Second Order Beliefs Models of Choice under Imprecise Risk:

Non-additive Second Order Beliefs vs. Nonlinear Second Order Utility*

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Abstract

This paper discusses models of choice under imprecise objective probabilistic information featuring beliefs about beliefs – second order beliefs. A new model, called Second Order Dual Expected Utility (SODEU) featuring non-additive second order beliefs is introduced, axiomatized and systematically contrasted with the leading alternative model of this kind, the Second Order Subjective Expected Utility (SOSEU) model (Klibanoff, Marinacci, and Mukerji, 2005; Nau, 2006; Seo, 2009) for which, for the sake of comparison, we provide a new axiomatization, dispensing with the complex constructs used in extant axiomatizations. Ambiguity attitude and attitude toward information in general are discussed and characterized.

Keywords: Imprecise probabilistic information, second order beliefs, non-additive probabilities, ambiguity aversion, Ellsberg paradox, Choquet integral.

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1 Introduction

This paper discusses models of choice under imprecise objective probabilistic information featuring beliefs about priors – second order beliefs, when preferences are defined on act-information pairs $(f, \mathscr{P})$, f mapping some state space S to some outcome space X and $\mathscr{P}$ being a set of priors on S summarizing the information available to the decision maker. We axiomatize a new functional to represent such preferences, *Second Order Dual Expected Utility (SODEU)*, defined as follows:

$$V_{SODEU}(f, \mathscr{P}) = \int_{\mathscr{P}} \int_S u \circ f \mathrm{d}P \ \mathrm{d}\nu^{\mathscr{P}}(P),$$

where u is a utility function on X, $\nu^{\mathscr{P}}$ is a nonadditive probability or *capacity*, defined on the closed convex hull of $\mathscr{P}$ and modeling second-order beliefs, and the outer integral is a Choquet integral.[1]

SODEU is similar to a version in the act-information setting of a more familiar model, Second Order Subjective Expected Utility (SOSEU)(Klibanoff et al., 2005; Nau, 2006; Seo, 2009; Rustichini, 1992), where preferences are represented by:

$$V_{SOSEU}(f, \mathscr{P}) = \int_{\mathscr{P}} \Phi \left(\int_S u \circ f \, \mathrm{d}P \right) \mathrm{d}\mu^{\mathscr{P}}(P),$$

where $\Phi : u(X) \to u(X)$ is nondecreasing and $\mu^{\mathscr{P}}$ is a probability measure on $\mathscr{P}$.

Both models are particular cases of the *Second Order Choquet Expected Utility (SOCEU)* model, whereby act-information pairs $(f, \mathscr{P})$ are evaluated by

$$V_{SOCEU}(f, \mathscr{P}) = \int_{\mathscr{P}} \Phi \left(\int_S u \circ f \, \mathrm{d}P \right) \mathrm{d}\nu^{\mathscr{P}}(P),$$

where $\Phi : u(X) \to u(X)$ is nondecreasing and $\nu^{\mathscr{P}}$ is a capacity on $\mathscr{P}$. In SOSEU the capacity is additive whereas in SODEU Φ is affine. In that sense, SODEU is dual to SOSEU,[2] hence its name. The second contribution of this paper is to make this duality explicit at the axiomatic level by providing a new axiomatization of SOSEU.[3]

[1] Both concepts of capacity and Choquet integral are formally defined in section 3.1.1.

[2] The use of the term dual here should not be understood as referring to an underlying theory of duality but to an analogy with the relationship between Expected Utility and Yaari's Dual theory of choice under risk (Yaari, 1987). Strictly speaking this duality concept has been developed for choice under risk, and, to the best of my knowledge, there is no agreement on what would be the analogue of the dual theory under uncertainty; therefore one might allow oneself a certain amount of freedom in using the term dual.

[3] Axiomatizing SOCEU in the same framework would require addressing very difficult and open technical problems regarding the extension of comonotonic additive functionals, so we leave this for further research.

In a nutshell, SOSEU satisfies (second order) independence but not reduction of compound lotteries while SODEU satisfies reduction but not independence.

Why should we be interested in the domain of act-information pairs rather than the standard domain of preferences, however? Why should we care about SODEU besides SOSEU? Why do we need a new axiomatization of SOSEU? Let us address these questions.

1.1 Objective Information in Decision Theory

In many decision making situations, people have *some* objective information about decision-relevant events (past cases, advice from experts, and so on), and ignoring this while modeling the decision maker's behavior may have undesirable consequences (Giraud and Tallon, 2010). For instance, this leads to an interpretation problem with the Maxmin Expected Utility model of decision under ambiguity (Gilboa and Schmeidler, 1989): because in this model acts are evaluated by taking the minimum expected utility with respect to a purely subjective set of priors, it is often interpreted as characterizing pessimistic behavior. However, the validity of this interpretation heavily relies on construing the set of priors as incorporating *all* information available to the decision maker: in this case, the minimum expected utility is truly the (*ex ante*) worst case scenario for this decision maker, and arguably basing one's decision on the worst case scenario is very pessimistic indeed. If the subjective set of priors is actually much smaller than the maximal one compatible with the available information, though, then taking the minimum expected utility with respect to this smaller set is not that pessimistic after all. Ghirardato and Marinacci (2002) have furthermore shown that in this model a decision maker is more ambiguity averse than another if and only if his set of priors is larger. But this makes sense only if the two decision makers have the same information: otherwise, one decision maker may have a larger set of priors because he *knows less* than another one, even if he is less ambiguity averse. Ambiguity aversion is a subjective trait that has nothing to do in principle with objective information.

In applications, the set of priors may or may not be regarded as subjective, depending on the problem at hand, but in any case, available information is usually not explicitly made clear. Therefore, the comparison of the size of the sets of priors cannot safely be interpreted as revealing more pessimism.[4] To do this we need to

[4]Note that Ghirardato, Maccheroni, and Marinacci (2004)'s axiomatization of a generalization of Gilboa and Schmeidler (1989)'s model, designed to differentiate ambiguity and ambiguity attitude, cannot completely succeed in settling the controversy. Indeed, even though it is immune from the accusation of excessive pessimism, it allows only (as the authors make perfectly clear) to differentiate between *revealed* or *perceived* ambiguity and attitude toward revealed ambiguity.

know where the set of priors comes from and how it relates to available information, and we cannot know this in the context of fully subjective axiomatizations for lack of a sufficiently rich setup. A similar problem arises in the Invariant Biseparable Model (Ghirardato et al., 2004). In this model, revealed beliefs are represented by a purely subjective set function called the willingness to bet. However, just as in Savage's model, in the standard framework there is no explicit link between these beliefs and the objective information available to the decision maker. Therefore, it is impossible in this model to distinguish information and attitude toward it; hence, notions like comparative ambiguity aversion, characterized through comparisons of the capacities of different decision makers, are likely to confound the possible discrepancies in the information available to each of them and attitude with respect to it.

A series of papers (Gajdos, Tallon, and Vergnaud, 2004; Gajdos, Hayashi, Tallon, and Vergnaud, 2008) have systematically addressed this interpretation problem for the Maxmin model by providing new foundations for it when preferences are defined over act-information pairs. We shall do the same for a model that is essentially the Invariant Biseparable Model (Ghirardato et al., 2004).

By incorporating objective information, the act-information setting allows for a clear separation between objective information and attitude toward it. To illustrate this point, consider Epstein (2010)'s claim that one of SOSEU's fundamental features and flaws is nonreduction of timeless compound lotteries. Epstein considers a decision maker facing two urns. One urn, called the second order urn, contains three balls, labeled r, b or g. A draw from this urn determines the composition of the other urn, the first order urn, a slightly modified three-color Ellsberg urn also containing three balls, one red and the other two either blue or green. If a ball labeled r is drawn from the second order urn, then there are one blue and one green ball in the first order urn; if the ball is labeled b, then there are two blue balls, and if it is labeled g then there are two green balls. The induced distributions on the set $S = \{R, B, G\}$ are thus

$$P_r = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right), \quad P_b = \left(\frac{1}{3}, \frac{2}{3}, 0\right) \quad \text{and} \quad P_g = \left(\frac{1}{3}, 0, \frac{2}{3}\right).$$

Define $\mathscr{P} = \{P_r, P_b, P_g\}$. Epstein discusses two situations, Situations I and II, where the decision maker behaves according to the SOSEU model with parameters (u^I, Φ^I, μ^I) and $(u^{II}, \Phi^{II}, \mu^{II})$ respectively, where μ^I and μ^{II} have support in $\mathscr{P}$. In Situation I, the decision maker knows nothing more than what was just described. Assuming by symmetry that $\mu^I_b = \mu^I_g$ (with $\mu^I_d := \mu^i(\{P_d\})$, $d \in \{r, b, g\}$), and normalizing u^I so that $u^I(0) = 0$ and $u^I(100) = 1$, preference for betting on red over

betting on blue in Situation I implies :

$$\Phi^I\left(\frac{1}{3}\right) > \mu_r^I \Phi^I\left(\frac{1}{3}\right) + \frac{1-\mu_r^I}{2}\Phi^I\left(\frac{2}{3}\right) + \frac{1-\mu_r^I}{2}\Phi^I(0). \tag{1}$$

In Situation II, the decision maker is told in addition that μ^I is actually the *true* second order distribution. Does this piece of information have any impact on behavior? There are two points of view, depending on whether we think that objective information affects the decision maker's beliefs or not.

According to the first point of view, which is Epstein's, "We would expect the announcement not to change risk preferences or preferences over acts defined *within* the second-order urn, nor to cause the individual to change his beliefs about that urn" (Epstein, 2010, p. 2094), hence $(u^I, \Phi^I, \mu^I) = (u^{II}, \Phi^{II}, \mu^{II})$ and equation (1) still holds in Situation II with $(u^{II}, \Phi^{II}, \mu^{II})$ instead of (u^I, Φ^I, μ^I): reduction of objective compound lotteries does not take place in Situation II.

From a different point of view (*e.g.* Klibanoff, Marinacci, and Mukerji, 2012, a reply to Epstein), however, one could argue that, although it makes perfect sense to assume that the *taste* parameters u and Φ are not affected by a change in objective information, so that Epstein's argument applies to them, it is more of a stretch to assume that information does not affect *beliefs*. The justification for the first point of view is that, if a decision maker behaves according to subjective expected utility (as it is the case here at the second order level) and if he learns that his subjective prior happens to coincide with the objective probability distribution, he has no reason to change it. But that doesn't mean he will *use* this prior in the same way if he *believes* it is right or if he *knows* it is. For instance, if the second order prior is possibly wrong but the first order prior is objective, the decision maker might consider that by reducing the compound lottery he might so to speak "contaminate" the latter with the possible falseness of the former. If both are objective, he can go ahead and proceed with the reduction without fear. If he does in Situation II, he will thus consider that the new objective information set is the singleton set $\{\rho^{\mu^I}\}$, where

$$\rho^{\mu^I} = \mu_r^I P_r + \frac{1-\mu_r^I}{2}P_b + \frac{1-\mu_r^I}{2}P_g = P_r.$$

The support of μ^{II} is thus $\{P_r\}$,[5] and

$$V^{II}(f_B, \{P_r\}) = \Phi^I\left(\frac{1}{3}\right) = \mu_r^{II}\Phi^I\left(\frac{1}{3}\right) + \frac{1-\mu_r^{II}}{2}\Phi^I\left(\frac{2}{3}\right) + \frac{1-\mu_r^{II}}{2}\Phi^I(0) = V^{II}(f_R, \{P_r\}).$$

The claim that there is an essential contradiction between the reduction of compound lotteries and SOSEU is therefore not valid without some extra assumption.

[5] μ^{II} is thus degenerate, as pointed out in Klibanoff et al. (2012).

When both stages of the compound lottery are *objective*, reduction can perfectly take place.

What is going on here? In the standard framework, the distinction between objective and subjective priors cannot formally be made, and thus a different treatment of them does not readily suggest itself to the modeler. On the other hand, in our framework, the distinction is not only formally possible but it can also be operationalized by implementing the different attitudes one may have regarding objective and subjective beliefs. To sum up, what our framework makes possible is to distinguish between the mathematical nature of a prior (which probability measure it is) and its cognitive status (is it objective information or a purely subjective belief?) and this additional distinction allows for a different treatment of this prior in the decision making process.

Another advantage of the act-information setting is that it makes it possible to distinguish two possible attitudes toward uncertainty (Gajdos et al., 2008): attitude toward ambiguity and attitude toward information. Attitude toward ambiguity is defined in the context of *fixed*, albeit imprecise (and generally unspecified), information on the likelihood of events. It is usually assessed by the ranking of more or less ambiguous *acts*. Thus, we might say that ambiguity is a property of acts, given an informational context. Attitude toward information itself, on the other hand, is assessed by the comparison of various pieces of information, *given a fixed act*. This allows us to study various properties of information described as sets of probability measures, such as how precise it is, given an act, and attitudes toward such properties, such as imprecision aversion. A contribution of this paper is to systematically explore this distinction in the context of the SODEU model. We will show that the second order capacity can be used to model attitude toward ambiguity, whereas the decision maker's willingness-to-bet can be used to model imprecision aversion.

1.2 Descriptive Motivation

A lot of the appeal of SOSEU to economists outside the field of decision theory comes from the fact that, under standard assumptions, the SOSEU functional is smooth, allowing one to apply the standard expected utility machinery to ambiguity (e.g., Gollier, 2011). However, smoothness limits its descriptive adequacy. For instance, Ahn, Choi, Gale, and Kariv (2011) estimate several models of choice under uncertainty on a rich data set of portfolio choices between ambiguous and risky assets and find that "approximately sixty percent of subjects are well approximated by SEU preferences" while the remainder of subjects exhibit ambiguity aversion of a sort

that "the smooth specification associated to REU[6] cannot [explain]."(Ahn et al., 2011, p. 25). Of course, one experimental study is not enough to discard smooth specifications of SOSEU altogether. But it suggests that it is perfectly sound to examine non-smooth alternative theories of second order beliefs, like SODEU.

Moreover, as shown in section 3.2.2, Example 1, SOSEU cannot account for certain ambiguity aversion-driven choices in a modified Ellsberg paradox. The construction of this modified Ellsberg paradox parallels the construction of modified Saint-Petersburg paradoxes. It is well known that, even though we can solve the latter by introducing a concave utility function for money, for any increasing and continuous Bernoulli utility function u, a new Saint-Petersburg paradox can be constructed by replacing the 2^n payoff by $u^{-1}(2^n)$. Similarly, even though the SOSEU model can account for the standard Ellsberg paradox, for any given (continuous and increasing) second order utility Φ, there exists a rescaling of first order expected utilities leading to a new Ellsberg paradox that it cannot account for. As will be made clear later in the paper, this happens because the SOSEU model satisfies a form of independence, called in the paper Second Order Independence.

1.3 Technical Motivation: first order versus second order objects

At a more technical level, our axiomatization has the advantage over extant ones (Klibanoff et al., 2005; Seo, 2009; Nau, 2006) of being based on standard first-order Anscombe-Aumann acts[7] only. We do not require the decision maker to have preferences on other somewhat complex objects like lotteries over Anscombe-Aumann acts (Seo, 2009) or conditional acts (Nau, 2006), and in particular on second order acts (Klibanoff et al., 2005).

While this might look like anecdotal, it actually has behavioral implications regarding second order versions of the three-color Ellsberg paradox, Epstein (2010)'s "Paradox for the 'smooth ambiguity' model". Epstein's argument to derive it relies on an assumption made by Klibanoff et al. (2005) about preferences on second order acts, and not from the SOSEU functional form itself, hence one may ask: does the paradox persist without this assumption? Axiomatizing SOSEU in a first-order context, dispensing with this assumption, is a first step toward answering this

[6]REU is the name given by Ahn et al. to SOSEU.

[7]We should distinguish between the *standard* Anscombe-Aumann approach, as standardized by Fishburn (1970), and the *original* Anscombe-Aumann approach, featuring lotteries on Anscombe-Aumann acts, as revived in Seo (2009) to provide a clean axiomatization of SOSEU and further and fruitfully exploited in Nascimento and Riella (2011) to generalize SOSEU.

question; we show in section 3.2.3 that while the paradox in the form proposed by Epstein does not follow from SOSEU *per se*, it still holds under slight perturbations.

1.4 Related Literature

The idea of generalizing the notion of risk, i.e., known probability distributions, to some notion of imprecise risk, i.e., imprecisely known probability distributions, dates back to Jaffray (1989), who applied the Von Neumann and Morgenstern (1947) axioms to belief functions[8] instead of lotteries to obtain a representation *à la* Arrow-Hurwicz (Arrow and Hurwicz, 1972). This idea was recently revived by several authors who proposed to model imprecise risk either by necessity measures (Rébillé, 2006) or by sets of lotteries (Stinchcombe, 2007; Olszewski, 2007; Ahn, 2008).

Our approach differs from the above mentioned in that we consider act-information pairs and do not assume that the decision maker is indifferent between two act-information pairs inducing the same set of lotteries. This approach originates in Wang (2003) where primitives are triples $(f, \mathscr{P}, P^*)$ with P^* a reference prior. Wang's main result is to provide axiomatic foundations for a general version of the minimum relative entropy principle of Anderson, Hansen, and Sargent (1999). In the same setting, Gajdos et al. (2004) axiomatize a generalized maxmin rule whereby the decision maker maximizes the minimum expected utility over a subset of the set of initial priors, the so-called *contraction model*. Such a rule is a special case of our model if and only if this subset is the core of a convex second order capacity; not all contraction representations have SODEU representations. Gajdos et al. (2008) generalize Gajdos et al. (2004) by endogenizing the reference prior.

Nehring (2007, 2009) studies a related framework where, along with the usual preference relation, the decision maker is endowed with a (potentially incomplete) comparative likelihood relation modeling his/her beliefs. He investigates the compatibility of revealed betting preferences with the decision maker's beliefs represented by the comparative likelihood relation. In turn, Sagi (2007) studies a form of multi-prior probabilistic sophistication that can be seen as the subjective counterpart to Nehring's objective approach.

Amarante (2009) studies the functional form that we call here SODEU, and we shall elaborate on the relationship with the present paper later on. Finally, the present paper has evolved out of a first attempt at axiomatizing the SODEU functional (Giraud, 2005).

[8] They roughly speaking correspond to sets of lotteries

1.5 Organization of the Paper

The paper is organized as follows: section 2 introduces the setup and the basic axioms, while in section 3 additional axioms and representation theorems for SODEU and SOSEU are stated and discussed. In section 4 we study ambiguity aversion and imprecision aversion in this setup. Section 5 contains concluding remarks. Proofs are gathered in the appendix.

2 The Model

2.1 Setup and Basic Definitions

The setup is Fishburn (1970)'s version of the Anscombe-Aumann framework. (S, Σ) is the measurable space of states of nature and $(X, \mathscr{B})$ the measurable space of outcomes. X is a convex subset of a vector space and $\mathscr{B}$ contains the singletons.

Let $\mathscr{F}$ be the set of simple acts, i.e., the set of finite-valued measurable functions f from S to X. As usual, we identify constant acts with elements of X. For $\alpha \in [0, 1]$, the α-mixture of two acts f and g is defined pointwise: $(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s)$. If $(A_i)_{i=1,\ldots,n}$ is a measurable partition of S , we write $f = (x_i, A_i)_{i=1,\ldots,n}$ whenever $f(s) = x_i$ for all $s \in A_i$.

Let $pc(\Sigma)$ be the set of all probability charges (finitely additive and normalized set functions) on Σ and $\mathfrak{P}$ be the set of all nonempty subsets $\mathscr{P}$ of $pc(\Sigma)$. Elements of a set $\mathscr{P} \in \mathfrak{P}$ will be referred to as *scenarios*. Here they will often play the role that states usually play in models of decision under uncertainty: for instance dominance will be considered scenario-wise rather than state-wise. In that case, we say that we are dealing with second order concepts.

An act-information pair $(f, \mathscr{P}) \in \mathscr{F} \times \mathfrak{P}$ corresponds to a situation where the decision maker considers choosing act f while $\mathscr{P}$ is the maximal set of priors consistent with the information available to him/her. Preferences are defined over act-information pairs and are denoted by $\succsim$. We refer to Gajdos et al. (2008) and section 1.1 for a justification of this setup.

2.2 Basic Axioms

In this section we introduce the axioms shared by both models we seek to axiomatize. We start with the standard ordering, continuity and nondegeneracy axioms, adapted to our setting:

Axiom 1 (Weak Order) $\succsim$ *is transitive and complete.*

Axiom 2 (Continuity) *For all $\mathscr{P} \in \mathfrak{P}$, for all $f, g, h \in \mathscr{F}$, if*

$$(f, \mathscr{P}) \succ (g, \mathscr{P}) \succ (h, \mathscr{P})$$

then there exist $\alpha, \beta \in (0, 1)$ such that:

$$(\alpha f + (1 - \alpha)h, \mathscr{P}) \succ (g, \mathscr{P}) \succ (\beta f + (1 - \beta)h, \mathscr{P}).$$

Axiom 3 (Non-Degeneracy) *There exists x^* and x_* in X such that for all $\mathscr{P} \in \mathfrak{P}$, $(x^*, \mathscr{P}) \succ (x_*, \mathscr{P})$.*

The next axiom is a dominance axiom saying that if f is preferred to g given all scenarios in $\mathscr{P}$, then it is preferred to g given $\mathscr{P}$. Formally:

Axiom 4 (Information Dominance) *For all $f, g \in \mathscr{F}$, for all $\mathscr{P} \in \mathfrak{P}$,*

$$[\forall P \in \mathscr{P}, (f, \{P\}) \succsim (g, \{P\})] \quad \Longrightarrow \quad (f, \mathscr{P}) \succsim (g, \mathscr{P}).$$

For simplicity we will require the existence of certainty equivalents.

Axiom 5 ($\mathscr{P}$-Certainty Equivalent) *For all $\mathscr{P} \in \mathfrak{P}$, for all $f \in \mathscr{F}$, there exists $x \in X$ such that $(f, \mathscr{P}) \sim (x, \mathscr{P})$.*

Given the first three axioms this axiom holds if preferences satisfy state-wise dominance (as opposed to scenario-wise). However, we do not assume this form of dominance since it is not needed for the axiomatizations we seek. For all $f \in \mathscr{F}$ and $\mathscr{P} \in \mathfrak{P}$, denote by $c(f, \mathscr{P})$ the set of certainty equivalents of f under objective information $\mathscr{P}$. We may sometimes abuse this notation by using it to denote *any* certainty equivalent of f under $\mathscr{P}$ when there is no need to distinguish between distinct certainty equivalents.

Finally, in both models information is irrelevant for constant acts.

Axiom 6 (Information Irrelevance for Constant Acts) *For all $x \in X$, for all $\mathscr{P}, \mathscr{Q} \in \mathfrak{P}$, $(x, \mathscr{P}) \sim (x, \mathscr{Q})$.*

3 Representations

3.1 Second Order Dual Expected Utility (SODEU)

3.1.1 Representation Concept

Recall the following definitions. Let $(\Omega, \mathscr{E})$ be a measurable space. A *capacity* on $(\Omega, \mathscr{E})$ is a function $\nu : \mathscr{E} \to \mathbb{R}$ such that $\nu(\varnothing) = 0$, $\nu(\Omega) = 1$ and $\nu(A) \leq \nu(B)$

whenever $A \subseteq B$. A mapping $\varphi : \Omega \to \mathbb{R}$ is *$\mathscr{E}$-measurable* if, for all $t \in \mathbb{R}$, $(\varphi \geq t) := \{\omega \in \Omega \mid \varphi(\omega) \geq t\} \in \mathscr{E}$. For any $\mathscr{E}$-measurable functions φ, the *Choquet integral* of φ with respect to ν is defined by:

$$\int_\Omega \varphi \, \mathrm{d}\nu := \int_{-\infty}^{0} [\nu(\varphi \geq t) - 1] \, \mathrm{d}t + \int_{0}^{+\infty} \nu(\varphi \geq t) \, \mathrm{d}t. \tag{2}$$

Definition 1 *A binary relation $\succsim$ on $\mathscr{F} \times \mathfrak{P}$ admits a* Second Order Dual Expected Utility (SODEU) representation *based on the utility function*[9] *$u : X \to \mathbb{R}$ and the family of capacities $(\nu^{\mathscr{P}})_{\mathscr{P} \in \mathfrak{P}}$, where $\nu^{\mathscr{P}}$ is defined on the closed convex hull $\overline{\mathrm{co}}(\mathscr{P})$ of $\mathscr{P}$ if*

$$(f, \mathscr{P}) \succsim (g, \mathscr{Q}) \iff \int_{\overline{\mathrm{co}}(\mathscr{P})} \int_S u \circ f dP \; d\nu^{\mathscr{P}}(P) \geq \int_{\overline{\mathrm{co}}(\mathscr{Q})} \int_S u \circ g dQ \; d\nu^{\mathscr{Q}}(Q).$$

According to SODEU, given some imprecise information objectively describable by a set of probability scenarios, the decision maker forms a not necessarily additive prior regarding the likelihood of each of the scenarios, computes the (Choquet) weighted average expected utility of the acts considered and chooses the act with higher average expected utility. Whenever the second order capacity is actually non-additive, this decision procedure is consistent with an intuitive account of the Ellsberg paradox according to which ambiguity aversion is a form of second order pessimism: the decision maker deems unfavorable scenarios more likely than favorable ones.

3.1.2 Axioms

We now introduce the axioms specific to SODEU preferences.

The first axiom restricts the possibility of hedging between acts to certain kinds of acts. Let us first introduce the relevant notion of mixing and the relevant category of acts for which hedging is useless.

Definition 2 *Let $f, g \in \mathscr{F}$, $\alpha \in [0, 1]$ and $\mathscr{P} \in \mathfrak{P}$. Then $h \in \mathscr{F}$ is a $\mathscr{P}$-*second order α-mixture *of f and g if, for all $P \in \mathscr{P}$, there exists $x, y \in X$ such that*

(i) $(f, \{P\}) \sim (x, \{P\})$,

(ii) $(g, \{P\}) \sim (y, \{P\})$,

(iii) $(h, \{P\}) \sim (\alpha x + (1 - \alpha) y, \{P\})$.

[9] Nothing is assumed here of u since we want our definition of SODEU to be general and not tied to a particular framework.

The (possibly empty) set of $\mathscr{P}$-second order α-mixtures of f and g is denoted $\alpha f \oplus_{\mathscr{P}} (1-\alpha)g$. The name of this concept is based on the fact that, given the basic axioms:

$$h \in \alpha f \oplus_{\mathscr{P}} (1-\alpha)g \text{ iff } c(h, \{P\}) = \alpha c(f, \{P\}) + (1-\alpha)c(g, \{P\}), \forall P \in \mathscr{P}.$$

Hence, a second order mixture of two acts may be regarded as their scenario-wise mixture.

Now, let us define the class of acts for which these mixtures provide no hedging gain:

Definition 3 *Let $f, g \in \mathscr{F}$ and $\mathscr{P} \in \mathfrak{P}$. f and g are* $\mathscr{P}$-comonotonic *if*

$$(f, \{P\}) \succ (f, \{Q\}) \quad \Longrightarrow \quad (g, \{P\}) \succsim (g, \{Q\}), \quad \forall P, Q \in \mathscr{P}.$$

The underlying intuition is that: each act induces a certain ranking of probabilistic scenarios based on how favorable a scenario is for this act. For instance, in the three color Ellsberg urn with 30 red balls and 60 black or yellow balls, if f_B is the act corresponding to betting on black, if $P = (\frac{1}{3}, \frac{2}{3}, 0)$ and $Q = (\frac{1}{3}, 0, \frac{2}{3})$, then $(f_B, \{P\}) \succ (f_B, \{Q\})$. Two acts are $\mathscr{P}$-comonotonic if the ranking of scenarios they induce are the same up to indifference.

Comonotonicity was introduced in the literature on decision under uncertainty for states of nature, not for scenarios. Given an information set $\mathscr{P}$, two acts f and g are *comonotonic* if, for all $s, s' \in S$,

$$(f(s), \mathscr{P}) \succ (f(s'), \mathscr{P}) \Longrightarrow (g(s), \mathscr{P}) \succsim (g(s'), \mathscr{P}).$$

It can readily be shown that in general these two notions are completely independent of one another (counterexamples and details are available upon request).

Now, if two acts are $\mathscr{P}$-comonotonic, since they order scenarios in the same way, they do not provide a hedging opportunity against each other, in the sense of compensating bad scenarios for one act with good scenarios for the other. Therefore, if f is preferred to g given information $\mathscr{P}$, mixing them (in the second order sense) with another act that is $\mathscr{P}$-comonotonic with both of them will result in two acts that induce the same ranking of scenarios as f and g. The preference ranking of the mixed acts should thus be the same as that of the original acts. This is what the next axiom requires:

Axiom 7 (Information-Comonotonic Second Order Independence) *For all $\mathscr{P} \in \mathfrak{P}$,*

for all $f, g, h \in \mathscr{F}$ pairwise $\mathscr{P}$-comonotonic, for all $\alpha \in (0, 1]$

$$(f, \mathscr{P}) \succsim (g, \mathscr{P}) \Longleftrightarrow (f', \mathscr{P}) \succsim (g', \mathscr{P}),$$

for any $f' \in \alpha f \oplus_{\mathscr{P}} (1-\alpha)h$ and $g' \in \alpha g \oplus_{\mathscr{P}} (1-\alpha)h$ whenever these sets are nonempty.

The last axiom, introduced by Gajdos et al. (2008), states that given a scenario, the certainty equivalent of an act is its (X-valued) expectation.

Axiom 8 (Reduction under Precise Information) *Let $f \in \mathscr{F}$ and $P \in pc(\Sigma)$. Then*

$$(f, \{P\}) \sim \left(\sum_{i=1}^{n} P(A_i) x_i, \{P\} \right) \quad \text{whenever } f = (x_i, A_i)_{i=1,\ldots,n}.$$

In the standard Anscombe-Aumann framework, X is a set of probability distributions, or roulette lotteries, over an outcome space Z, whereas $\mathscr{F}$ is the set of horse race lotteries. This axiom says that in the case of precise information, the decision maker does not care about the horse race stage and focuses on the induced roulette lottery. It holds for most decision under uncertainty models in the Anscombe-Aumann framework as long as beliefs are probabilistic. This axiom is the counterpart in our setup of the Reduction of Compound Lotteries axiom used by Seo (2009) and has essentially the same effect of delivering linearity at the first order level.

3.1.3 Representation Theorem

Theorem 1

A preference relation $\succsim$ satisfies axioms 1 through 8 if and only if there exist a nonconstant affine function $u : X \to \mathbb{R}$ and for all $\mathscr{P} \in \mathfrak{P}$, a capacity $\nu^{\mathscr{P}}$ defined on the closed convex hull of $\mathscr{P}$, $\overline{\text{co}}(\mathscr{P})$, such that $\succsim$ admits an SODEU representation based on them.

Moreover, if $(u, (\nu^{\mathscr{P}})_{\mathscr{P} \in \mathfrak{P}})$ and $(v, (\mu^{\mathscr{P}})_{\mathscr{P} \in \mathfrak{P}})$ both represent $\succsim$ in the previous sense then v is a positive affine transformation of u and

$$\int_{\overline{\text{co}}(\mathscr{P})} T \, \mathrm{d}\nu^{\mathscr{P}} = \int_{\overline{\text{co}}(\mathscr{P})} T \, \mathrm{d}\mu^{\mathscr{P}}$$

for each $\mathscr{P} \in \mathfrak{P}$ and each affine function $T : \overline{\text{co}}(\mathscr{P}) \to \mathbb{R}$.

The above theorem shows that the axioms are necessary and sufficient for the representation of preferences by an SODEU functional, and thus for the existence

of second order beliefs. However it falls short of pinning them down. Indeed, the proof involves defining a monotonic and comonotonic additive function on a subset of the space of bounded real functions on $\overline{\text{co}}(\mathscr{P})$ and extending it to a Choquet integral with respect to a certain capacity. Since the subset in question does not contain the indicator functions, this capacity cannot be uniquely identified.[10] This may be surprising as we should expect that under ideal circumstances, i.e., if we were able to observe preferences on every pair, we should be able to infer beliefs. But since we are talking about second order beliefs, observing preferences over first order acts is not necessarily sufficient to identify them. We can therefore hardly say that circumstances are ideal. Klibanoff et al. (2005)'s model is cast in this ideal framework, and this is why they can uniquely identify second order beliefs.[11]

Moreover, the theorem is silent about the relationship between the capacities $\nu^{\mathscr{P}}$ as $\mathscr{P}$ varies, and in particular nothing is said about updating $\nu^{\mathscr{P}}$ when new information about $\mathscr{P}$ arrives. There are two reasons for that. First, except for Dominance and Information Irrelevance for Constant Acts, our axioms imply conditions within the context of a particular information set, and therefore, they do not imply anything regarding the relationship between belief representations. Second, one could in principle introduce axioms that force a relationship between them based on standard updating rules for capacities. However, all axiomatizations in the literature are based on acts of the form fAg, that take the value $f(s)$ for $s \in A$ and $g(s)$ for $s \notin A$. To transpose these axiomatizations to the level of second order beliefs, one would need to use similar constructions for second order acts. Yet second order acts are not primitive objects in our setting, they are derived from first order acts in a way that does not give rise to a set closed under such act-mixing operations at the second order level. So this road seems to be barred.

3.2 Second Order Subjective Expected Utility (SOSEU)

3.2.1 Representation Concept

Definition 4 *A binary relation* $\succsim$ *on* $\mathscr{F} \times \mathfrak{P}$ *admits a* Second Order Subjective Expected Utility (SOSEU) representation *based on the utility function* $u : X \to \mathbb{R}$,

[10] Some authors (Zhou, 1999, 1998; Cerreia-Vioglio, Maccheroni, Marinacci, and Montrucchio, 2011) approach the uniqueness problem using continuity requirements, that deliver unique continuous (in a suitable sense) capacities; but this does not exclude the possibility that a non continuous capacity may also represent the functional.

[11] Actually they assume their existence and uniqueness, but axiomatizations of this assumption are readily available.

the evaluation function[12] $\Phi : u(X) \to \mathbb{R}$ *and the family of probability charges* $(\mu^{\mathscr{P}})_{\mathscr{P}\in\mathfrak{P}}$ *each defined on the respective* $\mathscr{P} \in \mathfrak{P}$ *if*

$$(f, \mathscr{P}) \succsim (g, \mathscr{Q}) \quad \Longleftrightarrow \quad \int_{\mathscr{P}} \Phi\left(\int_S u \circ f \,\mathrm{d}P\right) \mathrm{d}\mu^{\mathscr{P}}(P) \geq \int_{\mathscr{Q}} \Phi\left(\int_S u \circ g \,\mathrm{d}Q\right) \mathrm{d}\mu^{\mathscr{Q}}(Q).$$

3.2.2 Axioms

The axiomatization of SOSEU we propose relies on an independence axiom at the second order level and a weakening of the reduction axiom used for SODEU.

Axiom 7' (Second Order Independence) *For all* $\mathscr{P} \in \mathfrak{P}$*, for all* $f, g, f', g' \in \mathscr{F}$*, if*

$$(f, \mathscr{P}) \succsim (g, \mathscr{P}) \text{ and } (f', \mathscr{P}) \succsim (g', \mathscr{P}),$$

then

$$(h, \mathscr{P}) \succsim (h', \mathscr{P}),$$

for all $\alpha \in (0, 1]$*,* $h \in \alpha f \oplus_{\mathscr{P}} (1-\alpha)f'$ *and* $h' \in \alpha g \oplus_{\mathscr{P}} (1-\alpha)g'$*, whenever they exist, and the converse holds whenever* $f' = g'$*.*

This axiom is stronger than the standard independence axiom applied to second order α-mixtures, which would feature $f' = g'$ in the first part of the axiom. We use this version of the axiom because second order α-mixtures may not exist in certain cases, so that the full force for the standard independence axiom is not preserved. As mentioned in the introduction, section 1.2, Axiom 7' – the counterpart in our framework of Klibanoff et al. (2005)'s Assumption 2 – is violated by a modified version of the Ellsberg paradox, that we now present.

Example 1 *An urn contains 90 balls, either red, black or yellow. The state space is therefore* $S = \{R, B, Y\}$ *and the outcome space is* $\mathbb{R}$*. The decision maker knows that the composition of the urn can only be either one of* $C_1 = (29, 30, 31)$*,* $C_2 = (30, 60, 0)$ *and* $C_3 = (30, 0, 60)$*, where the first figure is the number of red balls, the second the number of black balls and the third the number of yellow balls. For a composition,* $C_i = (r_i, b_i, y_i)$*, let* $P_i = (r_i/3, b_i/3, y_i/3)$ *be the associated probability distribution and let* $\mathscr{P} = \{P_1, P_2, P_3\}$*. Assume preferences have an SOSEU representation based on* $(u, \Phi, (\mu^{\mathscr{P}}))$ *such that* $u(X) = \mathbb{R}$ *and* Φ *is strictly increasing and continuous. Consider the acts the expected utilities of which, with respect to the distributions in*

[12]Note that, here again, we give a general definition of SOSEU without tying it to a particular framework (e.g.,, the Anscombe-Aumann framework) and hence without assuming properties for u and Φ, like linearity, that would make sense only in such contexts.

Act	$\boldsymbol{f_1}$	$\boldsymbol{f_2}$	$\boldsymbol{f_3}$	$\boldsymbol{f_4}$	$\boldsymbol{f_5}$	$\boldsymbol{f_0}$	$\boldsymbol{f_{1,5}}$
EU w.r.t $\boldsymbol{P_1}$	$\Phi^{-1}\left(\frac{100}{3}\right)$	$\Phi^{-1}\left(\frac{100}{3}\right)$	$\Phi^{-1}\left(\frac{200}{3}\right)$	$\Phi^{-1}\left(\frac{200}{3}\right)$	$\Phi^{-1}\left(\frac{100}{3}\right)$	$\Phi^{-1}(0)$	$\Phi^{-1}\left(\frac{100}{3}\right)$
EU w.r.t. $\boldsymbol{P_2}$	$\Phi^{-1}\left(\frac{100}{3}\right)$	$\Phi^{-1}\left(\frac{200}{3}\right)$	$\Phi^{-1}\left(\frac{200}{3}\right)$	$\Phi^{-1}\left(\frac{100}{3}\right)$	$\Phi^{-1}(0)$	$\Phi^{-1}(0)$	$\Phi^{-1}\left(\frac{50}{3}\right)$
EU w.r.t. $\boldsymbol{P_3}$	$\Phi^{-1}\left(\frac{100}{3}\right)$	$\Phi^{-1}(0)$	$\Phi^{-1}\left(\frac{200}{3}\right)$	$\Phi^{-1}(100)$	$\Phi^{-1}\left(\frac{200}{3}\right)$	$\Phi^{-1}(0)$	$\Phi^{-1}(50)$

Table 1: Modified Ellsberg Paradox: Expected Utilities

$\mathscr{P}$, are as given in table 1.[13] *Examples of f_1 through f_4 are given in table 2 .*[14] *Since $f_{1,5} \in \frac{1}{2}f_1 \oplus_{\mathscr{P}} \frac{1}{2}f_5$ and $f_1 \in \frac{1}{2}f_2 \oplus_{\mathscr{P}} \frac{1}{2}f_5$,*[15] *Second Order Independence implies*[16]

$$(f_1, \mathscr{P}) \succsim (f_2, \mathscr{P}) \iff (f_{1,5}, \mathscr{P}) \succsim (f_1, \mathscr{P}).$$

Similarly, since $f_{1,5} \in \frac{1}{2}f_4 \oplus_{\mathscr{P}} \frac{1}{2}f_0$ and $f_1 \in \frac{1}{2}f_3 \oplus_{\mathscr{P}} \frac{1}{2}f_0$,[17] *it implies*[18]

$$(f_4, \mathscr{P}) \succsim (f_3, \mathscr{P}) \iff (f_{1,5}, \mathscr{P}) \succsim (f_1, \mathscr{P}).$$

Hence,

$$(f_1, \mathscr{P}) \succsim (f_2, \mathscr{P}) \iff (f_4, \mathscr{P}) \succsim (f_3, \mathscr{P}).$$

However, it may be argued that this behavior is contrary to what ambiguity aversion suggests. Indeed, first, the situation described is approximately the same as the simplified Ellsbergian context used in Epstein (2010) (Situation I described in the introduction, section 1.1).[19] *Moreover, up to the transformation by Φ^{-1} (an increasing function) the expected utilities of f_1, f_2, f_3 and f_4 are those of the bets*

[13] Their existence follows from lemma 1 in appendix A.1.

[14] We don't give the values of f_0 – actually 0 –, $f_{1,5}$ and f_5, in table 2, as they are only intermediate constructs used for the derivation of a prediction from Seconder Order Independence, so that knowing what they actually are would not provide any intuition about the fact we want to highlight and would be an unnecessary distraction of the reader's mind.

[15] Indeed, for any $P \in \mathscr{P}$,

$$\Phi\left(\int u \circ f_{1,5}\,\mathrm{d}P\right) = \frac{1}{2}\Phi\left(\int u \circ f_1\,\mathrm{d}P\right) + \frac{1}{2}\Phi\left(\int u \circ f_5\,\mathrm{d}P\right) = \frac{1}{2}\Phi\left(\int u \circ f_4\,\mathrm{d}P\right) + \frac{1}{2}\Phi\left(\int u \circ f_0\,\mathrm{d}P\right)$$

[16] Take $f = f_1$, $g = f_2$, $f' = g' = f_5$, $h = f_{1,5}$, $h' = f_1$ in the statement of the axiom.

[17] For any $P \in \mathscr{P}$

$$\Phi\left(\int u \circ f_1\,\mathrm{d}P\right) = \frac{1}{2}\Phi\left(\int u \circ f_2\,\mathrm{d}P\right) + \frac{1}{2}\Phi\left(\int u \circ f_5\,\mathrm{d}P\right) = \frac{1}{2}\Phi\left(\int u \circ f_3\,\mathrm{d}P\right) + \frac{1}{2}\Phi\left(\int u \circ f_0\,\mathrm{d}P\right).$$

[18] Take $f = f_4$, $g = f_3$, $f' = g' = f_0$, $h = f_{1,5}$, $h' = f_1$ in the statement of the axiom.

[19] For technical reasons, f_1, f_2, f_3 and f_4 as in table 1 may not exist for the set of probability distributions $\mathscr{P} = \{P_r, P_b, P_g\}$ corresponding to Situation I. See lemma 1 in the appendix.

on, respectively, Red, Black, not-Red and not-Black in Situation I. Hence, I suggest that – assuming that the composition $(29, 30, 31)$ *is close enough to the composition* $(30, 30, 30)$ *not to alter preferences – ambiguity aversion would lead to the standard Ellsberg paradox preference pattern* $(f_1, \mathscr{P}) \succ (f_2, \mathscr{P})$ *and* $(f_3, \mathscr{P}) \succ (f_4, \mathscr{P})$*. To*

$\alpha = 0.1$				$\alpha = .99$				$\alpha = 10.1$			
Acts	R	B	G	Acts	R	B	G	Acts	R	B	G
$u \circ f_1$	1.7×10^{15}	1.7×10^{15}	1.7×10^{15}	$u \circ f_1$	34.5	34.5	34.5	$u \circ f_1$	1.4	1.4	1.4
$u \circ f_2$	5.2×10^{19}	-2.3×10^{19}	-2.6×10^{19}	$u \circ f_2$	14.6	97.1	-7.3	$u \circ f_2$	-39.4	22	19.7
$u \circ f_3$	1.7×10^{18}	1.7×10^{18}	1.7×10^{18}	$u \circ f_3$	69.6	69.6	69.6	$u \circ f_3$	1.5	1.5	1.5
$u \circ f_4$	3×10^{21}	-1.5×10^{21}	-1.4×10^{21}	$u \circ f_4$	110.3	-3.4	102	$u \circ f_4$	0.4	1.9	2.2

Table 2: Modified Ellsberg Paradox: First-order Acts ($\Phi(t) = t^\alpha$)

support this intuition, consider the acts in table 2. For $\alpha = 0.1$*,* f_2 *and* f_4 *may be construed as bets on Red, while* f_1 *and* f_3 *are sure bets. A cautious*[20] *decision maker having chosen the constant* f_1 *over* f_2 *would be expected to also choose the constant* f_3 *over* f_4*. The choice of* f_1 *over* f_2 *and* f_4 *over* f_3 *predicted by SOSEU seems thus unlikely given the parallel structures of the two pairs of acts.*[21] *For* $\alpha = .99$*,* f_2 *may be construed as a bet on Black, while* f_4 *may be construed as a bet on not-Black. Moreover,* f_1 *(respectively* f_3*) is very close to the expected utility of* f_2 *(respectively* f_3*) with respect to the uniform distribution. Therefore a risk averse decision maker would be expected to choose* f_1 *over* f_2 *and* f_3 *over* f_4*, and a risk loving decision maker would make the opposite choices, but again the SOSEU prediction seems unlikely. Finally, when* $\alpha = 10.1$*,* f_2 *and* f_4 *may be construed as bets on not-Red,* f_1 *is (approximately) twice the expectation of* f_2 *w.r.t. the uniform distribution while* f_3 *is (up to rounding) the expectation of* f_4*, so again a risk averse decision maker would be expected to choose* f_1 *over* f_2 *and* f_3 *over* f_4*, not complying with the SOSEU predictions. The behavior suggested in these examples is thus precluded by Second Order Independence. On the other hand, since* f_2 *and* f_5 *are not* $\mathscr{P}$*-comonotonic, it is not precluded by Information Comonotonic Second Order Independence.*[22]

[20] We prefer to avoid here the term risk averse since we are not under precise risk so it wouldn't be rigorous. Actually one can show that there exists a distribution on $\{R, B, G\}$ such that f_1 is the expectation of f_2 and f_3 the expectation of f_4.

[21] Based on the order of magnitude of f_2 with respect to f_1, and of f_4 with respect to f_3, it is also possible that an adventurous decision maker would choose f_2 over f_1, but in that case he would probably also choose f_4 over f_3.

[22] SODEU predicts $(f_1, \mathscr{P}) \succ (f_2, \mathscr{P})$ and $(f_3, \mathscr{P}) \succ (f_4, \mathscr{P})$ whenever $\nu^{\mathscr{P}}(\{P_1, P_3\}) = \nu^{\mathscr{P}}(\{P_2, P_3\}) = \nu^{\mathscr{P}}(\{P_1, P_2\}) = \frac{1}{3}$, $\nu^{\mathscr{P}}(\{P_1\}) = \left(\Phi^{-1}\left(\frac{200}{3}\right) - \Phi^{-1}\left(\frac{100}{3}\right)\right) / 3\left(\Phi^{-1}(100) - \Phi^{-1}\left(\frac{200}{3}\right)\right)$, and $\nu^{\mathscr{P}}(\{P_2\}) = \nu^{\mathscr{P}}(\{P_3\}) = \left(\Phi^{-1}\left(\frac{100}{3}\right) - \Phi^{-1}(0)\right) / 3\left(\Phi^{-1}\left(\frac{200}{3}\right) - \Phi^{-1}\left(\frac{100}{3}\right)\right)$.

The next three axioms weaken Reduction under Precise Information. They are the counterpart in our setting of Assumption 1 in Klibanoff et al. (2005), which requires an expected utility representation for preferences on lottery acts.[23] Here we have something similar: an act together with a singleton information set generates a lottery and these axioms imply that induced preferences on lotteries have an expected utility representation.

For any $f \in \mathscr{F}$ and $P \in pc(\Sigma)$ let P^f denote the lottery induced by f on X: for all $B \in \mathscr{B}$, $P^f(B) = P(f^{-1}(B))$. Then, the first of the next three axioms states that, under risk, decision makers care only about the induced lotteries.

Axiom 9 (Probabilistic Sophistication under Precise Information) *For all $f, g \in \mathscr{F}$, $P, Q \in pc(\Sigma)$, $P^f = Q^g \Rightarrow (f, \{P\}) \sim (g, \{Q\})$.*

The second axiom states that given a fixed act, precise information can be slightly modified without affecting preferences.

Axiom 10 (Precise Information Continuity) *For all $f \in \mathscr{F}$, $P, Q, R \in pc(\Sigma)$, if*

$$(f, \{P\}) \succ (f, \{Q\}) \succ (f, \{R\}),$$

then there exist $\alpha, \beta \in (0, 1)$ such that

$$(f, \{\alpha P + (1-\alpha)R\}) \succ (f, \{Q\}) \succ (f, \{\beta P + (1-\beta)R\}).$$

The third axiom states that if, from the point of view of a specific act, a probability distribution P is more favorable than a probability distribution Q, then a common modification of these two distributions by their mixture with a third distribution should not change this preference.

Axiom 11 (Precise Information Independence) *For all $f \in \mathscr{F}$, $P, Q, R \in pc(\Sigma)$, $\lambda \in (0, 1]$,*

$$(f, \{P\}) \succsim (f, \{Q\}) \iff (f, \{\lambda P + (1-\lambda)R\}) \succsim (f, \{\lambda Q + (1-\lambda)R\}).$$

Finally, for technical reasons we must strengthen the Information Dominance axiom. Recall that $c(f, \mathscr{Q})$ stands for any certainty equivalent of f under information $\mathscr{Q}$.

Axiom 4' (Information Dominance for Second Order Mixtures) *Let $(f_i)_{i=1,\ldots,n}$,*

[23] Skipping details this is the subset of the first order acts such that for any probability distribution on the set of outcomes there exists an act in this subset that together with the Lebesgue measure on $(0, 1]$ generates the same probability distribution.

$(g_j)_{j=1,\dots,m}$ *be families of acts, and* $(\lambda_i)_{i=1,\dots,n} \in [0,1]^n$, $(\mu_j)_{j=1,\dots,m} \in [0,1]^m$ *such that* $\sum_{i=1}^n \lambda_i = 1$ *and* $\sum_{j=1}^m \mu_j = 1$ *and let* $\mathscr{P} \in \mathfrak{P}$. *Then,*

$$\left(\sum_{i=1}^n \lambda_i\, c(f_i, \{P\}), \{P\}\right) \succsim \left(\sum_{j=1}^m \mu_j\, c(g_j, \{P\}), \{P\}\right) \quad \forall P \in \mathscr{P}$$

$$\Longrightarrow \quad \left(\sum_{i=1}^n \lambda_i\, c(f_i, \mathscr{P}), \mathscr{P}\right) \succsim \left(\sum_{j=1}^m \mu_j\, c(g_j, \mathscr{P}), \mathscr{P}\right).$$

Consider a family of acts and a system of weights. By definition, if the second order mixture of this family, with the corresponding weights, exists,[24] then in any given scenario, its certainty equivalent is the convex combination, with the same weights, of the certainty equivalents for this scenario of the acts in the family. This axiom forces the "mixture-linearity" of the certainty equivalent mapping to also hold when the scenario is only imprecisely known. It imposes, more generally, that if a certain inequality between certainty equivalents of families of acts holds scenario-wise, then it should also hold when the scenario is not precisely known. Whenever certainty equivalents exist and the families of acts involved in this axiom have a single member, it reduces to Information Dominance .

3.2.3 Representation Theorem

These axioms together with the other basic axioms are necessary and sufficient for a representation of preferences by an SOSEU functional:

Theorem 2
A preference relation $\succsim$ *satisfies axioms 1, 2, 3, 4', 5, 6, 7', 9, 10 and 11 if and only if there exist a function* $u : X \to \mathbb{R}$, *an increasing function* $\Phi : u(X) \to \mathbb{R}$ *such that* $\Phi \circ u$ *is affine and, for all* $\mathscr{P} \in \mathfrak{P}$, *a probability* $\mu^{\mathscr{P}}$ *defined on* $\mathscr{P}$ *such that* $\succsim$ *admits an SOSEU representation based on them.*

Moreover, if $(u_1, \Phi_1, (\mu_1^{\mathscr{P}})_{\mathscr{P}\in\mathfrak{P}})$ *and* $(u_2, \Phi_2, (\mu_2^{\mathscr{P}})_{\mathscr{P}\in\mathfrak{P}})$ *both represent the preferences, then* u_1 *and* u_1 *are equal up to an affine transformation, so are* $\Phi_1 \circ u_1$ *and* $\Phi_2 \circ u_2$ *and*

$$\int_{\mathscr{P}} \Phi_1(T)\, \mathrm{d}\mu_1^{\mathscr{P}} = \int_{\mathscr{P}} \Phi_1(T)\, \mathrm{d}\mu_2^{\mathscr{P}}$$

for all affine functions $T : \overline{\mathrm{co}}(\mathscr{P}) \to \mathbb{R}$.

[24] Mixtures of more than two acts are defined in the standard way by repeated applications of the mixture operation whenever possible.

3.2.4 The Second Order Paradox and SOSEU

In section 1.2, we raised the issue of how the SOSEU model fares with respect to Epstein (2010)'s second order Ellsberg paradox. We claimed that if we restrict the analysis to first order acts then with Epstein's specific example there is no paradox anymore, but there still is one for slight perturbations of this example. Let us now prove this point formally. Let $\mathscr{P} = \{P_r, P_b, P_g\}$ be any set of probability distributions on the set of colors $S = \{R, B, G\}$. Which distribution is actually the case is determined by a draw from an urn containing three balls labelled r, b, or g, and exactly one red ball. Let F_r, F_b, F_{rg} and F_{bg} be bets on the label of these balls (table 3). Epstein argues that the problem of choosing between these second-order

	F_r	F_b	F_{rg}	F_{bg}
P_r	100	0	100	0
P_b	0	100	0	100
P_g	0	0	100	100

Table 3: Epstein's Second Order Ellsberg Paradox

bets is isomorphic to the standard three-color Ellsberg paradox, and thus that the ambiguity aversion-based prediction would be that $F_r \succ^2_{\mathscr{P}} F_b$ and $F_{bg} \succ^2_{\mathscr{P}} F_{rg}$. He also claims that SOSEU, as axiomatized in Klibanoff et al. (2005), because it assumes expected utility at the second order level, predicts that that $F_r \succ^2_{\mathscr{P}^\varepsilon} F_b$ if and only if $F_{rg} \succ^2_{\mathscr{P}^\varepsilon} F_{bg}$. For there to be a second order paradox, therefore, we need to examine whether this prediction is still valid when no direct assumption is made about second order preferences. Hence, in line with Klibanoff et al. (2005)'s assumption 2, define preferences on second order acts defined on $\mathscr{P}$ using the SOSEU model by letting

$$F \succsim^2_{\mathscr{P}} G \Longleftrightarrow \int_{\mathscr{P}} \Phi\left(F(P)\right) \mathrm{d}\mu^{\mathscr{P}}(P) \geq \int_{\mathscr{P}} \Phi\left(G(P)\right) \mathrm{d}\mu^{\mathscr{P}}(P).$$

Consistency (Klibanoff et al. (2005)'s assumption 3) thus requires that $(f, \mathscr{P}) \succsim (g, \mathscr{P})$ whenever

$$\int_S u \circ f \,\mathrm{d}P = F(P) \text{ and } \int_S u \circ g \,\mathrm{d}P = G(P), \forall P \in \mathscr{P}$$

and $F \succsim^2_{\mathscr{P}} G$.

Now, for fixed $\varepsilon \in [-\frac{1}{3}, \frac{1}{3}]$, consider the distributions

$$P_r^\varepsilon = \left(\frac{1}{3} - \varepsilon, \frac{1}{3}, \frac{1}{3} + \varepsilon\right), \quad P_b = \left(\frac{1}{3}, \frac{2}{3}, 0\right) \quad \text{and} \quad P_g = \left(\frac{1}{3}, 0, \frac{2}{3}\right)$$

and let $\mathscr{P}^\varepsilon = \{P_r^\varepsilon, P_b, P_g\}$. Table 4 gives,[25] for linear utility, the acts f_E, $E \in \{r, b, rg, bg\}$, such that, for all $P \in \mathscr{P}^\varepsilon$,

$$\int_S u \circ f_E \, \mathrm{d}P = F_E(P).$$

As can be seen, they exist if and only if $\varepsilon \neq 0$.

	f_r	f_b	f_{rg}	f_{bg}
R	$-\frac{200}{3\varepsilon}$	$\frac{100}{3\varepsilon}$	$\frac{300\varepsilon-100}{3\varepsilon}$	$\frac{300\varepsilon+200}{3\varepsilon}$
B	$\frac{100}{3\varepsilon}$	$\frac{450\varepsilon-50}{3\varepsilon}$	$-\frac{150\varepsilon-50}{3\varepsilon}$	$\frac{300\varepsilon-100}{3\varepsilon}$
G	$\frac{100}{3\varepsilon}$	$-\frac{50}{3\varepsilon}$	$\frac{300\varepsilon+50}{3\varepsilon}$	$\frac{300\varepsilon-100}{3\varepsilon}$

Table 4: Epstein's Second Order Ellsberg Paradox: First-Order Acts.

The case considered by Epstein is $\varepsilon = 0$. In that case, since the acts f_r, etc. do not exist, SOSEU makes no prediction, and there is therefore no genuine second order paradox: it does not follow from the functional form of SOSEU by itself, but from additional assumptions on second order acts. For $\varepsilon \neq 0$, however, the first order acts exist and SOSEU does predict that $F_r \succ^2_{\mathscr{P}^\varepsilon} F_b$ if and only if $F_{rg} \succ^2_{\mathscr{P}^\varepsilon} F_{bg}$. The vulnerability of SOSEU to second order paradoxes is therefore robust.[26]

3.3 Relationship with other Decision Models

3.3.1 Invariant Biseparable Preferences

Preferences are *biseparable* (Ghirardato and Marinacci, 2001) if[27] they can be represented by a monotonic functional V, unique up to increasing affine transformations

[25] Details are available upon request, but existence of these acts is guaranteed by lemma 1 in Appendix A.1.

[26] Klibanoff et al.'s reply (Klibanoff et al., 2012) to this second order paradox involves redefining the first order state-space to incorporate the draws from the second order urn and showing that an SOSEU model based on this first order state space and with concave utility will predict the behavior on second order acts hypothesized by Epstein. Their argument is that all information available to the decision maker should be modeled. I am obviously sympathetic with this claim, but doing it this way strikes me as rather circular, since then sources of ambiguity become both the support and the source of the same probability distributions. While more thought should be devoted to understanding the legitimacy of this procedure, I think the road taken here is logically safer.

[27] Leaving aside technicalities in the original definition related to the presence of nonnull events.

and a capacity $\rho : \Sigma \to [0,1]$ such that

$$V(xAy) = \rho(A)u_V(x) + (1-\rho(A))u_V(y) \quad \text{for all bets } xAy, \tag{3}$$

where $u_V(x) := V(x)$. ρ is actually independent of the choice of V and is called the *willingness to bet*, while V is called the *canonical representation*. A biseparable preference relation is *invariant* if its canonical representation can be written

$$V(f) = J(u_V \circ f),$$

where $J : B_0(\Sigma) \to \mathbb{R}$ is a constant linear[28] functional defined on the set of simple Σ-measurable functions $B_0(\Sigma)$.

Since the Choquet integral is constant linear, whenever $\succsim$ admits an SODEU representation, for each $\mathscr{P}$ its restriction to $\mathscr{F} \times \{\mathscr{P}\}$ is invariant biseparable, with willingness to bet $\rho^{\mathscr{P}}$ defined for all $A \in \Sigma$ by

$$\rho^{\mathscr{P}}(A) = \int_{\mathscr{P}} P(A)\ \mathrm{d}\nu^{\mathscr{P}}(P).^{29} \tag{4}$$

Now, Amarante (2009) showed conversely that any invariant biseparable preference relation can be represented by an SODEU functional, the second order capacity being defined on a subjective set of relevant priors. By contrast, in the present paper, the second order capacity is defined on the (closed convex hull of the) given set of priors representing objective information. These two approaches can be reconciled. Following Ghirardato et al. (2004), for each $\mathscr{P} \in \mathfrak{P}$, say that f *is unambiguously preferred to* g *given* $\mathscr{P}$, denoted $f \succsim^*_{\mathscr{P}} g$, if

$$(\lambda f + (1-\lambda)h, \mathscr{P}) \succsim (\lambda g + (1-\lambda)h, \mathscr{P}) \ \text{ for all } \lambda \in (0,1] \text{ and } h \in \mathscr{F}.$$

Then,

Corollary 1 *Let $\succsim$ satisfy axioms 1 to 8. Then, there exists a unique weak*-closed and convex set $\Gamma(\mathscr{P}) \subseteq \overline{\mathrm{co}}(\mathscr{P})$ such that*

$$f \succsim^*_{\mathscr{P}} g \ \Leftrightarrow \ \int_S u \circ f \,\mathrm{d}P \geq \int_S u \circ g \,\mathrm{d}P \ \ \textit{for all } P \in \Gamma(\mathscr{P})$$

[28] $J(\lambda\varphi + c) = \lambda J(\varphi) + c$, for all $\lambda > 0$, $\varphi \in B_0(\Sigma)$ and $c \in \mathbb{R}$.

[29] And with constant-linear functional $J_{\mathscr{P}}$ defined by $J_{\mathscr{P}}(\varphi) = \int_{\mathscr{P}} \int_S \varphi \,\mathrm{d}P \ \mathrm{d}\nu^{\mathscr{P}}(P)$.

and $\succsim$ admits an SODEU representation based on capacities $(\nu^{\mathscr{P}})_{\mathscr{P}\in\mathfrak{P}}$ such that

$$\nu^{\mathscr{P}}(\mathscr{Q}) = \nu^{\mathscr{P}}(\mathscr{Q} \cap \Gamma(\mathscr{P})),$$

for all $\mathscr{P} \in \mathfrak{P}$ and $\mathscr{Q} \subseteq \overline{\mathrm{co}}(\mathscr{P})$.

This corollary may be interpreted as identifying the set $\Gamma(\mathscr{P})$ as the *relevant subset* of $\overline{\mathrm{co}}(\mathscr{P})$, since priors outside this set are given zero probability. It can help identify perceived ambiguity as opposed to objective ambiguity.

3.3.2 Choquet Expected Utility and Expected Utility

When can preferences represented by an SODEU functional also be represented by a Choquet Expected Utility with respect to the willingness to bet? A necessary condition is obviously that they satisfy Schmeidler (1989)'s Comonotonic Independence axiom:

Axiom 12 (Comonotonic Independence) *For all $\mathscr{P} \in \mathfrak{P}$, for all $f, g, h \in \mathscr{F}$ pairwise comonotonic, for all $\alpha \in (0, 1]$*

$$(f, \mathscr{P}) \succsim (g, \mathscr{P}) \Longleftrightarrow (\alpha f + (1-\alpha)h, \mathscr{P}) \succsim (\alpha g + (1-\alpha)h, \mathscr{P}).$$

As it turns out, it is also sufficient:

Proposition 1
A preference relation $\succsim$ satisfies axioms 1 through 8 and 12 if and only if there exist an affine function $u : X \to \mathbb{R}$ and, for all $\mathscr{P} \in \mathfrak{P}$, a capacity $\nu^{\mathscr{P}}$ defined on $\mathscr{P}$ such that

$$(f, \mathscr{P}) \succsim (g, \mathscr{P}') \quad \Longleftrightarrow \quad \int_S u \circ f \,\mathrm{d}\rho^{\mathscr{P}} \geq \int_S u \circ g \,\mathrm{d}\rho^{\mathscr{P}'},$$

where for all $A \in \Sigma$ and $\mathscr{P} \in \mathfrak{P}$,

$$\rho^{\mathscr{P}}(A) = \int_{\overline{\mathrm{co}}(\mathscr{P})} P(A) \,\mathrm{d}\nu^{\mathscr{P}}.$$

Moreover, $\rho^{\mathscr{P}}$ (but not $\nu^{\mathscr{P}}$) is unique and u is unique up to an affine transformation.

Proposition 1 shades light on the link between objective information and subjective beliefs in the Choquet expected utility model: first order beliefs aggregate (in the sense of Choquet) the probabilistic information available to the decision maker. In particular, by standard properties of the Choquet integral, for all $A, B \in \Sigma$,

$$[\,P(A) \geq P(B), \forall P \in \mathscr{P}] \implies \rho^{\mathscr{P}}(A) \geq \rho^{\mathscr{P}}(B).$$

Thus, $\rho^{\mathscr{P}}$ aggregates available information in a way satisfying a unanimity property: if in all scenarios A is more likely than B, then the decision maker will be more willing to bet on A than to bet on B. Moreover, if $\nu^{\mathscr{P}}$ is additive, then so is $\rho^{\mathscr{P}}$, given its definition. This is incompatible with the occurrence of Ellsberg-type paradoxes. As a matter of fact, if $\nu^{\mathscr{P}}$ is additive then SODEU collapses to subjective expected utility with respect to $\rho^{\mathscr{P}}$. Furthermore, previous results show, roughly speaking, that axiomatizing SODEU is characterized by a strong reduction axiom and a weak independence axiom, whereas SOSEU is characterized by a weaker reduction axiom and a stronger independence condition. According to Proposition 2 below, combining both strong axioms delivers Expected utility.[30]

Proposition 2
A preference relation $\succsim$ satisfies axioms 1 through 6, 7' and 8 if and only if there exist an affine function $u : X \to \mathbb{R}$ and, for all $\mathscr{P} \in \mathfrak{P}$, a probability $\mu^{\mathscr{P}}$ defined on the power set of $\mathscr{P}$ such that

$$(f, \mathscr{P}) \succsim (g, \mathscr{P}') \quad \Longleftrightarrow \quad \int_S u \circ f \, \mathrm{d}\rho^{\mathscr{P}} \geq \int_S u \circ g \, \mathrm{d}\rho^{\mathscr{P}'},$$

where for all $A \in \Sigma$ and $\mathscr{P} \in \mathfrak{P}$,

$$\rho^{\mathscr{P}}(A) = \int_{\mathscr{P}} P(A) \, \mathrm{d}\mu^{\mathscr{P}}.$$

Moreover, $\rho^{\mathscr{P}}$ (but not $\mu^{\mathscr{P}}$) is unique and u is unique up to an affine transformation.

4 Analysis of Attitudes toward Information

When objective but imprecise probabilistic information is explicit, one can clearly distinguish between information itself and the attitude toward it. Thanks to this, Gajdos et al. (2008) have formally shown in the context of Gilboa and Schmeidler (1989)'s Maxmin EU model how attitude toward information affects perceived ambiguity. In this section, we pursue this analysis in the context of SODEU. Actually, in this model one can distinguish between attitude toward ambiguity and attitude toward imprecision. Attitude toward ambiguity is defined in the context of *fixed*, albeit imprecise, *information* on the likelihood of events, and usually assessed by the ranking of more or less ambiguous *acts*. Attitude toward imprecision, on the other hand, has to do with the quality of information. It is assessed by the comparison of more or less imprecise information, *given a fixed act*. Thus ambiguity is a property

[30] The simple proof is left to the reader.

of acts, relative to an informational context, whereas imprecision is a property of information, given a particular act.

4.1 Attitude toward Ambiguity

An agent is more ambiguity averse than another if whenever the first prefers a more ambiguous act to a less ambiguous one, so does the second. Without probabilistic information, the only unquestionably unambiguous acts are the constant acts,[31] hence Ghirardato and Marinacci (2002)'s definition of comparative ambiguity aversion:

Definition 5 *Let $\succsim_1$ and $\succsim_2$ be the preference relations of two decision makers. Then decision maker 1 is* more GM-ambiguity averse[32] than decision maker 2 at $\mathscr{P}$ *if and only if, for all $x \in X$, for all $f \in \mathscr{F}$:*

$$(f, \mathscr{P}) \succsim_1 (x, \mathscr{P}) \implies (f, \mathscr{P}) \succsim_2 (x, \mathscr{P})$$

and

$$(f, \mathscr{P}) \succ_1 (x, \mathscr{P}) \implies (f, \mathscr{P}) \succ_2 (x, \mathscr{P}).$$

With objective information, however, one can specify unambiguous acts: given information context $\mathscr{P}$, an act is $\mathscr{P}$*-unambiguous* if it induces the same lottery no matter which prior in $\mathscr{P}$ is chosen:

Definition 6 *For $\mathscr{P} \in \mathfrak{P}$, $f \in \mathscr{F}$ is a $\mathscr{P}$-unambiguous act if $P^f = Q^f, \forall P, Q \in \mathscr{P}$.*

Hence the definition of comparative ambiguity aversion given information $\mathscr{P}$:

Definition 7 *Let $\succsim_1$ and $\succsim_2$ be the preference relations of two decision makers. Then decision maker 1 is* more ambiguity averse than decision maker 2 given information $\mathscr{P}$ *if and only if, for any $\mathscr{P}$-unambiguous act k, for all $f \in \mathscr{F}$:*

$$(f, \mathscr{P}) \succsim_1 (k, \mathscr{P}) \implies (f, \mathscr{P}) \succsim_2 (k, \mathscr{P})$$

and

$$(f, \mathscr{P}) \succ_1 (k, \mathscr{P}) \implies (f, \mathscr{P}) \succ_2 (k, \mathscr{P}).$$

As it turns out, these two definitions are equivalent in the SODEU model and characterized by a property of the second order capacity:

[31] Provided state-independence holds of course.

[32] This is actually Ghirardato and Marinacci's definition of comparative uncertainty aversion; comparative ambiguity aversion adds to it a condition on the utility function that is automatically satisfied in the Anscombe-Aumann framework (Ghirardato and Marinacci, 2002, Proposition 11).

Proposition 3
Let $\succsim_1$ and $\succsim_2$ admit SODEU representations with utility functions u_1 and u_2 and second order capacities $\nu_1^{\mathscr{P}}$ and $\nu_2^{\mathscr{P}}$. Then the following are equivalent:

(i) Decision maker 1 is more GM-ambiguity averse than decision maker 2 at $\mathscr{P}$;

(ii) Decision maker 1 is more ambiguity averse than decision maker 2 given information $\mathscr{P}$;

(iii) u_2 is a positive affine transformation of u_1 and

$$\int_{\overline{\mathrm{co}}(\mathscr{P})} T \,\mathrm{d}\nu_2^{\mathscr{P}} \geq \int_{\overline{\mathrm{co}}(\mathscr{P})} T \,\mathrm{d}\nu_1^{\mathscr{P}}.$$

for any affine function $T : \overline{\mathrm{co}}(\mathscr{P}) \to \mathbb{R}$.

An affine function $T : \overline{\mathrm{co}}(\mathscr{P}) \to \mathbb{R}$ can be viewed as an expected utility profile (generated by of some unspecified act). This proposition therefore shows that a decision maker is more ambiguity averse than another if the former always expects to get less out of an expected utility profile than the latter; he is thus more pessimistic. This provides a rigorous foundation to the interpretation of ambiguity aversion as second order pessimism.

4.2 Imprecision Aversion

One expects an ambiguity averse decision maker to prefer, when making a decision, having precise rather than imprecise information about the probabilities. Following this intuition, Gajdos et al. (2008) introduced the notion of *(comparative) aversion to bet imprecision*:

Definition 8 *Let $\succsim_1$ and $\succsim_2$ be the preference relations of two decision makers. Then decision maker 1 is more averse to bet imprecision than decision maker 2 given information $\mathscr{P}$ if and only if, for all $A \in \Sigma$, $x, y \in X$ s.t. $(x, \mathscr{P}) \succ_i (y, \mathscr{P})$, $i = 1, 2$, and $P \in \overline{\mathrm{co}}(\mathscr{P})$,*

$$(xAy, \mathscr{P}) \succsim_1 (xAy, \{P\}) \quad \Longrightarrow \quad (xAy, \mathscr{P}) \succsim_2 (xAy, \{P\}).$$

In SODEU, aversion to bet imprecision is connected to the agent's *willingness to bet.*

Proposition 4
Let $\succsim_1$ and $\succsim_2$ be the preference relations of two decision makers admitting an

SODEU representation. Then, decision maker 1 is more averse to bet imprecision than decision maker 2 given information $\mathscr{P}$ if and only if $\rho_2^{\mathscr{P}} \geq \rho_1^{\mathscr{P}}$.

An immediate consequence of the characterizations of comparative ambiguity aversion (Proposition 3) and comparative imprecision aversion (Proposition 4) is the following corollary (the proof is straightforward and left to the reader):

Corollary 2 *Let $\succsim_1$ and $\succsim_2$ be the preference relations of two decision makers admitting an SODEU representation. Then if decision maker 1 is more ambiguity averse than decision maker 2, he or she is also more averse to bet imprecision.*

As noted above, there is, in principle, no relationship between ambiguity aversion, revealed by the comparison of different acts in the context of fixed information and aversion to (bet) imprecision, revealed by the comparison of different information sets relative to a fixed act; in the SODEU model, however, they are connected: the behavior of an SODEU decision maker exhibits a certain form of consistency across decision situations that are in principle unrelated. His or her ambiguity attitude spills over to his or her imprecision aversion. This is a testable implication of the SODEU model, and is therefore worth studying from an experimental point of view.

5 Conclusion

We axiomatized a model of decision making under ambiguous objective information (imprecise risk) where the decision maker maximizes the (Choquet) average expected utility of a given act with respect to some non-necessarily additive second order belief. We provided a parallel axiomatization of the SOSEU model. We discussed some special cases, among which the case where this functional form reduces to Choquet Expected Utility with respect to a capacity that is consistent with information in the sense that it ascribes higher likelihood to A than to B whenever for each prior A is more likely than B. This provides foundations for the intuition according to which decision makers facing imprecise risk aggregate information into one single likelihood measure in a way compatible with ambiguity aversion. We show how ambiguity aversion and imprecision aversion can be characterized in our model in terms of the different capacities that can be defined in the SODEU model, and how they are related to each other.

Some open problems remain. Our axiomatization of second order beliefs has in our opinion the advantage over some other axiomatizations of not taking as primitives second order objects like second order acts in Klibanoff et al. (2005) or lotteries over acts in Seo (2009). This, however, implies using a coarser language with a more

limited expressive power, and this has some drawbacks. First, as noted above, it makes it very difficult to study the question of updating the second order capacities in the light of new objective information about the probabilities. Second, since SODEU and SOSEU are special cases of Second Order Choquet Expected Utility, with nonlinear second order utility (the function Φ in SOSEU) and non-additive second order beliefs, it would be desirable to axiomatize this nesting model. This is left for further research.

Appendices

A Proofs

A.1 Modified Ellsberg paradox

The results rely on the following lemma:

Lemma 1 *Let*

$$\mathbf{P} = \begin{pmatrix} P_1(R) & P_1(B) & P_1(Y) \\ P_2(R) & P_2(B) & P_2(Y) \\ P_3(R) & P_3(B) & P_3(Y) \end{pmatrix}.$$

Assume $u : X \to \mathbb{R}$ is onto. Then, if $\mathbf{P}$ is nonsingular, then for all $\mathbf{E} = (e_k)_{k\in\{1,2,3\}} \in u(X)^3$, there exists $f : \{R, B, Y\} \to \mathbb{R}$, such that $\int_S u \circ f \, \mathrm{d}P_k = e_k$ for all $k \in \{1, 2, 3\}$.

Proof. If $\mathbf{P}$ is nonsingular, then the matrix equation $\mathbf{PU} = \mathbf{E}$ with unknown $\mathbf{U} = (u_i)_{i\in\{R,B,Y\}} \in \mathbb{R}^3$ has a solution. Then, since $\mathbf{E} \in u(X)^3$ and $u(X) = \mathbb{R}$, $\mathbf{U} = \mathbf{P^{-1}E} \in u(X)^3$, hence we can find $x_i \in X$ such that $u(x_i) = u_i$ and define $f(i) = x_i$. By construction this will ensure that $\int u \circ f \, \mathrm{d}P_k = e_k$ for any $k \in \{1, 2, 3\}$. □

A.2 Proof of Theorem 1

Proving the necessity of the axioms being routine, we focus on the proof of sufficiency. We thus assume that all the axioms hold. We proceed in several steps.

Step 1. Second order and first order mixtures. The following lemma will be used through the whole proof:

Lemma 2 *Let $f, g \in \mathscr{F}$ and $\alpha \in [0, 1]$. Then, $\alpha f + (1 - \alpha)g$ is a second order α-mixture of f and g.*

Proof. Suppose $f = (x_i, A_i)_{i=1,\dots,n}$ and $g = (y_j, B_j)_{j=1,\dots,m}$. For all i, j, let $C_{ij} = A_i \cap B_j$. We can rewrite $f = (x_i, C_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$ and $g = (y_j, C_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$. Then, $\alpha f + (1-\alpha) g = (\alpha x_i + (1-\alpha) y_j, C_{ij})_{\substack{i=1,\dots,n \\ j=1,\dots,m}}$. Therefore, by Reduction under precise information:

$$
\begin{aligned}
(\alpha f + (1-\alpha) g, \{P\}) &\sim \left(\sum_{\substack{i=1,\dots,n \\ j=1,\dots,m}} P(C_{ij})(\alpha x_i + (1-\alpha) y_j), \{P\} \right) \\
&\sim \left(\alpha \sum_{\substack{i=1,\dots,n \\ j=1,\dots,m}} P(C_{ij}) x_i + (1-\alpha) \sum_{\substack{i=1,\dots,n \\ j=1,\dots,m}} P(C_{ij}) y_j, \{P\} \right)
\end{aligned}
$$

But, by Reduction under precise information again,

$$
(f, \{P\}) \sim \left(\sum_{\substack{i=1,\dots,n \\ j=1,\dots,m}} P(C_{ij}) x_i, \{P\} \right) \quad \text{and} \quad (g, \{P\}) \sim \left(\sum_{\substack{i=1,\dots,n \\ j=1,\dots,m}} P(C_{ij}) y_j, \{P\} \right),
$$

proving the point. □

Because of this lemma, we can and will replace the second order mixture of f and g by their pointwise mixture everywhere in the rest of the proof.

Step 2. Construction of V.

The following lemma follows from Weak Order, Continuity and Second Order Information-Comonotonic Independence and the Mixture Space Theorem:

Lemma 3 *Let $\mathscr{P} \in \mathfrak{P}$ and let $\mathscr{M} \subseteq \mathscr{F}$ be such that for all $f, g \in \mathscr{M}$, for all $\alpha \in [0,1]$, $\alpha f + (1-\alpha) g \in \mathscr{M}$ and such that all acts in $\mathscr{M}$ are pairwise $\mathscr{P}$-comonotonic. Then there exists a mapping*

$$
V_{\mathscr{M}} : \mathscr{M} \times \{\mathscr{P}\} \to \mathbb{R}
$$

unique up to affine transformations such that

(i) For all $f, g \in \mathscr{M}$, for all $\alpha \in [0,1]$,

$$
V_{\mathscr{M}}(\alpha f + (1-\alpha) g, \mathscr{P}) = \alpha V_{\mathscr{M}}(f, \mathscr{P}) + (1-\alpha) V_{\mathscr{M}}(g, \mathscr{P});
$$

(ii) $(f, \mathscr{P}) \succsim (g, \mathscr{P}) \iff V_{\mathscr{M}}(f, \mathscr{P}) \geq V_{\mathscr{M}}(g, \mathscr{P})$, for all $f, g \in \mathscr{M}$.

Define $V : \mathscr{F} \times \mathfrak{P} \to \mathbb{R}$ by setting $V(f, \mathscr{P}) = V_{\mathscr{M}}(f, \mathscr{P})$ for some $\mathscr{M}$ satisfying the conditions of the lemma and such that $f \in \mathscr{M}$ and $X \subseteq \mathscr{M}$.

To show that this function is well-defined, we must show that:

(i) For all $f \in \mathscr{F}$, there exists $\mathscr{M}$ satisfying the conditions of the lemma such that $f \in \mathscr{M}$ and $X \subseteq \mathscr{M}$.

(ii) The value of V does not depend on the choice of $\mathscr{M}$.

For (i), when $\mathscr{P} = \{P\}$, $\mathscr{M} = \mathscr{F}$ satisfies the required conditions, and we can therefore let

$$V(f, \{P\}) := V_{\mathscr{F}}(f, \{P\}).$$

Moreover, let $v(x) = V(x, \{P\})$ for any P. Note that v is well defined since, by Axiom 6, $V(x, \{P\}) = V(x, \{Q\})$, for all $P, Q \in pc(\Sigma)$, $x \in X$.

Now, we need to show that V represents $\succsim$ on $\mathscr{F} \times \{\{P\} \mid P \in pc(\Sigma)\}$. Indeed, let $(f, \{P\})$ and $(g, \{Q\})$ be two act-information pairs. Axioms 5 and 6 imply:

$$\begin{aligned}
(f, \{P\}) \succsim (g, \{Q\}) \quad &\iff \quad (c(f, \{P\}), \{P\}) \succsim (c(g, \{Q\}), \{Q\}) \\
&\iff \quad (c(f, \{P\}), \{P\}) \succsim (c(g, \{Q\}), \{P\}) \\
&\iff \quad v(c(f, \{P\})) \geq v(c(g, \{Q\})) \\
&\iff \quad V(f, \{P\}) \geq V(g, \{Q\}).
\end{aligned}$$

When $\mathscr{P}$ is not a singleton, let

$$\mathscr{M}_f = \{\alpha f + (1 - \alpha)x \mid x \in X, \alpha \in [0, 1]\}.$$

Clearly $X \subseteq \mathscr{M}_f$. We will now show that $\mathscr{M}_f$ satisfies the conditions of lemma 3.

Let $g, g' \in \mathscr{M}_f$, with $g = \alpha f + (1 - \alpha)x$ and $g' = \alpha' f + (1 - \alpha')x'$. Let us first show that g and g' are $\mathscr{P}$-comonotonic. Let $P, Q \in \mathscr{P}$ such that $(g, \{P\}) \succ (g, \{Q\})$. Then by lemma 3, and because constants are $\mathscr{P}$

comonotonic with any act (because of axiom 6):

$$\begin{aligned}
& V(g,\{P\}) > V(g,\{Q\}) \\
\iff\ & \alpha V(f,\{P\}) + (1-\alpha)v(x) > \alpha V(f,\{Q\}) + (1-\alpha)v(x) \\
\iff\ & V(f,\{P\}) > V(f,\{Q\}) \\
\iff\ & \alpha' V(f,\{P\}) + (1-\alpha')v(x') > \alpha' V(f,\{Q\}) + (1-\alpha')v(x') \\
\iff\ & V(g',\{P\}) > V(g',\{Q\}) \\
\implies\ & (g',\{P\}) \succsim (g',\{Q\}).
\end{aligned}$$

It is routine to show that for all $\alpha \in [0,1]$, $\alpha g + (1-\alpha)g' \in \mathscr{M}_f$. This shows (i).

As for (ii), by axiom 3 (non-degeneracy), there exists x^* and x_* in X such that $(x^*, \mathscr{P}) \succ (x_*, \mathscr{P})$. For any sets $\mathscr{M}$ and $\mathscr{M}'$ satisfying the conditions above, $X \subseteq \mathscr{M} \cap \mathscr{M}'$, hence, since $V_{\mathscr{M}}$ and $V_{\mathscr{M}'}$ are unique up to affine transformations, they are completely determined upon setting $V_{\mathscr{M}}(x^*, \mathscr{P}) = V_{\mathscr{M}'}(x^*, \mathscr{P}) = 1$ and $V_{\mathscr{M}}(x_*, \mathscr{P}) = V_{\mathscr{M}'}(x^*, \mathscr{P}) = 0$. Since they both represent $\succsim$ on $\mathscr{M} \cap \mathscr{M}'$, after normalization they must be equal on this set, and since f necessarily belongs to this intersection $V_{\mathscr{M}}(f)$ does not depend on the choice of $\mathscr{M}$. Obviously V thus constructed represents $\succsim$ on $\mathscr{F} \times \{\mathscr{P}\}$.

Step 3. Construction of u. By Reduction under Precise Information (axiom 8), for $f = (x_1, A_1, \ldots, x_n, A_n)$,

$$V(f,\{P\}) = V\left(\sum_{i=1}^{n} P(A_i)x_i, \{P\}\right) = \sum_{i=1}^{n} P(A_i)V(x_i,\{P\}).$$

Thefore setting $u := v$, we have for all $f \in \mathscr{F}$ and all $P \in pc(\Sigma)$,

$$V(f,\{P\}) = \int_S u \circ f \,\mathrm{d}P.$$

Step 4. Let $\mathscr{P} \in \mathfrak{P}$. Let $\mathbb{B}(\overline{\mathrm{co}}(\mathscr{P}))$ be the set of all real-valued bounded functions on $\overline{\mathrm{co}}(\mathscr{P})$ (the closed convex hull of $\mathscr{P}$) endowed with the uniform convergence topology. For any $f \in \mathscr{F}$, let $\Psi(f) \in \mathbb{B}(\overline{\mathrm{co}}(\mathscr{P}))$ be defined by

$$\Psi(f)(P) = V(f,\{P\}) = \int_S u \circ f \,\mathrm{d}P.$$

Let $\mathbb{B}_0(\overline{\text{co}}(\mathscr{P})) := \Psi(\mathscr{F})$. Let us show that $\mathbb{B}_0(\overline{\text{co}}(\mathscr{P}))$ is convex. Let $\varphi, \psi \in \mathbb{B}_0(\overline{\text{co}}(\mathscr{P}))$. Let $f, g \in \mathscr{F}$ such that $\varphi = \Psi(f)$ and $\psi = \Psi(g)$. Let $\alpha \in [0,1]$. Then, for all $P \in pc(\Sigma)$

$$\begin{aligned}(\alpha\varphi + (1-\alpha)\psi)(P) &= (\alpha\Psi(f) + (1-\alpha)\Psi(g))(P) \\ &= \alpha\Psi(f)(P) + (1-\alpha)\Psi(g)(P)) \\ &= \alpha V(f, \{P\}) + (1-\alpha)V(g, \{P\}) \\ &= V(\alpha f + (1-\alpha)g, \{P\}).\end{aligned}$$

This implies that $\alpha\varphi + (1-\alpha)\psi = \Psi(\alpha f + (1-\alpha)g) \in \mathbb{B}_0$.

We will now show that there exists a mapping $I_{\mathscr{P}} : \mathbb{B}_0(\overline{\text{co}}(\mathscr{P})) \to \mathbb{R}$ such that $V(f, \mathscr{P}) = I_{\mathscr{P}}(\Psi(f))$.

If $\Psi(f) = \Psi(g)$, then $V(f, \{P\}) = V(g, \{P\})$ for all $P \in \mathscr{P}$. By Information Dominance, therefore, $(f, \mathscr{P}) \sim (g, \mathscr{P})$, hence $V(f, \mathscr{P}) = V(g, \mathscr{P})$. This shows the existence of $I_{\mathscr{P}}$.

Step 5. We claim that $I_{\mathscr{P}}$ has the following properties (we drop the reference to $\mathscr{P}$ for now):

(*i*) If $\varphi \geq \psi$, then $I(\varphi) \geq I(\psi)$.

(*ii*) If $\varphi = \Psi(f)$ and $\psi = \Psi(g)$ are comotonic, then $I(\alpha\varphi + (1-\alpha)\psi) = \alpha I(\varphi) + (1-\alpha)I(\psi)$ for all $\alpha \in (0,1)$.

(*i*) follows easily from Information Dominance. For (*ii*), given what we have shown above,

$$\begin{aligned}I(\alpha\Psi(f) + (1-\alpha)\Psi(g)) &= I(\Psi(\alpha f + (1-\alpha)g)) \\ &= V(\alpha f + (1-\alpha)g, \mathscr{P}) \\ &= \alpha V(f, \mathscr{P}) + (1-\alpha)V(g, \mathscr{P}) \\ &= \alpha I(\Psi(f)) + (1-\alpha)I(\Psi(g)),\end{aligned}$$

since if $\Psi(f)$ and $\Psi(g)$ are comonotonic, then f and g are $\mathscr{P}$-comonotonic.

Step 6. Construction of $\nu^{\mathscr{P}}$. We will now show that I restricted to $\mathbb{B}_0$ is the Choquet integral with respect to some capacity $\nu^{\mathscr{P}}$. I is monotonic and comonotonic additive for functions in $\mathbb{B}_0$. In order to apply Schmeidler (1986)'s representation theorem, we need to extend I to a monotonic and comonotonic additive function on $\mathbb{B}$. We will do this by applying Corollary

1 in Amarante (2009). Let $\mathbb{A}$ be the set of continuous affine functions on $\overline{\text{co}}(\mathscr{P})$. This corollary implies that any monotonic and comonotonic additive function defined on $\mathbb{A}$ can be extended to a monotonic and comonotonic additive function on $\mathbb{B}$. We must therefore show first that that I can be extended to a monotonic and comonotonic additive function on $\mathbb{A}$.

Consider first the (convex) cone generated by $\mathbb{B}_0$,

$$\mathbb{C}_0 = \{\lambda\varphi \mid \lambda > 0, \varphi \in \mathbb{B}_0\}.$$

We can extend I to a positively homogeneous function $\hat{I}$ on $\mathbb{C}_0$ by letting

$$\hat{I}(\lambda\varphi) = \lambda I(\varphi).$$

Let us show that $\hat{I}$ is well-defined. If $\lambda\varphi \leq \mu\psi$, assume without loss of generality that $\lambda \geq \mu$. Then $\varphi \leq \frac{\mu}{\lambda}\psi = \frac{\mu}{\lambda}\psi + (1 - \frac{\mu}{\lambda})0$. Notice that $0 \in \mathbb{B}_0$ since $0 = \Psi(x_*)$. Moreover, $I(0) = V(x_*, \mathscr{P}) = 0$. Therefore $I(\varphi) \leq \frac{\mu}{\lambda}I(\psi)$, hence $\lambda I(\varphi) = \mu I(\psi)$. It is furthermore easy to see that $\hat{I}$ is positively homogeneous and is the unique positively homogeneous extension of I. We therefore call it I again. In particular, for any $k > 0$, $I(k\mathbb{1}_{\mathscr{P}}) = kI(\mathbb{1}_{\mathscr{P}}) = kV(x^*, \mathscr{P}) = k$. It is easily seen to be comonotonic additive and monotonic. Similarly, we can extend I to the set

$$\mathbb{C}_1 = \{\varphi + k \mid \varphi \in \mathbb{C}_0 \text{ and } k \in \mathbb{R}\}$$

by setting

$$\tilde{I}(\varphi + k) = I(\varphi) + k.$$

If $\varphi + k \leq \varphi' + k'$, assume wlog that $k' \geq k$. Then $\varphi \leq \varphi' + k' - k$ and, since I is comonotonic additive and monotonic, this implies that

$$I(\varphi) \leq I(\varphi') + I((k' - k)\mathbb{1}_{\mathscr{P}}) = I(\varphi') + k' - k,$$

hence $I(\varphi) + k \leq I(\varphi') + k'$. Therefore $\tilde{I}$ is well-defined and monotonic. It is the unique monotonic and constant-additive extension of I to $\mathbb{C}_1$, therefore we call it I again. In particular, I is Lipschitz-continuous (w.r.t. the sup-norm) on $\mathbb{C}_1$ and can therefore be uniquely extended to the closure of $\mathbb{C}_1$, denoted $\mathbb{C}_2$.

Let us now show that $\mathbb{A} \subseteq \mathbb{C}_2$. Note first that $[0, 1] \subseteq u(X)$, since u is affine, $u(x^*) = 1$ and $u(x_*) = 0$. Now, take $\varphi \in \mathbb{A}$. By a standard result (Dunford

and Schwartz, 1958, p. 258), there exists a bounded Σ-measurable function $w : S \to \mathbb{R}$ such that for all $P \in \overline{\text{co}}(\mathscr{P})$,

$$\varphi(P) = \int_S w \, \mathrm{d}P.$$

Since w is bounded, it is the limit of a sequence (w_n) of simple Σ-measurable functions. Now, for each n, w_n is bounded, therefore it is possible to find $a_n > 0$, $b_n \in \mathbb{R}$ and a simple Σ-measurable function w'_n such that $0 \leq w'_n \leq 1$ and $w_n = a_n w'_n + b_n$. Letting, $w'_n = \sum_{i=1}^{k_n} t_{in} \mathbb{1}_{A_{in}}$ with $A_{in} \in \Sigma$ and $t_{in} \in [0,1]$ for all $i \in \{1, \ldots, k_n\}$, since $[0,1] \subseteq u(X)$ there exist $x_{1n} \ldots x_{k_n n}$ in X such that $u(x_{in}) = t_{in}$ for all i, hence, setting $f_n = (x_{in}, A_{in})_{1 \leq i \leq k}$, we have $w'_n = u \circ f_n$ and we can define $\varphi_n = a_n \Psi(f_n) + b_n \in \mathbb{C}_1$. Then,

$$\varphi_n(P) = \int_S w_n \, \mathrm{d}P$$

and

$$\begin{aligned}
\|\varphi - \varphi_n\|_\infty &= \sup_{P \in \mathscr{P}} \left| \int_S w - w_n \, \mathrm{d}P \right| \\
&\leq \sup_{P \in \mathscr{P}} \int_S |w - w_n| \, \mathrm{d}P \\
&\leq \sup_{P \in \mathscr{P}} \int_S \|w - w_n\|_\infty \, \mathrm{d}P \\
&= \|w - w_n\|_\infty,
\end{aligned}$$

hence $\varphi_n \to \varphi$ in the sup-norm, and therefore $\varphi \in \mathbb{C}_2$.

We can now apply Amarante (2009)'s result and Schmeidler (1986)'s theorem to show that there exists a capacity $\nu^{\mathscr{P}}$ on $\overline{\text{co}}(\mathscr{P})$ such that I is the Choquet integral with respect to it. In particular,

$$V(f, \mathscr{P}) = I(\Psi(f)) = \int_{\overline{\text{co}}(\mathscr{P})} \Psi(f) \, \mathrm{d}\nu^{\mathscr{P}}.$$

Step 7. We now need to show that V actually represents $\succsim$. We have:

$$\begin{aligned}
(f, \mathscr{P}) \succsim (g, \mathscr{P}') \quad &\iff \quad (c(f, \mathscr{P}), \mathscr{P}) \succsim (c(f, \mathscr{P}')), \mathscr{P}') \\
&\iff \quad v(c(f, \mathscr{P})) \geq v(c(g, \mathscr{P}')) \\
&\iff \quad V(f, \mathscr{P}) \geq V(g, \mathscr{P}').
\end{aligned}$$

where the second line follows from Axiom 6.

This completes the existence proof.

Step 8. For uniqueness, let V_1 be another SODEU representation. Let us show that there exists $a > 0$ and $b \in \mathbb{R}$ such $V_1 = aV + b$. Indeed, we know that since V and V_1 represent the same ordering there exists an increasing mapping $T : V(\mathscr{F} \times \mathfrak{P}) \to \mathbb{R}$ such that $V_1 = T \circ V$. Let us show that T is affine on its domain. Consider t and t' in the domain of T. Then there exists $(f, \mathscr{P})$ and $(g, \mathscr{Q})$ in $\mathscr{F} \times \mathfrak{P}$ such that $t = V(f, \mathscr{P})$ and $t' = V(g, \mathscr{Q})$. Let x and y be their respective certainty equivalents. Then, for any $\alpha \in [0, 1]$:

$$\begin{aligned}
T(\alpha t + (1-\alpha)t') &= T(\alpha V(f, \mathscr{P}) + (1-\alpha)V(g, \mathscr{Q})) \\
&= T(\alpha v(x) + (1-\alpha)v(y)) \\
&= T(v(\alpha x + (1-\alpha)y)) \\
&= v_1(\alpha x + (1-\alpha)y) \\
&= \alpha v_1(x) + (1-\alpha)v_1(y) \\
&= \alpha T(v(x)) + (1-\alpha)T(v(y)) \\
&= \alpha T(t) + (1-\alpha)T(t').
\end{aligned}$$

This implies in particular that u is defined up to a positive affine transformation.

For the uniqueness result regarding the capacity, fix an SODEU representation V with utility u and second order capacities $(\nu^{\mathscr{P}})_{\mathscr{P} \in \mathfrak{P}}$ and another V' with the same utility u and second order capacities $(\mu^{\mathscr{P}})_{\mathscr{P} \in \mathfrak{P}}$. Since we use the same utility, we have $V(f, \mathscr{P}) = V'(f, \mathscr{P})$ for all $(f, \mathscr{P})$, i.e.,

$$\int_{\overline{\mathrm{co}}(\mathscr{P})} \int_S u \circ f \, \mathrm{d}P \, \mathrm{d}\nu^{\mathscr{P}} = \int_{\overline{\mathrm{co}}(\mathscr{P})} \int_S u \circ f \, \mathrm{d}P \, \mathrm{d}\mu^{\mathscr{P}}$$

for all $(f, \mathscr{P})$. Since we have shown above (step 6) that for any affine function $T : \overline{\mathrm{co}}(\mathscr{P}) \to u(X)$ there exists $f \in \mathscr{F}$ such that $T = \Psi(f)$, this shows that for such affine functions,

$$\int_{\overline{\mathrm{co}}(\mathscr{P})} T \, \mathrm{d}\nu^{\mathscr{P}} = \int_{\overline{\mathrm{co}}(\mathscr{P})} T \, \mathrm{d}\mu^{\mathscr{P}}.$$

Letting the normalization of u vary, the same can be obtained for affine functions with values in any interval.

Corollary 1. As discussed in paragraph 4.2, if the preference relation $\succsim$ admits an SODEU representation, then its restriction to sets of the form $\mathscr{F} \times \{\mathscr{P}\}$ is invariant biseparable as defined in Ghirardato et al. (2004). Hence, by Ghirardato et al. (2004, Proposition 5), there exists a unique closed and convex set $\Gamma(\mathscr{P})$ such that

$$f \succsim^*_{\mathscr{P}} g \;\Leftrightarrow\; \int_X u \circ f \,\mathrm{d}P \geq \int_X u \circ g \,\mathrm{d}P \;\text{ for all } P \in \Gamma(\mathscr{P}).$$

Define the relation $\succsim^{**}_{\mathscr{P}}$ by

$$f \succsim^{**}_{\mathscr{P}} g \iff (f, \{P\}) \succsim (g, \{P\}) \text{ for all } P \in \mathscr{P}.$$

Then

$$f \succsim^{**}_{\mathscr{P}} g \iff \int_X u \circ f \,\mathrm{d}P \geq \int_X u \circ g \,\mathrm{d}P \;\text{ for all } P \in \mathscr{P}$$

and, by Information Dominance,

$$f \succsim^{**}_{\mathscr{P}} g \;\Rightarrow\; (f, \mathscr{P}) \succsim (g, \mathscr{P}).$$

This implies that $\succsim^{**}_{\mathscr{P}}$ is a sub-relation of $\succsim$ that satisfies independence and is represented by $\mathscr{P}$. By Ghirardato et al. (2004, Proposition 4) therefore:

$$f \succsim^{**}_{\mathscr{P}} g \;\Rightarrow\; f \succsim^*_{\mathscr{P}} g.$$

and, by Ghirardato et al. (2004, Proposition A.1), we have $\Gamma(\mathscr{P}) \subseteq \overline{\mathrm{co}}(\mathscr{P})$.

Now from Amarante (2009) we know that if the restriction of the preference relation $\succsim$ to sets of the form $\mathscr{F} \times \{\mathscr{P}\}$ is invariant biseparable, then it admits an SODEU representation. Specifically, there exists a capacity $\nu^*_{\mathscr{P}}$ defined on $\Gamma(\mathscr{P})$ such that preferences on $\mathscr{F} \times \{\mathscr{P}\}$ can be represented by the function $W^*_{\mathscr{P}}$ defined by

$$W^*_{\mathscr{P}}(f) = \int_{\Gamma(\mathscr{P})} \left(\int_S u \circ f \;\mathrm{d}P \right) \mathrm{d}\nu^*_{\mathscr{P}}(P).$$

Since for a given $\mathscr{P}$, $V(\cdot, \mathscr{P})$ and $W^*_{\mathscr{P}}$ are both invariant biseparable representations of preferences, we can assume (Ghirardato and Marinacci, 2001, Theorem 9) that their corresponding utility functions are the same, and we denote that function u. Normalize it so that $u^* = 1$ and $u_* = 0$. Let $f \in \mathscr{F}$ and $\mathscr{P} \in \mathfrak{P}$. There exists $x \in X$ such that $(f, \mathscr{P}) \sim (x, \mathscr{P})$. Therefore $V(f, \mathscr{P}) = u(x) = W^*_{\mathscr{P}}(f)$. Therefore, if we extend $\nu^*_{\mathscr{P}}$ to $\overline{\mathrm{co}}(\mathscr{P})$ by letting

$$\nu^*_{\mathscr{P}}(\mathscr{Q}) = \nu^*_{\mathscr{P}}(\mathscr{Q} \cap \Gamma(\mathscr{P}))$$

for all $\mathscr{Q} \subseteq \overline{\text{co}}(\mathscr{P})$, then

$$W^*_{\mathscr{P}}(f) = \int_{\overline{\text{co}}(\mathscr{P})} \left(\int_S u \circ f \, \mathrm{d}P \right) \mathrm{d}\nu^*_{\mathscr{P}}(P)$$

and represents preferences.

A.3 Proof of Theorem 2

Let us first prove the necessity of the axioms. We proceed in several steps.

Step 1. Construction of V. Note first that since $x \in c(x, \{P\})$ for any $x \in X$ and $P \in \mathscr{P}$, the following useful fact holds:

Fact 1 *For all $x, y \in X$, and for all $\alpha \in [0,1]$, $\alpha x + (1-\alpha)y$ is a second order α-mixture of x and y.*

As a consequence, by Weak Order, Continuity, Second Order Independence, Information Irrelevance for Constant Acts and the Mixture Space Theorem, there exists a mapping $v : X \to \mathbb{R}$ unique up to affine transformations such that

(i) For all $x, y \in X$, for all $\alpha \in [0,1]$,

$$v(\alpha x + (1-\alpha)y) = \alpha v(x) + (1-\alpha)v(y);$$

(ii) $(x, \mathscr{P}) \succsim (y, \mathscr{Q}) \iff v(x) \geq v(y)$, for all $x, y \in X$, $\mathscr{P}, \mathscr{Q} \in \mathfrak{P}$.

Now define $V : \mathscr{F} \times \mathfrak{P} \to \mathbb{R}$ by $V(f, \mathscr{P}) = v(c(f, \mathscr{P}))$.

Step 2. Let $\mathscr{P} \in \mathfrak{P}$. Let $\mathbb{B}(\mathscr{P})$ be the set of all real-valued bounded functions on $\mathscr{P}$ endowed with the uniform convergence topology. For any $f \in \mathscr{F}$, let $\Psi(f) \in \mathbb{B}(\mathscr{P})$ be defined by

$$\Psi(f)(P) = V(f, \{P\}).$$

Let $\mathbb{B}_0(\mathscr{P}) := \Psi(\mathscr{F})$. Contrary to what was the case in the proof of the previous theorem, $\mathbb{B}_0(\mathscr{P})$ is not necessarily convex.

By the same arguments as above (with Information Dominance replaced by Information Dominance for Second Order Mixtures) there exists a mapping $I_{\mathscr{P}} : \mathbb{B}_0(\mathscr{P}) \to \mathbb{R}$ such that $V(f, \mathscr{P}) = I_{\mathscr{P}}(\Psi(f))$.

Step 3. $I_{\mathscr{P}}$ has the following properties (we drop the reference to $\mathscr{P}$ for now):

(*i*) Let $(f_i)_{i=1,\dots,n} \in \mathscr{F}^n$, $(g_j)_{j=1,\dots,m} \in \mathscr{F}^m$, $(\lambda_i)_{i=1,\dots,n} \in [0,1]^n$ and$(\mu_j)_{j=1,\dots,m} \in [0,1]^m$ with $\sum_{i=1}^n \lambda_i = 1$ and $\sum_{j=1}^m \mu_j = 1$ and let $\mathscr{P} \in \mathfrak{P}$. Then, if

$$\sum_{i=1}^n \lambda_i \Psi(f_i) \geq \sum_{j=1}^m \mu_j \Psi(g_j),$$

then

$$\sum_{i=1}^n \lambda_i I(\Psi(f_i)) \geq \sum_{j=1}^m \mu_j I(\Psi(g_j).$$

(*ii*) If $\varphi = \Psi(f)$ and $\psi = \Psi(g)$, then $I(\alpha\varphi + (1-\alpha)\psi) = \alpha I(\varphi) + (1-\alpha)I(\psi)$ for all $\alpha \in (0,1]$ such that $\alpha\varphi + (1-\alpha)\psi \in \mathbb{B}_0(\mathscr{P})$.

The first property follows from Information Dominance for Second Order Mixtures: if

$$\sum_{i=1}^n \lambda_i \Psi(f_i) \geq \sum_{j=1}^m \mu_j \Psi(g_j),$$

then

$$\sum_{i=1}^n \lambda_i \Psi(f_i)(P) \geq \sum_{j=1}^m \mu_j \Psi(g_j)(P),$$

for all $P \in \mathscr{P}$, hence

$$\sum_{i=1}^n \lambda_i v(c(f_i, \{P\})) \geq \sum_{j=1}^m \mu_j v(c(g_j, \{P\})),$$

for all $P \in \mathscr{P}$, and therefore

$$v\left(\sum_{i=1}^n \lambda_i c(f_i, \{P\})\right) \geq v\left(\sum_{j=1}^m \mu_j c(g_j, \{P\})\right),$$

for all $P \in \mathscr{P}$, using the affinity of v, hence, by Information Dominance for Second Order Mixtures,

$$v\left(\sum_{i=1}^n \lambda_i c(f_i, \mathscr{P})\right) \geq v\left(\sum_{j=1}^m \mu_j c(g_j, \mathscr{P})\right),$$

and therefore

$$\sum_{i=1}^n \lambda_i v(c(f_i, \mathscr{P})) \geq \sum_{j=1}^m \mu_j v(c(g_j, \mathscr{P})),$$

which implies

$$\sum_{i=1}^{n} \lambda_i I(\Psi(f_i)) \geq \sum_{j=1}^{m} \mu_j I(\Psi(g_j)).$$

Let us prove the second property. The proof follows directly from the following facts:

Fact 2 *For all $h \in \mathscr{F}$, $\alpha\Psi(f) + (1-\alpha)\Psi(g) = \Psi(h)$ if and only if h is a second order α-mixture of f and g.*

Proof.

$$\begin{aligned}
& \alpha\Psi(f) + (1-\alpha)\Psi(g) = \Psi(h) \\
\iff\ & \forall P \in \mathscr{P},\ \alpha\Psi(f)(P) + (1-\alpha)\Psi(g)(P) = \Psi(h)(P) \\
\iff\ & \forall P \in \mathscr{P},\ \alpha v(c(f,\{P\})) + (1-\alpha)v(c(g,\{P\})) = v(c(h,\{P\})) \\
\iff\ & \forall P \in \mathscr{P},\ v(\alpha c(f,\{P\}) + (1-\alpha)c(g,\{P\})) = v(c(h,\{P\})) \\
\iff\ & \forall P \in \mathscr{P},\ (\alpha c(f,\{P\}) + (1-\alpha)c(g,\{P\}),\{P\}) \sim (c(h,\{P\}),\{P\}). \ \square
\end{aligned}$$

Fact 3 *For all $f, g \in \mathscr{F}$, for all $\alpha \in (0,1]$,*

$$(c(h,\mathscr{P}),\mathscr{P}) \sim (\alpha c(f,\mathscr{P}) + (1-\alpha)c(g,\mathscr{P}),\mathscr{P}), \quad \forall h \in \alpha f \oplus_{\mathscr{P}} (1-\alpha)g.$$

Proof. This follows directly from the Second Order Independence Axiom and the definition of certainty equivalents. $\square$

We may now prove the property. Let $\varphi = \Psi(f)$ and $\psi = \Psi(g)$ and $\alpha \in (0,1]$ such that $\alpha\varphi + (1-\alpha)\psi \in \mathbb{B}_0(\mathscr{P})$. Then, there exists $h \in \mathscr{F}$ such that $\alpha\varphi + (1-\alpha)\psi = \Psi(h)$. By fact 2, therefore, h is a second order α-mixture of f and g. Therefore:

$$\begin{aligned}
I(\alpha\varphi + (1-\alpha)\psi) &= I(\Psi(h)) \\
&= v(c(h,\mathscr{P})) \\
&= v(\alpha c(f,\mathscr{P}) + (1-\alpha)c(g,\mathscr{P})) \quad \text{by fact 3} \\
&= \alpha v(c(f,\mathscr{P})) + (1-\alpha)v(c(g,\mathscr{P})) \\
&= \alpha I(\Psi(f)) + (1-\alpha)I(\Psi(g)) \\
&= \alpha I(\varphi) + (1-\alpha)I(\psi).
\end{aligned}$$

Step 4. Construction of $\mu^{\mathscr{P}}$. We will now show that I restricted to $\mathbb{B}_0$ is the integral with respect to some probability charge $\mu^{\mathscr{P}}$. First, let us extend I

to the convex hull of $\mathbb{B}_0$. Let $\varphi \in co(\mathbb{B}_0)$. Consider

$$\varphi = \sum_{i=1}^{n} \lambda_i \varphi_i,$$

with $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$ and $\varphi_i \in \mathbb{B}_0$ for all i, a decomposition of φ. Note first that if such an extension $\hat{I}$ exists, it must satisfy

$$\hat{I}(\varphi) = \sum_{i=1}^{n} \lambda_i \hat{I}(\varphi_i) = \sum_{i=1}^{n} \lambda_i I(\varphi_i).$$

This shows in particular that such an extension is unique. We can therefore denote it I again. We must show that this formula consistently defines the extension. But this follows from property (i) above. Moreover, this property implies that the extension is monotonic. Since we can normalize $V(\cdot, \mathscr{P})$ so that $V(x, \mathscr{P}) = 0$ for some $x \in X$, I can be extended to a linear and monotonic (hence sup-norm continuous) mapping defined on $B(\mathscr{P})$. By the Riesz Representation Theorem, therefore, I is the integral w.r.t. some probability charge $\mu^{\mathscr{P}}$.

In particular,

$$V(f, \mathscr{P}) = I(\Psi(f)) = \int_{\mathscr{P}} \Psi(f) \, \mathrm{d}\mu^{\mathscr{P}} = \int_{\mathscr{P}} V(f, \{P\}) \, \mathrm{d}\mu^{\mathscr{P}}(P).$$

Step 5. Construction of u and Φ such that for all $f \in \mathscr{F}$ and all $P \in pc(\Sigma)$,

$$V(f, \{P\}) = \Phi\left(\int_S u \circ f \, \mathrm{d}P\right).$$

Consider the ordering $\succcurlyeq^{\ell}$ defined on the set $\Delta_0(X)$ of simple lotteries over elements of X defined by

$$\pi \succcurlyeq^{\ell} \pi' \iff (f, \{P\}) \succsim (f, \{Q\}),$$

for some $f \in \mathscr{F}$, $P, Q \in pc(\Sigma)$ such that $P^f = \pi$ and $Q^f = \pi'$. This ordering is well defined because of axiom 9. We want to show that there exists a utility function $u : X \to \mathbb{R}$ such that

$$\pi \succcurlyeq^{\ell} \pi' \iff \int_X u \, \mathrm{d}\pi \geq \int_X u \, \mathrm{d}\pi'.$$

For this it suffices to show that it satisfies all the axioms of the mixture

space theorem. We need a preliminary lemma:

Lemma 4 *For any triple (π, π', π'') of simple lotteries, there exist an act f and probability charges $P, Q, R \in pc(\Sigma)$ such that*

$$P^f = \pi, \quad Q^f = \pi' \quad \text{and} \quad R^f = \pi''.$$

The proof is fairly standard and available upon request.

Lemma 5 *$\succcurlyeq^\ell$ is a weak order.*

Proof. Transitivity follows from lemma 4, axiom 9 and transitivity of $\succsim$. Completeness follows from lemma 4 and completeness of $\succsim$. □

Lemma 6 *For all π, π', π'' such that $\pi \succ^\ell \pi' \succ^\ell \pi''$, there exist $\alpha, \beta \in (0,1)$ such that*

$$\alpha\pi + (1-\alpha)\pi'' \succ^\ell \pi' \succ^\ell \beta\pi + (1-\beta)\pi''.$$

Proof. This follows from lemma 4 and axiom 10. □

Lemma 7 *For all $\pi, \pi', \pi'' \in \Delta_0(X)$ and all $\alpha \in (0,1]$*

$$\pi \succcurlyeq^\ell \pi' \quad \text{iff} \quad \alpha\pi + (1-\alpha)\pi'' \succcurlyeq^\ell \alpha\pi' + (1-\alpha)\pi''.$$

Proof. This follows from lemma 4 and axiom 11. □

Given lemmas 5, 6 and 7, by the mixture space theorem, there exists $u : X \to \mathbb{R}$ such that

$$\pi \succcurlyeq^\ell \pi' \iff \int_X u \,\mathrm{d}\pi \geq \int_X u \,\mathrm{d}\pi'.$$

Now, axiom 9 implies that there exists a function $\sigma : \Delta_0(X) \to \mathbb{R}$ such that $V(f, \{P\}) = \sigma(P^f)$ for all $f \in \mathscr{F}$ and $P \in pc(\Sigma)$. Given the definition of $\succcurlyeq^\ell$, σ is a utility function for $\succcurlyeq^\ell$. Therefore, there exists a strictly increasing $\Phi : \mathbb{R} \to \mathbb{R}$ such that $\sigma(\pi) = \Phi\left(\int_X u \,\mathrm{d}\pi\right)$. Hence,

$$V(f, \{P\}) = \sigma(P^f) = \Phi\left(\int_X u \,\mathrm{d}P^f\right) = \Phi\left(\int_S u \circ f \,\mathrm{d}P\right).$$

Step 6. Cardinal uniqueness of u and $\Phi \circ u$ follows from standard arguments, and the uniqueness property of $\mu^{\mathscr{P}}$ follows from arguments similar to the ones used in the proof of Theorem 1.

A.4 Proofs of Propositions 1, 3, and 4.

Proposition 1. Let us first prove sufficiency of the axioms. First by theorem 1 we know the existence of u and $\nu^{\mathscr{P}}$ for all $\mathscr{P} \in \mathfrak{P}$. Moreover by Schmeidler (1989)'s representation theorem there exists an affine utility function v and for all $\mathscr{P} \in \mathfrak{P}$ a capacity $\zeta^{\mathscr{P}}$ such that V_C defined by

$$V_C(f, \mathscr{P}) := \int_S v \circ f \, \mathrm{d}\zeta^{\mathscr{P}}$$

represents preferences. Now both V (defined in theorem 1) and V_C are canonical biseparable representations of preferences, with utility indexes u and v and willingness to bet $\rho^{\mathscr{P}}$ and $\zeta^{\mathscr{P}}$. Therefore, u and v must be affine transformations of one another and $\rho^{\mathscr{P}}$ and $\zeta^{\mathscr{P}}$ must be identical. □

Proposition 3. Since $(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are straightforward, we need only show that $(i) \Rightarrow (iii)$. Fix $\mathscr{P} \in \mathfrak{P}$ and assume that 1 is weakly more ambiguity averse than 2 given $\mathscr{P}$. Since both preferences are c-linear and biseparable, it follows from Ghirardato and Marinacci (2001, Proposition 11) and Ghirardato and Marinacci (2001, Proposition 16) that there exist $a > 0$ and $b \in \mathbb{R}$ such that $u_2 = au_1 + b$ as wished. Without lost of generality, therefore, since u_i is defined up to positive affine transformations, we can choose $u_1 = u_2 := u$.

Now, let $f \in \mathscr{F}$. There exists $x \in X$ such that $(f, \mathscr{P}) \sim_1 (x, \mathscr{P})$. Therefore $V_1(f, \mathscr{P}) = u(x)$. But since 1 is weakly more ambiguity averse than 2, this implies $(f, \mathscr{P}) \succsim_2 (x, \mathscr{P})$, i.e., $V_2(f, \mathscr{P}) \geq u(x) = V_1(f, \mathscr{P})$. This implies, given the definition of the associated functionals I_1 and I_2, that $I_2 \geq I_1$ on $\mathbb{B}_0$. Therefore, we have

$$\int_{\overline{\mathrm{co}}(\mathscr{P})} T \, \mathrm{d}\nu_2^{\mathscr{P}} \geq \int_{\overline{\mathrm{co}}(\mathscr{P})} T \, \mathrm{d}\nu_1^{\mathscr{P}}.$$

for any affine function $T : \overline{\mathrm{co}}(\mathscr{P}) \to u(X)$. Letting the normalization of u vary, the same can be obtained for affine functions with values in any interval. □

Proposition 4. Necessity is straightforward. We prove sufficiency. For simplicity we drop the superscript $\mathscr{P}$. Suppose there exists $A \in \Sigma$ such that $\rho_1(A) > \rho_2(A)$. By definition of ρ, this implies

$$\min_{P \in \overline{\mathrm{co}}(\mathscr{P})} P(A) \leq \rho_2(A) < \rho_1(A) \leq \max_{P \in \overline{\mathrm{co}}(\mathscr{P})} P(A).$$

Therefore, there exists P_* and P^* in $\overline{\mathrm{co}}(\mathscr{P})$ such that $P_*(A) < \rho_2(A) < \rho_1(A) < P^*(A)$. Therefore, there exists $\alpha \in (0, 1)$ and $P = \alpha P^* + (1 - \alpha) P_* \in \overline{\mathrm{co}}(\mathscr{P})$ such

that

$$\rho_2(A) < P(A) < \rho_1(A).$$

But this contradicts the definition of imprecision aversion. □

References

Ahn, D., S. Choi, D. Gale, and S. Kariv (2011). Estimating ambiguity aversion in a portfolio choice experiment. Technical report, UC Berkeley.

Ahn, D. S. (2008). Ambiguity without a state-space. *Review of Economics Studies 75*(1), 3–28.

Amarante, M. (2009). Foundations of neo-Bayesian statistics. *Journal of Economic Theory 144*, 2146–2173. Columbia University.

Anderson, E., L. P. Hansen, and T. J. Sargent (1999). Robustness, detection and the price of risk. University of Chicago.

Arrow, K. J. and L. Hurwicz (1972). An optimality criterion for decision-making under ignorance. In C. F. Carter and J. Ford (Eds.), *Uncertainty and Expectations in Economics*. Oxford, England: Basil Blackwell & Mott Ltd.

Cerreia-Vioglio, S., F. Maccheroni, M. Marinacci, and L. Montrucchio (2011). Signed integral representations of comonotonic additive functionals. *Journal of Mathematical Analysis and Applications*.

Dunford, N. and J. T. Schwartz (1958). *Linear Operators*, Volume 1. Wiley Interscience.

Epstein, L. G. (2010). A paradox for the 'smooth ambiguity' model of preference. *Econometrica 78*(6), 2085–2099.

Fishburn, P. C. (1970). *Utility Theory for Decision Making*. Wiley and sons.

Gajdos, T., T. Hayashi, J.-M. Tallon, and J.-C. Vergnaud (2008). Attitude toward imprecise information. *Journal of Economic Theory 140*(1), 23–56.

Gajdos, T., J.-M. Tallon, and J.-C. Vergnaud (2004). Decision making with imprecise probabilistic information. *Journal of Mathematical Economics 40*(6), 647–681.

Ghirardato, P., F. Maccheroni, and M. Marinacci (2004). Differentiating ambiguity and ambiguity attitude. *Journal of Economic Theory 118*(2), 133–173.

Ghirardato, P. and M. Marinacci (2001). Risk, ambiguity, and the separation of utility and beliefs. *Mathematics of Operations Research 26*, 864–890.

Ghirardato, P. and M. Marinacci (2002, February). Ambiguity made precise: A comparative foundation. *Journal of Economic Theory 102*(2), 251–289.

Gilboa, I. and D. Schmeidler (1989). Maxmin expected utility with a non-unique prior. *Journal of Mathematical Economics 18*, 141–153.

Giraud, R. (2005). Objective imprecise probabilistic information, second order beliefs and ambiguity aversion: an axiomatization. In *ISIPTA '05, Proceedings of the 4th International Symposium on Imprecise Probabilities and Their Applications*, Pittsburgh, Pennsylvania.

Giraud, R. and J.-M. Tallon (2010). Are beliefs a matter of taste? A case for objective imprecise information. *Theory and Decision*, 1–9.

Gollier, C. (2011). Portfolio choices and asset prices: The comparative statics of ambiguity aversion. *The Review of Economics Studies 78*(4), 1329–1344.

Jaffray, J.-Y. (1989). Généralisation du critère de l'utilité à l'incertain régulier. *Operations Research/ Recherche Opérationnelle 23*(3).

Klibanoff, P., M. Marinacci, and S. Mukerji (2005, November). A smooth model of decision making under ambiguity. *Econometrica 73*(6), 1849–1892.

Klibanoff, P., M. Marinacci, and S. Mukerji (2012). On the smooth ambiguity model: A reply. *Econometrica 80*(3), 1303–1321.

Nascimento, L. and G. Riella (2011). Second-order ambiguous beliefs. *Economic Theory*, 1–33. 10.1007/s00199-011-0675-x.

Nau, R. F. (2006). Uncertainty aversion with second-order probabilities and utilities. *Management Science 52*(1), 136–145.

Nehring, K. (2007, April). Bernoulli without Bayes: A theory of utility-sophisticated preferences under ambiguity. Mimeo, University of California, Davis.

Nehring, K. (2009). Imprecise probabilistic beliefs as a context for decision-making under ambiguity. *Journal of Economic Theory*. Mimeo, University of California, Davis.

Olszewski, W. (2007). Preferences over sets of lotteries. *Review of Economics Studies 74*(04), 567–595.

Rébillé, Y. (2006, December). Decision making over necessity measures through the choquet integral criterion. *Fuzzy Sets and Systems 157*(23), 3025–3039.

Rustichini, A. (1992). Decision theory with higher order beliefs. In *Proceedings of TARK IV*.

Sagi, J. (2007). Modeling implications of source-invariance to Machina's 'almost objective fair bets'. Mimeo, Owen School of Management, Vanderbilt University.

Schmeidler, D. (1986). Integral representation without additivity. *Proceedings of the American Mathematical Society 97*(2), 255–261.

Schmeidler, D. (1989). Subjective probability and expected utility without additivity. *Econometrica 57*, 571– 587.

Seo, K. (2009). Ambiguity and second-order belief. *Econometrica 77*(5), 1575–1605.

Stinchcombe, M. B. (2007). Consequentialist ambiguity: choice under ambiguity as choice between sets of probabilities. mimeo.

Von Neumann, J. and O. Morgenstern (1947). *Theory of Games and Economic Behaviour*. Princeton: Princeton University Press.

Wang, T. (2001, revised 2003). A class of multi-priors preferences. University of British Columbia.

Yaari, M. E. (1987). The dual theory of choice under risk. *Econometrica 55*, 95–115.

Zhou, L. (1998). Integral representation of continuous comonotonically additive functionals. *Transactions of the American Mathematical Society 350*(5), 1811–1822.

Zhou, L. (1999). Subjective probability theory with continuous acts. *Journal of Mathematical Economics 32*(1), 121 – 130.