# Web Appendix: A Search-Theoretic Model of the Term Premium

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This web appendix contains three sections. In Section 1, we we formally solve the version of our model with money and real assets of two maturities. In Section 2, we describe the solution to a version of our model with money and nominal assets of two maturities, which deliver money instead of a real dividend when they mature. In Section 3, we analyze the case of $N > 2$ maturities.

# 1 Solution of the model with money and $N = 2$

In the main text of our paper, we already described the environment of our model (with money and without, in Section 2), and briefly highlighted our main results with money (in Section 4.1). Here, we provide more details. Sections 1.1-1.6 describe the optimal behavior of the agents, and define and characterize equilibrium. Among them, Section 1.4 provides an in-depth intuitive description of the money and asset demand curves in our model, which are at the heart of our results. Finally, Section 1.7 defines a few equilibrium objects used in the preceding analysis.

## 1.1 Value Functions

We begin with the description of the value functions in the $CM$. For a typical buyer, the state variables are the following. First, the units of money, $m$, that she brings into the CM. Second, the units of assets of maturity $N = 2$, $a_2$, that she bought in the previous period, and which will mature in the forthcoming period. Third, the dividend, $d$, that she received earlier in the period, i.e. before the LW market opened, and she did not spend in that market. The amount of real balances $d$ could have been delivered either from long term assets issued two periods ago, or from short term assets issued in the last period. The Bellman's equation is given by

$$W(m, d, a_2) = \max_{X,H,\hat{m},\hat{a}_1,\hat{a}_2} \left\{U(X) - H + \beta\mathbb{E}\left\{\Omega^i(\hat{m}, \hat{a}_1, \hat{a}_2)\right\}\right\}$$
$$\text{s.t. } X + \varphi\hat{m} + \psi_1(\hat{a}_1 - a_2) + \psi_2\hat{a}_2 = H + \varphi(m + \mu M) + d,$$

and subject to $\hat{a}_1 - a_2 \geq 0$. In the last expression, variables with hats denote next period's choices, and the term $\mathbb{E}$ denotes the expectations operator. The function $\Omega^i$ represents the value function in the OTC market for a buyer of type $i = \{C, N\}$, described in more detail below. It is important to highlight that we have defined $\hat{a}_1$ as the amount of *all* assets that mature in the next period (which is analogous to our definition of the supply of assets that mature in the next period). Hence, the amount of newly issued short term assets purchased by the agent is $\hat{a}_1 - a_2$, and we require $\hat{a}_1 - a_2 \geq 0$. This constraint simply enforces the assumption that agents cannot sell off-the-run short term asset in the CM (see the discussion in the main text). Later, we will focus only on equilibria where this constraint does not bind.

Some observations are in order. First, it can be easily verified that, at the optimum, $X = X^*$. Using this fact and replacing $H$ from the budget constraint into $W$ yields

$$\begin{aligned} W(m, d, a_2) &= U(X^*) - X^* + \varphi(m + \mu M) + d + \psi_1 a_2 \\ &+ \max_{\hat{m}, \hat{a}_1, \hat{a}_2} \left\{ -\varphi \hat{m} - \psi_1 \hat{a}_1 - \psi_2 \hat{a}_2 + \beta \mathbb{E} \left\{ \Omega^i(\hat{m}, \hat{a}_1, \hat{a}_2) \right\} \right\}. \end{aligned} \tag{1}$$

A standard feature of models that build on Lagos and Wright (2005) is that the optimal choice of the agent does not depend on the current state (due to the quasi-linearity of $\mathcal{U}$). This is also true here, with the exception that the range of admissible choices for $\hat{a}_1$ is restricted by the state variable $a_2$. Moreover, as is standard in this types of models, the CM value function is linear. In fact, $W$ is linear in the variable $z \equiv \varphi m + d$, which captures the total real balances of the buyer. This property will greatly simplify the analysis in what follows. We collect all the terms in (1) that do not depend on the state variables, and we write

$$W(z, a_2) = \Lambda + z + \psi_1 a_2, \tag{2}$$

where the definition of $\Lambda$ is obvious.

Next, consider a seller's value function in the CM. It is well-known that in monetary models where the identity of agents (as buyers or sellers) is fixed over time, sellers will typically not leave the CM with a positive amount of asset holdings.[1] Therefore, when a seller enters the CM, she will only hold real balances that she received as payment during trade in the preceding LW market, and her CM value function will be given by

$$\begin{gathered} W^S(z) = \max_{X,H} \left\{ U(X) - H + \beta V^S \right\} \\ \text{s.t. } X = H + z, \end{gathered}$$

where $V^S$ denotes the seller's value function in next period's LW market, to be discussed below.[2] Sellers also choose $X = X^*$, and $W^S$ will also be linear:

$$W^S(z) = \Lambda^S + z. \tag{3}$$

[1] The intuition behind this result is simple. In monetary models, assets will, in general, be priced above the "fundamental value", reflecting liquidity premia. Agents who know with certainty that they will not have an opportunity to consume in the forthcoming LW market (just like our sellers here) will not be willing to pay such premia. Here we take this result as given (for a detailed discussion, see Rocheteau and Wright (2005)).

[2] Since the seller leaves the CM with no assets, she will never visit the OTC market.

Consider now the value functions in the LW market. Let $q$ denote the quantity of special good produced, and $\pi$ the real value of *money and fruit* that change hands during trade in the LW market. These terms will be determined by a buyer-takes-all mechanism. The LW value function for a buyer who enters that market with portfolio $(z, a_2)$ is given by

$$V(z, a_2) = u(q) + W(z - \pi, a_2), \tag{4}$$

and the LW value function for a seller (who enters with no assets) is given by

$$V^S = -q + W^S(\pi).$$

Finally, consider the value functions in the OTC market. After leaving the CM, and before the OTC market opens, buyers learn whether they will have a chance to access this period's LW market (C-types) or not (N-types). This chance will occur with probability $\ell \in (0, 1)$. The expected value for the typical buyer, before she enters the OTC market, is given by

$$\mathbb{E}\left\{\Omega^i(m, a_1, a_2)\right\} = \ell\, \Omega^C(m, a_1, a_2) + (1 - \ell)\, \Omega^N(m, a_1, a_2). \tag{5}$$

In the OTC market, C-type buyers, who may want additional liquid assets, are matched with N-type buyers, who may hold liquid assets that they will not use in the current period. Hence, trade in the OTC involves C-types giving up long term assets for short term assets and cash. Given the matching function $f(\ell, 1 - \ell)$, define the matching probabilities for C-types and N-types as $\alpha_{_C} \equiv f(\ell, 1 - \ell)/\ell$ and $\alpha_{_N} \equiv f(\ell, 1 - \ell)/(1 - \ell)$, respectively. Let $\chi$ denote the units of long term assets that the C-type transfers to the N-type, and $\zeta$ the real value of liquid assets that the C-type receives in return. These terms will be determined by bargaining, and are fully analyzed in the main text. Then,

$$\Omega^C(m, a_1, a_2) = \alpha_{_C}\, V\,(z + \zeta, a_2 - \chi) + (1 - \alpha_{_C}) V(z, a_2), \tag{6}$$

$$\Omega^N(m, a_1, a_2) = \alpha_{_N}\, W\,(z - \zeta, a_2 + \chi) + (1 - \alpha_{_N})\, W(z, a_2). \tag{7}$$

Notice that N-type buyers proceed directly to the CM. Also, notice that our definition $z \equiv \varphi m + d$ allows us to write $V$ as a function of $(z, a_2)$ (recall that each unit of $a_1$ will deliver one unit of fruit between the OTC and LW subperiods).

## 1.2 Bargaining in the LW and OTC Markets

The bargaining problems and solutions in the LW and OTC markets are exactly identical to those described in the main text, once we expand the definition of real balances to include the real value of money in addition to the fruit dividend of the maturing assets. We therefore directly proceed to the analysis of a buyer's optimal behavior.

### 1.3 Objective Function and Optimal Behavior

In this sub-section, we characterize the optimal portfolio choice of the representative buyer. We will do so by deriving the buyer's objective function, i.e. a function that summarizes the buyer's cost and benefit from choosing any particular portfolio $(\hat{m}, \hat{a}_1, \hat{a}_2)$. Substitute (6) and (7) into (5), and lead the resulting expression by one period to obtain

$$\begin{aligned}\mathbb{E}\left\{\Omega^i(\hat{m}, \hat{a_1}, \hat{a_2})\right\} &= f\, V\,(\hat{z} + \zeta, \hat{a}_2 - \chi) + (\ell - f)\, V\,(\hat{z}, \hat{a}_2) \\ &\quad + fW(\hat{z} - \tilde{\zeta}, \hat{a}_2 + \tilde{\chi}) + (1 - \ell - f)\, W(\hat{z}, \hat{a}_2), \end{aligned} \tag{8}$$

where $f$ is a shortcut for $f(\ell, 1-\ell)$. Since each unit of asset that matures in the next period pays one unit of fruit before the LW market opens, it is understood that $\hat{z} = \hat{\varphi}\hat{m} + \hat{a}_1 = \hat{\varphi}\hat{m} + d$.

The four terms in (8) represent the benefit for a buyer who holds a portfolio $(\hat{m}, \hat{a_1}, \hat{a_2})$ and turns out to be a matched C-type (with probability $f$), an unmatched C-type (with probability $\ell - f$), a matched N-type (with probability $f$), or an unmatched N-type (with probability $1 - \ell - f$), respectively. The expressions $\chi, \zeta$, and $\tilde{\chi}, \tilde{\zeta}$ are implicitly described by the solution to the OTC bargaining problem. In particular,

$$\begin{aligned} \chi &= \chi(\hat{z}, \tilde{z}, \hat{a}_2), & \zeta &= \zeta(\hat{z}, \tilde{z}, \hat{a}_2), \\ \tilde{\chi} &= \chi(\tilde{z}, \hat{z}, \tilde{a}_2), & \tilde{\zeta} &= \zeta(\tilde{z}, \hat{z}, \tilde{a}_2). \end{aligned}$$

In these expressions, the first argument represents the C-type's real balances, the second argument represents the N-type's real balances, and the third argument stands for the C-type's long term asset holdings (recall from the main text that the N-type's long term asset holdings do not affect the bargaining solution). Terms with tildes stand for the representative buyer's beliefs about her potential counterparty's real balances and long term asset holdings in the OTC.[3]

Next, we substitute $W$ and $V$ from (2) and (4), respectively, into (8). We insert the term $\mathbb{E}\left\{\Omega^i(\hat{m}, \hat{a}_1, \hat{a}_2)\right\}$ into (1), and we focus on the terms inside the maximum operator of (1). We define the resulting expression as $J(\hat{m}, \hat{a}_1, \hat{a}_2)$, and we refer to it as the buyer's objective function. The objective function is further separated into a cost component and an expected-benefit component of carrying assets. We denote this expected benefit function by $G(\hat{z}, \hat{a}_2)$, recognizing that money and short term assets are perfect substitutes and combining them into a choice of

[3] For instance, $\tilde{\zeta} = \zeta(\tilde{z}, \hat{z}, \tilde{a}_2)$ stands for the amount of real balances that the agent will give away if she is a matched N-type. This term depends on her own real balances ($\hat{z}$), and the real balances ($\tilde{z}$) and long term asset holdings ($\tilde{a}_2$) of her trading partner (a C-type). The terms $\chi, \zeta$, and $\tilde{\chi}$ admit similar interpretations.

real balances, $\hat{z} \equiv \hat{\varphi}\hat{m} + \hat{a}_1$. After some manipulations, one can verify that

$$\begin{aligned} J(\hat{m}, \hat{a}_1, \hat{a}_2) &= -\varphi\hat{m} - \psi_1\hat{a}_1 - \psi_2\hat{a}_2 + \beta\, G(\hat{\varphi}\hat{m} + \hat{a}_1, \hat{a}_2), \\ G(\hat{z}, \hat{a}_2) &= f\, \left[u\left(\hat{z} + \zeta\right) + \hat{\psi}_1\left(\hat{a}_2 - \chi\right)\right] + (\ell - f)\left[u\left(\hat{z}\right) + \hat{\psi}_1\hat{a}_2\right] \\ &\quad + f\, \left[\hat{z} - \tilde{\zeta} + \hat{\psi}_1\left(\hat{a}_2 + \tilde{\chi}\right)\right] + (1 - \ell - f)\left(\hat{z} + \hat{\psi}_1\hat{a}_2\right). \end{aligned} \tag{9}$$

The negative terms in the definition of $J$ represent the cost of purchasing various amounts of the three assets available in the economy.[4] The four terms in the definition of $G$ admit similar interpretations as their counterparts in equation (8). For instance, the first term represents the expected benefit of a C-type buyer who matches in the OTC market. This agent will increase her LW consumption by an amount equal to $\zeta$, but she will also go to next period's CM with her long term assets reduced by $\chi$. In this event, the terms $\zeta, \chi$ will depend on her own choices $\hat{z}, \hat{a}_2$, and on her trading partner's (who is an N-type) real balances, $\tilde{z}$.

We can now proceed with the examination of the buyer's optimal choice of $(\hat{z}, \hat{a}_2)$. We will do so for any possible money and asset prices, and for any given beliefs about other agents' money and asset holdings. We focus on prices that satisfy $\varphi > \beta\hat{\varphi}$, since we know that this will be always true in steady-state monetary equilibria with $\mu > \beta - 1$ (unless $\varphi = \hat{\varphi} = 0$, interpreted as a non-monetary equilibrium). Also, the asset prices have to satisfy $\psi_1 \geq \beta$ and $\psi_2 \geq \beta\hat{\psi}_1$, since violation of these conditions would generate an infinite demand for the assets. The optimal behavior of the buyer is described formally in Lemma 1 below. Here, we provide an intuitive explanation of the buyer's optimal portfolio choice.

The objective function of the buyer depends on the terms $\chi, \zeta, \tilde{\chi}$, and $\tilde{\zeta}$, which, in turn, depend on the bargaining protocol in the OTC market. Given the buyer's beliefs $(\tilde{z}, \tilde{a}_2)$, she can end up in different branches of the bargaining solution, depending on her own choices of $(\hat{z}, \hat{a}_2)$. In general, the domain of the objective function can be divided into five regions in $(\hat{z}, \hat{a}_2)$-space, arising from three questions: (*i*) When the C-type and the N-type pool their real balances in the OTC market, can they achieve the first-best in the LW market? (*ii*) If I am a C-type, do I carry enough assets to compensate the N-type? (*iii*) If I am an N-type, do I expect a C-type to carry enough assets to compensate me? These regions are illustrated in Figure 2 of the main text, and are also described in detail there. Here, we directly continue to state the most important facts about the optimal choice of the representative buyer:

**Lemma 1.** *Taking prices,* $(\varphi, \hat{\varphi}, \psi_1, \hat{\psi}_1, \psi_2)$*, and beliefs,* $(\tilde{z}, \tilde{a})$*, as given, and assuming that* $\mu > \beta - 1$ *and* $\varphi > 0$*, then the optimal choice of the representative agent,* $(\hat{m}, \hat{a}_1, \hat{a}_2)$*, satisfies:*

[4] In the objective function, the term $-\psi_1\hat{a}_1$ appears as the cost of purchasing assets that mature in the next period. However, we know that the term $\psi_1 a_2$ is also present in the agent's value function (see equation (1)), so that, practically, the cost of leaving the CM with $\hat{a}_1$ units of assets that mature tomorrow is $-\psi_1(\hat{a}_1 - a_2)$. However, the term $\psi_1 a_2$ only has a level effect, and it does not change the optimal choice of $\hat{a}_1$, with the exception that any choice of the agent should respect the restriction $\hat{a}_1 - a_2 \geq 0$.

*a) Money and short term assets are perfect substitutes. If $\psi_1 > \varphi/\hat{\varphi}$, then $\hat{a}_1 = 0$, and if $\psi_1 < \varphi/\hat{\varphi}$, then $\hat{m} = 0$.*

*b) If the optimal choice $(\hat{z}, \hat{a}_2)$ is strictly within any region, or on the boundary of Region 1 with any other region, and if $\psi_1 = \varphi/\hat{\varphi}$, it satisfies the first-order condition $\nabla J = \mathbf{0}$, or equivalently, $\beta \nabla G = (\psi_1, \psi_2)$.*

*c) If $\varphi > \beta\hat{\varphi}$ and $\psi_2 = \beta\hat{\psi}_1$, the optimal $\hat{z}$ is unique, and any $\hat{a}_2$ is optimal as long as $(\hat{m}, \hat{a})$ is in Regions 1, 2, or 3 (or on their boundaries).*

*d) If $\varphi > \beta\hat{\varphi}$ and $\psi_2 > \beta\hat{\psi}_1$, the optimal choice is unique, and it lies in Regions 4 or 5 or on their boundaries with Regions 2 and 3.*

*Moreover, let $G^i(\hat{z}, \hat{a}_2)$, $i = 1, ..., 5$, denote the expected benefit function in Region $i$, and $G^i_k(\hat{z}, \hat{a}_2)$, $k = 1, 2$, its derivative with respect to the $k$-th argument. Then, we have:*

$$G^1_1(\hat{z}, \hat{a}_2) = 1 + (\ell - \lambda f)\left[u'\left(\hat{z}\right) - 1\right], \tag{10}$$

$$G^2_1(\hat{z}, \hat{a}_2) = 1 + (\ell - \lambda f)\left[u'\left(\hat{z}\right) - 1\right] + \lambda f\left[u'\left(\hat{z} + \tilde{z}\right) - 1\right], \tag{11}$$

$$G^3_1(\hat{z}, \hat{a}_2) = 1 + (\ell - \lambda f)\left[u'\left(\hat{z}\right) - 1\right] + f\left[u'\left(\hat{z} + \tilde{z}\right) - 1\right], \tag{12}$$

$$G^4_1(\hat{z}, \hat{a}_2) = 1 + \ell\left[u'\left(\hat{z}\right) - 1\right] + (1-\lambda) f\left[u'\left(\hat{z} + \tilde{z}\right) - 1\right] + \lambda f\,\frac{u'\left[\hat{z} + \zeta^a(\hat{z}, \hat{a}_2)\right] - u'\left(\hat{z}\right)}{(1-\lambda)u'\left[\hat{z} + \zeta^a(\hat{z}, \hat{a}_2)\right] + \lambda}, \tag{13}$$

$$G^5_1(\hat{z}, \hat{a}_2) = 1 + \ell\left[u'\left(\hat{z}\right) - 1\right] + \lambda f\,\frac{u'\left[\hat{z} + \zeta^a(\hat{z}, \hat{a}_2)\right] - u'\left(\hat{z}\right)}{(1-\lambda)u'\left[\hat{z} + \zeta^a(\hat{z}, \hat{a}_2)\right] + \lambda}, \tag{14}$$

$$G^1_2(\hat{z}, \hat{a}_2) = G^2_2(\hat{z}, \hat{a}_2) = G^3_2(\hat{z}, \hat{a}_2) = \hat{\psi}_1, \tag{15}$$

$$G^4_2(\hat{z}, \hat{a}_2) = G^5_2(\hat{z}, \hat{a}_2) = \hat{\psi}_1\left\{1 - f + f\,\frac{u'\left[\hat{z} + \zeta^a(\hat{z}, \hat{a}_2)\right]}{(1-\lambda)u'\left[\hat{z} + \zeta^a(\hat{z}, \hat{a}_2)\right] + \lambda}\right\}, \tag{16}$$

*where $\zeta^a(\,\cdot\,,\,\cdot\,)$ is the real balance trading volume in the OTC market in the case where long term assets are scarce, defined as part of the bargaining solution in the main text.*

*Proof.* Consider first the derivatives of the expected benefit function with respect to $\hat{z}$ and $\hat{a}_2$, i.e. equations (10)-(16). To obtain these conditions we substitute the appropriate solution to the bargaining problem (depending on the region in question) into (9), and we differentiate with respect to $\hat{z}$ or $\hat{a}_2$.

As an illustration, consider Region 2. Recall that in this region, $\hat{z} < q^* - \tilde{z}$, $\hat{a}_2 > \bar{a}(\hat{z}, \tilde{z})$, but $\tilde{a}_2 < \bar{a}(\tilde{z}, \hat{z})$. Based on this information, we have $\chi = \bar{a}(\hat{z}, \tilde{z})$, $\zeta = \tilde{z}$, $\tilde{\chi} = \tilde{a}_2$, and $\tilde{\zeta} = \zeta^a(\tilde{z}, \tilde{a}_2)$. Substituting these terms into the expected surplus function implies that

$$\begin{aligned} G^2(\hat{z}, \hat{a}_2) = &\ f\ \left\{u\left(\hat{z} + \tilde{z}\right) - \beta\psi_1\,\bar{a}(\hat{z}, \tilde{z})\right\} \\ &+ (\ell - f)\ u(\hat{z}) + f\ \left\{\left[\hat{z} - \zeta^a(\tilde{z}, \tilde{a}_2)\right] + \beta\psi_1\tilde{a}_2\right\} + (1 - \ell - f)\,\hat{z}. \end{aligned}$$

It is now straightforward to show that $G_1^2$ and $G_2^2$ are given by (11) and (15), respectively. The remaining derivations follow exactly the same steps.

Notice that we can solve $J_1^i = 0$, $i = 1, ..., 5$, with respect to either the term $\varphi/(\beta\hat{\varphi})$, which, in steady state equilibrium, is just one plus the nominal interest rate, or the term $\hat{\psi}_1$, whichever is smaller (unless they are equal). This will yield the demand for real balances as a function of their holding cost. For future reference, it is important to highlight that the demand for real balances is in fact continuous on the boundaries 1-2, and 1-5.[5] Similarly, we can solve $J_2^i = 0$, $i = 1, ..., 5$, with respect to $\psi_2/(\beta\hat{\psi}_1)$, in order to obtain the demand for long term assets. It can be easily verified that this function is continuous on the boundaries 1-2, 2-5, 2-3, and 4-5.

Some preliminary facts:

Next, let us state some facts about the surplus function $G : \mathbb{R}_+^2 \to \mathbb{R}$ and the objective function $J : \mathbb{R}_+^3 \to \mathbb{R}$:

*Fact 1:* $G$ (and therefore $J$) is continuous everywhere.

*Proof:* The solution to the OTC bargaining problem is continuous. One of the three constraints $\zeta \leq \tilde{z}$, $\zeta \leq q^* - z$, and $\chi \leq a_2$ must bind, together with the bargaining surplus sharing equation. Each of these is continuous in the choice variables. Therefore, $G$ is continuous.

*Fact 2:* $G$ (and therefore $J$) is differentiable within each of the five regions defined above.

*Proof:* As above, one of the constraints must bind together with the surplus sharing equation. Each of these is differentiable in the choice variables, and within a region of $G$, the binding constraint does not switch. Furthermore, $G$ is differentiable on those boundaries where both FOCs are continuous (see above).

*Fact 3:* $G$ is strictly concave in the first argument (real balances) whenever $z < q^*$.

*Proof:* As $G$ is continuous everywhere and differentiable within each region, $G_1$ is defined everywhere except at a finite number of boundary crossings. We need to show that $G_1$ is decreasing as a function of $\hat{z}$ within each region, and that $G_{1-} \geq G_{1+}$ on each boundary, where "–" denotes the left derivative and "+" denotes the right derivative.
That $G_1$ is strictly decreasing in $\hat{z}$ within Regions 1-3 follows immediately from equations (10)-(12), and the fact that $u'$ is strictly decreasing. In Regions 4 and 5, showing that $G_1$ is decreasing in $\hat{z}$ is less obvious. In Region 5 (where $\hat{z} + \zeta < q^*$), we have

$$G_1^5 = \ell\left[u'(\hat{z}) - 1\right] + \lambda f \frac{u'(\hat{z}+\zeta) - u'(z)}{(1-\lambda)u'(\hat{z}+\zeta) + \lambda}.$$

Since $\zeta$ is defined by the equation $(1-\lambda)\left[u\left(z+\zeta\right) - u(z)\right] + \lambda\zeta = \psi_1 a_2$, applying total differentiation in this equation yields

$$\frac{d\zeta}{d\hat{z}} = (1-\lambda)\frac{u'(\hat{z}) - u'(\hat{z}+\zeta)}{(1-\lambda)u'(\hat{z}+\zeta) + \lambda}.$$

[5] The demand for real balances is also continuous on the boundaries of the Regions 1-3 and 4-5 if $\tilde{a}_2 \geq \bar{a}(\tilde{z}, q^* - \tilde{z})$, in which case Region 2 does not exist.

Consequently,

$$\frac{\partial G_1^5}{\partial z} = \frac{1}{[(1-\lambda)u'(\hat{z}+\zeta)+\lambda]^3}\Big\{ f\lambda\left[(1-\lambda)u'(\hat{z})+\lambda\right]^2 u''(\hat{z}+\zeta) + \left[(\ell-f)\lambda+\ell(1-\lambda)u'(\hat{z}+\zeta)\right]\left[(1-\lambda)u'(\hat{z}+\zeta)+\lambda\right]^2 u''(\hat{z})\Big\}.$$

Since $u''(\,\cdot\,)<0$, the entire term $\partial G_1^5/\partial\hat{z}<0$. In Region 4, the only addition is a term involving $u'(\,\cdot\,)$, which is clearly decreasing too. Hence, $G_1^4$ is decreasing in $\hat{z}$ as well.

As we discussed above, $G_1$ is continuous across all the boundaries of the various regions, except the boundaries 2-3, 3-4, 4-5, 2-5, and the crossing 2-4. With some algebra, one can check that $G_1^2<G_1^3$, $G_1^3<G_1^4$, $G_1^4<G_1^5$, and $G_1^2<G_1^5$, across the respective boundaries. Also, $G_1^3>G_1^5$ at the crossing 2-3-4-5, establishing the chain $G_1^2<G_1^5<G_1^3<G_1^4$ at this crossing. Consequently, $G$ is concave in $\hat{z}$ throughout.

*Fact 4:* $G$ is concave in the second argument (long term assets), strictly in Regions 4 and 5.

*Proof:* As $G$ is continuous everywhere and differentiable within each region, $G_2$ is defined everywhere except at a finite number of boundary crossings. We need to show that $G_2$ is decreasing as a function of $\hat{a}_2$ within each region (strictly, in Regions 4 and 5), and that $G_{2-}\geq G_{2+}$ on each boundary, where "–" denotes the left derivative and "+" denotes the right derivative. In Regions 1-3, $G_2^i$ is constant, hence weakly concave. We now show that $G_2^i$ is strictly decreasing in $\hat{a}_2$ within Regions 4 and 5. Again applying total differentiation in equation $(1-\lambda)\left[u\left(z+\zeta\right)-u(z)\right]+\lambda\zeta=\psi_1 a_2$, yields

$$\frac{\partial\zeta}{\partial a_2}=\frac{\beta\hat{\psi}_1}{(1-\lambda)u'\left(\hat{z}+\zeta\right)+\lambda}.$$

Since this expression is clearly positive, and $u'$ is strictly decreasing, it follows that $\partial G_2^i/\partial\hat{a}_2<0$, for $i=4,5$.

Next, using the definitions of the regions, one can see that $G_2$ is continuous across the boundary 1-5, but not the boundaries 2-5 or 3-4. The term $u'(\hat{z}+\zeta)[(1-\lambda)u'(\hat{z}+\zeta)+\lambda]^{-1}$ is greater than $1$ in Regions 4 and 5, because $\hat{z}+\zeta<\min\{\hat{z}+\tilde{z},q^*\}$ (by definition of Regions 4 and 5), and therefore $u'(\,\cdot\,)>1$.

*Fact 5:* $G$ is weakly concave everywhere.

*Proof:* We need to show that $G_2$ is non-increasing as a function of $\hat{z}$ within each region, and across boundaries. First, $G_2$ depends on $\hat{z}$ only in Regions 4 and 5. There, $\zeta$ is strictly increasing in $\hat{z}$, therefore $u'(\hat{z}+\zeta)$ is strictly decreasing, and so is $u'(\hat{z}+\zeta)[(1-\lambda)u'(\hat{z}+\zeta)+\lambda]^{-1}$.

Now, the only boundaries where $G_2$ is not a continuous function of $\hat{z}$ are the boundaries of Regions 3 and 4, and 2 and 5, which are downward sloping in $(\hat{z},\hat{a}_2)$-space. On these boundaries, $G_{2-}>G_{2+}$ (see Fact 4). This is sufficient because an infinitesimal increase in $\hat{z}$ has the same effect as an infinitesimal increase in $\hat{a}_2$ (the definition of $G_{2+}$), and vice versa, as the boundaries are downward sloping in $(\hat{z}, \hat{a}_2)$-space.

We conclude that $G_2$ is weakly decreasing as a function of $\hat{z}$, therefore $G$ is submodular (real balances and long term assets are strategic substitutes). As $G$ is also weakly concave in each argument, it is weakly concave overall.

Proof of the statement of the Lemma:

a) If $\psi_1 > \varphi/\hat{\varphi}$, then $\partial J/\partial \hat{m} > \partial J/\partial \hat{a}_1$ for any $i = 1, \ldots, 5$, and vice versa.

b) Since $\psi_1 = \varphi/\hat{\varphi}$, $\partial J/\partial \hat{m} = \partial J/\partial \hat{a}_1$ for any $i = 1, \ldots, 5$; therefore, $\nabla J = \mathbf{0}$ is equivalent to $\beta\, \nabla G = (\psi_1, \psi_2)$. The fact that $\nabla J = \mathbf{0}$, follows from the fact that $G$ is weakly concave overall and differentiable within each region. So if the optimal choice $(\hat{z}, \hat{a}_2)$ is within a region, the first-order conditions must hold.

c) The fact that $\psi_2 = \beta\hat{\psi}_1$ rules out Regions 4 and 5. To see this point, notice from (16) that for any $(\hat{z}, \hat{a}_2)$ in the interior of these regions, $\psi_2 = \beta\hat{\psi}_1$ implies $\beta G_2^i > \psi_2$, for $i = 4, 5$. In Regions 1-3, demand for real balances is strictly decreasing, so the $\hat{z}$ satisfying $\varphi > \beta\hat{\varphi}$ is unique. But any $\hat{a}_2$ in Regions 1-3 satisfies $\beta G_2^i = \psi_2$, $i = 1, 2, 3$.

d) The fact that $\psi_2 > \beta\hat{\psi}_1$ rules out the interior of Regions 1-3 or the boundary 1-5. To see why, notice from (15), that for any $(\hat{z}, \hat{a}_2)$ in the regions in question, $\psi_2 > \beta\hat{\psi}_1$ implies $\beta G_2^i < \psi_2$, for $i = 1, 2, 3$. □

Lemma 1 formally describes the optimal behavior of the representative buyer. Given the results stated in the lemma, one can describe in detail the demand functions for the various assets, which we do in the following section. Although interesting, this analysis is not essential for understanding the main results of the paper, hence, the reader may skip ahead to Section 1.5 for our discussion of the steady-state equilibrium.

## 1.4 Analysis of Money and Asset Demand

In this section, we explore the implications of Lemma 1 for the buyers' demand for the various assets. Consider first the optimal choice of long term assets (i.e. $\hat{a}_2$). If the price of long term assets satisfies $\psi_2 = \beta\hat{\psi}_1$, the cost of carrying long term assets is zero and, therefore, it would be suboptimal for the buyer to be in a region where her long term assets would not allow her to afford the optimal quantity of liquid assets, when a C-type. As a result, when $\psi_2 = \beta\hat{\psi}_1$, the buyer never chooses a portfolio in the interior of Regions 4 and 5. If $\psi_2 > \beta\hat{\psi}_1$, carrying long term assets is costly. The optimal choice of the buyer is characterized by the first-order conditions and, graphically, it lies within Regions 4 or 5. For any set of prices which satisfy $\psi_1 = \varphi/\hat{\varphi} > \beta$, the optimal choice of real balances is uniquely characterized by the first-order condition with respect to either $\hat{m}$ or $\hat{a}_1$.

Next, we demonstrate the determination of the demand for real balances. This demand, $D_z$,

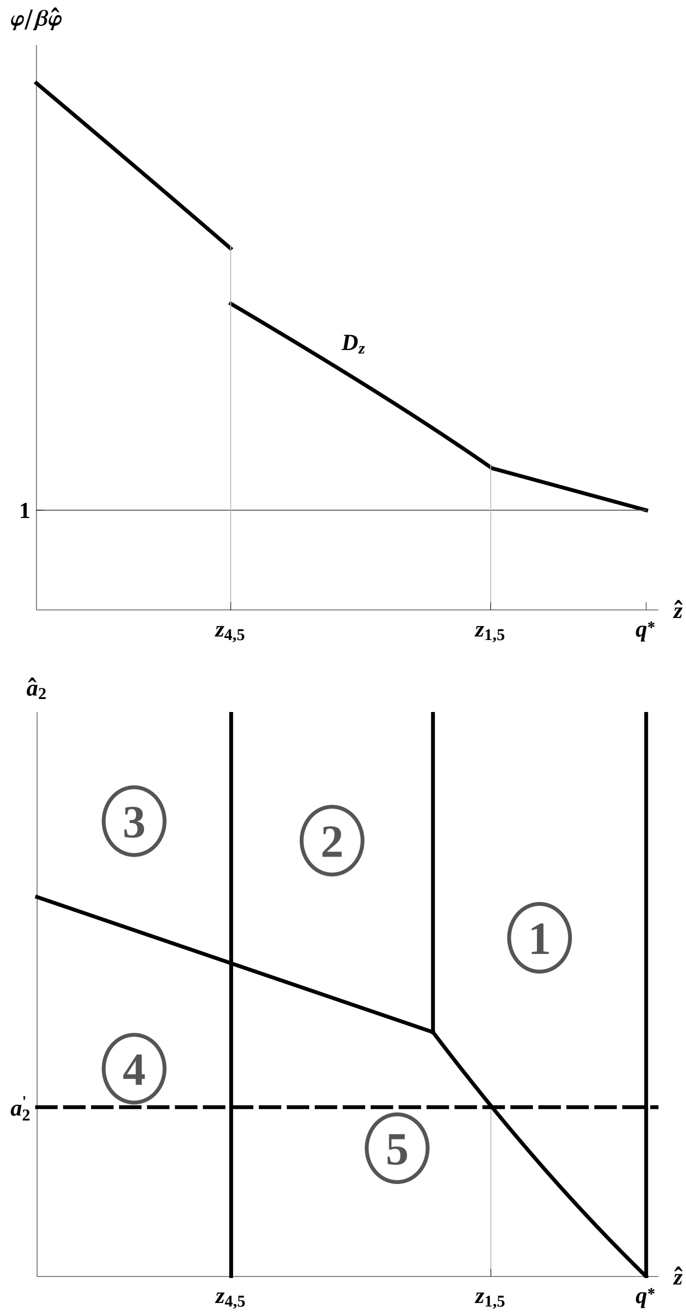


Figure 1: Demand for real balances given long term asset holdings $a_2'$.

is plotted in Figure 1 against the ratio $\varphi/(\beta\hat{\varphi})$, which captures the holding cost of real balances.[6] The level of long term asset holdings is kept fixed at $\hat{a}_2 = a_2'$ indicated in the lower panel of the figure. Notice that the lower panel of Figure 1 is identical to Figure 2 in the main text. Aligning the two plots vertically allows the reader to easily indicate which region the buyer will find herself in, for any choice of $\hat{z}$, and for a given value of $\hat{a}_2$. For $\hat{a}_2 = a_2'$, any $\hat{z} > \bar{z}_{1,5}$ implies that the buyer is in Region 1, and in this region one additional unit of real balances has

[6] More precisely, $\varphi/(\beta\hat{\varphi})$ captures the holding cost of money. However, in any equilibrium where $\hat{m}, \hat{a}_1 > 0$, the holding cost of the two liquid assets will necessarily be the same.

the following benefits: a) it serves as a store of value, if the buyer is an N-type; b) it allows the buyer to purchase more goods in the LW market, if she is an unmatched C-type; and c) it allows the buyer to reduce her demand for the N-type's real balances, if she is a matched C-type.

As $\hat{z}$ decreases below $\bar{z}_{1,5}$, the buyer finds herself in Region 5. The function $D_z$ is continuous and exhibits a kink at $\bar{z}_{1,5}$, and the slope of $D_z$ is steeper to the left of $\bar{z}_{1,5}$. To illustrate this property, consider how the marginal benefit of carrying one additional unit of real balances changes as the buyer moves from Region 1 to Region 5. Recall that, in Region 1, an additional unit of real balances has three effects. The effects indicated by (a) (store of value when N-type) and (b) (higher marginal utility when unmatched C-type) are still valid as we enter Region 5. What differs is the marginal benefit of real balances when the buyer is a matched C-type: in this event, an additional unit of $\hat{z}$ does not only allow her to reduce her demand for the N-type's real balances (effect (c) above), but it allows her to acquire extra purchasing power in the forthcoming LW market.[7] Hence, the slope of the demand function is higher (in absolute value) for $\hat{z}$ in the range $[\bar{z}_{4,5}, \bar{z}_{1,5})$ compared to $[\bar{z}_{1,5}, q^*)$. Also, from (10) and (14), we have

$$G_1^5 - G_1^1 = \lambda f \left\{ u'(\hat{z}) - 1 + \frac{u'\left[\hat{z} + \zeta^a(\hat{z}, \hat{a}_2)\right] - u'\left(\hat{z}\right)}{(1-\lambda)u'\left[\hat{z} + \zeta^a(\hat{z}, \hat{a}_2)\right] + \lambda} \right\},$$

which is what differentiates $D_z$ on the two sides of $\bar{z}_{1,5}$. When $\hat{z} = \bar{z}_{1,5}$, we have $\hat{z} + \zeta^a = q^*$, and it is easy to verify that $G_1^5 - G_1^1 = 0$. As a result, $D_z$ exhibits a kink but is continuous at $\bar{z}_{1,5}$.

Finally, $D_z$ exhibits a jump, at $\bar{z}_{4,5}$, the value of $\hat{z}$ that, given $\hat{a}_2 = a_2'$, brings the agent on the boundary of Regions 4 and 5 (and in the interior of Region 5 if and only if $\hat{z} > \bar{z}_{4,5}$). Consider the behavior of $D_z$ in a neighborhood of this point. In Region 5, an additional unit of real balances serves as a store of value, if the buyer is an N-type, and it allows the buyer to purchase more goods in the LW, if she is a C-type (matched or unmatched). These effects remain valid as we enter into Region 4. However, in Region 4 a new effect arises, which is relevant when the buyer is a matched N-type. In this region, the C-type counterparty can afford to buy all of the buyer's real balances, hence the buyer's choice of $\hat{z}$ affects the OTC terms of trade even when she is an N-type (assuming that $\lambda < 1$). Specifically, the less real balances the buyer brings, the more desperate the C-type will be for those real balances, and the more long term assets she will be willing to give up in order acquire them. Formally, (14) and (13) imply that

$$G_1^4 - G_1^5 = (1-\lambda) f \left[ u'\left(\hat{z} + \tilde{z}\right) - 1 \right].$$

[7] Put simply, in the event that the buyer is a matched C-type, if she is in Region 1, she will be able to buy $q^*$ anyway. Bringing more $\hat{z}$ will not change the quantity of LW consumption (it will still be equal to $q^*$), but it will allow her to rely less heavily on the N-type's liquid assets (which could be quite important, especially if the terms of trade are against her in the OTC market, i.e. if $\lambda$ is low). On the other hand, in Region 5, the matched C-type cannot buy $q^*$ even after purchasing all the real balances of the N-type that she can afford. In this case, bringing more $\hat{z}$ strictly increases her LW consumption.

Since $\bar{z}_{4,5} + \tilde{z} < q^*$, this term is strictly positive when $\hat{z} = \bar{z}_{4,5}$, provided that $\lambda < 1$. This gap between the values of $G_1^4$ and $G_1^5$ reflects the discontinuity of $D_z$ at $\bar{z}_{4,5}$.

## 1.5 Definition of Equilibrium and Preliminary Results

We restrict attention to symmetric steady-state equilibria, where all agents choose the same portfolios, and the real variables of the model remain constant over time. Since, in steady state, the real money balances do not change over time, we have $\varphi/\hat{\varphi} = 1 + \mu$ in any monetary equilibrium where $\hat{\varphi} > 0$. In such an equilibrium, we must also have $\psi_1 = \hat{\psi}_1 = 1 + \mu$, since money and short term assets are perfect substitutes.[8] Before stating the definition of a steady-state equilibrium, it is important to notice that symmetry rules out Regions 2 and 4 of the buyer's choice problem, since a C-type and an N-type buyer are ex ante identical.

In order to characterize equilibrium, we use three restrictions. First, aggregate real balances $Z$ are the combination of real money ($\varphi M$) and maturing short term bonds ($A_1$), so $Z \geq A_1$. Second, recall the constraint that agents cannot sell off-the-run short term assets in the CM; at most, they can refrain from buying newly issued short term bonds. So the post-CM holdings of short term bonds (equal to $A_1$ in symmetric equilibrium) must exceed the pre-CM holdings for every agent, including those of asset buyers in the preceding OTC market (equal to $A_2 + \chi(Z, Z, A_2)$ in symmetric equilibrium), thus $A_1 \geq A_2 + \chi(Z, Z, A_2)$. Together, these restrictions rule out combinations of low $Z$ and high $A_2$. With the following restriction on structural parameters, Region 3 is ruled out altogether:

$$\frac{1 + (1 - \lambda)\left[\frac{u(q^*) - u(q^*/2)}{q^*/2} - 1\right]}{1 + (\ell - \lambda f)\left[u'(q^*/2) - 1\right]} > \frac{\beta}{2}. \tag{17}$$

This restriction guarantees that, in Figure 2, the line $Z = 2A_2$ lies below the boundary of Regions 1,3, and 5 (or equivalently that the term $\bar{A}_1$, indicated in the figure and defined in (23), satisfies $\bar{A}_1 \geq q^*/2$). While it is possible to construct a counterexample, the restriction is satisfied for a wide range of utility functions if $f$ is close to $\ell$ (C-types have a high probability of matching). Henceforth, we assume that the model's parameters satisfy the inequality stated in (17).

With the above constraints satisfied, only two regions remain on aggregate:

1. Agents carry enough real balances and long term assets so that, when matched in the OTC market, the C-type can acquire sufficient liquidity in order to achieve the first-best in the LW market.

[8] To see how this simple relationship emerges, one just needs to equate the rate of return on money, $\hat{\varphi}/\varphi - 1 = (1 + \mu)^{-1} - 1$, with the rate of return on the short term assets, $\hat{\psi}^{-1} - 1$.

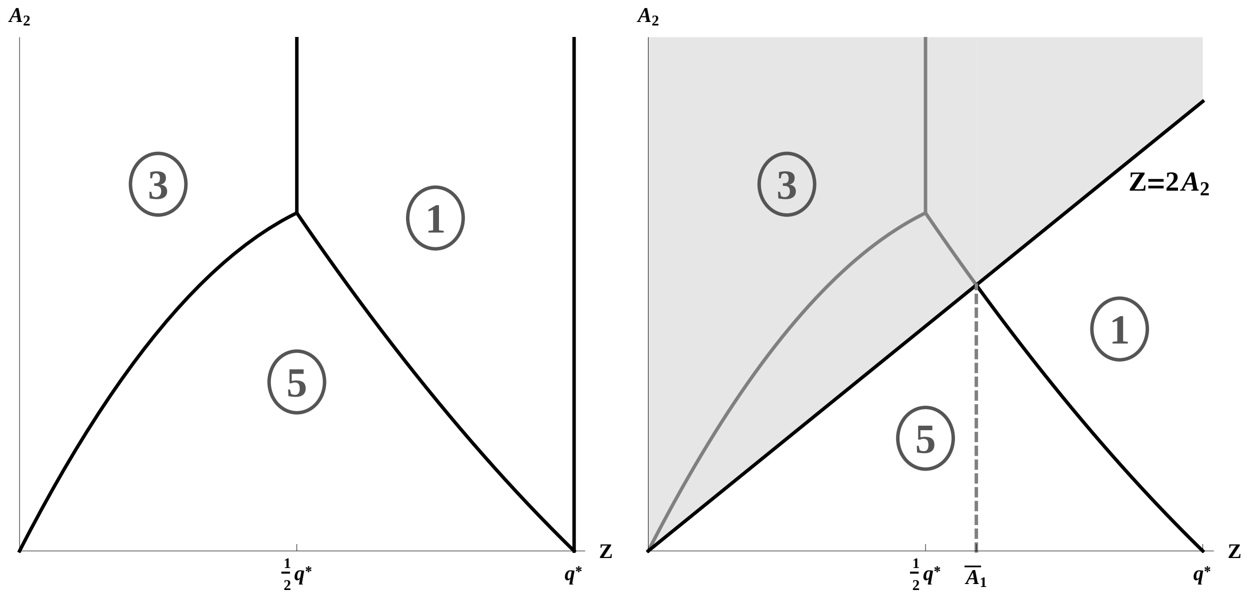

Figure 2: Aggregate regions of equilibrium, in terms of real balances.

5. Agents carry so few long term assets that, when matched in the OTC, the C-type will sell all of her long term assets but not obtain enough of the N-type's real balances in order to achieve the first-best in the LW market.

These regions are described in Figure 2, and we will refer to them as the "aggregate regions", as opposed to the "individual regions" of the buyer's optimal portfolio problem. In general, Region 1 represents the region of abundance of the long maturity asset, and Region 5 represents the region of scarcity.

**Definition 1.** A symmetric steady-state equilibrium is a list $\{\varphi, \psi_1, \psi_2, \chi, \zeta, Z, q_1, q_2\}$, where $Z = \varphi M + A_1$ represents the real balances, which are equal to the amount of good exchanged in the LW market when the buyer was not matched in the preceding OTC market, i.e. $q_1$. The term $q_2$ is the amount of good exchanged in the LW market when the buyer *was* matched. The equilibrium objects satisfy:

i. The representative buyer behaves optimally under the equilibrium prices $\psi_1, \psi_2, \varphi$, and, moreover, $\psi_1 = \hat{\psi}_1 = \varphi/\hat{\varphi} = 1 + \mu$ if $\hat{\varphi} > 0$.

ii. The equilibrium quantity $q_2$ is defined as the following function of $Z$:

$$q_2(Z) = \begin{cases} q^*, & \text{in Region 1,} \\ \tilde{q}(Z), & \text{in Region 5,} \end{cases}$$

where $\tilde{q}$ solves $(1-\lambda)\left[u(\tilde{q}) - u(Z)\right] + \lambda\left(\tilde{q} - Z\right) = \psi_1 A_2$.

iii. The terms of OTC trade $(\chi, \zeta)$ satisfy the bargaining solution evaluated at the aggregate quantities $Z$ and $A_2$.

iv. Markets clear at symmetric choices, and expectations are rational: $\hat{m} = (1+\mu)M$, $\hat{z} = \tilde{z} = Z$, $\hat{a}_1 = A_1$, and $\hat{a}_2 = \tilde{a}_2 = A_2$.

**Lemma 2.** *Define the function* $Z(\mu, A_1) \equiv \max\{A_1, \{Z : (1+\mu)/\beta = 1 + (\ell - \lambda f)(u'(Z) - 1)\}\}$. *If* $\mu > \beta - 1$ *and* $A_1 \geq A_2 + \chi[Z(\mu, A_1), Z(\mu, A_1), A_2]$ *are satisfied, then a symmetric steady-state equilibrium exists and is unique.*

*Proof.* The equilibrium objects $q_1, q_2, \chi$, and $\zeta$ are all deterministic functions of $Z$, so it suffices to focus on $Z$, $\psi_1$, and $\psi_2$. Since $\mu > \beta - 1$, we have $\varphi > \beta\hat{\varphi}$ if $\hat{\varphi} > 0$. Consequently, parts (c) and (d) of Lemma 1 apply, and an optimal $(\hat{z}, \hat{a}_2)$ exists and $\hat{z}$ is unique. The objects $\psi_1$ and $\hat{\varphi}$ (and a proportional $\varphi = (1+\mu)\hat{\varphi}$) must be chosen such that $\hat{z} = Z$ and $\hat{a}_1 = A_1$ satisfy the demand for real balances, $\beta G_1 = \psi_1$. If this equation is satisfiable for $\psi_1 = 1 + \mu$ and some $Z > A_1$, then $\varphi = \beta\hat{\varphi} > 0$ and $\psi_1 = \varphi/(\beta\hat{\varphi}) = 1 + \mu$. Otherwise, $Z = A_1$ and $\varphi = \hat{\varphi} = 0$, and $\psi_1 \leq 1 + \mu$.

Finally, set $\hat{a}_2 = A_2$. The assumption $A_1 \geq A_2 + \chi[Z(\mu, A_1), Z(\mu, A_1), A_2]$ guarantees that agents never need to sell assets in the CM; N-types held two-period assets $A_2$ at the end of the preceding period, which become one-period assets in the given period, and obtain $\chi$ more in the OTC market if they are matched. C-types and unmatched N-types will enter the CM with less than $A_2 + \chi$ one-period assets, so every agent can obtain the symmetric quantity of short term assets, $A_1$, by buying newly issued ones and not by selling previously-issued ones.

Additionally, if the parameters of the model satisfy inequality (17), then the equilibrium must be in Regions 1 or 5, as described in the text. Now examine the demand function for long term assets (equations (15) and (16)). It is constant in Regions 1 and strictly decreasing in $\hat{a}_2$ in Region 5 (also see the proof of Lemma 1, Fact 4), and is continuous on the boundary of Regions 1 and 5. If $(Z, A_2)$ lies in the interior of Region 5, then $\psi_2 > \beta\psi_1$ is unique. If $(Z, A_2)$ lies in the interior of Region 1 or on the boundary of Regions 1 and 5, then $\psi_2 = \beta\psi_1$, which is unique. □

Having formally described the definition of a steady-state equilibrium and guaranteed its existence and uniqueness, the next task is to characterize such equilibria. Ultimately, we wish to describe the equilibrium variables as functions of the exogenous supply parameters $A_1, A_2$ and the policy parameter $\mu$. Thus, before we state the main results, it is useful to describe the aggregate regions in terms of the parameter $\mu$ rather than $Z$. This task becomes easier with the help of Figure 3. An explicit description of the various curves that appear in this figure, as well as real balances $Z$ in terms of inflation $\mu$, is provided in Section 1.7 below. Here, we proceed with an intuitive interpretation. The following three observations are crucial.

a) The real balances $Z = \varphi M + A_1$ are decreasing in $\mu$, but also bounded below by $A_1$. Consequently, if inflation exceeds a certain level $\bar{\mu}(A_1, A_2)$ (indicated as a green piece-wise curve

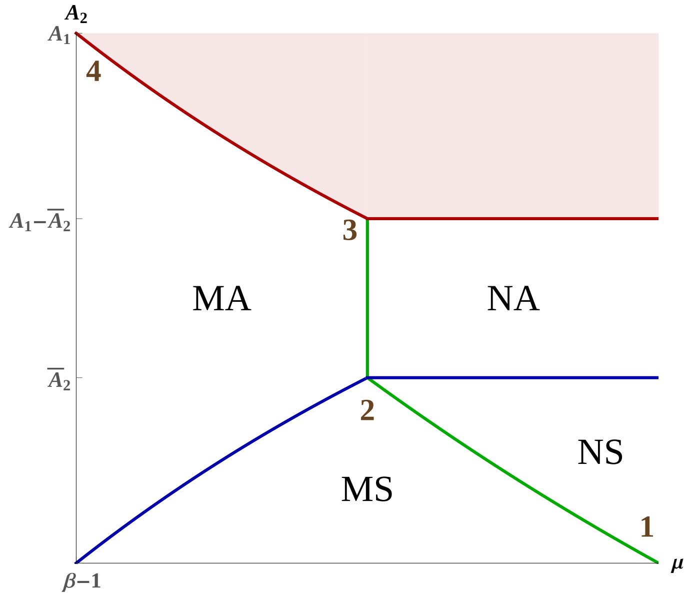


Figure 3: Aggregate regions of equilibrium, in terms of inflation, drawn under the assumption that $A_1 \in (\bar{A}_1, q^*)$, where $\bar{A}_1$ is defined in (23). If $A_1 \leq \bar{A}_1$, then Region $NA$ is empty and the horizontal segments of the blue and red lines coincide.

in Figure 3), then $\varphi = 0$, i.e. no monetary equilibrium exists. This critical level is a decreasing function of $A_1$. When $A_2$ is relatively plentiful, it does not affect the demand for real balances and the line $\mu = \bar{\mu}(A_1, A_2)$ is vertical (the segment between points 2 and 3). However, when $A_2$ is relatively scarce, it does affect the terms of trade in the OTC market and hence the demand for real balances. As $A_2$ decreases, C-types must increasingly rely on their own real balances, so that, despite an increasing cost of holding money (as $\mu$ increases), a monetary equilibrium still exits. Thus, the line $\mu = \bar{\mu}(A_1, A_2)$ is downward sloping for low $A_2$ (the segment between points 1 and 2). For any given $(A_1, A_2)$, increasing inflation beyond $\bar{\mu}$ has no effect on real balances.

b) The line between the origin and point 2 is the inverted image of the boundary of Regions 1 and 5 in Figure 2, i.e. it separates the parameter space in a way that for any $A_2$ north of the line, long term assets are abundant in the OTC market. This line slopes upwards because higher inflation both reduces the amount of real balances and increases the need to trade in the OTC market, hence making $A_2$ more likely to be scarce. As we move east of point 2, we enter the non-monetary region, and real balances are independent of $\mu$. Hence, the line that separates the space into the region of abundance or scarcity of the long term assets (in the OTC market) becomes a horizontal line (it depends only on the relative values of $A_1, A_2$, but not on $\mu$).

c) Furthermore, we need to consider the constraint that agents are unable to sell off-the-run short term assets in the CM. In other words, we need to guarantee that every agent enters the CM with an amount of off-the-run assets smaller than the amount of short term assets they leave the CM with. At the Friedman rule, i.e. for $\mu = \beta - 1$, no OTC trade will take place, and the relevant constraint is simply $A_2 \leq A_1$. Away from the Friedman rule, N-type agents will leave the OTC with an additional amount of assets, $\chi$, which increases with inflation, so the constraint $A_2 \leq A_1 - \chi$ becomes more binding (between points 3 and 4). For $\mu \geq \bar{\mu}$ (east of point 3), real balances and $\chi$ are unaffected by $\mu$ and the constraint becomes a horizontal line.

In summary, combinations of parameters $(A_2, \mu)$ that lie in the shaded region in Figure 3 are ruled out. In the remaining parameter space, every point that lies on the west (east) of the green piece-wise curve is associated with monetary (non-monetary) equilibrium. Similarly, every point that lies on the north (south) of the blue piece-wise curve is associated with equilibria where the long term assets are abundant in the OTC market. Thus, every equilibrium necessarily lies in one of four distinct regions clearly marked in Figure 3: i) $MA$ stands for *monetary* equilibrium where long term assets are *abundant* in the OTC market, ii) $MS$ stands for *monetary* equilibrium where long term assets are *scarce* in the OTC market, iii) $NA$ stands for *non-monetary* equilibrium where long term assets are *abundant* in the OTC market, and iv) $NS$ stands for *non-monetary* equilibrium where long term assets are *scarce* in the OTC market.

## 1.6 Characterization of Equilibrium

We are now ready to characterize equilibrium. We begin this subsection with an intuitive description of the results presented in Propositions 1, 2, and 3. The critical parameter in the analysis is the supply of maturing assets $A_1$. If this supply is plentiful, in a way to be made precise in Proposition 1, short term assets alone are enough to satisfy the liquidity needs of the economy (for trade in the LW market). In this case, there is no room for money and no role for OTC trade. On the other hand, if $A_1$ is insufficient to satisfy the liquidity needs of the economy (which we consider the interesting case), a role for money arises (the lower the value of $A_1$, the bigger that role). By no-arbitrage, the short term asset price will be fully determined by the policy parameter $\mu$, in particular $\psi_1 = 1+\mu$. Away from the Friedman rule, the equilibrium real balances will always be suboptimal ($Z < q^*$), and this has two important implications for asset prices. First, $\psi_1$ will carry a *liquidity premium* (i.e. $\psi_1 > \beta$), because the marginal unit of short term assets is not only a good store of value, but it can also increase consumption in the LW market. Second, with $Z < q^*$, trade in the OTC market becomes crucial. In this case, the long term assets can potentially also carry a liquidity premium, not because they can facilitate trade in the LW market, but because they can be used in the OTC market in order to purchase liquid assets. Naturally, $\psi_2$ will include a liquidity premium if the supply $A_2$ is relatively scarce, in the precise sense that the equilibrium falls in Regions $MS$ or $NS$ in Figure 3.

We now describe these results in a formal way, in propositions analogous to those in the main text.

**Proposition 1.** *If $A_1 \geq q^*$, the equilibrium is always non-monetary regardless of $\mu$, no trade occurs in the OTC market, and asset prices always equal their fundamentals: $\psi_i = \beta^i$ for $i = 1, 2$.*

*Proof.* We know that $Z \geq A_1$, therefore $\hat{z} \geq A_1$ in every equilibrium, and $G_1 = 1$. As $\mu > \beta - 1$, the cost of holding money is positive and $\beta G_1 = 1 + \mu$ is unsatisfiable. Therefore, $\varphi = 0$ (money has no value) and $Z = A_1$. OTC bargaining yields $\zeta = \chi = 0$. Optimal behavior yields $\psi_1 = \beta G_1 = \beta$ and $\psi_2 = \beta\psi_1 = \beta^2$. □

When $A_1 \geq q^*$, the supply of maturing short term assets suffices to cover the liquidity needs of the economy (i.e. the need for trade in the anonymous LW market). This has the following consequences. First, it is clear that in this economy there is no role for money: every LW meeting will always involve the exchange of the optimal amount of good, $q^*$. Second, since agents already bring with them sufficient liquidity in order to purchase $q^*$, there is no role for trade in the OTC market. Third, since assets are issued in a competitive market, $\psi_1$ will reflect the benefit of holding one additional unit of short term assets. But since here $A_1 \geq q^*$, the marginal unit of short term assets is only good as a store of value, and not as a facilitator of trade in the LW market. Thus, the unique equilibrium price must be $\psi_1 = \beta$. Finally, with no trade in the OTC market, long term assets cannot possibly be valued for any (direct or indirect) liquidity properties, which simply means that $\psi_2 = \beta^2$.

Henceforth, we maintain the assumption $A_1 < q^*$. Proposition 2 describes equilibrium prices and how they are affected by monetary policy. Proposition 3 does the same for the equilibrium value of production in the LW market. For this discussion, it is important to recall the definitions in equations (19)-(26) in Appendix 1.7.

**Proposition 2.** *The equilibrium price of short term assets is given by* $\psi_1 = \min\{1+\mu, 1+\bar{\mu}(A_1, A_2)\}$. *The equilibrium price of long term assets depends on the value of* $A_2$. *We have two cases:*

**Case 1:** *If* $A_2 \geq \bar{A}_2(A_1)$, *then* $\psi_2 = \beta\psi_1$.

**Case 2:** *If* $A_2 < \bar{A}_2(A_1)$, *then there exists a cutoff* $\tilde{\mu}(A_2)$ *such that:*

*a) For all* $\mu \in (\beta - 1, \tilde{\mu}(A_2)]$, *we have* $\psi_2 = \beta\psi_1$;

*b) For all* $\mu \in (\tilde{\mu}(A_2), \bar{\mu}(A_1, A_2))$, *we have* $\psi_2 = \beta\rho(\mu, A_2)\psi_1$, *where* $\rho(\mu, A_2) \in (1, (1+\mu)/\beta)$ *is a strictly increasing function of* $\mu$ *and a strictly decreasing function of* $A_2$;

*c) For all* $\mu \geq \bar{\mu}(A_1, A_2)$, *we have* $\psi_2 = \beta\rho(\bar{\mu}, A_2)\psi_1$.

*The term* $\rho$ *is given by*

$$\rho(\mu, A_2) = 1 + \lambda f \frac{u'(Z + \zeta^M) - 1}{(1-\lambda)u'(Z + \zeta^M) + \lambda}, \tag{18}$$

*where* $\zeta^M$ *is defined in (21).*

*Proof.* Proven jointly with Proposition 3 below. □

The results reported in Proposition 2 are highlighted in Figure 4. As pointed out earlier, in any monetary equilibrium (for $\mu < \bar{\mu}(A_1, A_2)$), by no-arbitrage, the rate of return on money and the short term asset has to be equal, implying that $\psi_1 = 1 + \mu$. An increase in $\mu$ makes the cost of holding money higher, and induces agents to replace money with the relatively cheaper short term asset, which is a perfect substitute. In equilibrium, this leads to an increase in the demand for short maturities and their price $\psi_1$. However, if the monetary authority increases $\mu$ beyond the threshold $\bar{\mu}(A_1, A_2)$, the equilibrium becomes non-monetary, and any further increase in $\mu$ has no effect on asset prices (or any other equilibrium variables). For any $\mu > \beta - 1$, the price of short term assets carries a liquidity premium (i.e. $\psi_1 > \beta$), which reflects the assets' property to mature in time to take advantage of consumption opportunities in the LW market.

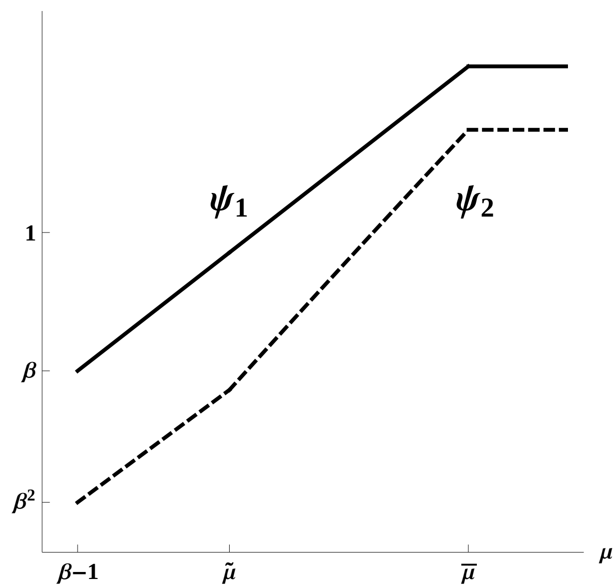


Figure 4: Equilibrium prices as functions of inflation.

The results that concern the equilibrium price of long term assets are even richer. Long term assets can be priced at a (liquidity) premium for two reasons:[9] first, because long term assets will become short term assets in the next period; second, because long term assets can be used in the OTC market in order to purchase liquid assets. In other words, the assets that do not mature today have *indirect* liquidity properties because they help agents bypass the cost of holding liquid assets (which is positive when $A_1 < q^*$ and $\mu > \beta - 1$). If equilibrium lies in Regions $MA$ or $NA$ (i.e. the regions of abundance of long term assets in OTC trade), $\psi_2 = \beta\psi_1 > \beta^2$, and long term assets sell at a premium, but only because they will become short term assets in the next period. In contrast, if $A_2 < \bar{A}_2(A_1)$ and $\mu > \tilde{\mu}(A_2)$, then equilibrium lies in the regions of relative scarcity of $A_2$ (Regions $MS$ or $NS$), and an additional unit of long term assets can help agents purchase essential liquidity in the OTC (i.e. liquidity that allows them to boost LW consumption). This property is valued by agents, who are now willing to buy long maturities at a price greater than $\beta\psi_1$. Thus, the term $\rho > 1$ represents a premium that reflects

[9] To be clear, the long term asset price will include a liquidity premium, whenever $\psi_2$ exceeds the price that it would obtain if we were to close down the LW market (and, therefore, shut off any liquidity channel in the model). Clearly, this price would be the so-called fundamental value $\psi_2 = \beta^2$.

the aforementioned indirect liquidity properties of long term assets. It is increasing in $\mu$ (within the regions of monetary equilibrium), precisely because the inflation tax that agents can avoid by holding long term assets is itself increasing in $\mu$. Similarly, $\rho$ is decreasing in $A_2$, because the service that long term assets provide (helping agents avoid the cost of holding liquid assets) becomes more valuable when $A_2$ is more scarce.

Consider now the equilibrium values of the quantity of good in the LW market.

**Proposition 3.** *The equilibrium value of $q_1$ is always equal to $Z$. When $\mu < \bar{\mu}(A_1, A_2)$, then $\partial q_1/\partial \mu < 0$, and when $\mu > \bar{\mu}(A_1, A_2)$, then $\partial q_1/\partial \mu = 0$. Regarding the equilibrium value of $q_2$:*

**Case 1:** *If $A_2 \geq \bar{A}_2(A_1)$, then $q_2 = q^*$ for any $\mu > \beta - 1$.*

**Case 2:** *If $A_2 < \bar{A}_2(A_1)$, then for the same cutoff $\tilde{\mu}(A_2)$ as in Proposition 2:*

*a) For all $\mu \in (\beta - 1, \tilde{\mu}(A_2)]$, $q_2 = q^*$;*

*b) For all $\mu \in (\tilde{\mu}(A_2), \bar{\mu}(A_1, A_2))$, $q_2 = Z + \zeta^M < q^*$ and $q_2$ is a strictly decreasing function of $\mu$;*

*c) For all $\mu \geq \bar{\mu}(A_1, A_2)$, $q_2 = A_1 + \zeta^N < q^*$ which does not depend on $\mu$.*

*Proof.* Proof of Propositions 2 and 3.

Recall that $A_1 < q^*$ is a maintained assumption throughout, and note that Region $NA$ is empty if and only if $A_1 \leq \bar{A}_1$. We begin with the statements that do not depend on Cases 1 or 2.

If the equilibrium is monetary, $\psi_1 = \varphi/\hat{\varphi} = 1 + \mu$. If the equilibrium is non-monetary, plug the definition of $\bar{\mu}(A_1, A_2)$ into the first-order condition $\psi_1 = \beta G_1^i$ for the appropriate region (1 if long term assets are abundant, 5 if they are scarce).

By the OTC bargaining solution, $q_1 = Z$. Among monetary equilibria, demand for real balances is downward-sloping in inflation; to see this, evaluate the first-order conditions at aggregate quantities. Among non-monetary equilibria, money is not valued, so $\mu$ is a mere number that does not affect equilibrium.

Case 1: Let $A_2 \geq \bar{A}_2(A_1)$. Then the equilibrium can only be in the aggregate Regions $MA$ or $NA$, or Region 1 in $(A_2, Z)$-space. By equation (15), the only solution to $\beta G_2 = \psi_2$ in Region 1 is $\psi = \beta\psi_1$. Furthermore, Region 1 is defined by the branch of the OTC bargaining solution where $\zeta = q^* - z$, so on aggregate, $q_2 = Z + \zeta(Z, Z, A_2) = q^*$.

Case 2: Let $A_2 < \bar{A}_2(A_1)$. Then the equilibrium is in Region $MA$ (corresponding to Region 1) if $\mu \in (\beta - 1, \tilde{\mu}(A_2))$, in Region $MS$ (corresponding to Region 5) if $\mu \in (\tilde{\mu}(A_2), \bar{\mu}(A_1, A_2))$, or in Region $NS$ (corresponding to Region 5, but $Z = A_1$ is now independent of $\mu$) if $\mu > \bar{\mu}(A_1, A_2)$.

a) In Region $MA$, the results of Case 1 apply.

b) In Region $MS$, the first-order conditions $\beta G_1^5 = 1 + \mu$ (money demand) and $\beta G_2^5 = \psi_2$ (demand for long term assets) apply, evaluated at aggregate quantities. Differentiating money demand and the equation (21) jointly, one can see that $Z$ is strictly decreasing in $A_2$, $Z + \zeta^M$ is strictly increasing in $A_2$, and both $Z$ and $\zeta^M$ are strictly decreasing in $\mu$. Therefore, $q_2$ is decreasing in $\mu$, and $q_2 < q^*$ is the very definition of Region 5. Finally, $\rho(\mu, A_2)$ is exactly $G_2^5/\psi_1$ evaluated at aggregate quantities, so $\psi_2 = \beta\rho(\mu, A_2)\psi_1$, and $\rho$ is strictly increasing in $\mu$ and decreasing in $A_2$ because $Z + \zeta^M$ is the opposite, and $u'(\cdot)$ is a strictly decreasing function.

c) In Region $NS$, the first-order conditions $\beta G_1^5 = \psi_1$ (demand for short term assets) and $\beta G_2^5 = \psi_2$ (demand for long term assets) apply, evaluated at aggregate quantities. Substituting the definitions of $\bar{\mu}(A_1, A_2)$ and $\zeta^N$ (equation 22) yields $\rho(\bar{\mu}(A_1, A_2), A_2) = G_2^5/\psi_1$ again. □

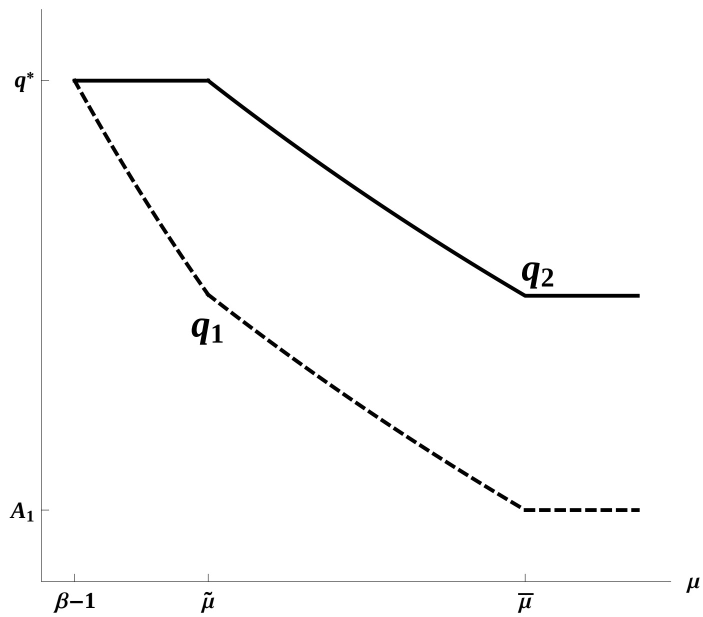


Figure 5: Equilibrium LW quantities as functions of inflation.

The results demonstrated in Proposition 3, and illustrated in Figure 5, are also very intuitive. Agents who did not match in the OTC have to rely exclusively on their own real balances. Hence, $q_1$ will always coincide with $Z$, and it will be a decreasing function of $\mu$, for $\mu < \bar{\mu}$. The equilibrium quantity $q_2$ represents the amount of good that the buyer can afford to purchase in the LW market, when she has previously traded in the OTC market. Hence, whenever equilibrium lies in the Regions $MA$ or $NA$, we have $q_2 = q^*$. In contrast, if equilibrium lies in the regions of scarcity of $A_2$ in OTC trade (Regions $MS$ or $NS$), the buyer will not be able to afford the first-best, and $q_2 < q^*$. In this case, $q_2$ is a decreasing function of (not affected by) $\mu$ if and only if equilibrium is monetary (non-monetary).

## 1.7 Some Equilibrium Objects

First, we explicitly describe equilibrium real balances $Z$ and the OTC trading volume in terms of real balances $\zeta$ (as opposed to in terms of long term assets $\chi$). In any nonmonetary region, real balances are equal to the supply of short term assets, $A_1$. In Region $MA$, real balances satisfy the money demand equation:

$$\frac{1+\mu}{\beta} = 1 + (\ell - \lambda f)\left[u'(Z) - 1\right]. \tag{19}$$

In Region $MS$, real balances and trading volume $\zeta^M$ jointly satisfy:

$$\frac{1+\mu}{\beta} = 1 + \lambda f \frac{u'(Z+\zeta^M) - 1}{(1-\lambda)u'(Z+\zeta^M) + \lambda} + \left[\ell - \lambda f \frac{1}{(1-\lambda)u'(Z+\zeta^M) + \lambda}\right]\left[u'(Z) - 1\right], \tag{20}$$

and:

$$(1-\lambda)\left[u(Z+\zeta^M) - u(Z)\right] + \lambda\zeta^M = (1+\mu)A_2. \tag{21}$$

Furthermore, $\zeta^N$ denotes the short term asset trading volume in the OTC market in the case of a non-monetary equilibrium, and it solves

$$\begin{aligned}(1-\lambda)&\left[u(A_1+\zeta^N) - u(A_1)\right] + \lambda\zeta^N = \\ &\beta A_2\left\{1 + f\lambda\tfrac{u'(A_1+\zeta^N)-1}{(1-\lambda)u'(A_1+\zeta^N)+\lambda} + \left[\ell - f\lambda\tfrac{1}{(1-\lambda)u'(A_1+\zeta^N)+\lambda}\right]\left[u'(A_1) - 1\right]\right\}.\end{aligned} \tag{22}$$

Next, we define cutoff levels of short term and long term asset supply that will separate classes of equilibria. First, the cutoff level for short term asset supply is:

$$\bar{A}_1 \equiv \left\{A_1 : \; \frac{1}{2}A_1 = \frac{(1-\lambda)\left[u(q^*) - u(A_1)\right] + \lambda(q^* - A_1)}{\beta + \beta(\ell - \lambda f)\left[u'(A_1) - 1\right]}\right\}. \tag{23}$$

Using condition (17), one can show that $\bar{A}_1 \in (q^*/2, q^*)$. Second, and in terms of the first, we define the cutoff level of long term asset supply for the non-monetary region (represented by the horizontal segment of the blue line in Figure 3; in the figure, $A_1 > \bar{A}_1$ so the second term in the minimum applies):

$$\bar{A}_2(A_1) \equiv \min\left\{\frac{1}{2}A_1, \frac{(1-\lambda)\left[u(q^*) - u(A_1)\right] + \lambda(q^* - A_1)}{\beta + \beta(\ell - \lambda f)\left[u'(A_1) - 1\right]}\right\}. \tag{24}$$

Finally, we define the upper bound of inflation consistent with monetary equilibrium (represented by the green piece-wise curve in Figure 3). If $A_1 > \bar{A}_1$ and $A_2 \geq \bar{A}_2(A_1)$, we have

$$\bar{\mu}(A_1, A_2) = \beta - 1 + \beta\left(\ell - \lambda f\right)\left[u'(A_1) - 1\right]. \tag{25}$$

On the other hand, if $A_2 < \bar{A}_2(A_1)$ (for any $A_1 < q^*$), we have

$$\bar{\mu}(A_1, A_2) = \beta - 1 + \beta\left[\ell - \frac{\lambda f}{(1-\lambda)u'(A_1+\zeta^N) + \lambda}\right]\left[u'(A_1) - 1\right] + \frac{\beta\lambda f\left[u'(A_1+\zeta^N) - 1\right]}{(1-\lambda)u'(A_1+\zeta^N) + \lambda}. \tag{26}$$

## 2 A version of the model with nominal instead of real assets

The structure of the economy is unchanged with only one modification. The real assets are replaced by nominal ones, which we refer to as "bonds" in this section. Notably, the timing structure is unchanged: a one-period bond issued in the CM subperiod pays one unit of currency in the OTC subperiod of the following period, and a two-period bond will pay one unit of currency in the OTC subperiod two periods thence, so Figure 1 is still valid. As a mechanical matter, bond redemption is financed by newly created money, which the government may choose to sterilize with lump-sum taxes in the CM of the same period (or accept the resulting inflation).[10] In order to distinguish this variation from the benchmark model in the paper, we denote the nominal bonds by the letter $B/b$ instead of $A/a$ (capital letters denote aggregate quantities), and the bond prices by the letter $p$ instead of $\psi$.

We begin with the OTC bargaining problem. Similar to the real model, the C-type's real balances are $z \equiv \varphi(m+b_1)$; $\varphi$ denotes the expected real price of money and $p_1$ denotes the expected nominal price of one-period bonds in the following CM (by definition of the timing, this is in the same period), and $m$ and $b_1$ denote holdings of money and one-period bonds by the C-type. Holdings of two-period bonds are denoted by $b_2$. For the N-type, the equivalents are $\tilde{z}$ and $\tilde{b}_2$.

Bargaining is over two outcomes of interest: the amount of real balances $\zeta$ that the N-type transfers to the C-type, and the amount of nominal long-term bonds $\chi$ that the C-type transfers in return. The bargaining problem can be written as follows:

$$\max_{\zeta,\chi} \{u(z+\zeta) - u(z) - \varphi p_1 \chi\} \tag{27}$$

$$\text{subject to:} \quad u(z+\zeta) - u(z) - \varphi p_1 \chi = \frac{\lambda}{1-\lambda}(-\zeta + \varphi p_1 \chi) \tag{28}$$

$$\chi \leq b_2 \tag{29}$$

$$\zeta \leq \tilde{z} \tag{30}$$

We can solve the bargaining problem by splitting it up into four different cases.

**No trade.** If $z \geq q^*$, there are no gains from trade and therefore $\chi = \zeta = 0$.

**Scarce long-term bonds.** $b_2$ is so small that constraint (29) binds. The solution is $\chi = b_2$ and

[10] Strictly speaking, the fact that money is introduced early in the period but taxes or transfers are assessed late in the period introduces a cyclical fluctuation into the money supply. As it is perfectly predictable to everyone in the economy, this fluctuation does not distort any prices or demand schedules. A quick examination of the equilibrium equations confirms this. If undesired, one could eliminate this feature by sterilizing the bond repayments early in the period with lump-sum taxes assessed simultaneously, but this would only add complexity to the model (as we need to make sure that all agents could in principle afford the taxes) without any additional insight.

$\zeta = \zeta^b$ which solves:

$$(1-\lambda)\left(u(z+\zeta^b) - u(z)\right) + \lambda\zeta^b = \varphi p_1 b_2 \tag{31}$$

This case will correspond to Region 5 in symmetric equilibria.

**Scarce real balances.** $\tilde{z}$ is so small that constraint (30) binds. The solution is $\zeta = \tilde{z}$ and:

$$\chi = \frac{1}{\varphi p_1}\left[(1-\lambda)\left(u(z+\tilde{z}) - u(z)\right) + \lambda\tilde{z}\right]$$

This case will correspond to Region 3 in symmetric equilibria.

**Abundance.** If neither of the above conditions hold, then $\zeta = q^* - z$ and:

$$\chi = \frac{1}{\varphi p_1}\left[(1-\lambda)\left(u(q^*) - u(z)\right) + \lambda\left(q^* - z\right)\right]$$

This case will correspond to Region 1 in symmetric equilibria.

In a symmetric general equilibrium, the aggregation conditions must hold:

$$b_2 = B_2$$
$$z = \tilde{z} = \varphi(M + B_1)$$

where $M$ denotes the supply of money and $B_1$ and $B_2$ denotes the supply of nominal bonds of maturity 1 and 2, respectively. As in the main text, the maturity index refers to the maturity remaining and not the maturity at issue, so that $B_1$ comprises both newly issued one-period bonds and last period's issue of two-period bonds.

And as in the main text, for our convenience we want to focus on the cases where real balances are abundant in OTC trade, i.e. the N-type always has enough real balances to satisfy the C-type's demand (whether this demand is limited by reaching the first-best $z + \zeta = q^*$, or by the C-type's ability to buy the liquidity with long-term bonds). Unlike in the model with real bonds, in the model with nominal bonds the boundary of aggregate regions 3 and 5 in $(M + B_1, B_2)$-space (corresponding to Figure 2 in Section 1 of this appendix) is downward sloping.[11] We can then verify that

$$B_2 \leq M + B_1 \tag{32}$$

[11] Proof: This boundary is given by

$$\varphi p_1 B_2 = (1-\lambda)\left(u(2z) - u(z)\right) + \lambda z$$
$$\Rightarrow \quad p_1 \frac{B_2}{M + B_1} = (1-\lambda)\frac{u(2z) - u(z)}{z} + \lambda$$

and the right-hand side is decreasing in $z$ by the concavity of $u$.

is a sufficient condition to rule out the scarce-money case (Region 3) in general equilibrium. This restriction replaces the more complicated restriction in the main text (Equation 25). If the reader finds a restriction on bond supply unpalatable, we could instead require that inflation is not too large, as we did in the main text. (The minimal restriction would involve a combination of inflation and the asset supplies. But here, we only need a sufficient one.)

We next turn to the buyer's optimal portfolio problem in the CM. Buyers take as given (current and expectations of future) prices $\varphi, \hat{\varphi}, p_1, \hat{p}_1, p_2$ and rationally forecast any bargaining in the OTC market. As in the main text, we abuse notation slightly and denote by $\zeta, \chi$ the expectations of a bargaining outcome in which the buyer enters as a C-type, and denote by $\tilde{\zeta}, \tilde{\chi}$ the expectations of a bargaining outcome in which the buyer enters as an N-type. By restriction (32) (which we maintain from now on), $\tilde{\zeta}, \tilde{\chi}$ will not be affected by any decisions the buyer makes in the preceding CM. She knows that her money or bond holdings will never be marginal in subsequent OTC trades where she enters as an N-type. The representative buyer's objective function, which we derive from the CM value function analogously to the main text, will therefore look as follows:

$$
\begin{aligned}
J\left(\hat{m}, \hat{b}_1, \hat{b}_2\right) &= -\varphi\left(\hat{m} + \hat{b}_1 + p_2 \hat{b}_2\right) + \beta\, G\left(\hat{\varphi}\left(\hat{m} + \hat{b}_1\right), \hat{b}_2\right), \\
G\left(\hat{z}, \hat{b}_2\right) &= f\,\left[u\left(\hat{z} + \zeta\right) + \hat{\varphi}\hat{\psi}_1\left(\hat{b}_2 - \chi\right)\right] + (\ell - f)\left(u\left(\hat{z}\right) + \hat{\varphi}\hat{\psi}_1 \hat{b}_2\right) \\
&\quad + f\,\left[\hat{z} - \tilde{\zeta} + \hat{\varphi}\hat{\psi}_1\left(\hat{b}_2 + \tilde{\chi}\right)\right] + (1 - \ell - f)\left(\hat{z} + \hat{\varphi}\hat{\psi}_1 \hat{b}_2\right).
\end{aligned}
$$

As in the model with real bonds, the negative terms in the definition of $J$ represent the cost of purchasing various amounts of the three assets available in the economy. The four terms in the definition of $G$ admit similar interpretations as their counterparts in equation (16) of the main text.

Taking first-order conditions, and imposing a symmetric steady-state equilibrium with $\hat{m} = M$, $\hat{b}_1 = B_1$, $\hat{b}_2 = B_2$, and $\varphi/\hat{\varphi} = 1 + \mu$, we can derive the following demand equations for the three assets:[12]

$$
\begin{aligned}
\frac{1+\mu}{\beta} &= 1 + \ell\,\left[u'(q_1) - 1\right] - \lambda\, f\, \frac{u'(q_1) - u'(q_2)}{(1-\lambda)u'(q_2) + \lambda} && \text{(money)} \\
p_1 &= 1 && \text{(short-term bonds)} \\
p_2 &= \frac{\beta}{1+\mu}\,\left[1 + \lambda\, f\, \frac{u'(q_2) - 1}{(1-\lambda)u'(q_2) + \lambda}\right] && \text{(long-term bonds)}
\end{aligned}
$$

[12] This way of writing the asset demand equations encompasses the cases of abundant and scarce long-term bonds; the former admits some simplification through $q_2 = q^*$ and therefore $u'(q_2) = 1$. We have ruled out the case of scarce real balances through restriction (32).

where $q_1 \equiv \varphi(M + B_1)$ denotes the amount of special goods purchased in the LW market by a C-type who did not trade in the OTC market, and $q_2 \equiv \min\{q^*, q_1 + \zeta(q_1, q_1, B_2)\}$ denotes the amount of special goods purchased in the LW market by a C-type who did trade in the OTC market, up to the first-best level $q^*$.[13]

It is apparent that the nature of the solution has not changed from the benchmark model with real bonds. The money demand equation (which determines $q_1$ and, through the bargaining solution, $q_2$) is identical. The prices of nominal bonds only differ from the real ones by the factor of expected inflation, $1 + \mu$ (the Fisher relationship holds). As before, if we want to make statements out of steady state, we can decompose the price of long-term nominal bonds into three components (hats denote expected future values):

$$p_2 = \underbrace{\frac{\beta}{1+\mu}}_{\substack{\text{monetary}\\\text{discounting}}} \times \underbrace{\hat{p}_1}_{\substack{\text{expected}\\\text{price of short-}\\\text{term bonds}}} \times \underbrace{\left[1 + \lambda f \frac{u'(\hat{q}_2) - 1}{(1-\lambda)u'(\hat{q}_2) + \lambda}\right]}_{\text{expected liquidity value}}$$

The money demand equation ensures that just like in the real bond case, the liquidity premium term is always less than $(1 + \mu)/\beta$, so that the price of nominal bonds is always less than $\hat{p}_1$ and the nominal forward holding return must be positive. Consequently, in any steady state we must have $p_2 < p_1^2$.

In conclusion, the term premium $p_2^{-0.5} - p_1^{-1}$ is positive, increasing in expected inflation, and decreasing in the liquidity $f$ of the OTC market, whether we look at real or nominal bonds. Result 2 also continues to hold: for an asset with $f = 0$, such as the CDs discussed in the main text, the long-term price $p_2$ is minimal and the term premium is maximal, holding all other parameters constant. Finally, since we derived Result 3 by comparing assets within the same period and with the same maturity date, our explanation for the on-the-run premium is not affected either by whether the assets in question are real or nominal.

# 3 Solution of the model with $N > 2$ maturities

In the main text of the paper, we provided a brief, verbal description of the model with a general number of maturities $N > 2$ (in Section 4.2). Here we analyze the model in more detail. We work with the monetary version of the model, but non-monetary results can be derived by

[13] Specifically, unless that solution would exceed $q^*$, $q_2$ solves:

$$(1 - \lambda)\left[u(q_2) - u(q_1)\right] + \lambda[q_2 - q_1] = \frac{B_2}{M + B_1} q_1.$$

replacing $Z$ with $A_1$ and using the result for $\psi_1$ from Proposition 2 *of the main text* instead of $\psi_1 = 1 + \mu$ (which comes from the analogous proposition for the model with money, namely, Proposition 2 of this web appendix).

With $N > 2$, there are many combinations of long term asset portfolios that a C-type can sell in order to obtain additional liquidity in the OTC market. We choose to not place any restrictions on which assets can be traded for liquidity. That is, we assume that in any OTC meeting the C-type can exchange any portfolio of long term assets (assets that do not mature in the current period) for a portfolio of liquid assets (money and they yield of assets that mature in the current period). In that sense, even though $N > 2$, the interesting distinction is still between assets that mature now (and are therefore liquid) and assets that do not mature now (but can be traded for liquid assets in the OTC).

We now generalize Proposition 2 for the case of $N$ maturities and money. As before, we only focus on equilibria in which agents are always able to obtain the representative portfolio in the CM without selling off-the-run assets. A simple sufficient condition, which we maintain for Proposition 4, is that $A_1 \geq 2A_2 \geq \ldots \geq 2^{N-1}A_N$.[14] Recall that the threshold level relevant for the abundance or scarcity of long term assets can be expressed using the definition of the OTC bargaining solution $\chi$, evaluated at aggregate quantities. We also use the definition of $\bar{\mu}$ as described in Section 1 for the inflation threshold after which equilibrium becomes nonmonetary, and we always assume $\mu > \beta - 1$.

**Proposition 4.** *Define the supply of long term assets relevant for abundance in OTC trade as*

$$A_L \equiv \beta^{N-2}A_N + \ldots + A_2.$$

*If $\mu < \bar{\mu}(A_1, A_L)$, the equilibrium price of one-period assets is given by $\psi_1 = 1 + \mu$, and the equilibrium price of long term assets (i.e. $\psi_i$, $i \geq 2$) depends on the value of $A_L$. We have two cases:*

**Case 1:** *If $A_L \geq \chi(A_1, A_1, A_L)$, then $\psi_i = \beta^{i-1}\psi_1$, for all $i \geq 2$.*

**Case 2:** *If $A_L < \chi(A_1, A_1, A_L)$, then there exists a cutoff $\tilde{\mu}(A_L)$ such that:*

*a) For all $\mu < \tilde{\mu}(A_L)$, we have $\psi_i = \beta^{i-1}\psi_1$;*

*b) For all $\mu > \tilde{\mu}(A_L)$, we have $\psi_i = \left(\beta(1+\rho^L)\right)^{i-1}\psi_1$, for all $i \geq 2$, where $\rho^L \in (0, (1+\mu-\beta)/\beta)$ is a strictly increasing function of $\mu$ and a strictly decreasing function of any $A_i$, $i \geq 2$.*

*The term $\rho^L$ is defined jointly with $\zeta^L$ (the real balance trading volume in the OTC market) as a function*

[14] This condition generalizes the $A_2 \leq 1/2A_1$ segment in the right panel of Figure 3 in the main text of the paper. It is tighter than necessary, but simpler than the condition we used to make the domain of the baseline equilibrium as general as possible.

*of equilibrium real balances $Z$ and the price of one-period assets $\psi_1$:*

$$\rho^L = \lambda f \frac{u'(Z+\zeta^L)-1}{(1-\lambda)u'(Z+\zeta^L)+\lambda}$$

$$\beta\psi_1 \sum_{i=2}^{N} \left(\beta(1+\rho^L)\right)^{i-2} A_i = (1-\lambda)\left[u(Z+\zeta^L)-u(Z)\right]+\lambda\zeta^L$$

*Proof.* The assumption $A_1 < q^*$ guarantees that $\bar{\mu} > \beta - 1$, and $2^{N-1}A_N \leq \ldots \leq 2A_2 \leq A_1$ guarantees that no agent can enter the CM with more units of any bond than the aggregate supply of that bond, so the constraint that agents cannot sell off-the-run assets in the CM is satisfiable in symmetric equilibrium.

Trade in the LW market is unchanged from the model with two maturities. In the OTC market, C-type agents want to obtain real balances (short term assets about to mature, plus money) and are willing to offer any longer term asset in return. In general, the bargaining solution may be indeterminate, but if any one longer term asset is scarce (the C-type gives up all of it but would still like more real balances), all of them are. Consequently, all assets that do not mature in the very next period are perfect substitutes as agents choose their portfolios in the CM.

The rest of the proof is very similar to that of Proposition 2. Short term assets are perfect substitutes for money and must have the same rate of return if both are valued; therefore, $\psi_1 = 1+\mu$. Regarding longer term assets, cases 1 and 2 are identical to the model with $N = 2$, with two exceptions. First, any occurence of $A_2$ must be replaced with $A_L$. Second, the value of longer-term assets in the CM depends on their scarcity in the subsequent OTC market, measured by $\rho^L$. But the total value of the supply of longer-term assets, which determines their scarcity, is itself determined by their value in the future CM and affected by $\rho^L$. This was not an issue in the model with two maturities because only the price of two-period assets was affected by $\rho$, but not that of one-period assets. With three or maturities, the definition of $\rho$ becomes more complicated (hence the index $\rho^L$).

Using total differentiation again in the scarce case 2b), one can show that both $Z$ and $\zeta$ decrease as functions of $\mu$, while $Z+\zeta$ increases as a function of $A_L$. Hence, $\rho^L$ is an increasing function of $\mu$ and a decreasing function of $A_L$, and therefore, it is a decreasing function of $A_i$ for all $i \geq 2$. $\square$

Proposition 4 reveals that the results in the case of a general $N > 2$ are qualitatively very similar to the ones in the $N = 2$ case. In particular, one-period assets are "in a class of their own", since they are the only assets that are (direct) substitutes to money. Hence, in monetary equilibrium, we obtain $\psi_1 = 1+\mu$. The price of longer term assets, $\psi_i$, $i \geq 2$, always carries a liquidity premium because these assets will eventually also become short term assets in future periods. Moreover, if the supply of longer term assets is relatively scarce (Case 2-b of the proposition), the price $\psi_i$, $i \geq 2$, will also contain an *indirect* liquidity premium, $\rho$, which reflects the assets' property to be traded for liquid assets in the OTC market. Naturally, the premium $\rho$ is increasing in inflation (in monetary equilibria) and decreasing in the supply of long term assets (in the regions of scarcity), because high inflation or asset scarcity make the service that long term assets provide more valuable.

It is straightforward to check that a positively sloped yield curve will also arise here regardless of the region of equilibrium. Consider for instance a monetary equilibrium with relatively abundant supply $A_L$ (the argument for the case of scarce supply is similar). In this case, we have $\psi_1 = 1 + \mu$ and $\psi_2 = \beta(1 + \mu)$, and we have already shown that $r_2 > r_1$. Thus, focus on $i \in \{2, ..., N - 1\}$, and consider the term $r_{i+1} - r_i$. It can be easily verified that

$$r_{i+1} > r_i \Leftrightarrow \left[\frac{1}{\beta^i(1+\mu)}\right]^{\frac{1}{i+1}} > \left[\frac{1}{\beta^{i-1}(1+\mu)}\right]^{\frac{1}{i}} \Leftrightarrow \left(\frac{1}{\beta}\right)^{\frac{1}{i(i+1)}} > \left(\frac{1}{1+\mu}\right)^{\frac{1}{i(i+1)}},$$

which is always satisfied, since by assumption $\mu > \beta - 1$.

We conclude that the model with $N > 2$ maturities delivers an upward sloping yield curve throughout the domain $i = 1, ..., N$. This result emerges even though any two assets with lifetime $i, j \geq 2$ are qualitatively similar, in that neither of them can serve as a direct substitute to money, a property that only one-period assets have. Nevertheless, assets with maturity $i \geq 2$ are, in a sense, still more liquid than assets with maturity $i + 1$ because the former will become one-period assets (and perfect substitutes to money) *earlier* than the latter.

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