Online Supplementary Appendix to:

Dynamic Markets for Lemons: Performance, Liquidity, and Policy Intervention

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Contents:

1 Proofs of Lemmas 1 and 2

Proof of Lemma 1: Let $t \in \{1, \ldots, T\}$. We prove $L1.1$. Write $\bar{p} = \max\{r_t^H, r_t^L\}$, and suppose that $\lambda_t(\bar{p}) < 1$. Then there is $\hat{p} > \bar{p}$ in the support of λ_t. Since $I(\bar{p}, r_t^\tau) = I(\hat{p}, r_t^\tau) = 1$ for $\tau \in \{H, L\}$, we have

$$\begin{aligned} V_t^B &\geq \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(\bar{p}, r_t^\tau)(u^\tau - \bar{p}) + \left[1 - \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(\bar{p}, r_t^\tau)\right] \delta V_{t+1}^B \\ &= \alpha \sum_{\tau \in \{H,L\}} q_t^\tau \left(u^\tau - \bar{p}\right) + (1-\alpha)\, \delta V_{t+1}^B \\ &> \alpha \sum_{\tau \in \{H,L\}} q_t^\tau \left(u^\tau - \hat{p}\right) + (1-\alpha)\, \delta V_{t+1}^B \\ &= \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(\hat{p}, r_t^\tau)(u^\tau - \hat{p}) + \left[1 - \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(\hat{p}, r_t^\tau)\right] \delta V_{t+1}^B, \end{aligned}$$

which contradicts $DME.B$.

We prove $L1.2$ by induction. Because $V_{T+1}^\tau = 0$ for $\tau \in \{B, H, L\}$, then $DME.H$ and $DME.L$ imply

$$r_T^H = c^H + \delta V_{T+1}^H = c^H > c^L = r_T^L = c^L + \delta V_{T+1}^L.$$

Hence $\lambda_T(c^H) = 1$ by $L1.1$, and therefore $V_T^H = 0$ and $V_T^L \leq c^H - c^L$. Also, if $q_T^H > \bar{q}$, then offering the high price $r_T^H = c^H$ yields a payoff $u(q_T^H) - c^H > u(\bar{q}) - c^H > 0$, and if $q_T^H \leq \bar{q}$, then $q_T^L > 0$, and therefore offering the low price $r_T^L = c^L$ yields a payoff $q_T^L \left(u^L - c^L\right) > 0$. Hence in either case $V_T^B > 0$. Let $k \leq T$, and assume that $L1.2$ holds for $t \in \{k, \ldots, T\}$; we show that it holds for $k-1$. Since $V_k^H = 0$, $DME.H$ implies $r_{k-1}^H = c^H + \delta V_k^H = c^H$. Since $V_k^L \leq c^H - c^L$ and $\delta < 1$, then $DME.L$ implies $r_{k-1}^L = c^L + \delta V_k^L \leq (1-\delta)c^L + \delta c^H < c^H$. Hence $\lambda_k(c^H) = 1$ by $L1.1$, and therefore $V_{k-1}^H = 0$. Also since $\lambda_t(c^H) = 1$ for $t \geq k-1$, then $V_{k-1}^L \leq c^H - c^L$. Finally, $V_{k-1}^B \geq \delta V_k^B > 0$.

In order to prove $L1.3$, note that $L1.2$ implies $\lambda_t^H \leq \lambda_t^L$. Hence

$$q_{t+1}^H = \frac{m_{t+1}^H}{m_{t+1}^H + m_{t+1}^L} = \frac{(1 - \alpha\lambda_t^H) m_t^H}{(1 - \alpha\lambda_t^H) m_t^H + (1 - \alpha\lambda_t^L) m_t^L} \geq \frac{m_t^H}{m_t^H + m_t^L} = q_t^H.$$

As for $L1.4$, it is a direct implication of $L1.1$ and $L1.2$.

We prove $L1.5$. Suppose that $\lambda_t(p) > \lambda_t(r_t^L)$ for some $p \in (r_t^L, r_t^H)$. Then there is $\hat{p}$ in the support of λ_t such that $r_t^L < \hat{p} < r_t^H$. Since $I(\hat{p}, r_t^L) = 1$ and $I(\hat{p}, r_t^H) = 0$, then

$$\begin{aligned}
V_t^B &\geq \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(r_t^L, r_t^\tau)(u^\tau - r_t^L) + \left[1 - \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(r_t^L, r_t^\tau)\right] \delta V_{t+1}^B \\
&= \alpha q_t^L (u^L - r_t^L) + (1 - \alpha q_t^L) \delta V_{t+1}^B \\
&> \alpha q_t^L (u^L - \hat{p}) + (1 - \alpha q_t^L) \delta V_{t+1}^B \\
&= \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(\hat{p}, r_t^\tau)(u^\tau - \hat{p}) + \left[1 - \alpha \sum_{\tau \in \{H,L\}} q_t^\tau I(\hat{p}, r_t^\tau)\right] \delta V_{t+1}^B,
\end{aligned}$$

which contradicts $DME.B$. $\square$

Proof of Lemma 2: We prove $L2.1$. Assume by way of contradiction that the claim does not hold, and let $\bar{t}$ be the first date such that $\rho_{\bar{t}}^H = 1$. By $P2.2$, $\bar{t} > 1$. We show that $\rho_{\bar{t}-1}^H = 1$, which contradicts that $\bar{t}$ is the first date for which $\rho_{\bar{t}}^H = 1$. Since $\rho_{\bar{t}}^H = 1$ and $V_t^L \geq 0$ for all t, we have

$$V_{\bar{t}}^L = \alpha(c^H - c^L) + (1-\alpha)\, \delta V_{\bar{t}+1}^L \geq \alpha(c^H - c^L).$$

Since frictions are small, then $\alpha\delta(c^H - c^L) > u^L - c^L$, and therefore

$$r_{\bar{t}-1}^L = c^L + \delta V_{\bar{t}}^L \geq c^L + \alpha\delta(c^H - c^L) > c^L + u^L - c^L = u^L.$$

Hence offering $r_{\bar{t}-1}^L$ at date $\bar{t}-1$ is suboptimal, i.e., $\rho_{\bar{t}-1}^L = 0$. Moreover, $q_{\bar{t}-1}^H = q_{\bar{t}}^H$. Since offering $r_{\bar{t}}^H$ at date $\bar{t}$ is optimal we have

$$V_{\bar{t}}^B = \alpha(u(q_{\bar{t}}^H) - c^H) + (1-\alpha)\, \delta V_{\bar{t}+1}^B,$$

and $u(q_{\bar{t}}^H) - c^H \geq \delta V_{\bar{t}+1}^B > 0$ (by $L1.2$). Thus, offering $r_{\bar{t}-1}^H = c^H$ ($L1.2$) at date $\bar{t}-1$ yields

$$\alpha(u(q_{\bar{t}-1}^H) - c^H) + (1-\alpha)\, \delta V_{\bar{t}}^B = \alpha(u(q_{\bar{t}}^H) - c^H)\,(1 + (1-\alpha)\delta) + (1-\alpha)^2\, \delta^2 V_{\bar{t}+1}^B.$$

Then we have

$$\begin{aligned}
\alpha(u(q_{\bar{t}-1}^H) - c^H) + (1-\alpha)\, \delta V_{\bar{t}}^B - \delta V_{\bar{t}}^B &= \alpha(u(q_{\bar{t}}^H) - c^H)\,(1 - \alpha\delta) - (1-\alpha)\, \delta^2 \alpha V_{\bar{t}+1}^B \\
&\geq \alpha(u(q_{\bar{t}}^H) - c^H)\,(1 - \delta)\,(1 + \delta(1-\alpha)) \\
&> 0.
\end{aligned}$$

Hence offering a negligible price at date $\bar{t}-1$ is suboptimal, i.e., $1-\rho^L_{\bar{t}-1}-\rho^H_{\bar{t}-1}=0$. Since $\rho^L_{\bar{t}-1}=0$, then $\rho^H_{\bar{t}-1}=1$.

We prove $L2.2$. We first show that $\rho^L_t<1$ for $t<T$. Assume by way of contradiction that $\rho^L_t=1$ for some $t<T$. Then $L1.3$ and$\bar{\rho}/\alpha\delta<1$ by the inequality $F.2$ imply

$$q^H_T \geq q^H_{t+1} = g(q^H_t, 0) > g(q^H, \bar{\rho}/\alpha\delta) > \hat{q}.$$

Hence

$$q^H_T u^H + q^L_T u^L - c^H > \hat{q} u^H + (1-\hat{q})\, u^L - c^H = (1-\hat{q})\,(u^L - c^L) > q^L_T(u^L - c^L),$$

i.e., offering $r^L_T = c^L$ at date T is suboptimal, and therefore $\rho^L_T = 0$. Thus, $\rho^H_T = 1$ by $P2.3$, which contradicts $L2.1$.

We show that $\rho^L_T<1$. Assume that $\rho^L_T=1$. Then $q^H_T\leq\hat{q}$ (since otherwise an offer of r^L_T is suboptimal), $V^L_T=0$ and $V^B_T=\alpha q^L_T(u^L-c^L)$. Hence $r^L_{T-1}=c^L$ by $DME.L$, and

$$\begin{aligned}
q^L_{T-1}(u^L - r^L_{T-1}) + q^H_{T-1}\delta V^B_T &= q^L_{T-1}(u^L - c^L) + (1-q^L_{T-1})\delta V^B_T \\
&> q^L_{T-1}\delta V^B_T + (1-q^L_{T-1})\delta V^B_T \\
&= \delta V^B_T,
\end{aligned}$$

i.e., the payoff to offering r^L_{T-1} at date $T-1$ is greater than that of offering a negligible price. Therefore $\rho^L_{T-1}+\rho^H_{T-1}=1$. Since $q^H_{T-1}\leq q^H_T$ by $L1.3$ and $q^H_T\leq\hat{q}$, then the payoff to offering $r^H_{T-1}=c^H$ at $T-1$ is

$$\begin{aligned}
q^H_{T-1}u^H + q^L_{T-1}u^L - c^H &\leq q^H_T u^H + q^L_T u^L - c^H \\
&\leq q^L_T(u^L - c^L) \\
&\leq q^L_{T-1}(u^L - c^L) \\
&< q^L_{T-1}(u^L - c^L) + q^H_{T-1}\delta V^B_T,
\end{aligned}$$

where the last term is the payoff to offering $r^L_{T-1}=c^L$ at $T-1$. Hence $\rho^H_{T-1}=0$, and therefore $\rho^L_{T-1}=1$, which contradicts that $\rho^L_t<1$ for all $t<T$ as shown above. Hence $\rho^L_T<1$.

We prove $L2.3$. By $P2.3$, $L2.1$ and $L2.2$, we have $\rho^H_T>0$ and $\rho^L_T>0$. Since both high price offers and low price offers are optimal at date T, and reservation prices are

$r_T^H = c^H$ and $r_T^L = c^L$, we have

$$q_T^H u^H + q_T^L u^L - c^H = q_T^L(u^L - c^L).$$

Thus, using $q_T^L = 1 - q_T^H$ and solving for q_T^H yields

$$q_T^H = \frac{c^H - c^L}{u^H - c^L} = \hat{q}.$$

We prove $L2.4$ by induction. By $L2.3$, $V_T^L = \alpha\rho_T^H \left(c^H - c^L\right) > 0$. Since $V_t^L \geq \delta V_{t+1}^L$ for all $t \leq T$, then $V_t^L \geq \delta^{T-t} V_T^L > 0$.

We prove $L2.5$. Suppose by way of contradiction that $\rho_t^L = 0$ for some t. Since $\rho_T^L > 0$ by $L2.3$, then $t < T$. Also $\rho_t^L = 0$ implies $\rho_t^H > 0$ by $P2.1$. Since $\rho_t^H < 1$ by $L2.1$, then buyers are indifferent at date t between offering c^H or a negligible price, i.e.,

$$q_t^H u^H + q_t^L u^L - c^H = \delta V_{t+1}^B.$$

We show that $\rho_{t+1}^H = 0$. Suppose that $\rho_{t+1}^H > 0$; then

$$V_{t+1}^B = \alpha(q_{t+1}^H u^H + q_{t+1}^L u^L - c^H) + (1-\alpha)\delta V_{t+2}^B.$$

Hence $\delta < 1$ and $V_{t+1}^B > 0$ by $L1.2$ imply

$$q_t^H u^H + q_t^L u^L - c^H = \delta V_{t+1}^B < V_{t+1}^B = \alpha(q_{t+1}^H u^H + q_{t+1}^L u^L - c^H) + (1-\alpha)\delta V_{t+2}^B,$$

But $\rho_t^L = 0$ implies that $q_{t+1}^H = q_t^H$, and therefore

$$q_{t+1}^H u^H + q_{t+1}^L u^L - c^H < \delta V_{t+2}^B,$$

i.e., offering c^H at date $t+1$ yields a payoff smaller than offering a negligible price, which contradicts that $\rho_{t+1}^H > 0$.

Since $\rho_{t+1}^H = 0$, then $DME.L$ implies

$$V_{t+1}^L = \alpha\rho_{t+1}^L(r_{t+1}^L - c^L) + (1-\alpha\rho_{t+1}^L)\delta V_{t+2}^L = \delta V_{t+2}^L.$$

Since $V_{t+1}^L > 0$ by $L2.4$, then $V_{t+2}^L > 0$, and therefore $DME.L$ and $\delta < 1$ imply

$$r_t^L = c^L + \delta V_{t+1}^L = c^L + \delta^2 V_{t+2}^L < c^L + \delta V_{t+2}^L = r_{t+1}^L.$$

i.e., $r_t^L < r_{t+1}^L$. We show that this inequality cannot hold, which leads to a contradiction.

Since $\rho_t^H < 1$ by $L2.1$, then $\rho_t^L = 0$ implies $1 - \rho_t^H - \rho_t^L > 0$; i.e., negligible price offers are optimal at date t. Hence at date t the payoff to offering r_t^L must be less than or equal to the payoff to offering a negligible price, i.e.,

$$q_t^H \delta V_{t+1}^B + q_t^L (u^L - r_t^L) \leq \delta V_{t+1}^B.$$

Using $q_t^H = 1 - q_t^L$ we may write this inequality as

$$u^L - r_t^L \leq \delta V_{t+1}^B.$$

Likewise, $\rho_{t+1}^H = 0$ implies $0 < \rho_{t+1}^L < 1$ by $P2.1$ and $L2.2$, and therefore $1 - \rho_{t+1}^H - \rho_{t+1}^L > 0$. Hence low and negligible price offers are both optimal at date $t+1$, and therefore

$$V_{t+1}^B = \alpha q_{t+1}^L (u^L - r_{t+1}^L) + (1 - \alpha q_{t+1}^L) \delta V_{t+2}^B = \delta V_{t+2}^B.$$

Hence

$$V_{t+1}^B = u^L - r_{t+1}^L.$$

Thus, $\delta < 1$ and $V_{t+1}^B > 0$ by $L1.2$ imply

$$u^L - r_t^L \leq \delta V_{t+1}^B < V_{t+1}^B = u^L - r_{t+1}^L.$$

Therefore $r_t^L > r_{t+1}^L$, which contradicts $r_t^L < r_{t+1}^L$.

We prove $L2.6$. For $t \in \{1, \ldots, T\}$, since $V_t^L \geq 0$, and $r_t^L - c^L = \delta V_{t+1}^L$ by $DME.L$, we have

$$\begin{aligned} V_t^L &= \alpha \left(\rho_t^H (c^H - c^L) + \rho_t^L (r_t^L - c^L) \right) + \left(1 - \alpha (\rho_t^H + \rho_t^L) \right) \delta V_{t+1}^L \\ &\geq \alpha \rho_t^H (c^H - c^L). \end{aligned}$$

By $P2.2$, we have $\rho_1^H = 0 < \bar{\rho}/\alpha\delta$. For $1 < t \leq T$, since $\rho_{t-1}^L > 0$ by $L2.5$ (i.e., low price offers are optimal at date $t-1$) and $V_{t-1}^B > 0$ by $L1.2$, then $u^L > r_{t-1}^L$. Hence

$$u^L - c^L > r_{t-1}^L - c^L = \delta V_t^L \geq \alpha \delta \rho_t^H \left(c^H - c^L \right),$$

and therefore

$$\rho_t^H < \frac{u^L - c^L}{\alpha \delta \left(c^H - c^L \right)} = \bar{\rho}/\alpha\delta.$$

Finally, we prove ($L2.7$). Let $t \in \{1, \ldots, T-1\}$. We proceed by showing that (i) $\rho_t^H > 0$ implies $\rho_t^H + \rho_t^L < 1$, and (ii) $\rho_t^H + \rho_t^L < 1$ implies $\rho_{t+1}^H > 0$. Then $L2.7$ follows by induction: Since $\rho_1^H = 0$ by $P2.2$ and $\rho_1^L < 1$ by $L2.2$, then $\rho_1^H + \rho_1^L < 1$, and therefore $\rho_2^H > 0$ by (ii). Assume that $\rho_k^H + \rho_k^L < 1$ and $\rho_{k+1}^H > 0$ holds for some $1 \le k < T - 1$; we show that $\rho_{k+1}^H + \rho_{k+1}^L < 1$ and $\rho_{k+2}^H > 0$. Since $\rho_{k+1}^H > 0$, then $\rho_{k+1}^H + \rho_{k+1}^L < 1$ by (i), and therefore $\rho_{k+2}^H > 0$ by (ii).

We establish (i), i.e., $\rho_t^H > 0$ implies $\rho_t^H + \rho_t^L < 1$. Suppose not; let $t < T$ be the first date such that $\rho_t^H > 0$ and $\rho_t^H + \rho_t^L = 1$. Since $q_t^H \geq q_1^H = q^H$ by $L1.3$, and $\rho_t^H < \bar{\rho}/\alpha\delta$ by $L2.6$, then $g(q^H, \bar{\rho}/\alpha\delta) > \hat{q}$ (by $F.2$) and $L2.3$ imply

$$q_{t+1}^H = g(q_t^H, \rho_t^H) > g(q^H, \bar{\rho}/\alpha\delta) > \hat{q} = q_T^H,$$

which contradicts $L1.3$.

Next we prove (ii), i.e., $\rho_t^H + \rho_t^L < 1$ implies $\rho_{t+1}^H > 0$. Suppose by way of contradiction that $\rho_t^H + \rho_t^L < 1$ and $\rho_{t+1}^H = 0$ for some $t < T$. Since $\rho_t^L > 0$ by $L2.5$, then low and negligible offers are optimal at date t. Hence

$$u^L - r_t^L = \delta V_{t+1}^B.$$

Since $\rho_{t+1}^H = 0$, then

$$V_{t+1}^L = \delta V_{t+2}^L.$$

Since $V_{t+1}^L > 0$ by $L2.4$ and $\delta < 1$, we have

$$r_{t+1}^L = c^L + \delta V_{t+2}^L = c^L + V_{t+1}^L > c^L + \delta V_{t+1}^L = r_t^L.$$

Since $0 < \rho_{t+1}^L < 1$ by $L2.2$ and $L2.5$ and $\rho_{t+1}^H = 0$, then $1 - \rho_{t+1}^H - \rho_{t+1}^L > 0$; i.e., low and negligible offers are optimal at $t + 1$. Therefore

$$u^L - r_{t+1}^L = \delta V_{t+2}^B.$$

Thus, $V_{t+1}^B > 0$ by $L1.2$ and $\delta < 1$ imply

$$u^L - r_t^L = \delta V_{t+1}^B < V_{t+1}^B = \delta V_{t+2}^B = u^L - r_{t+1}^L,$$

i.e., $r_t^L > r_{t+1}^L$, which contradicts the inequality above. $\square$

2 Policy Intervention Formulae

The Public-Private Investment Program for Legacy Assets

As noted in Section 4, the introduction of a small PPIP subsidy $s > 0$ in a market where $1 < T < \infty$ affects the equilibrium sequences of probabilities of high price offers ρ^H and the reservation prices of low quality sellers r^L, as well as the traders' payoffs and surplus, via its impact on $\hat{q}(s)$, where

$$\hat{q}(s) = \frac{c^H - c^L - s}{u^H - c^L - s},$$

and hence via the functions $\bar{\phi}(s) = (1 - \hat{q}(s))\left(u^L - c^L\right)$, and $\phi_t(s) = \alpha\delta^{T-t}\bar{\phi}(s)$. The formulae describing the sequence of probabilities of low price offers ρ^L is

$$\rho_1^L(s) = \frac{\phi_2(s) - (u(q^H) - c^H) - (1 - q^H)s}{\alpha(1 - q^H)(c^H - u^L - s + \phi_2(s))},$$

and $\rho_T^L = 1 - \rho_T^H$. If $T > 2$, then

$$\rho_t^L(s) = (1 - \alpha\rho_t^H(s))\frac{(1 - \delta)\phi_{t+1}(s)}{\alpha(c^H - u^L - s + \phi_{t+1}(s))}\frac{u^H - u^L - s}{u^H - c^H - \phi_t(s)}$$

for all $1 < t < T - 1$, and

$$\rho_{T-1}^L(s) = (1 - \alpha\rho_{T-1}^H(s))\frac{u(\hat{q}(s)) - c^H + (1 - \hat{q}(s))s - \phi_{T-1}(s)}{\alpha\hat{q}(s)(u^H - c^H - \phi_{T-1}(s))}.$$

As δ approaches one, the high price is offerred with positive probability only at date T. Hence the cost of the subsidy $C(s)$ is

$$\begin{aligned}
C(s) &= s\alpha\rho_T^H(s)m_T^L(s) \\
&= s\frac{u^L - c^L - \alpha\bar{\phi}(s)}{c^H - c^L}m^H\frac{(1 - \hat{q}(s))}{\hat{q}(s)} \\
&= sm^H\frac{u^L - c^L - \alpha\bar{\phi}(s)}{c^H - c^L}\frac{u^H - c^H}{c^H - c^L - s}.
\end{aligned}$$

The net surplus, $NS(s)$, is

$$\begin{aligned}
NS(s) &= [\tilde{S}^{DME}(s) - C(s)] - \tilde{S}^{DME}(0) \\
&= m^H\alpha(u^L - c^L)\left(\hat{q}(0) - \hat{q}(s)\right) - sm^H\frac{u^L - c^L - \alpha\bar{\phi}(s)}{c^H - c^L}\frac{u^H - c^H}{c^H - c^L - s} \\
&= \frac{sm^H\left(u^H - c^H\right)\left(u^L - c^L\right)}{u^H - c^L - s}\Gamma(\alpha),
\end{aligned}$$

where

$$\Gamma(\alpha) := \frac{\alpha}{u^H - c^L} - \frac{u^H - c^L - s - \alpha\left(u^H - c^H\right)}{\left(c^H - c^L - s\right)\left(c^H - c^L\right)}.$$

Since

$$\Gamma(1) = -\frac{1}{c^H - c^L}\frac{u^H - c^H}{u^H - c^L} < 0,$$

and $d\Gamma(\alpha)/d\alpha > 0$, then $\Gamma(\alpha) < 0$ for all α. Therefore $NS(s) < 0$ for all $s > 0$.

The Effect of a Subsidy Conditional on Trading at a Low Price

With a subsidy $s > 0$ to either buyers or sellers who trade the good at a low price $p < c^H$ the fraction of high quality in the market at the last date solves the equation

$$q_T^H u^H + \left(1 - q_T^H\right) u^L - c^H = \left(1 - q_T^H\right)\left(u^L - c^L + s\right).$$

Solving for q_T^H yields

$$q_T^H = \check{q}(s) = \frac{c^H - c^L + s}{u^H - c^L + s}.$$

Hence

$$\frac{d\check{q}(s)}{ds} = \frac{u^H - c^H}{\left(u^H - c^L + s\right)^2} > 0.$$

Also the role played by the functions $\bar{\phi}$ and ϕ_t in Proposition 3, is played by the functions $\check{\phi}(s) := (1 - \check{q}(s))\left(u^L - c^L + s\right)$ and $\check{\phi}_t(s) := \alpha\delta^{T-t}\check{\phi}(s)$. Hence

$$\frac{d\check{\phi}(s)}{ds} = -(u^L - c^L + s)\frac{d\check{q}(s)}{ds} + 1 - \check{q}(s) = \frac{\left(u^H - c^H\right)\left(u^H - u^L\right)}{\left(u^H - c^L + s\right)^2} \in (0, 1),$$

and

$$\frac{d\check{\phi}_t(s)}{ds} = \alpha\delta^{T-t}\frac{d\check{\phi}(s)}{ds} \in (0, 1).$$

The formulae for the probabilities of low price offers at each date are obtained by replacing $\hat{q}$, $\bar{\phi}$ and ϕ_t in the formulae given in Proposition 3 with $\check{q}(s)$, $\check{\phi}(s)$ and $\check{\phi}_t(s)$, respectively. However, the formulae describing the sequence of probabilities of high price offers and the traders' payoffs and surplus are as follows:

High Price Offers: $\rho_1^H = 0$,

$$\rho_t^H(s) = \frac{1 - \delta}{\alpha\delta}\frac{u^L - c^L + s}{c^H - u^L - s + \check{\phi}_t(s)},$$

for all $1 < t < T$, and

$$\rho_T^H(s) = \frac{u^L - c^L + s - \check{\phi}_{T-1}(s)}{\alpha\delta\left(c^H - c^L\right)}.$$

Payoffs and Surplus: $V_1^B(s) = \check{\phi}_1(s)$, $V_1^L(s) = u^L - c^L + s - \check{\phi}_1(s)$, and

$$S^{DME}(s) = m^L(u^L - c^L) + m^H \alpha \delta^{T-1} \check{\phi}(s) + sm^L.$$

Reservation prices: $r_t^L(s) = u^L + s - \check{\phi}_t(s)$ for all $t < T$ and $r_T^L(s) = c^L$ if the subsidy is given to buyers, and $r_t^L(s) = u^L - \check{\phi}_t(s)$ and $r_T^L(s) = c^L - s$ if it is given to sellers.

Corollary 6 follows readily by differentiating these formulae. We have

$$\frac{d\rho_T^H(s)}{ds} = \frac{1 - \frac{d\check{\phi}_{T-1}(s)}{ds}}{\alpha\delta\left(c^H - c^L\right)} > 0,$$

and

$$\begin{aligned}
\frac{d\rho_1^L(s)}{ds} &= \frac{1}{\alpha(1-q^H)(c^H - u^L + \check{\phi}_2(s))}\left(1 - \frac{\check{\phi}_2(s) + c^H - u(q^H)}{c^H - u^L + \check{\phi}_2(s)}\right)\frac{d\check{\phi}_2(s)}{ds} \\
&= \frac{u(q^H) - u^L}{\alpha(1-q^H)(c^H - u^L + \check{\phi}_2(s))^2}\frac{d\check{\phi}_2(s)}{ds} \\
&> 0.
\end{aligned}$$

Also $dV_1^B(s)/ds = d\check{\phi}_1(s)/ds > 0$ and $dV_1^L(s)/ds = 1 - d\check{\phi}_1(s)/ds > 0$. The effect on the net surplus is positive, since the cost of the subsidy is at most sm^L, while the subsidy increases the surplus by $m^H\alpha\delta^{T-1}(\check{\phi}(s) - \bar{\phi}) + sm^L > sm^L$.

If $T = \infty$, then

$$\hat{\rho}_1^L(s) = \frac{c^H - u(q^H)}{\alpha(1-q^H)(c^H - u^L)},$$

and $\hat{\rho}_t^L(s) = 0$ for $t > 1$. Also $\hat{\rho}_1^H = 0$, and

$$\hat{\rho}_t^H(s) = \frac{1-\delta}{\alpha\delta}\frac{u^L - c^L + s}{c^H - u^L - s}.$$

for $t > 1$. Thus, the subsidy increases the liquidity of both qualities. Moreover, the surplus is

$$\hat{S}^{DME}(s) = m^L(u^L - c^L) + sm^L,$$

the cost of the subsidy is $\alpha\hat{\rho}_1^L(s)sm^L$, and hence the net surplus increases by $\left(1 - \alpha\hat{\rho}_1^L(s)\right)sm^L > 0$.

The Effect of a Subsidy Conditional on Trading at the High Price

With a subsidy $s > 0$ to either buyers or sellers who trade at the high price c^H the fraction of high quality in the market at the last date solves the equation

$$q_T^H u^H + \left(1 - q_T^H\right)u^L - c^H + s = \left(1 - q_T^H\right)\left(u^L - c^L\right).$$

Solving for q_T^H yields

$$q_T^H = \breve{q}(s) = \frac{c^H - c^L - s}{u^H - c^L}.$$

Hence

$$\frac{d\breve{q}(s)}{ds} = -\frac{1}{u^H - c^L} < 0.$$

The role played by the functions $\bar{\phi}$ and ϕ_t in Proposition 3 is now played by $\breve{\phi}(s) := (1 - \breve{q}(s))(u^L - c^L)$ and $\breve{\phi}_t(s) := \alpha\delta^{T-t}\breve{\phi}(s)$, respectively. Hence

$$\frac{d\breve{\phi}(s)}{ds} = -(u^L - c^L)\frac{d\breve{q}(s)}{ds} = \frac{u^L - c^L}{u^H - c^L} \in (0, 1),$$

and

$$\frac{d\breve{\phi}_t(s)}{ds} = \alpha\delta^{T-t}\frac{d\breve{\phi}(s)}{ds} \in (0, 1).$$

The formulae for the probabilities of high price offers at each date, and the traders' payoffs and surplus are obtained by replacing $\hat{q}$, $\bar{\phi}$ and ϕ_t in the formulae given in Proposition 3 with $\breve{q}(s)$, $\breve{\phi}(s)$ and $\breve{\phi}_t(s)$, respectively. However, the formulae describing the sequence of probabilities of low price offers are as follows:

$$\rho_1^L(s) = \frac{c^H - u(q^H) - s + \breve{\phi}_2(s)}{\alpha(1 - q^H)(c^H - u^L + \breve{\phi}_2(s))},$$

and $\rho_T^L(s) = 1 - \rho_T^H(s)$. If $T > 2$, then

$$\rho_t^L(s) = (1 - \alpha\rho_t^H(s))\frac{(1 - \delta)\breve{\phi}_{t+1}(s)}{\alpha(c^H - u^L - s + \breve{\phi}_{t+1}(s))}\frac{u^H - u^L}{u^H - c^H + s - \breve{\phi}_t(s)}$$

for $t \in \{2, \dots, T - 2\}$, and

$$\rho_{T-1}^L(s) = (1 - \alpha\rho_{T-1}^H(s))\frac{(1 - \alpha\delta)\,\breve{\phi}(s)}{\alpha\breve{q}(s)(u^H - c^H + s - \breve{\phi}_{T-1}(s))}.$$

Corollary 7 readily follows by differentiating these formulae. Differentiating ρ_t^H for $t \in \{2, \dots, T - 1\}$ yields

$$\frac{d\rho_t^H(s)}{ds} = -\frac{1 - \delta}{\alpha\delta}\frac{u^L - c^L}{(c^H - u^L + \breve{\phi}_t(s))^2}\frac{d\breve{\phi}_t(s)}{ds} < 0.$$

Also

$$\frac{d\rho_T^H(s)}{ds} = \bar{\rho}\frac{d\breve{q}(s)}{ds} < 0.$$

For low price offers,

$$\frac{d\rho_1^L(s)}{ds} = -\frac{1}{\alpha(1-q^H)(c^H - u^L + \breve{\phi}_2(s))}\left(1 - \frac{d\breve{\phi}_2(s)}{ds} + \frac{c^H - u(q^H) - s + \breve{\phi}_2(s)}{c^H - u^L + \breve{\phi}_2(s)}\frac{d\breve{\phi}_2(s)}{ds}\right) < 0.$$

Finally, $dV_1^B(s)/ds = d\breve{\phi}_1(s)/ds > 0$ and $dV_1^L(s)/ds = -d\breve{\phi}_1(s)/ds < 0$, and

$$\frac{dS^{DME}(s)}{ds} = \alpha\delta^{T-1}m^H\frac{d\breve{\phi}(s)}{ds} > 0.$$

Thus, for $T = \infty$ the subsidy has no impact on either the payoffs or the surplus, and is purely wasteful.

Government Purchases

Assume that at the market open the government offers to buy β units of the good, e.g., via a uniform price auction. In equilibrium, the government acquires β units of low quality at a price equal to the reservation price of low quality sellers in the market that follows, i.e., r_1^L. In this market, after the government purchase, the measure of buyers exceeds the measure of sellers by β. We assume that the probability that a buyer is matched at date t is $\alpha\theta_t$, where

$$\theta_t = \frac{m_t^H + m_t^L}{m_t^H + m_t^L + \beta}$$

is the market tightness at date t.

Let us consider a market that opens over two dates, i.e., $T = 2$. A small government intervention does not affect the basic structure of the DME; specifically, at date 1 buyers only offer low and negligible prices with positive probability, and at date 2 only offer high and low prices with positive probability.

Since at date 2 buyers are indifferent between low and high price offers, then

$$q_2^H u^H + (1 - q_t^H)u^L - c^H = (1 - q_2^H)(u^L - c^L).$$

Thus, in equilibrium $q_2^H = \hat{q}$. At date 1, buyers are indifferent between offering low and negligible prices, i.e.,

$$u^L - r_1^L = \delta V_2^B = \delta\alpha\theta_2\bar{\phi},$$

which implies

$$r_1^L = u^L - \delta\alpha\theta_2\bar{\phi}.$$

Also by $DME.L$ the reservation price of low quality sellers satisfies

$$r_1^L = c^L + \delta V_2^L = c^L + \delta\alpha\rho_2^H(c^H - c^L).$$

Solving for ρ_2^H in the system of equations involving r_1^L yields

$$\rho_2^H = \frac{u^L - c^L - \delta\alpha\theta_2\bar{\phi}}{\delta\alpha(c^H - c^L)}.$$

Since the high price is offered with probability zero at date 1, then $m_2^H = m_1^H = m^H$. Also $m_2^L = (1-\alpha\rho_1^L)m_1^L$ and $m_1^L = m^L - \beta$. Hence

$$\frac{m_2^H}{m_2^H + m_2^L} = \frac{m^H}{m^H + (1-\alpha\rho_1^L)(m^L - \beta)} = \hat{q},$$

and therefore

$$m_2^L = (1-\alpha\rho_1^L)(m^L - \beta) = \frac{1-\hat{q}}{\hat{q}}m^H,$$

and

$$\theta_2 = \frac{m_2^H + m_2^L}{m_2^H + m_2^L + \beta} = \frac{m^H + \frac{1-\hat{q}}{\hat{q}}m^H}{m^H + \frac{1-\hat{q}}{\hat{q}}m^H + \beta} = \frac{m^H}{m^H + \hat{q}\beta}.$$

(We assume $\beta \leq m^L - \frac{1-\hat{q}}{\hat{q}}m^H$ to ensure that $\rho_1^L \geq 0$.) Note that m_2^L, and therefore the measure of low quality sellers that trades at date 1, is independent of β. Since all low quality sellers matched at date 2 trade, then the liquidity of low quality and the volume of trade of low quality are also independent of β.

Substituting the expression for m_2^L into the expression for ρ_2^H gives

$$\rho_2^H = \frac{u^L - c^L - \delta\alpha\frac{m^H + m_2^L}{m^H + m_2^L + \beta}\bar{\phi}}{\delta\alpha(c^H - c^L)} = \frac{u^L - c^L - \delta\alpha\frac{m^H}{m^H + \hat{q}\beta}\bar{\phi}}{\delta\alpha(c^H - c^L)}.$$

Payoffs are

$$V_1^L = u^L - c^L - \delta\alpha\theta_2\bar{\phi},$$

and

$$V_1^B = \delta\alpha\theta_2\bar{\phi}.$$

Let ε be the amount by which the government values low quality less than buyers. The net surplus is

$$(m^H + m^L)V_1^B + m^L V_1^L + \beta(u^L - \varepsilon - r_1^L) = m^L(u^L - c^L) + (m^H + \beta)\delta\alpha\frac{m^H}{m^H + \hat{q}\beta}\bar{\phi} - \beta\varepsilon.$$

Differentiating this expression with respect to β and setting $\beta = 0$ yields $(1-\hat{q})\alpha\delta\bar{\phi}-\varepsilon$. Hence net surplus is increasing in β at $\beta = 0$ so long as $(1-\hat{q})\alpha\delta\bar{\phi} > \varepsilon$.

As an example, consider the market of Example 1 with $T = 2$ and $\alpha = \delta = .95$. Note that β cannot exceed $m^L - \frac{1-\hat{q}}{\hat{q}} m^H = .8 - .2 = 0.6$. Net surplus is increasing at $\beta = 0$ so long as

$$\alpha\delta(1-\hat{q})\bar{\phi} = (.95)^2(.5)(.1) = .045125 > \varepsilon.$$

Figure 1 below shows net surplus as a function of β for $\varepsilon = 0$ (solid line), $\varepsilon = .025$ (dashed line), and $\varepsilon = .05$ (dotted line).

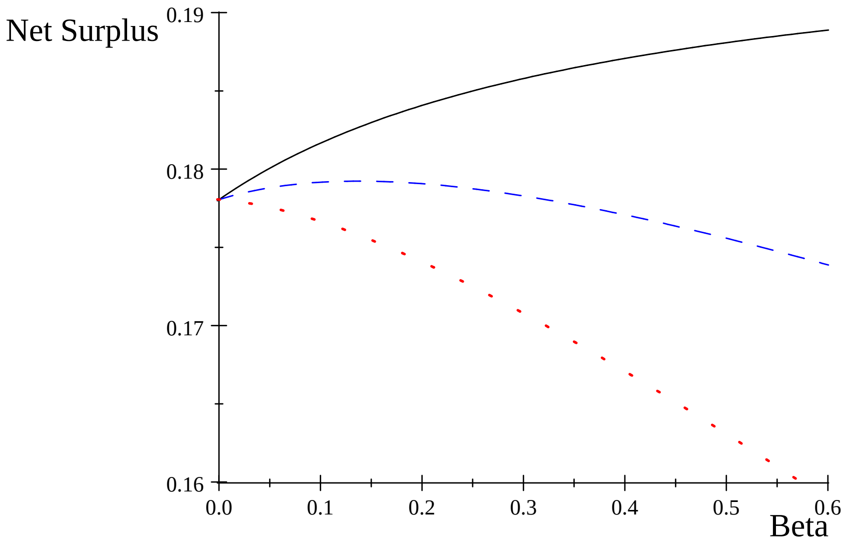

Figure 1: The effect of government purchases on net surplus

3 Dynamic Competitive Equilibrium

We study the market described in Section 2 when trade is *centralized*, i.e., trade is multilateral and agents are price takers. The market opens for T consecutive dates, and the traders' discount rate is $\delta \in (0, 1]$.

The supply and demand schedules are defined as follows. Let $p = (p_1, \ldots, p_T) \in \mathbb{R}^T_+$ be a sequence of prices. The utility to a seller of quality $\tau \in \{H, L\}$ who supplies at date t is $\delta^{t-1}(p_t - c^\tau)$. Hence the maximum utility that a τ-quality seller may attain is

$$v^\tau(p) = \max_{t \in \{1,\ldots,T\}} \{0, \delta^{t-1}(p_t - c^\tau)\}.$$

The *supply of τ-quality good*, denoted by $S^\tau(p)$, is the set of sequences $s^\tau = (s^\tau_1, \ldots, s^\tau_T) \in \mathbb{R}^T_+$ satisfying:

$(S.1)$ $\sum_{t=1}^T s^\tau_t \leq m^\tau$,

$(S.2)$ $s^\tau_t > 0$ implies $\delta^{t-1}(p_t - c^\tau) = v^\tau(p)$, and

$(S.3)$ $\left(\sum_{t=1}^T s^\tau_t - m^\tau\right) v^\tau(p) = 0$.

Condition $S.1$ requires that no more of good τ than is available, m^τ, be supplied. Condition $S.2$ requires that supply be positive only at dates where it is optimal to supply. Condition $S.3$ requires that the total amount of good τ available be supplied when τ-quality sellers may attain a positive utility (i.e., when $v^\tau(p) > 0$).

Denote by $u_t \in [u^L, u^H]$ the expected value to buyers of a unit supplied at date t. Then the utility to a buyer who demands a unit of the good at date t is $\delta^{t-1}(u_t - p_t)$. If the sequence of buyers' expected values is $u = (u_1, \ldots, u_T)$, then the maximum utility a buyer may attain is

$$v^B(p, u) = \max_{t \in \{1,\ldots,T\}} \{0, \delta^{t-1}(u_t - p_t)\}.$$

The *market demand*, denoted by $D(p, u)$, is the set of sequences $d = (d_1, \ldots, d_T) \in \mathbb{R}^T_+$ satisfying:

$(D.1)$ $\sum_{t=1}^T d_t \leq m^B$,

$(D.2)$ $d_t > 0$ implies $\delta^{t-1}(u_t - p_t) = v^B(p, u)$, and

$(D.3)$ $\left(\sum_{t=1}^T d_t - m^B\right) v^B(p, u) = 0$.

Condition $D.1$ requires that the total demand not exceed the measure of buyers. Condition $D.2$ requires that the demand be positive only at dates where buying is optimal. Condition $D.3$ requires that demand be equal to the measure of buyers when buyers may attain a positive utility (i.e., when $v^B(p, u) > 0$).

We define dynamic competitive equilibrium along the lines in the literature – see e.g., Wooders (1998), and Janssen and Roy (2002).

Definition. A *dynamic competitive equilibrium (DCE)* is a profile (p, u, s^H, s^L, d) such that $s^H \in S^H(p)$, $s^L \in S^L(p)$, $d \in D(p, u)$, and for each t:

$(DCE.1)$ $s_t^H + s_t^L = d_t$, and

$(DCE.2)$ $s_t^H + s_t^L = d_t > 0$ implies $u_t = \dfrac{u^H s_t^H + u^L s_t^L}{s_t^H + s_t^L}$.

Condition $DCE.1$ requires that the market clear at each date, and condition $DCE.2$ requires that the expectations described by the vector u be correct whenever there is trade. For a market that opens for a single date (i.e., if $T = 1$), our definition reduces to Akerlof's. The surplus generated in a DCE may be calculated as

$$S^{DCE} = \sum_{\tau \in \{H,L\}} \sum_{t=1}^{T} s_t^\tau \delta^{t-1}(u^\tau - c^\tau). \tag{1}$$

In lemmas 3 and 4 we establish some properties of dynamic competitive equilibria.

Lemma 3. *In every DCE, (p, u, s^H, s^L, d), we have $\sum_{\{t|s_t^H>0\}} s_t^L < m^L$.*

Proof. Let (p, u, s^H, s^L, d) be a DCE. For all t such that $s_t^H > 0$ we have

$$\delta^{t-1}(p_t - c^H) = v^H(p) \geq 0$$

by $(S.2)$. Hence $p_t \geq c^H$. Also $d_t > 0$ by $DCE.1$, and therefore

$$v^B(p) = \delta^{t-1}(u_t - p_t) \geq 0$$

implies $0 \leq u_t - p_t \leq u_t - c^H$, i.e., $u_t \geq c^H = u(\bar{q})$. Thus

$$\frac{s_t^H}{s_t^H + s_t^L} \geq \bar{q},$$

i.e.,

$$(1 - \bar{q}) \sum_{\{t|s_t^H>0\}} s_t^H \geq \bar{q} \sum_{\{t|s_t^H>0\}} s_t^L.$$

Since $\sum_{\{t|s_t^H>0\}} s_t^H \leq m^H$, then

$$(1-\bar{q})m^H \geq (1-\bar{q}) \sum_{\{t|s_t^H>0\}} s_t^H \geq \bar{q} \sum_{\{t|s_t^H>0\}} s_t^L.$$

Since $q^H = m^H/(m^H + m^L) < \bar{q}$ by assumption, then

$$\sum_{\{t|s_t^H>0\}} s_t^L \leq \frac{1-\bar{q}}{\bar{q}} m^H < \frac{1-q^H}{q^H} m^H = \frac{\dfrac{m^L}{m^H+m^L}}{\dfrac{m^H}{m^H+m^L}} m^H = m^L. \ \square$$

Lemma 4 shows that low quality must trade before high quality.

Lemma 4. *Let (p, u, s^H, s^L, d) be a DCE. If $s_t^H > 0$ for some t, then there is $t' < t$ such that $s_{t'}^L > 0 = s_{t'}^H$ and $\delta^{t'-1}(u^L - c^L) \geq \delta^{t-1}(c^H - c^L)$.*

Proof. Let (p, u, s^H, s^L, d) be a DCE, and assume that $s_t^H > 0$. Then $\delta^{t-1}(p_t - c^H) = v^H(p) \geq 0$ by $S.2$, and therefore $p_t \geq c^H$. Hence $v^L(p) \geq \delta^{t-1}(p_t - c^L) \geq \delta^{t-1}(c^H - c^L) > 0$, and therefore $\sum_{k=1}^T s_k^L = m^L$ by $S.3$. Since

$$\sum_{\{k|s_k^H>0\}} s_k^L < m^L$$

by Lemma 3, then there is t' such that $s_{t'}^L > 0 = s_{t'}^H$. Hence $d_{t'} > 0$ by $DCE.1$, which implies $u_{t'} = u^L$ by $DCE.2$, and $p_{t'} \leq u^L$ by $D.2$. Also $s_{t'}^L > 0$ implies $v^L(p) = \delta^{t'-1}(p_{t'} - c^L) \geq \delta^{t-1}(p_t - c^L)$ by $S.2$. Thus

$$\delta^{t'-1}(u^L - c^L) \geq \delta^{t'-1}(p_{t'} - c^L) \geq \delta^{t-1}(p_t - c^L) \geq \delta^{t-1}(c^H - c^L).$$

Since $u^L < c^H$ this inequality implies $t' < t$. $\square$

Proposition 6 establishes that there is a DCE where all low quality units trade at date 1 at the price u^L, and none of the high quality units ever trade. Moreover, if the market opens over a sufficiently short horizon, then every DCE has these properties. Specifically, the horizon T must be less than $\overline{T}$, which is defined by the inequality

$$\delta^{\overline{T}-2}(c^H - c^L) > u^L - c^L \geq \delta^{\overline{T}-1}(c^H - c^L).$$

Since $\overline{T}$ approaches infinity as δ approaches one, for a given T the condition $T < \overline{T}$ holds when δ is near one, i.e., when traders are sufficiently patient.

Proposition 6. *There are DCE in which all low quality units trade immediately at the price u^L and none of the high quality units trade, e.g., (p, u, s^H, s^L, d) given by $p_t = u_t = u^L$ for all t, $s_1^L = d_1 = m^L$, and $s_1^H = s_t^H = s_t^L = d_t = 0$ for $t > 1$ is a DCE. In these DCE the payoff to low quality sellers is $u^L - c^L$, the payoff to high quality sellers and buyers is zero, and the surplus is $\bar{S}$. Moreover, if $T < \overline{T}$, then every DCE has these properties.*[1]

Proof. The profile in Proposition 6 is clearly a DCE. We show that every DCE, (p, u, s^H, s^L, d), satisfies $p_1 = u_1 = u^L$, $s_1^L = d_1 = m^L$ and $s_1^H = s_t^H = s_t^L = d_t = 0$ for $t > 1$.

We first show that $s_t^H = 0$ for all $t \in \{1, \ldots, T\}$. Suppose that $s_t^H > 0$ for some t. Then Lemma 4 implies that there is $t' < t$ such that

$$u^L - c^L \geq \delta^{t'-1}(u^L - c^L) \geq \delta^{t-1}(c^H - c^L) \geq \delta^{T-1}(c^H - c^L),$$

which is a contradiction.

We show that $p_t \geq u^L$ for all t. If $p_t < u^L$ for some t, then

$$v^B(p, u) = \max_{t \in \{1,\ldots,T\}} \{0, \delta^{t-1}(u_t - p_t)\} > 0,$$

and therefore $\sum_{t=1}^T d_t = m^B = m^H + m^L$. However, $s_t^H = 0$ for all t implies

$$\sum_{t=1}^T (s_t^H + s_t^L) \leq m^L < m^L + m^H = \sum_{t=1}^T d_t,$$

which contradicts $DCE.1$.

Since $p_t \geq u^L$ for all t, then

$$v^L(p) = \max_{t \in \{1,\ldots,T\}} \{0, \delta^{t-1}(p_t - c^L)\} > 0,$$

and therefore $\sum_{t=1}^T s_t^L = m^L$ by $S.3$.

We show that $p_1 = u^L$ and $s_1^L = d_1 = m^L$ and $s_t^L = 0$ for $t > 1$. Let t be such that $s_t^L > 0$. Then $s_t^H = 0$ implies $u_t = u^L$. By $DCE.1$ we have $d_t = s_t^L > 0$ and thus

$$\delta^{t-1}(u_t - p_t) = \delta^{t-1}(u^L - p_t) \geq 0$$

[1] Janssen and Roy (2002)'s definition of competitive equilibrium requires additionally that the expected value to buyers of a random unit at dates when there is no trade is at least the value of the lowest quality for which there is a positive measure of unsold units. When $T < \bar{T}$ no CE with this property exists.

by $D.2$. This inequality and $p_t \geq u^L$ imply that $p_t = u^L$. Hence for all t such that $s_t^L > 0$ we have $p_t = u^L$.

Let $t > 1$ and assume that $s_t^L > 0$. Then $p_t = u^L$. Since $\delta < 1$ and as shown above $p_1 \geq u^L$, then

$$p_1 - c^L > \delta^{t-1}(u^L - c^L) = \delta^{t-1}(p_t - c^L),$$

which contradicts $S.2$. Hence $s_t^L = 0$ for $t > 1$, and therefore $\sum_{t=1}^T s_t^L = m^L$ implies $s_1^L = d_1 = m^L > 0$, and $p_1 = u^L$. □

The intuition for why high quality does not trade when $T < \overline{T}$ is clear: If high quality were to trade at $t \leq T$, then p_t must be at least c^H. Hence the utility to low quality sellers is at least $\delta^{t-1}(c^H - c^L)$. Since

$$\delta^{t-1}(c^H - c^L) \geq \delta^{T-1}(c^H - c^L) \geq \delta^{\overline{T}-2}(c^H - c^L) > u^L - c^L > 0,$$

then all low quality sellers trade at prices greater than u^L. But at a price $p \in (u^L, c^H)$ only low quality sellers supply, and therefore the demand is zero. Hence all trade is at prices of at least c^H. Since $u(q^H) < c^H$ by assumption, and since in equilibrium all low quality is supplied, there must be a date at which there is trade and the expected value of a random unit supplied is below c^H. This contradicts that there is demand at such a date. Thus, high quality is not supplied in a DCE. Consequently, low quality sellers capture the entire surplus, i.e., the price is u^L, as low quality sellers are the short side of the market.

By Propositions 3 the surplus realized in a decentralized market is greater than the competitive surplus, i.e., $S^{DME} > \bar{S}$, while a dynamic competitive market that opens over a finite horizon generates the competitive surplus, i.e., $S^{DCE} = \bar{S}$, by Proposition 6. Thus, *decentralized markets perform better than centralized markets when the horizon is finite.* This continues to be the case even as frictions vanish by Proposition 4.

Proposition 7 below establishes that in a centralized market that opens over a sufficiently long horizon there are dynamic competitive *separating* equilibria in which all low quality units trade immediately and all high quality units trade with delay. Specifically, the horizon T must be at least $\widetilde{T}$, which is defined by the inequality

$$\delta^{\widetilde{T}-2}(u^H - c^L) > u^L - c^L \geq \delta^{\widetilde{T}-1}(u^H - c^L).$$

Since $u^H > c^H$, then $\widetilde{T} \geq \overline{T}$.

Proposition 7. *If $T \geq \widetilde{T}$, then there are DCE in which all low quality units trade at date 1 and all high quality units trade at date $\widetilde{T}$. Such DCE yield a surplus of*

$$S^{DCE} = m^L(u^L - c^L) + m^H \delta^{\tilde{T}-1}(u^H - c^H) > \bar{S}.$$

Moreover, if $T = \infty$, then

$$\lim_{\delta \to 1} S^{DCE} = \tilde{S}^{DME}.$$

Proof. Assume that $T \geq \widetilde{T}$. We show that the profile (p, u, s^H, s^L, d) given by $p_t = u_t = u^L$ for $t < \widetilde{T}$, and $p_t = u_t = u^H$ for $t \geq \widetilde{T}$, $s_1^H = 0$, $s_1^L = m^L = d_1$, $s_{\widetilde{T}}^L = 0$, $s_{\widetilde{T}}^H = d_{\widetilde{T}} = m^H$, and $s_t^H = s_t^L = d_t = 0$ for $t \notin \{1, \widetilde{T}\}$ is a DCE.

Since $p_{\widetilde{T}} = u^H > c^H$, then $v^H(p) \geq \delta^{\widetilde{T}-1}(p_{\widetilde{T}} - c^H) > 0$. Further, since $\delta < 1$ then

$$\delta^{\widetilde{T}-1}(p_{\widetilde{T}} - c^H) = \delta^{\widetilde{T}-1}(u^H - c^H) > \delta^{t-1}(p_t - c^H)$$

for $t \neq \widetilde{T}$. Hence $s^H \in S^H(p)$. For low quality sellers, $\delta < 1$ and $u^L - c^L \geq \delta^{\widetilde{T}-1}(u^H - c^H)$ imply

$$v^L(p) = p_1 - c^L = u^L - c^L \geq \delta^{t-1}(p_t - c^H)$$

for $t > 1$. Hence $s^L \in S^L(p)$. For buyers,

$$v^B(p, u) = \delta^{t-1}(u_t - p_t) = 0$$

for all t. Hence $d \in D(p, u)$. Finally, $s_t^L + s_t^H = d_t$ for all t, and therefore $DCE.1$ is satisfied, and $u_1 = u^L$ and $u_{\widetilde{T}} = u^H$ satisfy $DCE.2$. Thus, the profile defined is a DCE. The surplus in this DCE is

$$S^{DCE} = m^L(u^L - c^L) + m^H \delta^{\widetilde{T}-1}(u^H - c^H).$$

Assume that $T = \infty$, and let $\delta < 1$. The surplus at the DCE of Proposition 7 is

$$S^{DCE}(\delta) = q^L(u^L - c^L) + q^H \delta^{\widetilde{T}(\delta)-1}(u^H - c^H).$$

By definition $\widetilde{T}(\delta)$ satisfies

$$\delta^{\widetilde{T}(\delta)-1}(u^H - c^L) \leq u^L - c^L < \delta^{\widetilde{T}(\delta)-2}(u^H - c^L).$$

i.e.,

$$\delta < \frac{u^H - c^L}{u^L - c^L}\delta^{\widetilde{T}(\delta)-1} \leq 1$$

Hence

$$\lim_{\delta\to 1}\delta = \frac{u^H - c^L}{u^L - c^L}\lim_{\delta\to 1}\delta^{\widetilde{T}(\delta)-1} = 1,$$

i.e.,

$$\lim_{\delta\to 1}\delta^{\widetilde{T}(\delta)-1} = \frac{u^L - c^L}{u^H - c^L} = (1-\hat{q})\frac{u^L - c^L}{u^H - c^H}.$$

Substituting, we have

$$\lim_{\delta\to 1}\hat{S}^{DCE}(\delta) = \left[m^L + m^H(1-\hat{q})\right](u^L - c^L) = \tilde{S}^{DME}. \square$$

Centralized markets that open over a sufficiently long horizon eventually *recover* from adverse selection, i.e., have equilibria in which high quality trades and the surplus is above the competitive surplus. Consequently, *when the horizon is infinite, centralized markets may outperform decentralized markets* – which by Proposition 5 yield the competitive surplus.[2]

In the proof of Proposition 7 we show that

$$\lim_{\delta\to 1}\delta^{\widetilde{T}-1} = \frac{u^L - c^L}{u^H - c^L},$$

and therefore that the surplus realized from trading high quality in this equilibrium approaches

$$m^H\frac{u^L - c^L}{u^H - c^L}(u^H - c^H) = m^H(1-\hat{q})(u^L - c^L).$$

Thus, as δ approaches one, the surplus approaches $\tilde{S}^{DME}$, which is also the surplus realized in the DME when $T < \infty$ as α and δ approach one – see Proposition 4. This result reveals that the same incentive constraints are at play in both centralized and decentralized markets: In a separating DCE, high quality trades with a sufficiently long delay that low quality sellers prefer trading immediately at a low price to waiting and trading at a high price. Likewise, in a DME, high price offers are made with sufficiently low probability that low quality sellers accept a low price offer.

[2] When $\overline{T} \leq T < \widetilde{T}$ there are no separating CE, but there are *partially pooling* CE in which high quality trades. In the most efficient of these CE, in which some low quality trades at date 1 while the remaining low quality and all the high quality trade at date T, the surplus is greater than $\bar{S}$.

Policy Intervention and Liquidity

As noted earlier, the effect of a subsidy or tax is akin to that of a change of the value of the good, i.e., of u^L or u^H. Marginal changes in these values do not affect the value of $\overline{T}$ or $\widetilde{T}$ generically, and hence do not affect the net surplus in a centralized market. If $T < \infty$ and δ is near one, then subsidies have no impact on net surplus. If $T = \infty$, a subsidy on low quality or tax on high quality that reduces $\widetilde{T}$ increases net surplus in the separating DCE since high quality trades earlier.

When $T < \overline{T}$, low quality is liquid as it trades immediately, while high quality is illiquid as it never trades. When $T = \infty$ all units trade in the separating DCE, but high quality trades with delay, and therefore is less liquid than low quality, which trades immediately.

4 The Public-Private Investment Program

Public-Private Investment Program

$500 Billion to $1 Trillion Plan to Purchase Legacy Assets

(White Paper released by the U.S. Treasury on March 23, 2009)

Overview

Troubled real estate-related assets, comprised of legacy loans and securities, are at the center of the problems currently impacting the U.S. financial system. The Financial Stability Plan, announced on February 10th, outlined a broad approach to address this issue via the formation of Public-Private Investment Funds ("PPIFs"). Today Treasury is announcing the Public-Private Investment Program under which it will make targeted investments in multiple PPIFs that will purchase legacy real estate-related assets.

Addressing the problems created by legacy assets should help to improve the health of the financial institutions where they are held, leading to an increased flow of credit throughout the economy, and helping improve market functioning in the near-term. Investments made by Treasury under the Public-Private Investment Program are intended to complement the other components of the Financial Stability Plan that have been announced, including the Capital Assistance Program, the Homeowner Affordability and Stability Plan, and the Consumer and Business Lending Initiative, continuing the Obama Administration's efforts to improve the stability and functioning of the financial system.

The Legacy Asset Problem

A variety of troubled legacy assets are currently congesting the U.S. financial system. An initialf undamental shock associated with the bursting of the housing bubble and deteriorating economic conditions generated losses for leveraged investors including banks. This shock was compounded by the fact that loan underwriting standards used by some originators had become far too lax and by the proliferation of structured credit products, some of which were ill understood by some market participants.

The resulting need to reduce risk triggered a wide-scale deleveraging in these markets and led to fire sales. As prices declined further, many traditional sources of capital exited these markets, causing declines in secondary market liquidity. As a result, we have been in a vicious cycle in which declining asset prices have triggered further deleveraging and reductions in market liquidity, which in turn have led to further price declines. While fundamentals have surely deteriorated over the past 18-24 months, there is evidence that current prices for some legacy assets embed substantial liquidity discounts.

The discounts currently embedded in some legacy asset prices are a significant strain on the economic capital of U.S. financial institutions and have reduced their ability to engage in new credit formation. At the same time, the difficulty of obtaining private financing on reasonable terms to purchase these assets has limited the ability of investors to reduce these discounts. The lack of clarity about the value of these legacy assets has made it difficult for some financial institutions to raise new private capital.

The Public-Private Investment Program is designed to draw new private capital into the market for these assets by providing government equity co-investment and attractive public financing. This program should facilitate price discovery and should help, over time, to reduce the excessive liquidity discounts embedded in current legacy asset prices. This in turn should free up capital and allow U.S. financial institutions to engage in new credit formation. Furthermore, enhanced clarity about the value of legacy assets should increase investor confidence and enhance the ability of financial institutions to raise new capital from private investors.

The primary areas of focus for the government's troubled legacy asset programs are the residential and commercial mortgage sectors, including both whole loans and securitizations backed by loan portfolios. These troubled assets are held by all types of financial institutions, including those that predominantly hold them in the form of loans, such as banks, and those that predominantly hold securities, such as insurers, pension funds, mutual funds and individual retirement accounts. While the program may initially target real estate-related assets, it can evolve, based on market demand, to include other asset classes.

The Public-Private Investment Plan: A Comprehensive Solution

A key principle of the chosen approach is to use private capital and private fund managers to help provide a market mechanism for valuing the troubled assets. By creating partnerships with private investors, this approach should serve to both protect the interests of taxpayers over the long-term and help restore liquidity and enable price discovery in the markets for troubled assets in the short-term.

The two key elements of the plan are:

- **Legacy Loans Program**: a program to combine an FDIC guarantee of debt financing with equity capital from the private sector and the Treasury to support the purchase of troubled loans from insured depository institutions.

- **Legacy Securities Program**: a program to combine financing from the Federal Reserve and Treasury through the Term Asset-Backed Securities Loan Facility ("TALF") with equity capital from the private sector and the Treasury to address the problem of troubled securities.

The equity co-investment component of these programs has been designed to well align public and private investor interests in order to maximize the long-run value for U.S. taxpayers. Specifically, while the plan is designed to help reduce the liquidity discounts contained in legacy asset prices in the near-term, the most important way to protect taxpayers is to ensure that the government is not paying more for assets than their long-run value as determined by private investors. Since TARP funds will be invested alongside private capital on similar terms, this reduces the likelihood that taxpayers will be overpaying. At the same time, taxpayers will have the opportunity to participate in the asset's upside along with private investors. Similarly, the debt financing components of these programs have been structured to protect taxpayer dollars and the FDIC's Deposit Insurance Fund from credit losses to the greatest extent possible.

Together, these two programs should help to restart markets for troubled assets, begin the process of repairing balance sheets, and eventually lead to increased lending in comparison with levels that would have occurred without this effort.

The Legacy Loans Program

In order to help cleanse bank balance sheets of troubled legacy loans and reduce the overhang of uncertainty associated with these assets, the FDIC and Treasury are launching the Legacy Loans Program. This program will attract private capital to purchase eligible loan assets from participating banks through the provision of FDIC debt guarantees and Treasury equity coinvestment. A wide array of investors are expected to participate. The program will particularly encourage the participation of individuals, mutual funds, pension plans, insurance companies,and other long-term investors. The program is intended to boost private demand for distressed assets that are currently held by banks and facilitate market-priced sales of troubled assets.

The FDIC will provide oversight for the formation, funding, and operation of a number of PPIFs that will purchase assets from banks. The Treasury and private investors will invest equity capital in Legacy Loans PPIFs and the FDIC will provide a guarantee for debt financing issued by the PPIFs to fund asset purchases. The FDIC's guarantee will be collateralized by the purchased assets and the FDIC will receive a fee in return for its guarantee. The Treasury will manage its investment on behalf of taxpayers to ensure the public interest is protected. The Treasury intends to provide 50% of the equity capital for each PPIF, but private investors will retain control of asset management, subject to rigorous oversight from the FDIC.

Institutions of all sizes will be eligible to sell assets under the Legacy Loans Program. To start the process, banks will identify to the FDIC the assets, typically a pool of loans, that they wish to sell. Assets eligible for purchase will be determined by the participating banking organizations, including the primary banking regulators, the FDIC, and the Treasury. In order to protect taxpayer dollars from credit losses, the FDIC will employ contractors to analyze the pools and will determine the level of debt to be issued by the PPIF that it is willing to guarantee. This will not exceed a 6-to-1 debt-to-equity ratio. An eligible pool of loans, with committed financing, will then be auctioned by the FDIC to qualified bidders. Private investors will bid for the opportunity to contribute 50% of the equity for the PPIF with the Treasury contributing the remainder. The winning bid for this equity stake together with the amount of debt the FDIC is willing to guarantee (based on a predetermined debt-to-equity ratio), will define the price offered to the selling bank. The bank would then decide whether to accept the offer price.

Once the initial transaction has been completed, the private capital partners will control and manage the assets until final liquidation, subject to strict oversight from the FDIC. The FDIC will play an ongoing reporting, oversight and accounting role on behalf of the FDIC and Treasury. The exact requirements and structure of the Legacy Loans Program will be subject to notice and comment rulemaking.

Example

If a bank has a pool of residential mortgages with \$100 face value that they are seeking to divest, the bank would approach the FDIC. The FDIC would determine, according to the above process, that they would be willing to leverage the pool at a 6-to-1 debt-to-equity ratio. The pool would then be auctioned by the FDIC, with several private buyers submitting bids. The highest bid from the private sector – in this example, \$84 – would define the total price paid by the private investors and the Treasury for the mortgages. Of this \$84 purchase price, the Treasury and the private investors would split the \$12 equity portion. The new PPIF would issue debt for the remaining \$72 of the price and the debt would be guaranteed by the FDIC. This guarantee would be secured by the purchased assets. The private investor would then manage the servicing of the asset pool and the timing of its disposition on an ongoing basis – using asset managers approved and subject to oversight by the FDIC. Through transactions like this, the Legacy Loans Program is designed to use private sector pricing to cleanse banks' balance sheets of troubled assets and create a more healthy banking system.

The Legacy Securities Program

The Legacy Securities Program consists of two related parts. This program is designed to draw private capital into the markets for legacy securities by providing matching equity capital under the Treasury's Public-Private Investment Program and debt financing from the Federal Reserve and Treasury under the TALF. However, any private investor, even those who do not partner with Treasury under the Public-Private Investment Program, will also be able to access the TALF to purchase legacy securities. The goal is to restart the market for these legacy securities, which will allow banks and other financial institutions to free up economic capital and stimulate the extension of new credit. The resulting process of price discovery should also reduce the uncertainty surrounding financial institutions holding these securities, potentially enabling them to raise new private capital.

Expansion of TALF for Legacy Securities

The Treasury and the Federal Reserve are creating a lending program that is targeted at the broken market for legacy securities tied to residential real estate, commercial real estate, and consumer credit. The intention is to incorporate this program into the previously announced TALF, which may total as much as $1 trillion.

Through this expansion of the TALF, non-recourse loans will be made available to investors to fund purchases of legacy securitization assets. Eligible assets are expected to include certain non-agency residential mortgage-backed securities ("RMBS") that were originally rated AAA, and outstanding and commercial mortgage-back securities ("CMBS") and ABS that are rated AAA. Borrowers will need to meet certain eligibility criteria. Haircuts will be determined at a later date and will reflect the riskiness of the assets provided as collateral. Lending rates, minimum loan sizes, and loan durations have not yet been determined. These and other terms of the program will be informed by discussions with market participants. As with securitizations backed by new originations of consumer and business credit already included in the TALF, the provision of leverage through this program should give investors greater confidence to purchase these assets, thus increasing market liquidity.

Legacy Securities PPIFs

In conjunction with these efforts, the Treasury is also announcing a program to partner with private fund managers to support the market for legacy securities. Under this program, private investment managers will have the opportunity to apply for qualification as a Fund Manager ("FM"). Applicants will be pre-qualified based upon criteria that are expected to include a demonstrable historical track record in the targeted asset classes, a minimum amount of assets under management in the targeted asset classes, and detailed structural proposals for the proposed Legacy Securities PPIF. Treasury expects to approve approximately 5 FMs and may consider adding more depending on the quality of applications received. Approved FMs will have a period of time to raise private capital to target the designated asset classes and will receive matching equity capital from Treasury. FMs will be required to submit a fundraising plan to include retail investors, if possible. Treasury equity capital will be invested on a fully side-by-side basis with these private investors in each PPIF.

Furthermore, FMs will have the ability, to the extent their fund structures meet certain guidelines, to subscribe to Treasury for senior debt for the PPIFs in the amount of up to 50% of a fund's total equity capital, and Treasury will consider requests for senior debt for the PPIFs in the amount of up to 100% of a fund's total equity capital subject to further restrictions on asset level leverage, redemption rights, disposition priorities, and other factors Treasury deems relevant. This senior debt will have the same duration as the underlying fund and will be repaid on a pro-rata basis as principal repayments or disposition proceeds are realized by the PPIF. These senior loans will be structurally subordinated to any financing extended by the Federal Reserve to these PPIFs via the TALF.

Treasury expects the PPIFs to initially target non-agency RMBS and CMBS originated prior to 2009 with a rating of "AAA" at origination.

Example

Treasury will launch the application process for managers interested in the Legacy Securities Program. An interested FM would submit an application and be pre-qualified to raise private capital to participate in joint investment programs with Treasury. Treasury would agree to provide a one-for-one equity match for every dollar of private capital that the FM raises and provide fund-level leverage for the proposed PPIF. The FM would commence the sales process for the PPIF and raise \$100 of private capital for the PPIF. Treasury would provide \$100 of equity capital to be invested on side-by-side basis with private capital and would provide up to a \$100 loan to the PPIF if the fund met certain guidelines. Treasury would also consider requests from the FM for an additional loan of up to \$100 subject to further restrictions. As a result, the FM would have \$300 (or, in some cases, up to \$400) in total capital and would commence a purchase program for targeted securities. The FM would have full discretion in investment decisions, although the PPIFs will predominately follow a long-term buy and hold strategy. Depending on the amount of loans provided directly from Treasury, the PPIF would also be eligible to take advantage of the expanded TALF program for legacy securities when that program is operational.