},b_{\sigma}) = (c^*_{\sigma},b^*_{\sigma})$. The control rights structures (c,b) and (c^*,b^*) describe the same matching mechanism: $\psi^{c,b} = \psi^{c^*,b^*}$.

Proof First observe that c^*, b^* satisfies Pycia and Unver's [9] (R1)-(R6), so ψ^{c^*,b^*} is indeed a trading cycles mechanism. To see that $\psi^{c,b}(\omega) = \psi^{c^*,b^*}(\omega)$

holds for all states, fix an arbitrary ω . I show by induction that any submatching σ which reached under $[\psi^{c,b},\omega]$ must also be reached under $[\psi^{c^*,b^*},\omega]$. Of course, \emptyset is reached under both. Assume that σ^* is reached under $[\psi^{c,b},\omega]$ as well as under $[\psi^{c^*,b^*},\omega]$. If $(c_{\sigma^*},b_{\sigma^*})=(c_{\sigma^*}^*,b_{\sigma^*}^*)$ then both mechanisms prescribe the same ownership at σ^* , and both reach the same σ' . So suppose we had $(c_{\sigma^*},b_{\sigma^*})\neq(c_{\sigma^*}^*,b_{\sigma^*}^*)$ and let σ' be the next submatching reached after σ^* in $[\psi^{c,b},\omega]$. Since $(c_{\sigma^*},b_{\sigma^*})\neq(c_{\sigma^*}^*,b_{\sigma^*}^*)$ there is exactly one owner, say i^* and one broker, say j at σ^* under $\psi^{c,b}$, let g be the brokered house. So we either have $\sigma'=\sigma^*\cup\{(i^*,h)\}$ or $\sigma'=\sigma^*\cup\{(i^*,g),(j,h)\}$ for some $h\in\overline{H}_{\sigma^*}\setminus\{g\}$.

In either case agent i^* is matched with his most preferred house in \overline{H}_{σ^*} . As the owner of all these houses under (c^*, b^*) at σ^* agent i^* appropriates the same house, meaning that σ' is reached after σ^* under $[\psi^{c^*,b^*},\omega]$ in the first case. In the second case Pycia and Unver's [9] (R6) implies that agent j becomes the sole owner of all houses at $\sigma^* \cup \{(i^*,g)\}$. The fact that j points to h at σ^* under $[\psi^{c,b},\omega]$ implies that j prefers h to all other houses in $\overline{H}_{\sigma^*}\setminus\{g\}$, meaning that he chooses h at $\sigma^*\cup\{(i^*,g)\}$. So also in this last case σ' is reached under $[\psi^{c^*,b^*},\omega]$. Since $\psi^{c,b}(\omega)$ is reached under $[\psi^{c,b},\omega]$ it must also be reached under $[\psi^{c^*,b^*},\omega]$. Since any mechanism reaches exactly one matching for a fixed profile of preferences, we can conclude that $\psi^{c,b}(\omega) = \psi^{c^*,b^*}(\omega)$ holds as desired.

The proof of Lemma 2 follows immediately from this Lemma:

Proof of Lemma 2: Take any control rights structure (c, b) such that there is never more than one owner at any given σ . Define (c^*, b^*) as in Lemma 5 so that $\psi^{c,b} = \psi^{c^*,b^*}$. The assumption that there is never more than one owner at any given σ implies that ψ^{c^*,b^*} is a path-dependent serial dictatorship.

Proof of Lemma 1: Lemma 2 implies that the set of reachable submatchings σ such that $c_{\sigma}(h) \neq c_{\sigma}(d)$ with $b_{\sigma}(h) = b_{\sigma}(d) = ow$ for some $h, d \in \overline{H}_{\sigma}$ is non-empty. Let σ^* be minimal in this set. Assume w.l.o.g that $(c_{\sigma^*}, b_{\sigma^*})(k) = (1, ow)$ and $(c_{\sigma^*}, b_{\sigma^*})(g) = (2, ow)$ for houses $k, g \in \overline{H}_{\sigma^*}$, that there exists a sole owner $i_{\sigma} = c_{\sigma}(\overline{H}_{\sigma})$ at any reachable $\sigma \subseteq \sigma^*$, and that

that 1 has to announce his preferences before he learns the announcement of agent 2 according to the c-tree t.

Define the housing problem \mathcal{H}^A as follows. Fix a submatching σ' such that $I_{\sigma'} = \overline{I}_{\sigma^*} \setminus \{1,2\}$ and $H_{\sigma'} = \overline{H}_{\sigma^*} \setminus \{k,g\}$. The preferences of all agents $i \notin \{1,2\}$ are known. For agents $i \in I_{\sigma^*}$ $\omega^i_{\sigma^*(i)} = 1 \ge \omega^i_h$ holds for all $h \in H$; for agents $i \in I_{\sigma'}$ $\omega^i_k > \omega^i_g > \omega^i_{\sigma'(i)} = 1 > \omega^i_h$ holds for all $h \in H \setminus \{g, k, \sigma'(i)\}$. The values $0 > \omega^i_h$ that agents i = 1, 2 assign to houses h other than g and k are known. Restricted to agents 1,2 and houses k, g, the housing problem \mathcal{H}^A is identical to the housing problem \mathcal{H}^a as defined in Example 4.

No agent learns in the unique equilibrium of $\langle \psi^{c,b}, t \rangle(\mathcal{H}^A)$. To see this I show first that the matching μ with $\sigma^* \subset \mu$, $\sigma' \subset \mu$, $\mu(1) = k$ and $\mu(2) = g$ is obtained when all agents reveal their ex-ante preferences. Since $\omega^i_{\sigma^*(i)} = 1 \ge \omega^i_h$ holds for all $h \in H$ and all $i \in I_{\sigma^*}$ and since σ^* is reachable, it must be reached. Once σ^* is reached, agents 1 and 2 are endowed with house k and g respectively. At σ^* all remaining agents prefer k to all remaining houses implying that the submatching $\sigma^* \cup \{(1,k)\}$ is reached next. By by continuity, Pycia and Unver's [9] (R4), agent 2 continues to own house g at $\sigma^* \cup \{(1,k)\}$. Moreover, at $\sigma^* \cup \{(1,k)\}$ all remaining agents prefer g to all remaining houses implying that the submatching $\sigma^* \cup \{(1,k),(2,g)\}$ is reached next. The Pareto optimality of the submechanism of $\psi^{c,b}$ at $\sigma^* \cup \{(1,k),(2,g)\}$, together with the fact that σ' is the unique Pareto optimal submatching of the agents $I_{\sigma'}$ to the houses $H_{\sigma'}$, implies that all remaining agents (the agents in $I_{\sigma'}$) are matched in accordance with μ .

Consider any agent $i \in I_{\sigma^*}$, fix the strategies of all other agents in I_{σ^*} to truthtelling and fix the strategies of the agents in \overline{I}_{σ^*} arbitrarily. To see that telling the truth is a best reply for agent i, observe that the submatching σ^* is reached if i tells the truth and all other agents follow the fixed strategy profile. So if agent i tells the truth he obtains house $\sigma^*(i)$, his most preferred house among all houses in H. In sum, truthtelling is a best response for all agents in I_{σ^*} no matter what we assume about the strategies of the agents in

²¹For σ^* to be reachable one agent $i_{\emptyset} \in I_{\sigma^*}$ must be the initial dictator. Since this initial agent prefers $\sigma^*(i_{\emptyset})$ to all other houses, he appropriates $\sigma^*(i_{\emptyset})$. The reachability of σ^* then requires that at the submatching $\{(i_{\emptyset}, \sigma^*(i_{\emptyset}))\}$ an agent $i \in I_{\sigma^*} \setminus \{i_{\emptyset}\}$ turns into the next dictator. The fact that σ^* is reached follows by induction.

 \overline{I}_{σ^*} . Given that no $i \in I_{\sigma^*}$ has the choice to learn, the submatching σ^* must be reached in any equilibrium of $\langle \psi^{c,b}, t \rangle (\mathcal{H}^A)$.

Next consider the submechanism of $\psi^{c,b}$ at σ^* . The strategic situation faced by agents 1 and 2 in that submechanism is nearly identical to the one they face in the top trading cycles mechanism constructed in Example 4, the only difference is that they have some additional strategies in the present mechanism: pointing to houses other than g or k. However, all these additional strategies are dominated for any outcome of learning given that according to \mathcal{H}^A $\omega_k^i, \omega_g^i > \omega_h^i$ holds for i = 1, 2, all $h \in H \setminus \{g, k\}$, and all $\omega \in \Omega$. The fact that not learning is the unique equilibrium in Example 4, implies that conditioning on all agents in I_{σ^*} telling the truth, agents 1 and 2 best respond by not learning and truthfully revealing their ex-ante preferences, so the submatching $\sigma^* \cup \{(1,k),(2,g)\}$ must be reached in any equilibrium.

The submechanism of $\psi^{c,b}$ at $\sigma^* \cup \{(1,k),(2,g)\}$ is a trading cycles mechanism and therefore strategy proof. This implies that the truthful revelation of their known preferences a is best reply for any agent in $I_{\sigma'}$, given that the agents in $\overline{I}_{\sigma'}$ follow the best reply strategies described so far. In sum we can conclude that not learning and telling the truth is the unique equilibrium of $\langle \psi^{c,b}, t \rangle (\mathcal{H}^A)$. So the profile of ex-ante utilities implemented by $\langle \psi^{c,b}, t \rangle$ in \mathcal{H}^A is $(4; 2; 1; \dots; 1)$.

Now consider the serial dictatorship $\langle \delta^*, t^* \rangle (\mathcal{H}^A)$. Since agents 1 and 2 are the first two dictators under δ^* and since they prefer houses k and g to all other houses in any state ω , their equilibrium behavior is the same as in the serial dictatorship discussed in Example 4. The submechanism after the assignment of 1 and 2 is a serial dictatorship which matches each of the remaining agents i with $\mu(i)$, since each i of these agents prefers $\mu(i)$ to all remaining houses. So the unique profile of expected utilities implemented by $\langle \delta^*, t^* \rangle (\mathcal{H}^A)$ is $(4.2; 3; 1; \dots; 1)$, which Pareto-dominates the unique outcome of $\langle \psi^{c,b}, t \rangle (\mathcal{H}^A)$.

Proof of Lemma 3: Fix a path-dependent serial dictatorship $\tilde{\gamma} = \psi^{c,b}$ that is not a simple serial dictatorship. Let σ^* be a minimal reachable submatching in $\psi^{c,b}$ such that the dictator at the following submatching de-

pends on the choice of the current dictator. Formally assume that there exist $g, k, d \in \overline{H}_{\sigma^*}$ such that $c_{\sigma^*}(h) = 1$ for all $h \in \overline{H}_{\sigma^*}$, $c_{\tilde{\sigma}}(h) = 2$ for all $h \in \overline{H}_{\tilde{\sigma}}$ where $\tilde{\sigma} = \sigma^* \cup \{(1,g)\}$ and $c_{\hat{\sigma}}(h) = 3$ for all $h \in \overline{H}_{\hat{\sigma}}$ where $\hat{\sigma} = \sigma^* \cup \{(1,k)\}$.²²

Define the housing problem \mathcal{H}^C as follows. Fix a submatching σ' such that $\sigma^* \subset \sigma'$, $I_{\sigma'} = I \setminus \{1, 2, 3\}$, and $H_{\sigma'} = H \setminus \{g, k, d\}$. The preferences of all agents $i \neq 1, 2, 3$ are known with $\omega^i_{\sigma'(i)} = 1 \geq \omega^i_h$ holding for all $h \in H$. The values $0 > \omega^i_h$ that agents i = 1, 2, 3 assign to houses h other than g, k and d are known. Restricted to agents 1,2,3 and houses g, k, d, the housing problem \mathcal{H}^C is identical to the housing problem \mathcal{H}^c as defined in Example 5.

The proof that the truth-telling strategy profile according to which agent 1 learns is the only equilibrium is nearly identical to its counterpart for the preceding Lemma. All agents but agent 1 know their preferences and therefore have a unique truthtelling strategy. Following the arguments in the preceding proof all agents in I_{σ^*} best respond by telling the truth, σ^* must be reached in any equilibrium of $\langle \tilde{\gamma}, t \rangle (\mathcal{H}^C)$. At σ^* agent 1 becomes the dictator. His choice problem is nearly identical to that in $\langle \gamma, t^{\gamma} \rangle (\mathcal{H}^c)$: in addition to $\{g, k, d\}$ (his choice set in $\langle \gamma, t^{\gamma} \rangle (\mathcal{H}^c)$) there might be some more houses in \overline{H}_{σ^*} ; however agent 1 strictly prefers g, k and d to any of these additional houses for any state $\omega \in \Omega$. So agent 1's optimal behavior in $\langle \tilde{\gamma}, t \rangle (\mathcal{H}^C)$ is implied by his optimal behavior in $\langle \gamma, t^{\gamma} \rangle (\mathcal{H}^c)$ in Example 5: agent 1 is better off learning his preferences than not. The preferences of all remaining agents are known, truth-telling is therefore a best reply for them in the submechanism following agent 1's choice. The ex-ante utilities of the agents are $(1.9; 1; \dots; 1)$.

Just as in that Example 5, it would be a Pareto improvement for agents 2 and 3 to "switch". So the alternative path-dependent serial dictatorship $\langle \tilde{\gamma}', t^{\tilde{\gamma}'} \rangle$ where $\tilde{\gamma}$ and $\tilde{\gamma}'$ are identical except that agents 2 and 3 switch roles, formally $\tilde{\gamma}' = \psi^{c',b'}$ with $c_{\sigma}(h) = 2 \Rightarrow c'_{\sigma}(h) = 3$, $c_{\sigma}(h) = 3 \Rightarrow c'_{\sigma}(h) = 2$, and $c_{\sigma}(h) \notin \{2,3\} \Rightarrow c_{\sigma}(h) = c'_{\sigma}(h)$ for all σ, h , ex-ante Pareto-dominates the given mechanism $\langle \tilde{\gamma}, t \rangle$ at the housing problem \mathcal{H}^{C} , the game $\langle \tilde{\gamma}', t^{\tilde{\gamma}'} \rangle (\mathcal{H}^{C})$

 $^{^{22}}$ I omit the definition of the second component b_{σ} since in a path-dependent serial dictatorship we have that $b_{\sigma}(h) = ow$ for all reachable σ . Since agents 2 and 3 have a choice there must be at least one house other than g and k.

Proof of Lemma 4: Consider the set of all reachable submatchings σ for which t prescribes that the agents in I_{σ} announce their preferences in the order in which they become dictators. Let σ^* be maximal in this set. This implies that if the first I_{σ^*} agents' declarations lead to the submatching σ^* , then t prescribes that some agent i who is not the dictator at σ^* , must according to t reveal his preference before learning the announcement of $c_{\sigma^*}(\overline{H}_{\sigma^*})$, the dictator at σ^* . Assume that i=2 and $1=c_{\sigma^*}(\overline{H}_{\sigma^*})$. Since both 1 and 2 need to declare their preferences, their choices must be followed by at least one more agent, say 3. Define the housing problem \mathcal{H}^B like \mathcal{H}^C in the preceding proof with the one exception that restricted to agents 1,2,3 and houses k, g, d, the housing problem \mathcal{H}^B is identical to the housing problem \mathcal{H}^b as defined in Example 2.

Given the parallel setup of the preceding and the current mechanism, truthtelling is a best reply for all agents in $I_{\sigma'}$ in $\langle \delta, t \rangle (\mathcal{H}^B)$. This implies that σ^* must be reached in any equilibrium of $\langle \delta, t \rangle (\mathcal{H}^B)$. The problem faced by agents 1, 2 and 3 at σ^* is nearly identical to that in Example 2, the only difference being the availability of some inferior houses $\overline{H}_{\sigma^*} \setminus \{g, k, d\}$. Agents 1 and 2 will therefore both learn their value of house d in the unique equilibrium of $\langle \delta, t \rangle (\mathcal{H}^B)$.

To see that $\langle \delta^*, t^* \rangle$ dominates $\langle \delta, t \rangle$ at \mathcal{H}^B observe that - just as in Example 2 - agent 2 will only acquire information in $\langle \delta^*, t^* \rangle (\mathcal{H}^B)$ when it is relevant to his decision. So agent 2 will only become informed when agent 1 does not choose house d. The equilibria of $\langle \delta, t \rangle (\mathcal{H}^B)$ and $\langle \delta^*, t^* \rangle (\mathcal{H}^B)$ induce identical the mappings from states ω to matchings μ . The equilibrium outcome functions differ only in one respect: there are some states ω under which agent 2 acquires information in the unique equilibrium of $\langle \delta, t \rangle (\mathcal{H}^B)$ but does not do so in the unique equilibrium of $\langle \delta^*, t^* \rangle (\mathcal{H}^B)$. So all agents but agent 2 obtain the same ex-ante utility in the respective equilibria of $\langle \delta, t \rangle (\mathcal{H}^B)$ and $\langle \delta^*, t^* \rangle (\mathcal{H}^B)$. The 3S dictatoship $\langle \delta^*, t^* \rangle$ Pareto dominates $\langle \delta, t \rangle$ at \mathcal{H}^B since agent 2's expected cost of information acquisition is lower in the equilibrium of the former.

Proof of Theorem 2 and the "simultaneous" part of Remark 1:

This proof consists in a few amendments of the proof of Theorem 1. The solution concept, which requires that all agents' ex post preferences lead to unique choices of houses from sets remains well-defined: by assumption any (possible) ex post preferences of any agent strictly rank any two different houses. However simple serial dictatorship might not have a unique equilibrium when all agents are forced to learn simultaneously. Some agents might have multiple optimal partitions - given all other agents' learning choices. To show that $\langle \delta^*, t^s \rangle$ cannot be dominated by any $\langle \varphi, t^s \rangle$ at any housing problem \mathcal{H} , I show that a selected equilibrium of $\langle \delta^*, t^s \rangle(\mathcal{H})$ cannot be dominated by any of the equilibria of $\langle \varphi, t^s \rangle(\mathcal{H})$.

This equilibrium is selected as follows. Let i be the first dictator who strictly prefers some equilibrium of $\langle \delta^*, t^s \rangle(\mathcal{H})$ to another. Discard all equilibria that are inferior according to i's preference.²³ If only one equilibrium survives terminate the process, if not use the preferences of the next dictator who strictly ranks any two of the remaining equilibria to reduce the set yet further. Continue this process until only a single equilibrium survives or until all agents are indifferent between all surviving equilibria. The application of the proof of the "if" part of Theorem 1 to the selected equilibrium yields the desired result: simultaneous simple serial dictatorship is Pareto optimal in the set of all simultaneous matching mechanisms.

The proof of "only if" part of Theorem 2 starts with Lemma 2, according to which any trading cycles mechanism $\psi^{c,b}$ that is not a path-dependent serial dictatorship must have at least two owners at some reachable submatching σ . Now fix any $\langle \psi^{c,b}, t^s \rangle$ such that $\psi^{c,b}$ is not a path-dependent serial dictatorship and define \mathcal{H}^A as in the proof of in Lemma 1. Since only one agent learns in the unique equilibrium of $\langle \delta^*, t^* \rangle (\mathcal{H}^A)$, this strategy profile is also the unique equilibrium of $\langle \delta^*, t^s \rangle (\mathcal{H}^A)$. This together with the observation that $\langle \delta^*, t^* \rangle$ dominates $\langle \psi^{c,b}, t^s \rangle$ at \mathcal{H}^A implies that $\langle \delta^*, t^s \rangle$ dominates $\langle \psi^{c,b}, t^s \rangle$ at \mathcal{H}^A . The proof of Lemma 3 which shows that any path-dependent but not simple serial dictatorship is dominated by a 3S dictatorship at some housing problem directly applies to the simultaneous case. The reason is

 $^{^{23}}$ This first dictator i cannot be agent 1, since the no-indifference condition implies that he has a unique optimal plan of choice from the set H.

that only one agent has the choice to learn in \mathcal{H}^C , the housing problem constructed to prove Lemma 3, so $\langle \psi^{c,b}, t \rangle (\mathcal{H}^C) = \langle \psi^{c,b}, t^s \rangle (\mathcal{H}^C)$ holds for all mechanisms $\langle \psi^{c,b}, t \rangle$. In sum, we have that for any $\psi^{c,b}$ not a simple serial dictatorship there exists a housing problem \mathcal{H} and a path-dependent serial dictatorship γ such that $\langle \psi^{c,b}, t^s \rangle$ is dominated by $\langle \gamma, t^s \rangle$ at \mathcal{H} , proving the "only if" part of Theorem 2 as well as the part of Remark 1 that pertains to simultaneous mechanisms.

Let me finally reconsider the strengthening of Remark 1 to the claim that the dominating mechanism can always be a simple serial dictatorship. To this end suppose that the path-dependent serial dictatorship $\langle \gamma, t^{\gamma} \rangle$ as defined in the Example 5 was dominated by a 3S-dictatorship $\langle \delta, t^{\delta} \rangle$ in some housing problem \mathcal{H} . Following the arguments in the proof that serial dictatorship cannot be dominated (Section 7) agent 1 would have to be the first dictator under δ to keep his utility at least as high as under γ .²⁴

Now consider the case with agent 2 as the second dictator under δ . In set $\{k,d\}$ agent 2 will choose just as he does under $\langle \gamma, t^{\gamma} \rangle$. If he replicates the outcome of $\langle \gamma, t^{\gamma} \rangle(\mathcal{H})$ for the other two choice sets $\{g, k\}$ and $\{g, d\}$, then $\langle \delta, t^{\delta} \rangle(\mathcal{H})$ leads to exactly the same matchings as $\langle \gamma, t^{\gamma} \rangle(\mathcal{H})$ and can therefore not be dominating. So agent 2's choices in these two sets must lead to different matchings. Agent 3 would consequently sometimes obtain a house he would not have chosen. Given that the agents' preferences are independently drawn, agent 3 would be made worse off. The alternative 3S-dictatorship with agent 3 as the second dictator can be ruled out mutatis mutandis. In sum, there is no housing problem such that the path-dependent serial dictatorship $\langle \gamma, t^{\gamma} \rangle$ is dominated by a simple serial dictatorship $\langle \delta, t^{\delta} \rangle$ at that housing problem. Remark 1 cannot be strengthened in the conjectured way in the case of dynamic mechanisms. The same applies to the case of simultaneous mechanisms given that all of the above arguments continue to hold if we replace t^{γ} and t^{δ} by t^{s} throughout the preceding paragraph.

²⁴Actually if the housing problem was such that agent 1's most preferred house in all possible states is the other two agents' least preferred house in all states, agent 1 could also be made the second or third dictator. But in that case the serial dictatorship with agent 1 as the first dictator keeping the order between the other two fixed yields the same outcome in the housing problem.

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