Theoretical Economics 12 (2017), 533–554 1555-7561/20170533

On the equivalence of large individualized and distributionalized games

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The theory of large one-shot simultaneous-play games with a biosocial typology has been presented in both the individualized and distributionalized forms—large individualized games (LIG) and large distributionalized games (LDG), respectively. Using an example of an LDG with two actions and a single trait in which some Nash equilibrium distributions cannot be induced by the Nash equilibria of the representing LIG, this paper offers three equivalence results that delineate a relationship between the two game forms. Our analysis also reveals the different roles that the Lebesgue unit interval and a saturated space play in the theory.

Keywords. Distributionalized games, individualized games, Nash equilibrium distribution, Nash equilibrium, representation, equivalence, weak equivalence, quasi-equivalence, realization, similarity, symmetry, countability, saturation.

JEL classification. C62, D50, D82, G13.

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Some of the results reported here were presented at the IMS Workshop on Probabilistic Impulse in Modern Economic Theory at the National University of Singapore, January 11–18, 2011, at the 13th Conference of the Society for the Advancement of Economic Theory at MINES Paris Tech, July 22–26, 2013, and at the AMES, Singapore, August 2–4, 2013. The authors thank Hülya Eraslan, Kevin Reffet, Larry Samuelson, Xiang Sun, Yeneng Sun, Metin Uyanik, and Rajiv Vohra for stimulating conversation and/or correspondence. They are also indebted to the co-editor and an anonymous referee for their comments that led to this extensively revised version of the original submission. An earlier version was completed when Khan was a Visiting Research Fellow at the Australian National University, February 15–April 15, 2016. Yu's research is supported by SHHRC IDG (430-2014-00007); Zhang's research is supported by NSFC (11201283).

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DOI: 10.3982/TE1806

1. Introduction

It is well understood by students of probability and statistical theory that rather than the random variable itself, it is the distribution of the random variable that is the useful and relevant object to be investigated; the sample space of the random variable is usually not of any real consequence for the problem at hand. This idea has not fully caught on in Walrasian general equilibrium theory and in game theory with a continuum of players. Despite Hildenbrand's conception of an economy as a distribution on the space of agent characteristics in the context of his asymptotic implementation of Aumann's theorems,[1] the first explicit treatment of a large game and its Nash equilibrium, both as distributions, is due to Mas-Colell (1984). His discussion built on Milgrom and Weber (1985) and presupposed the formulation of a game and its equilibrium concept as a random variable, as in Schmeidler (1973).[2]

The domain of the random variable—the sample space—in both general equilibrium theory and in large games refers to the space of players' names,[3] and it would not be far-fetched to refer to the approach based on random variables as *individualized* microeconomic and that based on distributions as *distributionalized* macroeconomic. In the first approach, each characteristic and each equilibrium action is tied to an individual player and is, as such, named and non-anonymous, whereas in the second approach, one can only speak of proportions of players having a given set of characteristics and equilibrium actions, and is, as such, anonymous. This anonymous conception, with its focus on aggregates, as in macroeconomics, is silent on questions having to do with the cardinality of the set of players and their individual identities.[4] To be sure, the individualistic formulation makes explicit the connection between individual responses and the resulting social outcomes; but it is the distributional formulation that delivers existence proofs effortlessly.[5]

[1]Hildenbrand conceived an economy to be a distribution of a random variable on the space of agent characteristics, and the convergence of a sequence of economies to a limit economy as the corresponding weak-star limit of the distributions of their characteristics; see Hildenbrand (1974) and his text for earlier papers of Kannai, Hart and Kohlberg, in addition to Hildenbrand himself. So as to minimize references, we send the reader to the relevant texts and surveys.

[2]In Milgrom and Weber (1985), it is not the specification of the game, but rather its Bayesian-Nash equilibrium, that is formulated as a joint distribution on actions and information.

[3]This presupposes a setting of complete information; in an incomplete information setting, the domain could also include the space of information, but such a description and an analysis of an infinite game with incomplete information is not yet available in the literature; but see Section 4 below.

[4]This terminology is not standard in the literature. In their survey, Khan and Sun (2002) conform to the above usage, but other authors take note of the fact that it is a statistical summary of the actions of the other players, rather than the actions of *each* individual, that enters as an argument in a player's payoff, be he anonymous or non-anonymous in the sense of the terms here, and that it is this that renders a game anonymous. In Kalai (2004), for example, the term "semi-anonymous" is also used for a large but finite setting that a player has a name as well as a trait, and the externality argument takes both strategies and traits into account; also see Kalai and Shmaya (2013) and its references. To avoid confusion, we shall avoid this anonymity terminology and talk of large individualized and large distributionalized games (LIG and LDG).

[5]The fact that the distributional approach offers the most promise for dynamics was realized early on by Jovanovic and Rosenthal (1988) and pursued in Bergin (1992), Bergin and Bernhardt (1992, 1995).

Thus, an analyst has two formulations of a large game at her disposal, and a natural question arises as to how they correspond to each other. Since the formulation of a game based on a random variable induces that based on the distribution of that random variable, one is asking how the equilibrium of one induces that of the other, and vice versa.[6]

In this paper, we offer a systematic investigation of this question[7] in the context of the recent generalization of the theory of large games to admit a biosocial typology. With this broadening of the notion of player interdependence to include a dependence on societal summaries of *both* players' actions and players' traits, we obtain a richer and more relevant notion of social interaction, and thereby are able to explore further what is commonly referred to as *externalities* in Walrasian general equilibrium theory. The individualized and distributionalized components of the original theory carry over in a natural and straightforward way to this generalization where an agent has a name as well as a trait, and the size of a coalition is not the only consideration.[8] We conduct our investigation in the generalized vernacular.

The context we work in involves both a common (compact) action set as well as a common (compact) set of traits. A trait, and a payoff function defined on actions and distributions on traits and action, constitute the characteristic of a player. A *large distributionalized game* (LDG henceforth) is a (probability) measure on the space of characteristics. A *large individualized game* (LIG henceforth) is a (measurable) mapping from an atomless probability space of players to the space of characteristics. The corresponding equilibrium notions of these two games are Nash equilibrium distribution (NED) and Nash equilibrium (NE), defined in the usual manner. Given an LDG, its *representation* is an LIG that induces it. This is simply to say that given a distribution, we find a random variable whose distribution it is. In general, an LDG has a multitude of representations, even if the space of players is fixed. We consider three notions of equivalence between an LIG and an LDG.

An LIG and an LDG are considered *equivalent* if (a) each NE of the LIG induces an NED of the LDG and (b) each NED of the LDG is induced by some NE of the LIG. Whereas the first condition always holds between a representing LIG and an LDG (Lemma 1), the second, in general, is not always fulfilled. Example 1 demonstrates this in a game with two actions and a single trait. This suggests that equivalence is perhaps too demanding a notion. Given the failure of equivalence in general, one needs to relax the conditions to some extent to explore the relationships that hold. An NED of an LDG is a *symmetric Nash equilibrium distribution* (SNED) if players with the same characteristics take the same action. An LDG and an LIG are *weakly equivalent* if the LIG represents the LDG,

[6]It is worth stating that Mas-Colell (1984, p. 203) had already asked this question and observed a potential discrepancy, but did not pursue it further; his concern was with existence issues.

[7]In the context of a large game with finite actions, the issue was undertaken in Rath (1995); its main result is subsumed in Theorem 2 below.

[8]See Khan et al. (2013a, 2013b), and Qiao and Yu (2014); also see Kalai (2004) for its large but finite analogs where the interdependence assumption is called *semi-anonymity*. The canonical models of large games without traits (Schmeidler 1973 and Mas-Colell 1984) are special cases of the generalization investigated here when the space of traits is reduced to a singleton.

and each SNED of the LDG is induced by some NE of the LIG. Weak equivalence always holds: an LDG and an LIG that represents it are weakly equivalent (Theorem 1).

Even though the weak equivalence result is satisfying, it applies only to SNEDs, a subset of NEDs, and as such suffers from a limitation. As a result, when there is no SNED in an LDG, weak equivalence holds between the LDG and an LIG that represents it, and in addition, the LIG may not have an NE even though there always exists an NED in an LDG.[9] To encompass all NEDs, we consider quasi-equivalence that relies on the concept of similarity between two NEDs. Two NEDs of an LDG are *similar* if their marginals on the product space of traits and actions are identical. Since the payoff of a player depends on the choice of action and the distribution on traits and actions, two similar NEDs have the same best response sets and are alike in important game theoretic aspects. An LIG and an LDG are *quasi-equivalent* if corresponding to each NED of the LDG there is an NE of the LIG that induces an NED similar to the given one. The countability of the trait and action spaces are necessary and sufficient for an LIG that represents an LDG to be quasi-equivalent to it (Theorem 2).

In terms of the relationships among these three notions of equivalence, equivalence implies quasi-equivalence and quasi-equivalence implies weak equivalence. In either case, the converse relation is false in general. This leads to our turning to the case where equivalence holds with some additional assumption on the representing LIG. We show that the original notion of equivalence of an LIG and an LDG is obtained if the LIG *saturatedly*[10] represents the LDG (Theorem 3).

The paper then is organized as follows. In Section 2, we present the two game forms and their solution concepts, and offer an example that equivalence is far too much to hope for without additional assumptions and refinements. Section 3 turns to these additional considerations, and offers other equivalence notions and spaces that are saturated. Section 4 concludes the paper with a summary and two remarks on possible extensions. Technical details of examples and proofs of the results are relegated to Appendixes A and B.

2. Two canonical formulations

Let A be a compact metric space with at least two elements representing the set of actions, and let T be a compact metric space representing the space of traits.[11] Let $\mathscr{M}(T \times A)$ be the set of Borel probability measures on $T \times A$ equipped with its Borel σ-algebra $\mathscr{B}(T \times A)$, and metrized by the topology of weak-star convergence. Let $\mathscr{U}_{(A,T)}$ be the space of real-valued continuous functions on $A \times \mathscr{M}(T \times A)$ representing the

[9]The existence of an NED in a large game is a general result, but that of an NE or an SNED requires more restrictive assumptions. This aspect is discussed in detail in Section 3; see the discussion before Example 2 and footnote 19 in particular.

[10]Note that the notion of one game form saturatedly representing another, as in Definition 9, is based on the saturation property, first used to study large games in 2002; see the Acknowledgements in Keisler and Sun (2009). We provide a detailed discussion of the property in Section 3.

[11]Earlier work in Khan et al. (2013a, 2013b) suggests that the compactness requirement on T can be relaxed to the weaker requirement of complete separability, or a Polish space. We leave an investigation of this for future work.

space of payoff functions, metrized by the supremum norm. The characteristics of an individual player then consist of a trait and a payoff function, and thus the space of characteristics is $T \times \mathscr{U}_{(A,T)}$, a complete separable metric space with its Borel σ-algebra $\mathscr{B}(T \times \mathscr{U}_{(A,T)})$. In the sequel, $(I, \mathscr{I}, \lambda)$ denotes an atomless probability space and $\mathrm{Meas}((I, \mathscr{I}, \lambda); (T \times \mathscr{U}_{(A,T)}))$ denotes the space of measurable functions from $(I, \mathscr{I}, \lambda)$ to $(T \times \mathscr{U}_{(A,T)})$. When there is no possibility of confusion we abbreviate the latter by $\mathrm{Meas}(I; T \times \mathscr{U}_{(A,T)})$. We also denote the unit Lebesgue interval by $([0,1], \mathscr{B}([0,1]), \ell)$, where $\mathscr{B}([0,1])$ is the Borel σ-algebra on $[0,1]$ and ℓ is the Lebesgue measure on it. Next we develop the basic notions underlying the two formulations of a large game.

Definition 1. A *large distributionalized game* (LDG) is a Borel probability measure μ on the space of characteristics, $T \times \mathscr{U}_{(A,T)}$, which is to say that $\mu \in \mathscr{M}(T \times \mathscr{U}_{(A,T)})$. A *Nash equilibrium distribution* (NED) of an LDG μ is a Borel probability measure τ on the space of characteristics and actions, $T \times \mathscr{U}_{(A,T)} \times A$, such that the marginal of τ on the space of characteristics $T \times \mathscr{U}_{(A,T)}$ is μ and $\tau(B(\tau)) = 1$, where

$$B(\tau) = \big\{(t, u, a) \in T \times \mathscr{U}_{(A,T)} \times A : u(a, \tau_{T\times A}) \geq u(x, \tau_{T\times A}) \text{ for all } x \in A\big\}.$$

We denote the set of Nash equilibrium distributions of an LDG μ by $\mathrm{NED}(\mu)$.

We now turn to a large individualized game and its equilibria. For this, in addition to the ingredients A, T, and $\mathscr{U}_{(A,T)}$, we need one more constituent object $(I, \mathscr{I}, \lambda)$, a name space of players.

Definition 2. A *large individualized game* (LIG) is a measurable function $\mathscr{G}$ from the space of players' names I to the space of characteristics, $T \times \mathscr{U}_{(A,T)}$, which is to say that $\mathscr{G} \in \mathrm{Meas}(I; T \times \mathscr{U}_{(A,T)})$. A (pure) strategy profile f of $\mathscr{G}$ is a measurable function from I to the action set A, and is said to be a *(pure strategy) Nash equilibrium* (NE) if for λ-almost all $i \in I$,

$$v_i\big(f(i), \lambda(\alpha, f)^{-1}\big) \geq v_i\big(a, \lambda(\alpha, f)^{-1}\big) \quad \text{for all } a \in A,$$

with v_i abbreviated for $\mathscr{G}_2(i)$ and α abbreviated for $\mathscr{G}_1$, where $\mathscr{G}_k$ is the projection of $\mathscr{G}$ on its kth coordinate, $k = 1, 2$. We denote the set of Nash equilibria of an LIG $\mathscr{G}$ by $\mathrm{NE}(\mathscr{G})$.

It should be noted that these notions of LIGs and LDGs are taken from Qiao and Yu (2014), and have advantage over the corresponding notions in Khan et al. (2013a, 2013b) in that they do not depend on a given marginal.[12] We now connect these notions by saying that an LIG *represents* an LDG if the LIG induces the same distribution on the space of characteristics as the LDG, which is defined formally as follows.

[12]Specifically, in Khan et al. (2013a, 2013b), a payoff function is a continuous real-valued function defined on $A \times \mathscr{M}^\rho(T \times A)$ rather than on $A \times \mathscr{M}(T \times A)$ as defined here, where $\rho \equiv \lambda\alpha^{-1}$, and $\mathscr{M}^\rho(T \times A)$ is the subspace of $\mathscr{M}(T \times A)$ such that for any $\tau \in \mathscr{M}^\rho(T \times A)$, its marginal probability on T, $\tau_T = \rho$; see the discussion in Qiao and Yu (2014). To repeat footnote 11 in this connection, compactness on T in this paper can be relaxed to the weaker requirement of a Polish space by appealing to the argumentation in Khan et al. (2013a, 2013b).

Definition 3. An LIG $\mathcal{G}$ *represents* an LDG μ if $\mu = \lambda\mathcal{G}^{-1}$.

Given an LDG $\mu \in \mathcal{M}(T \times \mathcal{U}_{(A,T)})$ and an atomless probability space $(I, \mathcal{I}, \lambda)$, one can show that there always exists[13] an LIG $\mathcal{G} \in \text{Meas}(I; T \times \mathcal{U}_{(A,T)})$ that represents μ. In general, an LDG has a multiplicity of LIGs that represent it.

We now consider the equivalence of an LIG and an LDG in terms of their equilibria.

Definition 4. An LIG $\mathcal{G}$ and an LDG μ are *equivalent* if the following statements hold:

(i) For every $f \in \text{NE}(\mathcal{G})$, $\lambda(\mathcal{G}, f)^{-1} \in \text{NED}(\mu)$.

(ii) For every $\tau \in \text{NED}(\mu)$, there is $f \in \text{NE}(\mathcal{G})$ such that $\tau = \lambda(\mathcal{G}, f)^{-1}$.

Note that in our setup, for any LDG μ, $\text{NED}(\mu)$ is nonempty.[14] Therefore, if an LIG and an LDG are equivalent, then (ii) implies that the LIG must represent the LDG and the LIG must have an NE. With this notion of equivalence of the two game forms in place, our first observation establishes a particular kind of correspondence between the NED of an LDG and NE of any given LIG that represents it.

Lemma 1. *If an LIG $\mathcal{G}$ represents an LDG μ, $f \in \text{Meas}(I; A)$ and $\tau = \lambda(\mathcal{G}, f)^{-1}$, then $f \in \text{NE}(\mathcal{G})$ if and only if $\tau \in \text{NED}(\mu)$.*

Lemma 1 establishes that if we consider a strategy profile f (i.e., a measurable mapping from the name space to the action set) of an LIG $\mathcal{G}$, then the joint distribution induced by $\mathcal{G}$ and f is an NED of the LDG that $\mathcal{G}$ represents if any only if f satisfies the Nash equilibrium condition. This shows that Definition 4(i) is always satisfied between an LDG and any LIG that represents it. Any equilibrium of an LIG, an individualized microeconomic form, automatically induces an equilibrium in its macroeconomic counterpart.

Before investigating Definition 4(ii), we provide the following observation.

Lemma 2. *For any $\tau \in \text{NED}(\mu)$ and an atomless probability space $(I, \mathcal{I}, \lambda)$, there exists an LIG $\mathcal{G} \in \text{Meas}(I; T \times \mathcal{U}_{(A,T)})$ that represents μ and an NE f of $\mathcal{G}$ such that $\tau = \lambda(\mathcal{G}, f)^{-1}$.*

The content of Lemma 2 can be phrased as follows. Given an NED of an LDG, and given an atomless name space of players, there exists an LIG based on that name space that induces the given LDG and that has an NE that induces the given NED of the LDG. It provides the microfoundation for a particular given NED of an LDG when the name space is available. What it does not do is to establish that an LIG is equivalent to the LDG that it represents, i.e., Lemma 2 does not ensure that the requirement expressed in Definition 4(ii) is fulfilled. The question then is whether Lemma 2 can be strengthened

[13]See Lemma 2.1(ii) in Keisler and Sun (2009), for example.

[14]This claim on the existence of an NED in an LDG can be established through the fixed-point argument in Mas-Colell (1984); also see Khan et al. (2013b).

so that the requirement in Definition 4(ii) holds as well? Example 1 offered below gives a decisively negative answer to this question. It involves only two actions and a single trait, and essentially involves a single-payoff function, one on which societal responses have no effect under the first action and have an "inverse" effect under the second action.

Example 1. Let the space of players $(I, \mathscr{I}, \lambda)$ be the Lebesgue unit interval $([0,1], \mathscr{B}([0,1]), \ell)$, let their common action space be $A = \{a_1, a_2\}$, and let $\mathscr{G}$ bea function such that for all $i \in [0,1]$, $\mathscr{G}(i) = iu$, where for any $\nu \in \mathscr{M}(A)$,

$$u(a, \nu) = \begin{cases} 1/2 & \text{for } a = a_1, \\ 1 - \nu(a_2) & \text{for } a = a_2. \end{cases}$$

The space of traits T is a singleton in this example and we abbreviate $T \times \mathscr{U}_{(A,T)}$ to $\mathscr{U}_A$.

Let $\mu = \ell\mathscr{G}^{-1}$. Clearly, $\mathscr{G}$ is an LIG, μ is an LDG, and $\mathscr{G}$ represents μ. It is easy to verify that f is a Nash equilibrium of $\mathscr{G}$ if and only if $\ell f^{-1}(a_1) = \ell f^{-1}(a_2) = 1/2$, i.e., in equilibrium, both actions are best responses. Moreover, $\mathrm{NED}(\mu)$ is the set of all Borel probability measures τ on $\mathscr{U}_A \times A$ such that $\tau_{\mathscr{U}_A} = \mu$ and $\tau_A = (\delta_{a_1} + \delta_{a_2})/2$, where δ_a is the Dirac measure on $\{a\}$.

Define f_1 and f_2 as

$$f_1(i) = a_1 \quad \text{if } i < 1/2 \quad \text{and} \quad f_1(i) = a_2 \quad \text{if } i \geq 1/2,$$
$$f_2(i) = a_2 \quad \text{if } i < 1/2 \quad \text{and} \quad f_2(i) = a_1 \quad \text{if } i \geq 1/2.$$

Both f_1 and f_2 are Nash equilibria of $\mathscr{G}$ and $\ell f_1^{-1} = \ell f_2^{-1}$. Let $\tau^1 = \ell(\mathscr{G}, f_1)^{-1}$ and $\tau^2 = \ell(\mathscr{G}, f_2)^{-1}$. By Lemma 1, both τ^1 and τ^2 are NEDs of μ. Observe that $\tau^1(\{(\mathscr{G}(i), a_1) : i < 1/2\}) = 1/2$ and $\tau^2(\{(\mathscr{G}(i), a_1) : i < 1/2\}) = 0$, so $\tau^1 \neq \tau^2$. Now consider $\tau^\theta = \theta\tau^1 + (1 - \theta)\tau^2$, $\theta \in (0,1)$. Clearly, $\tau^\theta_{\mathscr{U}_A} = \mu$ and $\tau^\theta_A = \tau^1_A = \tau^2_A$. Therefore, $B(\tau^\theta) = B(\tau^1) = B(\tau^2)$. Since $\tau^1(B(\tau^\theta)) = \tau^2(B(\tau^\theta)) = 1$, $\tau^\theta(B(\tau^\theta)) = 1$. So each τ^θ is an NED of μ.

The important point is that given $\theta \in (0,1)$, there is *no* $f \in \mathrm{NE}(\mathscr{G})$ such that $\tau^\theta = \ell(\mathscr{G}, f)^{-1}$. Hence, $\mathscr{G}$ and μ are not equivalent.[15] We defer the details of the substantiation of this claim to Appendix A; suffice it to say that it relies essentially on the extreme point characterization of an SNED[16] offered in Khan and Sun (1995).

The results that we present below can be categorized under three headings: (i) weak equivalence under which the focus is on SNEDs, (ii) quasi-equivalence under which the equality of two distributions is weakened to an equality of their marginals, an,d finally, (iii) equivalence requiring additional assumptions on the space of players of the representing LIG. Under (i), the set of equilibrium distributions is restricted; under (ii), equality is weakened to similarity; under (iii), we invoke an additional assumption on a property of the space of players that has already proved to be of consequence for the subject of atomless games. These notions overcome the aforementioned deficiency of Lemma 2.

[15] It is also worth noting that in Example 1, for any fixed $\theta \in (0,1)$, we can construct another LIG that represents μ and has an NE that does induce the NED τ^θ. The case when $\theta = 1/2$ is done to illustrate Lemma 2 and is Example 4 in Appendix A.

[16] See Definition 5 below for its formal definition.

3. Equivalence results

There is an interesting feature of Example 1 that deserves to be noted. It is simply that even though there exists an NED of an LDG that cannot be induced by an NE of a given representing LIG, there is an identifiable subset[17] of NEDs, such as τ^1 and τ^2 in the example, that can be so induced. This observation leads to the following statement.

Definition 5. In an LDG μ, a $\tau \in \mathrm{NED}(\mu)$ is *symmetric* if there exists a measurable function $h : T \times \mathscr{U}_{(A,T)} \longrightarrow A$ such that $\tau(\text{graph of } h) = 1$. We denote the set of symmetric Nash equilibrium distributions of an LDG μ by $\mathrm{SNED}(\mu)$.

We then build on the idea of an SNED to present the following weakening of Definition 4 and our first substantive result based on it. Unlike Definition 4, we are now explicit in making it a requirement that an LIG represents the LDG.

Definition 6. An LIG $\mathscr{G}$ and an LDG μ are *weakly equivalent* if $\mathscr{G}$ represents μ and in Definition 4(ii), $\mathrm{NED}(\mu)$ is replaced by $\mathrm{SNED}(\mu)$ to read as follows:

(ii′) For every $\tau \in \mathrm{SNED}(\mu)$, there is $f \in \mathrm{NE}(\mathscr{G})$ such that $\tau = \lambda(\mathscr{G}, f)^{-1}$.

The equivalence of an LIG and an LDG implies that the LIG is weakly equivalent to the LDG. However, the converse may not hold. For instance, in Example 1, the LIG $\mathscr{G}$ and the LDG μ are not equivalent, but they are weakly equivalent. If $\tau \in \mathrm{SNED}(\mu)$ and $\tau(\text{graph of } h) = 1$, then $\ell(\{i : h(iu) = a_1\}) = \ell(\{i : h(iu) = a_2\}) = 1/2$. Consider the function $f : [0, 1] \longrightarrow \{a_1, a_2\}$ defined as $f(i) = a_1$ if and only if $h(iu) = a_1$ for all $i \in [0, 1]$. It is clear that f is an NE of $\mathscr{G}$ and $\tau = \ell(\mathscr{G}, f)^{-1}$.

We are ready to present our first theorem.

Theorem 1. *Any LIG that represents an LDG is weakly equivalent to it.*

Intuitively, Theorem 1 works because an SNED has the individualistic microfoundation in itself: according to Definition 5, at any SNED in an LDG, players with the same characteristics take the same action. Thus, when an SNED exists for a given LDG, an NE for an LIG representing it can be obtained in a way such that players with the same characteristics take the same actions.

However, Theorem 1 has not fully established a correspondence between NEDs of an LDG and the NE of any of its individualized form, since it only focuses on the set of SNEDs, a subset of NEDs. In particular, it is well known that an (atomic) LDG may not have an SNED, even when the action set is finite and the trait space is a singleton; see Rath (1995, Example 2).[18] And thus, Theorem 1 still trivially holds for any LDG without an SNED and an LIG that represents the LDG. Even worse, it is possible that Theorem 1

[17] See Lemmas 3 and 4 below in this regard.

[18] To be sure, as shown in Mas-Colell (1984, Theorem 2), an SNED does exist for an atomless LDG with finite actions and a single trait. Such a result still holds when the action set is countable; see the nontrivial generalization in the 1995 papers of Khan and Sun, referenced in Khan and Sun (2002), where an NE is also connected to an SNED.

still holds when an LIG does not have any NE even though there *always* exists an NED in an LDG.[19]

It is well understood that if the action set is uncountable, there may not exist an NE in an LIG; see Khan et al. (1997). We now also understand that with an uncountable set of traits, there may not exist an NE in an LIG; see Qiao and Yu (2014). In the first case, the set of traits can be a singleton, and in the second case, the action set can contain only *two* elements. These examples are thus decisive in dashing hopes of any progress with sets of uncountable cardinality. Two examples along these lines are provided below briefly for the reader's convenience.

Example 2. Let the space of players be the Lebesgue unit interval $([0,1], \mathscr{B}([0,1]), \ell)$, let the action set A be the interval $[-1,1]$, and let $\mathscr{G}$ be a function on I such that for each player $i \in [0,1]$, for $a \in A$, $\nu \in \mathscr{M}(A)$,

$$\mathscr{G}(i)(a,\nu) = \int_{-1}^{1} w_i(a,x)\,\mathrm{d}\nu(x), \quad \text{where } w_i(a,x) = -\big|i - |a|\big| + (i-a)h(i,x)$$

and the function $h : [0,1] \times [-1,1] \longrightarrow \mathbb{R}$ is such that for all $t \in [0,1]$,

$$h(i,a) = \begin{cases} a & \text{if } 0 \le a \le i, \\ i & \text{if } i < a \le 1, \\ -h(i,-a) & \text{if } a < 0. \end{cases}$$

Let $\mu = \ell\mathscr{G}^{-1}$. It is clear that $\mathscr{G}$ is an LIG, with the space of traits being a singleton, that μ is an LDG and that $\mathscr{G}$ represents μ. While μ has an NED, $\mathscr{G}$ does not have an NE.[20]

Example 3. Let the space of players be the Lebesgue unit interval $([0,1], \mathscr{B}([0,1]), \ell)$, let the space of traits T be the unit interval $[0,1]$, and let the common action space be $A = \{a_1, a_2\}$. Let α be the identity mapping on $[0,1]$. For any $a \in A$ and $\nu \in \mathscr{M}(T \times A)$, let the payoff function for player i be

$$v_i(a,\nu) = \int_{T\times A} (t-i) 1_{[0,i)\times\{a\}}(t,x)\,\mathrm{d}\nu.$$

Let $\mathscr{G}$ be a function on $[0,1]$ such that for all $i \in [0,1]$, $\mathscr{G}(i) = (\alpha(i), v_i)$, where $\alpha(i)$ and v_i are specified as above. Let $\mu = \ell\mathscr{G}^{-1}$. It is clear that the LIG $\mathscr{G}$ represents the LDG μ. Whereas μ has an NED, it is shown in Qiao and Yu (2014) that $\mathscr{G}$ is an LIG without any NE.

[19] The existence of an NED in an LDG is a remarkably robust result, is true for any complete separable metric space of traits and true for action sets that are not even metrizable but only compact Hausdorff, and with no non-atomic requirements on the measure space of payoff functions; see Mas-Colell (1984, Theorem 1) and its generalization in Khan (1989) referenced in Khan and Sun (2002); also see Khan et al. (2013b). However, an NE may fail to exist in an LIG; see Examples 2 and 3. In this connection, note that the first example of nonexistence of an NE is a large game with uncountable actions but without traits is due to Rath-Sun-Yamashige in 1995; see the precise reference and its reproduction in Khan and Sun (2002).

[20] This example is originally in a 1997 paper of Khan, Rath, and Sun, discussed and referenced as Example 3 in Khan and Sun (2002).

To summarize and to repeat, these examples suggest that in general, when the action set A or the space of traits T is uncountable in a large game, connecting equilibria in an LDG and an LIG can be problematic: the latter may not have an equilibrium while the former always does.[21] Therefore, so as to address the entire set of NEDs (rather than the set of SNEDs that could be vacuous) and the set of NEs, and given that the equivalence does not hold as well, the next modification is to draw on the notion of similarity introduced in Rath (1995) for games without traits.

Definition 7. Two NEDs τ and τ' of μ are *similar* if $\tau_{T\times A} = \tau'_{T\times A}$. This is denoted $\tau \simeq \tau'$.

In a game, from the perspective of players' payoffs, what is most relevant is the marginal distribution on traits and actions. Since an NED is a joint distribution on traits, payoffs, and actions, the externality part is the marginal distribution on T and A, and it is this that regulates the best responses. If two NEDs are similar, they generate the same best response sets and are alike in important game theoretic aspects. Based on similarity, we modify equivalence to quasi-equivalence below.

Definition 8. An LIG $\mathscr{G}$ and an LDG μ are *quasi-equivalent* if in Definition 4, (ii) is changed so as to weaken the equality to similarity:

(ii$''$) For every $\tau \in \mathrm{NED}(\mu)$, there is $f \in \mathrm{NE}(\mathscr{G})$ such that $\lambda(\mathscr{G}, f)^{-1} \simeq \tau$.

The equivalence of an LIG and an LDG implies that they are quasi-equivalent. The converse may not hold: Example 1 shows the quasi-equivalence of an LDG and an LIG that represents it cannot be strengthened to the equivalence in Definition 4 even with finite traits and actions. Quasi-equivalence implies weak equivalence. The nonemptiness of the set of NEDs and quasi-equivalence imply that the given LIG represents the given LDG; see the discussion following Definition 4. Weak equivalence then follows from Theorem 1. Weak equivalence does not imply quasi-equivalence, as illustrated by Examples 2 and 3 in which an NE fails to exist for some LIG when the action set or trait set is uncountable.

We are now ready to present our theorem on the characterization of quasi-equivalence.

Theorem 2. *The following two statements are equivalent:*

(i) Both A and T are countable.

(ii) Any LIG that represents an LDG in $\mathscr{M}(T \times \mathscr{U}_{(A,T)})$ is quasi-equivalent to it.

The quasi-equivalence relation between an LDG and an LIG that represents it is thus fully characterized by the countability of both the action set and the space of traits.[22]

[21] In terms of the existence of an NE, this observation also suggests the possible dissonance of two LIG representations of the same LDG: in each example above, whereas the given LIG does not have any NE, one can always find another LIG that represents the given LDG and that has an NE (by Lemma 2).

With A and T being countable, given any NED of an LDG, one can always find another NED, similar to the original one, that is induced by an NE of the LIG. What is a surprise, however, is that the countability assumption is also necessary for such a quasi-equivalence. From a substantive point of view, it is after all the (marginal) equilibrium distribution on the trait and action sets that constitutes the raison d'etre of the theory. It is a little surprising that this situation of "approximate" equality is characterized by a setting of countable traits and countable actions. In a phrase, countability of the relevant sets implies and is implied by quasi-equivalence. From a technical point of view, Theorem 2 is a nontrivial consequence of a nontrivial theorem.[23] Note also that the implication of quasi-equivalence, or of equivalence for that matter, is not shared by weak equivalence for the existence theory. Since weak equivalence is focused on a subset of equilibria, it has little to say regarding the existence of equilibrium in one game form implying that in the other.

Some additional discussion of the "externality" may be warranted. For the externality notion that has been used in this paper, one generated by traits *and* actions, it is the uncountable cardinality of the juxtaposition of the two spaces that is responsible for the difficulties. Under the countability restriction of the trait and action spaces of an LDG, it follows from Theorem 2 that there exists an NE in a given LIG that represents a given LDG. As the counterexamples suggest, this is no longer true if T or A is uncountable (i.e., a measurable selection from individualized best responses that induces the externality part of an NED may fail to exist). If there is to be no cardinality restrictions on T and A, one has to impose some other structure on an LIG, the name space, for example, that guarantees the existence of an NE. That is the exact equivalence characterization that we are after and achieve.

The literature of a large game with uncountable actions and/or uncountable traits has already provided alternative hypotheses to address the existence of an NE in an LIG; see, for examples, games with countable characteristics as in Carmona (2008), games with "many players of every type" as in Noguchi (2009), and games with a saturated name space[24] as in Keisler and Sun (2009), Carmona and Podczeck (2009), and Khan et al. (2013a). We note that under these hypotheses, the notion of equivalence proposed in Definition 4 does hold. Given our interest in exact equivalence, we would like to extract a feature shared by these classes of games. To do this, we now focus on the representing LIG based on a name space that has a certain feature related to the following property.[25]

[22] For the special case of the similarity result (i.e., from (i) to (ii) in Theorem 2) in a finite-action setting without traits; see Rath (1995) and the 1995 Tokyo Conference Volume paper of Khan and Sun referenced in Khan and Sun (2002).

[23] This is the Ballobas–Varapoulos generalization of the marriage lemma. For its formal statement and its application in large games to relax the finiteness assumption on the action set, see the 1995 papers of Khan and Sun referenced in Khan and Sun (2002).

[24] The necessity of a saturated name space for the existence of NE is considered in Keisler and Sun (2009) and Khan et al. (2013a). The latter also considers the relation of a saturated name space and the closed-graph property of equilibrium correspondence in large games; also see Qiao and Yu (2014).

[25] See Keisler and Sun (2009) and their reference to Hoover-Keisler (1984).

Local Saturation Property. An atomless probability space $(I, \mathscr{I}, \lambda)$ is said to have the *local saturation property for a Borel probability measure* ρ on the product of complete separable metric spaces $X \times Y$ such that if $\rho_X = \lambda f^{-1}$ for a measurable mapping $f : I \longrightarrow X$, then there is a measurable mapping $g : I \longrightarrow Y$ such that $\lambda(f, g)^{-1} = \rho$.

For instance, if a Borel probability measure ρ on the product of two complete separable metric spaces $X \times Y$ has its marginal on X being purely atomic, then by Proposition 2.3 in Keisler and Sun (2009), any atomless probability space has the local saturation property for ρ. Note also that in Example 1, the Lebesgue unit interval has the local saturation property for τ^1 and τ^2 therein, but does not have the local saturation property for τ^θ with $\theta \in (0, 1)$.

A probability space $(I, \mathscr{I}, \lambda)$ is *saturated*, or has the *global saturation property* if it has the local saturation property for every Borel probability measure ν on the product of any two complete separable metric spaces.[26] For instance, an atomless Loeb probability space is saturated. Note also that every saturated probability space is an atomless space, that the unit Lebesgue interval is not saturated, and that every nonsaturated atomless probability space admits a saturated extension.[27] The definition below is stronger than the local saturation property, but weaker than the global saturated property.

Definition 9. Given an LDG μ, a representing LIG based on the name space $(I, \mathscr{I}, \lambda)$ is said to *saturatedly represent* μ if $(I, \mathscr{I}, \lambda)$ has the local saturation property for each $\rho \in \mathscr{M}(T \times \mathscr{U}_{(A,T)} \times A)$ with $\rho_{T \times \mathscr{U}_{(A,T)}} = \mu$.

We now present a basic result on the equivalence of an LDG and an LIG.

Theorem 3. *Any LIG that saturatedly represents an LDG is equivalent to it.*

As Example 1 already shows, a representing LIG with the Lebesgue name space may not in general saturatedly represent the given LDG. We consider next two cases related to Theorem 3. The first one concerns the local saturation property. If an LDG $\mu \in \mathscr{M}(T \times \mathscr{U}_{(A,T)})$ is purely atomic, as pointed out earlier, any atomless probability space has the local saturation property for each $\rho \in \mathscr{M}(T \times \mathscr{U}_{(A,T)} \times A)$ with $\rho_{T \times \mathscr{U}_{(A,T)}}$ equal to μ. Thus, this fact directly yields the following corollary to Theorem 3.

Corollary 1. *If an LDG is purely atomic, then any LIG that represents it is equivalent to it.*

If the population has at most countable characteristics, the corresponding LDG must be purely atomic. Thus, Corollary 1 implies that when we model a large game with countable characteristics in its LIG form, the choice of the LIG form does not matter, and every LIG that represents this LDG is equivalent to it.

[26]The distinction between the local and global saturation properties of a probability space is emphasized in Keisler and Sun (2009); also see Khan et al. (2013a, Proposition A) and the discussion therein.

[27]The first two claims are standard, and the last claim is not difficult to show; details are available from the authors on request.

Second, since a saturated probability space has the global saturation property, the following corollary is also directly implied by Theorem 3.

Corollary 2. *An LIG with a saturated name space that represents an LDG is equivalent to it.*

The result above suggests that for any suitable A, T, and $\mathscr{U}_{(A,T)}$, there is always a "universally" equivalent name space such that even before knowing the μ, we are sure that an equivalent LIG with that name space can be constructed. Observe that if the name space of LIGs in Examples 1, 2, and 3 is extended to a saturated one,[28] we obtain equivalence between the given LDGs and the modified LIGs.

It is important to understand that these results do not render Theorems 1 and 2 obsolescent based on arbitrary atomless probability spaces (of which the Lebesgue interval is the prototype) and charting out the different versions of equivalence. Indeed, since the mid-sixties, the Lebesgue interval has been the workhorse space for the set of players' names in general equilibrium theory, and from the mid-seventies, in nonatomic game theory. Even leaving this aside, one can ask about the relationship between the two sets of results: those based on the Lebesgue interval as opposed to those based on saturated spaces. The answer lies in the fact that the two can be directly related to symmetric and nonsymmetric NEDs, respectively, in an atomless LDG. We saw in Theorem 1 that any LIG that represents a given LDG has an NE that induces a given SNED of the LDG; also see the discussion preceding Example 1. This naturally suggests the question, "What transpires when the given NED of an LDG is not necessarily symmetric?" To answer this question, one can relate the choice of the name space of the LIGs such that all representing LIGs on *that* space have NE that induce the given NED.[29] But first, we need the following concept.

Definition 10. Let τ be an NED of an LDG μ. An atomless probability space $(I, \mathscr{I}, \lambda)$ *realizes* τ if for *every* LIG $\mathscr{G} \in \text{Meas}(I; T \times \mathscr{U}_{(A,T)})$ that represents μ has an NE f such that $\tau = \lambda(\mathscr{G}, f)^{-1}$.

We are now ready to present a result,[30] whose first assertion is implied directly by Lemma 3 in Appendix A and Keisler and Sun (2009, Proposition 2.4), and whose second assertion is a direct consequence of Keisler and Sun (2009, Theorem 2.7).

Corollary 3. *In an atomless LDG, (i) an NED is symmetric if and only if it can be realized by the Lebesgue unit interval; (ii) a nonsymmetric NED can be realized by an atomless probability space $(I, \mathscr{I}, \lambda)$ if and only if $(I, \mathscr{I}, \lambda)$ is saturated.*

We conclude this section by noting some implications of our findings for the existence theory of large games with traits. Since there *always* exists an NED in an LDG,

[28] We defer to future investigation the possible application of the nonsaturated measure space studied in Khan and Zhang (2012).

[29] Such a connection is explicit in Keisler and Sun (2009); also see Noguchi (2009).

[30] Such a result has been reported in a similar setup in Khan et al. (2013b); see Khan et al. (2013b, Theorem 4, Corollary 1) and the remark after Khan et al. (2013b, Corollary 1).

Theorems 2 and 3 yield two sufficient conditions to guarantee the existence of an NE in an LIG. To be more specific, there exists an NE in an LIG $\mathscr{G} : (I, \mathscr{I}, \lambda) \longrightarrow T \times \mathscr{U}_{(A,T)}$ if (i) both T and A are countable or (ii) $\mathscr{G}$ saturatedly represents $\lambda\mathscr{G}^{-1}$. Furthermore, one can apply Theorem 1 and Corollary 1 to obtain a characterization of saturated spaces in terms of the existence of an NE in an LIG. In particular, if either of A or T is uncountable, then all LIGs based on $(I, \mathscr{I}, \lambda)$ as its name space, and with A as its action set or T as its space of traits, have NE *if and only if* $(I, \mathscr{I}, \lambda)$ is saturated. We sketch briefly the intuition of the argument.[31] Given an uncountable set of actions or traits, one can transfer the nonexistence claim of Example 2 or Example 3 to another LIG $\mathscr{G}'$ without an NE based on the Lebesgue unit interval as its name space but with the same given actions or traits, and show that there is only one NED in the reduced LDG. Theorem 1 implies that the induced LDG $\ell\mathscr{G}'^{-1}$ does not have any SNED. The claim follows as a direct consequence of Corollary 3.

It is also worth underscoring the fact that one can translate the existence results, discussed above for an LIG, to those (a) for Bayes-Nash equilibria (henceforth, BNE) in a Bayesian game with diffused and disparate information, and (b) for approximate equilibria in large but finite games with traits. For (a), with the standard diffuseness and mutual independence assumptions, as shown in Fu and Yu (2015), by treating a *real* player, together with her type in a Bayesian game, as an *artificial* player, and using the real player's name as the *trait* of the artificial player in the induced large game, one can transfer a Bayesian game with private information to a large game and establish that a BNE exists in the original Bayesian game *if and only if* a Nash equilibrium exists in the induced large game. This shows the relevance of the sufficient conditions on the existence of an NE in an LIG for Bayesian game theory.

For (b), the methodology for an asymptotic implementation draws on nonstandard analysis, and is by now relatively well understood. The existence of an NE in an LIG with a saturated name space can be used to study approximate equilibria in large but finite games with traits as a consequence of the observation that an atomless Loeb space is saturated, and, therefore, results for it translated (the term of art is "transferred") to an increasing sequence of large but finite games.[32] It is worthy of emphasis that the other way does not work: in general, even the exact NE of a convergent sequence of finite games may not even imply that the limit LIG has an NE; see Qiao and Yu (2014). The use of a saturated space to model the name space of the limit game is thus shown to be not only sufficient, but also necessary to avoid such a dissonance.

4. Concluding remarks

In summary, we chart the relationship between two formulations of the theory of large games with a biosocial typology, and draw on the notions of similarity, countability, realizability, and saturation to connect to the antecedent literature. We leave it for future

[31] The details are reported in Qiao et al. (2016), which in turn draws and builds on Khan et al. (2013a). Such a result in a similar setup has been reported through direct constructions.

[32] For details, we refer the reader to Khan et al. (2013a, Section 6). The ϵ ex post property of mixed-strategy Nash equilibria in semi-anonymous games can be obtained through the ex post property of well defined mixed-strategy Nash equilibria as in Khan et al. (2015), where the exact law of large numbers of Sun (2006) is used. Also see footnote 4 above.

work to take these ideas to questions of stochastic dynamics, especially those arising in macroeconomic settings.[33] We conclude by commenting on two other directions of work.

First, there is little doubt that the methodology that is implicit in this paper can be used to develop results on individualized representation of large economies with externalities. Such results have been presented only in a distributionalized form and an example of an economy without an individualized equilibrium; see Nogushi and Zame (2006). The point of course is not to present a taxonomy of results, but to see how the resolution of the dissonance between the two formulations sheds insight into the substantive problems that either formulation is attempting to articulate and come to terms with.

Our second point concerns the question of a player's trait entering as an argument of a player's payoff function. In the framework analyzed here, each trait is pegged to an individual name, which is to say that the trait function $\alpha : I \longrightarrow T$ is a deterministic function. One can of course consider a situation when the traits are random or, to think in terms of a dynamic context, a function of chosen actions, but all this is very much outside the scope of this work. We refer the reader to Balbus et al. (2015) for a consideration of this enriched context in its distributionalized form, and leave the consideration of its individualized form for future investigation.

Appendix A: Auxiliary results and details of examples

We offer two results on the characterization of SNEDs in an LDG. The second is used in the discussion of examples and the first is used to prove the second. Given an LIG $\mathscr{G} \in \mathrm{Meas}(I; T \times \mathscr{U}_{(A,T)})$, let $\sigma(\mathscr{G}) = \{\mathscr{G}^{-1}(V) : V \in \mathscr{B}(T \times \mathscr{U}_{(A,T)})\}$. It is clear that $\sigma(\mathscr{G})$ is the smallest σ-algebra on I with respect to which $\mathscr{G}$ is measurable.

Lemma 3. *Suppose that μ is an LDG and $\mathscr{G} \in \mathrm{Meas}(I; T \times \mathscr{U}_{(A,T)})$ represents μ. Then $\tau \in \mathrm{SNED}(\mu)$ if and only if $\tau = \lambda(\mathscr{G}, f)^{-1}$ for a $\sigma(\mathscr{G})$-measurable NE f of $\mathscr{G}$.*

Given any probability space $(I, \mathscr{I}, \lambda)$, a function on I is called *almost one-to-one* if it is one-to-one on I except for some λ-null set of $\mathscr{I}$. If μ is atomless, the existence of an almost one-to-one LIG $\mathscr{G} \in \mathrm{Meas}(([0,1], \mathscr{B}([0,1]), \ell); T \times \mathscr{U}_{(A,T)})$ that represents μ can be deduced from Bogachev (2007, Theorem 9.6.3). The next result brings out the implications and the importance of an almost one-to-one LIG with Lebesgue name space. It is important to note that this concept has a natural interpretation in that it entails a society that is heterogeneous in the extreme: except for a null set, every agent that differs in name also differs in characteristics.

Lemma 4. *Let μ be an LDG and let $\mathscr{G} \in \mathrm{Meas}(([0,1], \mathscr{B}([0,1]), \ell); T \times \mathscr{U}_{(A,T)})$ be an LIG that represents μ. Assume that $\mathscr{G}$ is almost one-to-one. If $f \in \mathrm{NE}(\mathscr{G})$, then $\ell(\mathscr{G}, f)^{-1} \in$ $\mathrm{SNED}(\mu)$.*

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[33] For stochastic dynamics and global games, see Balbus et al. (2015), Bergemann and Morris (2013), Bergin (1992), Bergin and Bernhardt (1992, 1995), Jovanovic and Rosenthal (1988), Morris and Shin (2003) and their references; for other settings, see Guesnerie and Jara-Moroni (2011), Jara-Moroni (2012).

We now complete the discussion pertaining to Example 1.

Example 1 (Revisited). Observe that $\mathscr{G}$ is one-to-one on $[0,1]$ and that μ is atomless. By Lemma 4, both τ^1 and τ^2 are SNEDs of μ.

For any NED τ of μ, let

$$\Gamma(\tau)=\{\rho\in\mathscr{M}(\mathscr{U}_A\times A):\rho_{\mathscr{U}_A}=\mu \text{ and } \rho_A=\tau_A\}.$$

The extreme point characterization of an SNED of an atomless game as given in Khan and Sun (1995) is that τ is an SNED of μ if and only if τ is an extreme point of $\Gamma(\tau)$. In the present context, $\Gamma(\tau^1)=\Gamma(\tau^2)=\Gamma(\tau^\theta)$ for any $\theta\in(0,1)$. It follows that τ^1 and τ^2 are extreme points of $\Gamma(\tau^\theta)$. Given any $\theta\in(0,1)$, since τ^θ is a strict convex combination of τ^1 and τ^2 and $\tau^1\neq\tau^2$, it cannot be an extreme point of $\Gamma(\tau^\theta)$. Therefore, τ^θ for $\theta\in(0,1)$ is an NED of μ, but cannot be symmetric. Lemma 4 now implies that for any $f\in\operatorname{Meas}([0,1];A)$, $\ell(\mathscr{G},f)^{-1}\neq\tau^\theta$.

Another feature of this example is noteworthy. We have seen that no NE of $\mathscr{G}$ can induce τ^θ for $\theta\in(0,1)$. Notice also that each τ^θ is similar to both τ^1 and τ^2. So an NE of $\mathscr{G}$ does induce an NED similar to τ^θ. This emphasizes the fact that one cannot go beyond similarity in Theorem 2.

In light of the preceding discussion, and given Lemma 2, there must exist an LIG that represents μ above and an NE of that LIG that induces τ^θ for a given $\theta\in(0,1)$. In the next example, we find an LIG that represents μ and an NE of that LIG that induces τ^θ for $\theta=1/2$.

Example 4. In this example, u, μ, $\mathscr{G}$, f_1, f_2, τ^1, and τ^2 are as in Example 1. Define an LIG $\mathscr{H}:([0,1],\mathscr{B}([0,1]),\ell)\longrightarrow\mathscr{U}_A$ as

$$\mathscr{H}(i)=2iu \quad \text{if } i<\frac{1}{2} \quad \text{and} \quad \mathscr{H}(i)=2[i-(1/2)]u \quad \text{if } i\geq\frac{1}{2},$$

and define a strategy profile $f:[0,1]\longrightarrow A$ as $f(i)=a_1$ if $i\in[0,1/4)\cup(3/4,1]$ and as $f(i)=a_2$ if $i\in[1/4,3/4]$. Since $f^{-1}(a_1)=f^{-1}(a_2)=1/2$, $f\in\operatorname{NE}(\mathscr{H})$. From Lemma 1, $\eta=\ell(\mathscr{H},f)^{-1}$ is an NED of $\ell\mathscr{H}^{-1}$. We will show that $\eta=\tau^\theta$ for $\theta=1/2$. It follows that $\ell\mathscr{H}^{-1}=\eta_{\mathscr{U}_A}=\tau^\theta_{\mathscr{U}_A}=\mu$. Thus, $\mathscr{H}$ represents μ and has an NE f that induces the NED τ^θ for $\theta=1/2$.

Let $B_1=\{(\mathscr{G}(i),a_1):i\in L_1\}$, $B_2=\{(\mathscr{G}(i),a_2):i\in L_1\}$, $C_1=\{(\mathscr{G}(i),a_1):i\in L_2\}$, and $C_2=\{(\mathscr{G}(i),a_2):i\in L_2\}$, where $L_1=(0,1/4)\cup(1/4,1/2)$ and $L_2=(1/2,3/4)\cup(3/4,1)$. Then $\tau^1(B_1)=\tau^1(C_2)=\tau^2(B_2)=\tau^2(C_1)=1/2$ and $\tau^1(B_2)=\tau^1(C_1)=\tau^2(B_1)=\tau^2(C_2)=0$.

For any Borel subset E of $\mathscr{U}_A\times A$, let $F_1=E\cap B_1$, $F_2=E\cap B_2$, $G_1=E\cap C_1$, and $G_2=E\cap C_2$. It is immediate that $\tau^1(F_2)=\tau^1(G_1)=\tau^2(F_1)=\tau^2(G_2)=0$. One has $(\mathscr{H},f)^{-1}(F_1)=\{i/2:i\in(\mathscr{G},f_1)^{-1}(F_1)\}$, $(\mathscr{H},f)^{-1}(F_2)=\{(i+1)/2:i\in(\mathscr{G},f_2)^{-1}(F_2)\}$, $(\mathscr{H},f)^{-1}(G_1)=\{(i+1)/2:i\in(\mathscr{G},f_2)^{-1}(G_1)\}$, and $(\mathscr{H},f)^{-1}(G_2)=\{i/2:i\in$

$(\mathscr{G}, f_1)^{-1}(G_2)\}$. So $\eta(F_1) = \tau^1(F_1)/2$, $\eta(F_2) = \tau^2(F_2)/2$, $\eta(G_1) = \tau^2(G_1)/2$, $\eta(G_2) = \tau^1(G_2)/2$, and, therefore,

$$\eta(E) = \left[\tau^1(E) + \tau^2(E)\right]/2.$$

We have now shown that $\eta = \tau^\theta$ for $\theta = 1/2$.

Appendix B: Proofs of the results

Proof of Lemma 1. It is easy to show that $\tau_{T\times\mathscr{U}_{(A,T)}} = \mu$ and $\tau_{T\times A} = \lambda(\alpha, f)^{-1}$. Let $v \equiv \mathscr{G}_2$.

Assume that $f \in \mathrm{NE}(\mathscr{G})$. Let $I' = \{i \in I : v_i(f(i), \lambda(\alpha, f)^{-1}) \geq v_i(a, \lambda(\alpha, f)^{-1})$ for all $a \in A\}$. Since $f \in \mathrm{NE}(\mathscr{G})$, $\lambda(I') = 1$. The fact that $\tau_{T\times A} = \lambda(\alpha, f)^{-1}$ implies that $(\mathscr{G}(i), f(i)) \in B(\tau)$ for all $i \in I'$. Therefore, $\tau(B(\tau)) = \lambda(\mathscr{G}, f)^{-1}(B(\tau)) = \lambda(\{i \in I : (\mathscr{G}(i), f(i)) \in B(\tau)\}) \geq \lambda(\{i \in I' : (\mathscr{G}(i), f(i)) \in B(\tau)\}) = \lambda(I') = 1$. This shows that $\tau \in \mathrm{NED}(\mu)$.

Let $\tau \in \mathrm{NED}(\mu)$. Then $\tau(B(\tau)) = 1$. Since $\tau = \lambda(\mathscr{G}, f)^{-1}$, $\tau(B(\tau)) = \lambda(\{i \in I : (\mathscr{G}(i), f(i)) \in B(\tau)\}) = 1$. Moreover, $\tau_{T\times A} = \lambda(\alpha, f)^{-1}$ and $B(\tau) = \{(t, u, a) : u(a, \tau_{T\times A}) \geq u(x, \tau_{T\times A})$ for all $x \in A\}$ imply that $v_i(f(i), \lambda(\alpha, f)^{-1}) \geq v_i(x, \lambda(\alpha, f)^{-1})$ for all $x \in A$ and for λ-almost all $i \in I$, i.e., $f \in \mathrm{NE}(\mathscr{G})$. □

Proof of Lemma 2. Since τ is a probability measure on $T \times \mathscr{U}_{(A,T)} \times A$, a complete separable metric space, we can appeal to Keisler and Sun (2009, Lemma 2.1(ii)) to assert that there is a measurable mapping $z : I \longrightarrow T \times \mathscr{U}_{(A,T)} \times A$ such that $\tau = \lambda z^{-1}$. Let $\mathscr{G}$ and f be the projections of z on $T \times \mathscr{U}_{(A,T)}$ and A, respectively. Then $\mu = \lambda\mathscr{G}^{-1}$ and $\tau = \lambda(\mathscr{G}, f)^{-1}$. Lemma 1 shows that $f \in \mathrm{NE}(\mathscr{G})$. □

Proof of Theorem 1. Let μ be an LDG and let $\mathscr{G} \in \mathrm{Meas}(I; T \times \mathscr{U}_{(A,T)})$ represent μ. It is easy to see that Definition 6(ii′) trivially holds if μ does not have an SNED. Therefore, suppose that μ has an SNED. If $\tau \in \mathrm{SNED}(\mu)$, then there exists a measurable $h : T \times \mathscr{U}_{(A,T)} \longrightarrow A$ such that $\tau(\text{graph of } h) = 1$. Define $f : I \longrightarrow A$ by $f(i) = h(\mathscr{G}(i))$. If M is any Borel subset of $T \times \mathscr{U}_{(A,T)} \times A$, then $\tau(M) = \tau(M \cap \text{graph of } h) = \tau(\{(t, u, h(t, u)) : (t, u, h(t, u)) \in M\}) = \mu(\{(t, u) : (t, u, h(t, u)) \in M\}) = \lambda(\{i : (\mathscr{G}(i), f(i)) \in M\}) = \lambda(\mathscr{G}, f)^{-1}(M)$. So $\tau = \lambda(\mathscr{G}, f)^{-1}$. That $f \in \mathrm{NE}(\mathscr{G})$ follows from Lemma 1. □

Proof of Theorem 2. We first show that the countability of A and T implies quasi-equivalence. Let μ be an LDG in $\mathscr{M}(T \times \mathscr{U}_{(A,T)})$, let $\mathscr{G}$ be an LIG that represents μ, and let $\tau \in \mathrm{NED}(\mu)$. We begin the proof for the case where μ is atomless. Suppose that μ is atomless. By an argument similar to the proof of Khan et al. (2013b, Theorem 2), we can show that there exists $\tau^* \in \mathrm{SNED}(\mu)$ such that $\tau^* \simeq \tau$. Theorem 1 now implies that there exists $f \in \mathrm{NE}(\mathscr{G})$ and $\tau^* = \lambda(\mathscr{G}, f)^{-1} \simeq \tau$.

We now turn to the case with atoms. Assume that μ has atoms. As T is countable, μ is atomless if and only if $\mu_{\mathscr{U}_{(A,T)}}$ is atomless. Let $\mathscr{U}_{(A,T)} = \mathscr{U}_0 \cup \mathscr{U}_1$, where $\mathscr{U}_0$ is the atomless part and $\mathscr{U}_1$ is the set of atoms of $\mu_{\mathscr{U}_{(A,T)}}$. Write $\tau = \tau^0 + \tau^1$, where τ^0 and τ^1 are, respectively, the restrictions of τ to $T \times \mathscr{U}_0 \times A$ and $T \times \mathscr{U}_1 \times A$. Denote $\alpha \equiv \mathscr{G}_1$ and $v \equiv \mathscr{G}_2$.

Let $I_0 = \{i \in I : v(i) \in \mathscr{U}_0\}$ and $I_1 = \{i \in I : v(i) \in \mathscr{U}_1\}$. Let λ_0 and λ_1 be the restrictions of λ to I_0 and I_1. Since a measure space has countable number of atoms, $\mathscr{U}_1 = \{u_j^* : j \in J\}$, where J is countable. As T is countable, we can write $T = \{t_k : k \in K\}$, where K is countable. Further partition I_1 to sets I_1^{jk}, where $I_1^{jk} = \{i \in I_1 : v(i) = u_j^*, \alpha(i) = t_k\}$, $j \in J$, $k \in K$. Fix any $j \in J$ and $k \in K$, and define a measure σ^{jk} on A by $\sigma^{jk}(P) = \tau^1(\{t_k\} \times \{u_j^*\} \times P)$, where $P \in \mathscr{B}(A)$. Let $B_j = \{a \in A : u_j^*(a, \tau_{T\times A}) \geq u_j^*(x, \tau_{T\times A})$ for all $x \in A\}$. The B_j is a closed set and since $\tau(B(\tau)) = 1$, σ^{jk} is concentrated on B_j. It is easy to verify that $\sigma^{jk}(B_j) = \lambda_1(I_1^{jk})$. By rescaling λ_1 on I_1^{jk} and σ^{jk}, one obtains a measurable function $z_{jk} : I_1^{jk} \longrightarrow B_j$ such that $\lambda_1 z_{jk}^{-1} = \sigma^{jk}$. Define $z : I_1 \longrightarrow A$ by $z(i) = z_{jk}(i)$ if $i \in I_1^{jk}$. We will verify that $\tau^1_{T\times A} = \lambda_1(\alpha, z)^{-1}$.

Let C be a Borel subset of $T \times A$. Let $K' = \{k \in K : (t_k, a) \in C$ for some $a \in A\}$ and $\{t_k : k \in K'\}$ be the projection of C on T. Let $C_k = \{a : (t_k, a) \in C\}$ for $k \in K'$:

$$\begin{aligned}
\tau^1_{T\times A}(C) &= \sum_{k\in K'} \tau^1_{T\times A}\bigl(\{t_k\} \times C_k\bigr) = \sum_{k\in K'} \tau^1\bigl(\{t_k\} \times \mathscr{U}_1 \times C_k\bigr) \\
&= \sum_{k\in K'} \sum_{j\in J} \tau^1\bigl(\{t_k\} \times \bigl\{u_j^*\bigr\} \times C_k\bigr) \\
&= \sum_{k\in K'} \sum_{j\in J} \sigma^{jk}(C_k) = \sum_{k\in K'} \sum_{j\in J} \lambda_1 z_{jk}^{-1}(C_k) \\
&= \lambda_1(\alpha, z)^{-1}(C).
\end{aligned}$$

We now construct a game $\mathscr{H}$, related to $\mathscr{G}$, which has an atomless distribution. By Bogachev (2007, Proposition 9.1.11), there is a measurable $g : I \longrightarrow [0, 1]$ such that $\ell = \lambda g^{-1}$. For $r \in [0, 1]$, let $C^*(r)$ denote the function in $\mathscr{U}_{(A,T)}$ that is identically r on $A \times \mathscr{M}(T \times A)$. Define $w : I \longrightarrow \mathscr{U}_{(A,T)}$ as

$$w(i) = \begin{cases} v(i) & \text{if } i \in I_0, \\ C^*\bigl(g(i)\bigr) & \text{if } i \in v^{-1}\bigl(\bigl\{C^*(0)\bigr\}\bigr) \text{ and } C^*(0) \text{ is an atom of } \mu_{\mathscr{U}_{(A,T)}}, \\ g(i)\dfrac{v(i)}{\|v(i)\|} & \text{otherwise.} \end{cases}$$

It is clear that w is measurable and $\|w(i)\| = g(i)$ on I_1. Now define $\mathscr{H} : I \longrightarrow T \times \mathscr{U}_{(A,T)}$ as $\mathscr{H}(i) = (\alpha(i), w(i))$ for all $i \in I$.

Let $\mu' = \lambda\mathscr{H}^{-1}$. Using the fact that λg^{-1} is atomless, it is easy to check that the LDG μ' is atomless. From the construction of $\mathscr{H}$, for each $i \in I_1$, $(\mathscr{H}(i), z(i)) \in B(\tau)$. Consider the mappings $\mathscr{H}$ and z from I_1, and let $\rho^1 = \lambda_1(\mathscr{H}, z)^{-1}$ and $\rho = \tau^0 + \rho^1$. The marginal of ρ on $T \times \mathscr{U}_{(A,T)}$ is μ' as $\mu' = \lambda\mathscr{H}^{-1} = \lambda_0\mathscr{G}^{-1} + \lambda_1\mathscr{H}^{-1} = \tau^0_{\mathscr{U}_{(A,T)}} + \rho^1_{\mathscr{U}_{(A,T)}} = \rho_{\mathscr{U}_{(A,T)}}$. We now show that ρ is an NED of μ'.

Note that $\rho_{T\times A} = \tau^0_{T\times A} + \rho^1_{T\times A} = \tau^0_{T\times A} + \lambda_1(\alpha, z)^{-1} = \tau^0_{T\times A} + \tau^1_{T\times A} = \tau_{T\times A}$. Thus, $B(\rho) = B(\tau)$. Therefore, to show that ρ is an NED of μ', it suffices to show that $\rho(B(\tau)) = 1$. Since $\rho^1 = \lambda_1(\mathscr{H}, z)^{-1}$, $\rho^1(B(\tau)) = \lambda_1(\{i \in I_1 : (\mathscr{H}(i), z(i)) \in B(\tau)\}) = \lambda(I_1)$. Moreover, $\tau^0(B(\tau)) = \tau(B(\tau) \cap (T \times \mathscr{U}_0 \times A)) = \tau(T \times \mathscr{U}_0 \times A) = \tau_{\mathscr{U}_{(A,T)}}(\mathscr{U}_0) = \lambda(I_0)$. So $\rho(B(\tau)) = \tau^0(B(\tau)) + \rho^1(B(\tau)) = \lambda(I_0) + \lambda(I_1) = 1$. Thus, $\rho \in \mathrm{NED}(\mu')$.

Therefore, we can now appeal to the argument in the first paragraph of the proof to assert that there exists $f \in \mathrm{NE}(\mathscr{H})$ such that $\lambda(\mathscr{H}, f)^{-1} \simeq \rho$ since μ' is atomless. From the construction, it is obvious that $\mathscr{H}$ and $\mathscr{G}$ have the same set of NE. So $f \in \mathrm{NE}(\mathscr{G})$. Let $\tau^* = \lambda(\mathscr{G}, f)^{-1}$. By Lemma 1, we know that $\tau^* \in \mathrm{NED}(\mu)$. Furthermore, since it follows from the construction that $\tau^*_{T\times A} = \lambda(\alpha, f)^{-1} = \rho_{T\times A} = \tau_{T\times A}$, we also have that $\tau^* \simeq \tau$. We have now proved that (i) implies (ii).

We next show that (ii) implies (i). Suppose (i) does not hold. Then either A or T is uncountable. If A is uncountable, then one can always construct a counterexample (see, for example, Keisler and Sun 2009) in which an LIG having A as its action set does not have any equilibrium. So A must be countable. If T is uncountable, then one can construct a counterexample (see, for example, Qiao et al. 2016) in which an LIG having T as its space of traits does not have any equilibrium. This suggests that T must be countable too. The proof is now complete. □

Proof of Theorem 3. Let $\mathscr{G} \in \mathrm{Meas}(I; T \times \mathscr{U}_{(A,T)})$ be an LIG that saturatedly represents an LDG μ, and let τ be an NED of μ. Since $\lambda\mathscr{G}^{-1} = \mu = \tau_{T\times\mathscr{U}_{(A,T)}}$, the fact that $(I, \mathscr{I}, \lambda)$ has the saturated property for τ implies that $\tau = \lambda(\mathscr{G}, f)^{-1}$ for a measurable function $f : I \longrightarrow A$. Lemma 1 yields that f is an NE of $\mathscr{G}$. □

Proof of Lemma 3. Suppose that $\tau \in \mathrm{SNED}(\mu)$. Then as in the proof of Theorem 1, one can construct $f \in \mathrm{NE}(\mathscr{G})$ such that $\tau = \lambda(\mathscr{G}, f)^{-1}$. It is easy to verify that f in that proof is $\sigma(\mathscr{G})$-measurable.

Next suppose that $\tau = \lambda(\mathscr{G}, f)^{-1}$ for a $\sigma(\mathscr{G})$-measurable NE f of $\mathscr{G}$. Lemma 1 shows that $\tau \in \mathrm{NED}(\mu)$. From Aliprantis and Border (2006, Theorem 4.41), $f = h \circ \mathscr{G}$ for some measurable function h from $T \times \mathscr{U}_{(A,T)}$ to A. (Note that we have replaced the range space $\mathbb{R}$ in the cited result by A, because every Borel subset of a complete separable metric space is isomorphic to a Borel subset of the Cantor set; see Parthasarathy 1967, Theorem I.2.3.) It remains to show that $\tau(\text{graph of } h) = 1$. Since $\tau = \lambda(\mathscr{G}, f)^{-1}$, $\tau(\text{graph of } h) = \lambda(\mathscr{G}, f)^{-1}(\text{graph of } h) = \lambda(\{i \in I : (\mathscr{G}(i), f(i)) \in \text{graph of } h\}) = \lambda(\{i \in I : (\mathscr{G}(i), h(\mathscr{G}(i))) \in \text{graph of } h\}) = \lambda(\{i \in I : \mathscr{G}(i) \in T \times \mathscr{U}_{(A,T)}\}) = 1$. So $\tau \in \mathrm{SNED}(\mu)$. □

Proof of Lemma 4. Let L' be a Borel subset of $[0, 1]$ such that $\ell(L') = 1$ and $\mathscr{G}$ is one-to-one on L'. We can assume without loss of generality that both $\mathscr{G}$ and f are constant on $L \setminus L'$. If f is an NE of $\mathscr{G}$, then Lemma 1 shows that $\ell(\mathscr{G}, f)^{-1} \in \mathrm{NED}(\mu)$. We will show that it is symmetric. For a Borel subset B of A, let $C = f^{-1}(B) \cap L'$. Since f is measurable, C is measurable. By Parthasarathy (1967, Theorem I.3.9), $\mathscr{G}(C)$ is a Borel subset of $T \times \mathscr{U}_{(A,T)}$, so $C \in \sigma(\mathscr{G})$. Lemma 3 implies that $\ell(\mathscr{G}, f)^{-1} \in \mathrm{SNED}(\mu)$. □

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Co-editor Faruk Gul handled this manuscript.

Manuscript received 24 March, 2014; final version accepted 4 July, 2016; available online 7 July, 2016.