Online appendix to 'The folk theorem with imperfect public information in continuous time'

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Abstract

Appendix O.1 contains coloured panels of Figures 3 and 4 in Section 3.4. In Appendix O.2 we show how the results and the proofs in the main paper need to be adapted when players are restricted to pure strategies.

O.1 Coloured panels of Figures 3 and 4 in Section 3.4

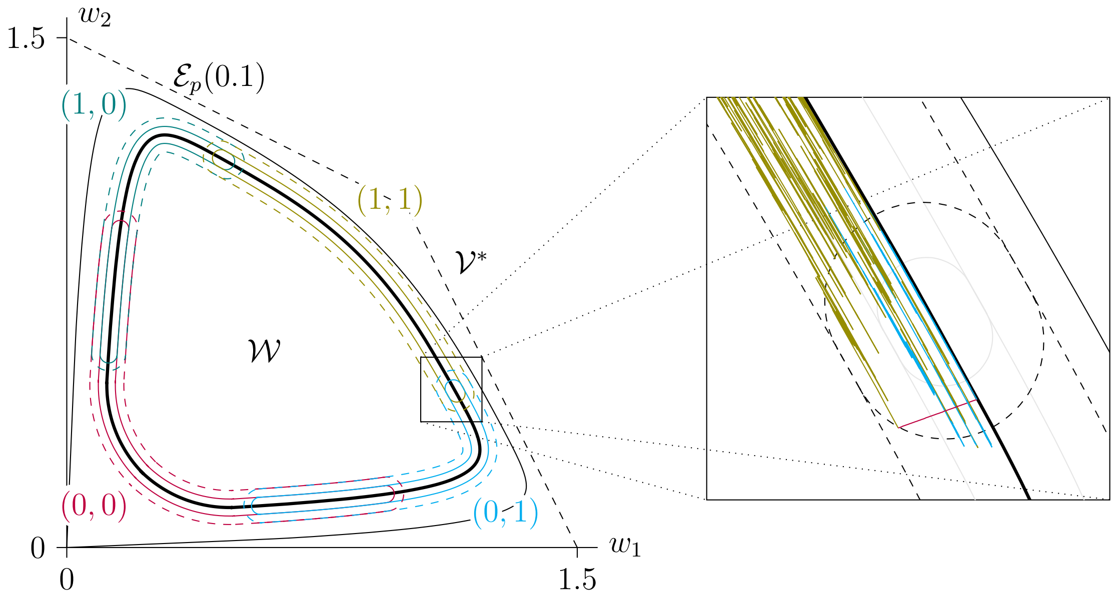

Figure O.1: The left panel shows a cover of $\partial\mathcal{W}$ (bold black line) into four overlapping sets (solid coloured lines), such that payoffs in a band of width ε around the sets (dashed coloured lines) can be decomposed with respect to the same pure action profile for discount rate $r = 0.1$. The cover of $\mathcal{W}$ is completed by playing the static Nash equilibrium in the interior of $\mathcal{W}$. Also depicted is $\partial\mathcal{E}_p(0.1)$ (thin black line) constructed with the techniques in Sannikov [2]. The right panel shows the simulation of the continuation value of a PPE in a zoom-in of the left panel. Lines in olive, cyan and red mean that action profiles $(1, 1)$, $(0, 1)$ and $(0, 0)$, respectively, are played. When the continuation value leaves the band around the cover of $\partial\mathcal{W}$, the static Nash equilibrium is played until the boundary of $\mathcal{W}$ is reached.

O.2 Pure strategies

Observe first, that none of the preliminary results of Lemma 4.10 require mixing. All these results, including the time-change and monotonicity of the equilibrium payoff set therefore translate to games in pure strategies. We will use the following notations.

Definition.

1. $\underline{v}_i^p := \min_{a_{-i} \in \mathcal{A}^{-i}} \max_{a_i \in \mathcal{A}^i} g(a_i, a_{-i})$ is the *pure-action minmax payoff* of player i.
2. $\mathcal{V}_p^* := \{w \in \mathcal{V} \mid w_i \geq \underline{v}_i^p \;\forall i\}$ are the *pure-action individually rational payoffs.*
3. We denote the set of payoffs achievable in pure-strategy PPE by $\mathcal{E}_p(r)$.

A first difference to strategies in mixed actions is that the stage game need not have a Nash equilibrium. We used this stage-game Nash equilibrium in two places, namely to construct an equilibrium profile achieving payoffs in the interior of $\mathcal{W}$ in the proof of Lemma 4.10 and in the Nash-threat version of the folk theorem. While it is clear that we will need to assume existence of a stage-game Nash equilibrium in pure actions for the corresponding version of the folk theorem, we can modify the proof of Lemma 4.10 such that the decomposition in the interior of $\mathcal{W}$ does not rely on it. The argument is similar to Proposition 9.2.2 of Mailath and Samuelson [1], namely, the interior can be decomposed in the same way as the boundary. Indeed, choose an $\varepsilon > 0$ such that $B_{2\varepsilon}(w) \subseteq \operatorname{int} \mathcal{W}$ and let the role of $\mathcal{S}_w$ be taken by any hyperplane H through w. Let N be orthogonal to H and let a be any enforceable action profile such that $g(a) \notin H$ and (a, N) satisfy a condition of Lemma 4.7. Proceeding in the same way as on the boundary, we obtain continuations W that remain in $B_{2\varepsilon}(w) \subseteq \mathcal{W}$. Note that this does not affect the nature of the strategies, i.e., the resulting equilibrium profiles are constant on the sets $[\![k\tilde{\tau}, (k+1)\tilde{\tau}[\![$.

A more significant difference to strategies in mixed actions is that we lose the approximation Lemmas C.2 and C.3 for the decomposition of a set $\mathcal{W}$. While we do not need Lemma C.3 because individual full rank of all pure action profiles implies enforceability of the minmax profile on the relevant coordinate hyperplane directly by Lemma 4.6, the loss of Lemma C.2 is more grievous. For the decomposition of regular payoffs, existence of a pairwise identifiable action profile (or a Nash equilibrium) with inward pointing drift is necessary. Without the approximation in Lemma C.2, we can only hope to decompose payoff sets $\mathcal{W}$ in the interior of

$$\mathcal{V}_p^{pw} := \operatorname{conv} g\big(\{a \in \mathcal{A} \mid a \text{ is enforceable and pairwise identifiable}\} \cup \mathcal{A}^{\mathcal{N}}\big).$$

Suppose first that there exists a stage-game Nash equilibrium a^e in pure actions and let $\mathcal{V}_p^0$ denote the convex hull of $g(a^e)$ and the Pareto-efficient pure action payoffs Pareto-dominating $g(a^e)$. Lemma C.4 in conjunction with the assumption on pairwise

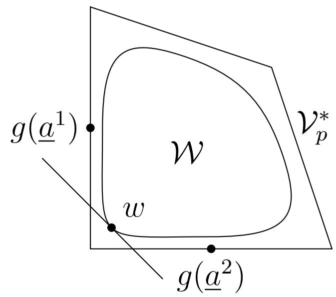

Figure O.2: Restricting players to pure strategies may cause problems for the decomposition of payoffs in the lower corner of $\mathcal{W}$.

identifiability of Pareto-efficient action profiles leads to $\mathcal{V}_p^0 \subseteq \mathcal{V}_p^{pw}$, hence the pure-strategy Nash-threat folk theorem looks very similar to the corresponding result with strategies in mixed actions.

Theorem O.1 (Nash-threat folk theorem in pure strategies)**.** *Suppose that g is affine in m, that there exists a Nash equilibrium a^e in pure actions and that Pareto-efficient action profiles are pairwise identifiable. Then for any smooth set $\mathcal{W}$ in the interior of $\mathcal{V}_p^0$, there exists a discount rate $\tilde{r} > 0$ such that $\mathcal{W} \subseteq \mathcal{E}_p(r)$ for all $r \in (0, \tilde{r})$.*

To establish a minmax version of the folk theorem, we need to decompose payoffs in the lower corner of a set $\mathcal{W}$, which can cause trouble as seen in Figure O.2 if we do not have access to a density argument as in Lemma C.2. It is difficult to find a sufficient condition for $\mathcal{V}_p^* \subseteq \mathcal{V}_p^{pw}$ that applies in full generality, other than the strong condition that all action profiles achieving extremal payoffs have pairwise full rank. We state the following weaker version of the minmax folk theorem, together with a corollary for the result to apply as usual.

Theorem O.2. *Suppose that for every player i,*

1. *the minmax profile $\underline{a}^i$ is enforceable and it is either pairwise identifiable or satisfies the unique best response property for player i,*
2. *there exists an enforceable action profile $a^{i,*} \in \mathcal{A}^{(i)}$ that is either pairwise identifiable or satisfies the unique best response property for player i.*

Then for any smooth set $\mathcal{W} \subseteq \operatorname{int} \mathcal{V}_p^* \cap \mathcal{V}_p^{pw}$, there exists a discount rate $\tilde{r} > 0$ such that $\mathcal{W} \subseteq \mathcal{E}_p(r)$ for all $r \in (0, \tilde{r})$.

Corollary O.3 (Minmax folk theorem in pure strategies)**.** *Suppose that all pure action profiles achieving extremal payoffs have pairwise full rank and that conditions 1 and 2 of Theorem O.2 are met. Then for any smooth set $\mathcal{W} \subseteq \operatorname{int} \mathcal{V}_p^*$, there exists a discount rate $\tilde{r} > 0$ such that $\mathcal{W} \subseteq \mathcal{E}_p(r)$ for all $r \in (0, \tilde{r})$.*

We conclude with an example where the conditions of Corollary O.3 are not met but Theorem O.2 still applies. In Sannikov [2], games where m equals the identity function are called a game with a *special signal structure*. A game with a special

	1	2	3	4
1	$(-8,-6)$	$(-4,12)$	$(0,2)$	$(4,4)$
2	$(9,0)$	$(8,4)$	$(7,5)$	$(6,6)$
3	$(20,6)$	$(12,-4)$	$(4,8)$	$(-4,8)$
4	$(0,12)$	$(4,-12)$	$(8,11)$	$(12,10)$

Table O.1: Payoff table of a minimal example to Theorem O.2.

signal structure is a special case of a game with a product structure where $d_i = 1$ for all i and hence $d = n$. As a result all action profiles are pairwise identifiable but no action profile has individual full rank for player i unless $|\mathcal{A}^i| = 2$. Nevertheless, a folk theorem in pure strategies may apply via Theorem O.2.

One main advantage of the special signal structure is the fact that the enforceability condition (6) for player i depends on a_{-i} only through $g_i(\,\cdot\,, a_{-i})$. This leads to the following characterization of enforceable pure action profiles via the graphs $g_i(\,\cdot\,, a_{-i})$ for the individual players. It has been observed in Sannikov [2] already and is reproduced here for the sake of exposition.

Lemma O.4. *In a game with a special signal structure, for a pure action profile $a \in \mathcal{A}$ the following are equivalent:*

(a) a is enforceable,

(b) $g_i(a)$ is contained in the concave envelope of $g_i(\,\cdot\,, a_{-i})$ for all i.

Proof. Fix a player i. Being in the concave envelope of $g_i(\,\cdot\,, a_{-i})$ means that there exists a $\lambda_i \in \mathbb{R}$ such that $g_i(\tilde{a}_i, a_{-i}) \leq g_i(A) + \lambda_i(\tilde{a}_i - a_i)$ for all $\tilde{a}_i \in \mathcal{A}^i$. But then β^i with $\beta^i_i = -\lambda_i$ fulfills (6). The argument also works the other way around. □

Consider the game with a special signal structure, where the stage-game payoffs are given by the matrix in Table O.1. In this game, any payoff in the interior of $\mathcal{V}^*_p$ can be achieved by a pure-strategy PPE for a discount rate r small enough, even though no action profile has full rank, there is no stage-game Nash equilibrium in pure actions and not all of the extremal payoffs correspond to enforceable action profiles. Hence it justifies why we stated Theorem O.2 in this weak version.

The payoffs can be generated by $c_i \equiv 0$ and

$$b_1 = \begin{pmatrix} -12 & 5 & \frac{28}{3} & -1 \\ 4 & -1 & -8 & 4 \end{pmatrix}, \qquad b_2 = \begin{pmatrix} 6 & -8 & 3 & 2 \\ -12 & 10 & -\frac{1}{3} & \frac{1}{2} \end{pmatrix},$$

for $m(a) = a$, where the j^{th} column of b_i corresponds to $b_i(j)$. This shows that the payoff table is consistent with a special signal structure. As a result, all action profiles

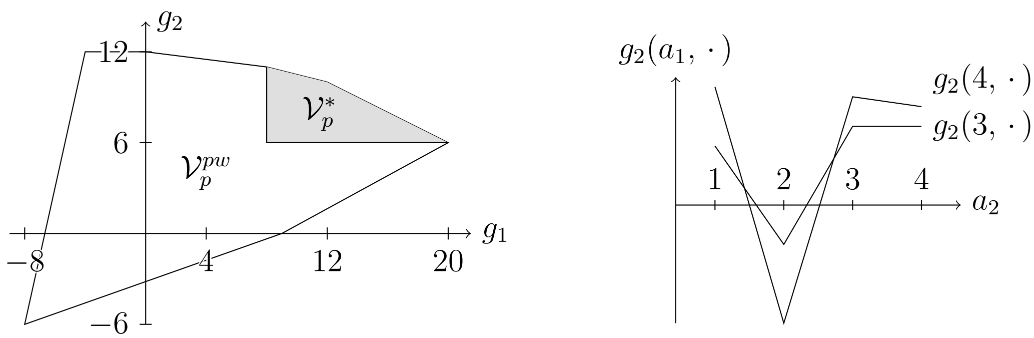

Figure O.3: The right panel shows together with Lemma O.4 that the action profiles $(3,2)$ and $(4,2)$ achieving the extremal payoffs $(12,-4)$ and $(4,-12)$ are not enforceable. Nevertheless, the folk theorem holds via Theorem O.2 because the minmax profiles $\underline{a}^1 = (4,3)$ and $\underline{a}^2 = (2,4)$ are enforceable and $\mathcal{V}_p^* \subseteq \mathcal{V}_p^{pw}$, as indicated in the left panel.

are pairwise identifiable and hence condition 2 of Theorem O.2 is satisfied. Next, we check with Lemma O.4 which action profiles are enforceable to determine $\mathcal{V}_p^{pw}$. This is done exemplarily in the right panel of Figure O.3 for the action profiles $(3,2)$ and $(4,2)$ achieving the extremal payoffs $(12,-4)$ and $(4,-12)$ respectively that fail to be enforceable. The only other action profiles that fail to be enforceable are $(1,3)$, $(3,3)$ and $(3,4)$, thus we obtain $\mathcal{V}_p^{pw}$ as indicated in the left panel of Figure O.3. In particular, the minmax profiles $\underline{a}^1 = (4,3)$ and $\underline{a}^2 = (2,4)$ are enforceable, hence condition 1 of Theorem O.2 is satisfied and $\mathcal{V}_p^* \subseteq \mathcal{V}_p^{pw}$. Therefore, Theorem O.2 applies, hence any payoff in the interior of $\mathcal{V}_p^*$ can be achieved in equilibrium for r small enough.

References

[1] George J. Mailath and Larry Samuelson: *Repeated Games and Reputations*, Oxford University Press, 2006

[2] Yuliy Sannikov: Games with imperfectly observable actions in continuous time, *Econometrica*, **75** (2007), 1285–1329