

# Supplementary Appendix to “Incentive Compatible Allocation and Exchange of Discrete Resources”<sup>49</sup>

Marek Pycia (UCLA) and M. Utku Ünver (Boston College)

## Appendix D Examples

The lemmas in Appendix C.2 show how requirements R1–R6 are driven by group strategy-proofness and Pareto efficiency. The following examples further illustrate the role of the consistency requirements R4–R6.

Requirement R4 is needed to ensure the individual strategy-proofness of the TC mechanisms. This requirement is also needed to ensure the individual strategy-proofness of the TTC mechanisms, and it is embedded in Pápai’s (2000) inheritance-tree construction.<sup>50</sup>

**Example 2. *Why do we need R4 to prevent individual manipulation?*** Consider three agents  $i_1, i_2, i_3$  and three houses  $h_1, h_2, h_3$ . Agent  $i_1$  owns house  $h_1$ , and agent  $i_2$  owns  $h_2$  and  $h_3$  at submatching  $\emptyset$ . Suppose that at submatching  $\{(i_1, h_1)\}$ , agent  $i_3$  owns  $h_2$  and  $h_3$ , thus violating R4 for  $i_2$ . Suppose that all agents rank houses as follows  $h_1 \succ_i h_2 \succ_i h_3$ . When  $i_2$  submits his true preferences, he is matched with  $h_3$  in the third round of the algorithm; indeed, in the first round  $i_1$  and  $h_1$  are matched, and in the second round  $i_3$  and  $h_2$  are matched. When  $i_2$  submits the ranking  $h_2 \succ_i h_1 \succ_i h_3$ , he is matched with  $h_2$  in the first round. Because  $i_2$  prefers  $h_2$  to  $h_3$ , the individual strategy-proofness is violated.

Requirement R5 states that a brokerage right persists when we move from smaller to larger submatchings, provided two or more owners from the smaller submatching remain unmatched at the larger submatching; it also specifies who takes over control rights over the brokered house when the broker loses them. The following example illustrates why we need this requirement to keep TC individually strategy-proof.

**Example 3. *Why do we need R5 to prevent individual manipulation?*** Consider four agents  $i_1, \dots, i_4$ . Assume that at the empty submatching agent  $i_2$  brokers a house and other agents own one house each. Denote by  $h_k$  the house controlled by agent  $i_k$ . We maintain R1–R3, R4, and R6, and violate R5 by assuming that  $h_2$  is owned by  $i_4$  at submatching  $\{(i_1, h_1)\}$ . R5 is violated because two previous owners,  $i_3$  and  $i_4$ , are unmatched at  $\{(i_1, h_1)\}$ , but  $i_2$  is no longer a broker. Consider a preference profile such that  $h_1$  is  $i_1$ ’s and  $i_2$ ’s mutual first-choice

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<sup>50</sup>This requirement is also embedded in the original Gale’s construction (Shapley and Scarf, 1974) and in other TTC constructions, cf. Abdulkadiroğlu and Sönmez (1999), Pápai (2000), Ehlers, Klaus, and Pápai (2002), Abdulkadiroğlu and Sönmez (2003), Ehlers and Klaus (2003), Roth, Sönmez, and Ünver (2004), Hakimov and Kesten (2014), and Morrill (2014b).

house,  $h_2$  is the first choice of the other agents, and  $h_3$  is the second choice of  $i_2$  and  $i_3$ . Under this preference profile and control-rights structure, in the first round of the TC algorithm,  $i_1$  and  $i_2$  point to  $h_1$ , while  $i_3$  and  $i_4$  point to  $h_2$ . House  $h_1$  points to its owner  $i_1$  and  $h_2$  points to its broker  $i_2$ . There is a unique cycle  $h_1 \rightarrow i_1 \rightarrow h_1$ , and the submatching  $\{(i_1, h_1)\}$  is matched. In the second round,  $h_2$  is owned by  $i_4$ , and all remaining agents point to it. There is a unique cycle  $h_2 \rightarrow i_4 \rightarrow h_2$ , and  $i_4$  is matched with  $h_2$ . Agent  $i_2$  is matched with neither his first nor second choices. On the other hand, if  $i_2$  declares  $h_3$  to be his first choice, then in the first round of TC, he would point to  $h_3$ , completing a cycle  $h_3 \rightarrow i_3 \rightarrow h_2 \rightarrow i_2 \rightarrow h_3$ . This misreporting allows  $i_2$  to be matched to his second-choice house,  $h_3$ .

R6 complements R5 by specifying the rights of a broker who loses his brokerage right (the broker-to-heir transition). The following two examples illustrate why we need R6 to keep TC individually strategy-proof and non-bossy.

**Example 4. *Why do we need R6 to prevent individual manipulation?*** Consider four agents  $i_1, \dots, i_4$ . Assume that at the empty submatching agent  $i_2$  brokers  $h_2$ , agent  $i_1$  owns  $h_1$  and  $h_4$ , and agent  $i_3$  owns  $h_3$ . At submatching  $\{(i_1, h_1)\}$ , agent  $i_3$  owns  $h_2$  and  $h_3$ . While  $i_2$  loses his brokerage rights at  $\{(i_1, h_1)\}$ , R5 is not violated because there is a single previous owner unmatched at  $\{(i_1, h_1)\}$ , and he is  $i_3$ . We further assume that at  $\{(i_1, h_1), (i_3, h_2)\}$  agent  $i_4$  owns  $h_3$ . Thus, R6 is violated, as at the submatching  $\{(i_1, h_1), (i_3, h_2)\}$ ,  $i_2$  does not own  $h_3$ . Suppose that, other than this violation, the control rights structure satisfies R1–R6. Consider the preference profile at which agents’ first choices are house  $h_1$  for agents  $i_1$  and  $i_2$ , house  $h_2$  for agent  $i_3$ , and house  $h_3$  for agent  $i_4$ . If agent  $i_2$ ’s second choice is  $h_3$ , then  $i_2$  would benefit by ranking  $h_3$  first.

**Example 5. *Why do we need R6 to prevent bossiness?*** Consider the same control-rights structure as in Example 4. Consider the preference profile at which  $i_1$  and  $i_3$ ’s first choices are  $h_1$ , and  $i_2$  and  $i_4$ ’s first choices are  $h_3$ , and the second choice of  $i_3$  is  $h_2$ . Agent  $i_3$  is bossy. Indeed, agent  $i_3$  receives house  $h_2$  both when he ranks  $h_2$  first and when he ranks  $h_1$  first and  $h_2$  second. However, in the former case,  $i_2$  receives  $h_3$ , while in the latter case, he receives  $h_4$ .

**Example 6. *Why can we not replace R5–R6 by a simpler (and stronger) persistence of brokerage?*** Consider the following requirement: “if  $|\sigma'| < |I| - 1$  and agent  $i$  brokers house  $h$  at  $\sigma$  and is unmatched at  $\sigma' \supset \sigma$ , then  $i$  brokers  $h$  at  $\sigma'$ ” (an analogue of R4 for brokers). The following example shows that we cannot replace R5–R6 with this stronger requirement. There are four agents,  $i_1, i_2, i_3, i_4$ , four houses,  $h_1, h_2, h_3, h_4$ , and a TC mechanism  $\psi^{c,b}$  whose control rights structure  $(c, b)$  is explained below and illustrated by the table in Figure 1.

Houses  $h_1, h_3$  are owned by agent  $i_1$  (denoted by “o” next to  $i_1$  in the figure); he continues owning them as long as he is unmatched (R4 is satisfied). When  $i_1$  is matched, the unmatched of the two houses is owned by  $i_3$  (if he is still unmatched). When both  $i_1$  and  $i_3$  are matched

$h_1$	$h_2$	$h_3$	$h_4$
$i_{1,o}$	$i_{2,o}$	$i_{1,o}$	$i_{4,b}$
$i_{3,o}$	$(i_2, h_4) \swarrow \searrow$	$i_{3,o}$	$(i_1, h_1) \swarrow \searrow$
	<i>otherwise</i>		<i><math>i_2</math> and <math>i_3</math> are matched, and <math>i_1</math> is matched with <math>h_2</math> or <math>h_3</math></i>
$i_{2,o}$	$i_{4,o}$	$i_{1,o}$	$i_{2,o}$
$i_{4,o}$	$i_{3,o}$	$i_{3,o}$	$i_{4,o}$
		$i_{4,o}$	$i_{3,o}$

Figure 1: A control rights structure with broker-to-heir transition

and  $h_1$  or  $h_3$  is unmatched, the house is owned by  $i_2$ . When all agents are matched and one of the houses  $h_1$  or  $h_3$  is unmatched, the house is owned by  $i_4$ .

House  $h_2$  is owned by  $i_2$ . When  $i_2$  is matched but  $h_2$  is not then  $h_2$  is inherited by one of the unmatched agents; who inherits  $h_2$  depends on the submatching. If  $i_1$  is matched with  $h_1$  and  $i_2$  is matched with  $h_4$ , then the next owners of  $h_2$  are  $i_4$  and  $i_3$ , in this order. In all other cases, the order of next owners of  $h_2$  is  $i_1, i_3$ , then  $i_4$ .

House  $h_4$  is initially brokered by agent  $i_4$  (denoted by “b” next to  $i_4$  in the figure). Agent  $i_4$  continues to broker  $h_4$  as long as he is unmatched with two exceptions: (i) if  $i_1$  is matched with  $h_1$  then  $i_4$  loses the brokerage right, and  $h_4$  becomes an owned house with the order of owners  $i_2, i_4$ , then  $i_3$ ; and (ii) if  $i_4$  is the only remaining agent, then he owns  $h_4$ . The second exception is dictated by R2. We explain in detail how the first exception occurs and why it is consistent with our conditions. At  $\sigma = \emptyset$ ,  $i_4$  brokers  $h_4$  and  $i_2$  owns  $h_2$ . At  $\sigma' = \{(i_1, h_1)\}$ ,  $i_2$ , the only remaining  $\sigma$ -owner, owns  $h_4$ , and  $i_4$  loses his brokerage rights. At submatching  $\{(i_2, h_4)\}$ ,  $i_4$  owns  $h_2$  in line with R6, and, by R4,  $i_4$  owns  $h_2$  at  $\{(i_2, h_4)\} \cup \sigma' = \{(i_1, h_1), (i_2, h_4)\}$ . This is an instance of the broker-to-heir transition.

The TC mechanism defined by this control-rights structure is different from all TC mechanisms with consistent control rights structures in which the simple analogue of R4 for brokers holds true: “if  $|\sigma'| < |I| - 1$  and agent  $i$  brokers house  $h$  at  $\sigma$  and is unmatched at  $\sigma' \supset \sigma$ , then  $i$  brokers  $h$  at  $\sigma'$ .” Indeed, by way of contradiction, assume that there is a TC mechanism  $\psi$  with a control-rights structure satisfying the above strong form of brokerage persistence and produces the same allocation as  $\psi^{c,b}$  for each profile of agents’ preferences.

First, notice that at the empty submatching,  $i_4$  is the broker of  $h_4$  in  $\psi$ . This is so because  $h_4$  is not owned by any agent at the empty submatching  $\emptyset$  as  $(\psi[\succ])^{-1}(h_4) = (\psi^{c,b}[\succ])^{-1}(h_4)$  varies with  $\succ \in \mathbf{P}$  (that is, across profiles at which all agents rank  $h_4$  first). Hence, there is an agent who has the brokerage right over  $h_4$ , and it must be  $i_4$ , as  $\psi[\succ](i_4) = \psi^{c,b}[\succ](i_4) = g$  for all  $\succ \in \mathbf{P}$  such that all agents rank  $h_4$  first and any  $g \in \{h_1, h_3, h_2\}$  second.

Second, consider the submatching  $\sigma = \{(i_1, h_1)\}$  and a preference profile  $\succ \in \mathbf{P}$  such that  $i_1$  ranks  $h_1$  first, and other agents rank  $h_4, h_3, h_2$ , and  $h_1$  in this order. In mechanism  $\psi$ , agent  $i_4$  would continue to be the broker of  $h_4$  at  $\sigma$ , and thus  $\psi[\succ](i_4) = h_3$ . However,  $\psi^{c,b}[\succ](i_4) = h_2$ .

This contradiction shows that indeed the TC mechanism of the example cannot be represented by a control-rights structure in which brokerage satisfies the analogue of R4 for brokers (in particular it cannot be represented without brokers).

A broker's loss of the brokerage right introduces a subtlety in our otherwise standard proof of non-bossiness. The cycles of three agents or more are the same under any two preference profiles  $\succ$  and  $\succ'$  that differ only in a ranking of an agent assigned the same house under both of them; this property is implied by R4 and R5. However, the cycles of one or two agents can be different under these two profiles. This subtlety and its resolution can be illustrated in the setting of Example 6. Consider a preference profile in which agents  $i_1$  and  $i_3$  rank houses  $h_1 \succ_{i_1, i_3} h_4 \succ_{i_1, i_3} h_2 \succ_{i_1, i_3} h_3$ , and agents  $i_2$  and  $i_4$  rank houses  $h_4 \succ_{i_2, i_4} h_2 \succ_{i_2, i_4} h_3 \succ_{i_2, i_4} h_1$ . Under this preference profile,  $\succ_{\{i_1, i_2, i_3, i_4\}}$ , in the first round, broker  $i_4$  obtains object  $h_2$  in a cycle  $i_4 \rightarrow h_2 \rightarrow i_2 \rightarrow h_4 \rightarrow i_4$ . However, if  $i_2$  submitted instead preference ranking  $\succ'_{i_2}$  identical to  $\succ_{i_1, i_3}$ , then  $i_4$  and  $i_2$  would not swap houses in the first round. They would both remain unmatched in round 2, and  $i_4$  would have lost his brokerage right; house  $h_4$  would now be owned by  $i_2$ .<sup>51</sup> Agent  $i_2$  would then match with  $h_4$  in round 2. In round 3, agent  $i_4$  would become the owner of  $h_2$  (R6's broker-to-heir transition). Thus, in round 3 agent  $i_4$  would match with  $h_2$ . While the cycles are different, the allocations are the same. Beyond the example, R6's broker-to-heir transition implies that a similar scenario is bound to happen whenever one- or two-agent cycles are different under  $\succ$  and  $\succ'$ .

## Appendix E Proof of the Pareto Efficiency of TC

We prove Pareto efficiency by a simple recursion. Consider the TC algorithm. Each agent matched in the first round of the algorithm gets his first- or second-choice house and is matched with a house controlled by an agent matched in the same round. Moreover, if an agent  $i$  gets his second choice, then  $i$ 's first choice is being assigned to another agent for whom it is the first choice; thus, assigning to  $i$  his first choice would harm this other agent.

In general, each agent matched in the  $r$ -th round of the algorithm is matched with a house controlled by an agent matched in the same round. Moreover, if any of these agents does not get his first choice among houses unmatched in this round, then the house this agent prefers is assigned to another agent for whom it is the first choice. Thus, if an agent matched in the  $r$ -th round were given a better house, this would harm some other agent matched in the same or earlier round. **QED**

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<sup>51</sup>Condition R5 requires that  $i_2$  owns  $h_4$  when  $i_1$  becomes matched and  $i_4$  loses the brokerage right.

## Appendix F An Extension: Relaxing the Assumption That There Are More Houses Than Agents

The assumption that there are more houses than agents simplifies the exposition, but our insights do not hinge on it. In fact, the key insight (Theorem 2) that each house is either owned or brokered remains true with no change in its proof.

**Theorem 7.** *For any group strategy-proof and Pareto efficient mechanism, for any submatching  $\sigma$ , and for any house  $h \in \overline{H}_\sigma$ , there is either a unique agent who owns\*  $h$  at  $\sigma$  or else there is a unique agent who brokers\*  $h$  at  $\sigma$ .*

The owner and broker mechanisms can again be implemented through a recursive trading cycles algorithm. The control-rights structures  $(c, b)$  are defined as before. Consistency requirements R2–R6 are as before, and the consistency requirement R1 is amended so that for any  $\sigma \in \overline{\mathcal{M}}$  we have:<sup>52</sup>

**(R1)** There is at most one brokered house at  $\sigma$ , or  $|\overline{H}_\sigma| = 3$  and all remaining houses are brokered.

As a preparation to implement multiple brokers via trading cycles, recall that in each round of the basic TC algorithm introduced in Section 3, we force brokers not to point to their brokered houses. We can equivalently let brokers point to brokered houses but postpone matching trivial cycles of a broker and his brokered house until there are no trading cycles that contain only owners. Only then would we force the brokers not to point to their brokered houses and only when there is an owner who also points to the brokered house. After forcing the broker to point to his second choice, we would then clear the broker’s cycle. While this second approach leads to a slower clearing of cycles, it facilitates running TC with multiple brokers.<sup>53</sup>

Notice that postponing matching the brokers’ cycles in this way has no impact on the outcome of the TC algorithm. In general, the order in which we match cycles in the mechanism of Section 3 does not matter. Indeed, our proof that TC is Pareto efficient and group strategy-proof does not rely on the order in which we clear cycles, thus any clearing strategy gives us a Pareto efficient and group strategy-proof mechanism. The definitions of owners\* and brokers\* from the proof of Theorem 1 and the equivalence argument from Appendix C.3 then show that

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<sup>52</sup>We continue to refer to this requirement as R1 because in the context of Section 3 it is equivalent to R1 introduced there. Indeed, in Section 3 we assumed that  $|H| > |I|$ , and, thus when there are exactly three houses left unmatched, the number of unmatched agents is strictly less than 3, and hence, by R3, it is not possible that all three houses are brokered. See also footnote 31.

<sup>53</sup>As discussed in earlier drafts of this paper, postponing matching the brokers until there are no owners-only cycles ensures that TC is group strategy-proof and Pareto efficient in settings with outside options. Note also that this slower way of running the TC algorithm allows us to dispense with requirement R2.

all of those mechanisms have the same control-rights structure, and hence are equivalent.<sup>54</sup>

This preparation allows us to describe the TC algorithm of Section 3 in the following equivalent way:

**The TC algorithm (extended version).** The algorithm starts with empty submatching  $\sigma^0 = \emptyset$ , and in each round  $r = 1, 2, \dots$  it matches some agents with houses. By  $\sigma^{r-1}$  we denote the submatching of agents matched before round  $r$ . If  $\sigma^{r-1} \in \overline{\mathcal{M}}$ , then the algorithm proceeds with the following steps of round  $r$ :

*Step 1. Pointing.* Each house  $h \in \overline{H_{\sigma^{r-1}}}$  points to the agent who controls it at  $\sigma^{r-1}$ . Each agent  $i \in \overline{I_{\sigma^{r-1}}}$  points to his most preferred outcome in  $\overline{H_{\sigma^{r-1}}}$ .

*Step 2(a). Matching Simple Trading Cycles.* A cycle

$$h^1 \rightarrow i^1 \rightarrow \dots \rightarrow h^n \rightarrow i^n \rightarrow h^1$$

in which  $n \in \{1, 2, \dots\}$ , agents  $i^\ell \in \overline{I_{\sigma^{r-1}}}$  point to houses  $h^{\ell+1} \in \overline{H_{\sigma^{r-1}}}$ , and houses  $h^\ell$  point to agents  $i^\ell$  is simple when one of the agents is an owner (here  $\ell = 1, \dots, n$  and superscripts are added modulo  $n$ ). Each agent in each simple trading cycle is matched with the house he is pointing to.

*Step 2(b). Forcing Brokers to Downgrade Their Pointing.* If there are no simple trading cycles in the preceding Step 2(a), and only then, we proceed as follows (otherwise we proceed to step 3):

- If there is a cycle in which a broker  $i$  points to a brokered house, and there is another broker or owner that points to this house, then we force broker  $i$  to point to his next choice, and we return to Step 2(a).<sup>55</sup>
- Otherwise, we clear all trading cycles by matching each agent in each cycle with the house he is pointing to.

*Step 3.* Submatching  $\sigma^r$  is defined as the union of  $\sigma^{r-1}$  and the set of newly matched agent-house pairs. When all agents or all houses are matched under  $\sigma^r$ , then the algorithm terminates and gives matching  $\sigma^r$  as its outcome.

The analogue of Theorem 1 holds true:

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<sup>54</sup>This observation is analogous to the well-known fact that in TTC the order in which we match the cycles of agents does not matter (cf. Roth and Postlewaite 1977). We can also prove this observation by noticing that, if different orders of eliminating cycles changed the outcome of the algorithm, then there would be a group of agents that would be able to improve their outcomes by submitting preferences in which they put houses they could not obtain at the top and thus endogenously affecting the order in which cycles are cleared. The drafts of this paper from 2010–2011 provide the details of this alternative argument.

<sup>55</sup>Broker  $i$  is unique by R1.

**Theorem 8.** *A mechanism is group strategy-proof and Pareto efficient if and only if it is a TC mechanism.*

The proof of this result follows the same steps as the proof of Theorem 1, except that one needs to check the case of three houses and three agents in our inductive arguments; see Appendix G below.

The analogues of Theorems 3, 4, 5 and Corollaries 1 and 2 remain true with no changes in their statements.<sup>56</sup> The proofs of Theorems 3, 4, and 5 go through word-for-word (with Theorem 8 in lieu of Theorem 1). The proof of Corollary 1 goes through after we recognize that in any neutral trading-cycles mechanism  $\psi^{c,b}$  there is at most one broker in every round of TC. This is so because, by R1, the only other possibility is  $|\overline{H_\sigma}| = 3$  and three brokers, in which case  $\psi^{c,b}$  is not neutral as its allocation depends on the relabeling of houses when the unmatched agents submit the same preference ranking. The proof of Corollary 2 goes through after we recognize that any TC mechanism with three brokers at some  $\sigma$  such that  $|\overline{H_\sigma}| = 3$  violates reallocation-proofness. Indeed, suppose that  $i, j, k$  are  $\sigma$ -brokers of houses  $h_i, h_j, h_k \in \overline{H_\sigma}$ , respectively, and consider a preference profile  $\succ \in \mathbf{P}[\sigma]$  such that  $\succ_i \in \mathbf{P}_i[\sigma; h_k]$ ,  $\succ_j \in \mathbf{P}_j[\sigma; h_j, h_i]$ , and  $\succ_k \in \mathbf{P}_k[\sigma; h_j, h_i]$ . Then, the deviation  $\succ'_i \in \mathbf{P}_i[\sigma; h_j, h_k]$  and  $\succ'_j \in \mathbf{P}_j[\sigma; h_k, h_i]$  violates reallocation proofness.

We finish this appendix with an example illustrating how the TC algorithm is run when there are multiple brokers.

**Example 7.** Suppose  $H = \{h_1, h_2, h_3\}$  and  $I = \{i_1, i_2, i_3\}$ . Consider a TC mechanism such that  $i_\ell$  brokers  $h_\ell$  for each  $\ell \in \{1, 2, 3\}$  at submatching  $\emptyset$ . Suppose the preferences of agents are given as follows:

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<sup>56</sup>As mentioned in Section 2, our insights also extend to the case with outside options and any numbers of agents and houses. The analogue of Theorem 2 remains true, and all group strategy-proof and Pareto efficient mechanisms can be implemented via recursive trading among brokers and owners. To get a sense of why our insights are robust, consider the simple extension in which we allow for outside options but the domain of preferences is restricted so that all agents prefer any house to being unmatched. In this case, Theorems 1/8, 3, 4, and 5 remain true with no changes in their statements nor their proofs. Theorem 2/7 remains true when restricted to submatchings on the path of the candidate TC mechanism. The empty submatching is on the path of any TC mechanism, and at this submatching we can identify the control rights in the same way as in Appendix C.1. Knowing the control rights at the empty submatching allows us to replicate the analysis of Appendix C and to identify the submatchings that may form in the first round of the candidate TC mechanism for any profile of preferences  $\succ$ . Any subset  $\sigma$  of such a first-round submatching is on the path, and the following observation holds true:  $\sigma \subseteq \phi[\succ]$  for  $\succ \in \mathbf{P}[\sigma]$ . This observation allows us to replicate the argument of Appendix C.1 and identify the control rights at any subset  $\sigma$  of a first-round submatching. We then establish what submatchings may form by the second round of the candidate TC for any  $\succ$ . Proceeding in this way, we define a submatching  $\sigma$  to be on the path if there exists a profile  $\succ$  such that  $\sigma \subseteq \phi[\succ]$  matches all agents matched under the candidate TC before some round  $r = 1, 2, \dots$  and some (possibly none) agents matched in round  $r$ , and no other agents. The above observation holds true for all submatchings on the path and allows us to replicate the arguments of Appendix C for such submatchings. The general analysis of outside options from earlier drafts will form the core of a companion paper. See also Pycia and Ünver (2007) and Pycia and Ünver (2011).

$\succ_{i_1}$	$\succ_{i_2}$	$\succ_{i_3}$
$h_1$	$h_2$	$h_1$
$h_2$	$h_1$	$\vdots$
$h_3$	$h_3$	

Then the only round of the TC algorithm runs as follows:

**Step 1.** Agent  $i_1$  and  $i_3$  point to  $h_1$ , while  $i_2$  points to  $h_2$ .

**Step 2 (1st iteration).** There are two trading cycles  $h_1 \rightarrow i_1 \rightarrow h_1$  and  $h_2 \rightarrow i_2 \rightarrow h_2$ , and neither one is simple. Because there were no simple cycles in Step 2(a), we enter Step 2(b). Broker  $i_1$  and another broker point to house  $h_1$  brokered by  $i_1$ , and thus broker  $i_1$  is forced to point to his second choice  $h_2$ , and we return to Step 2.

**Step 2 (2nd iteration).** The only cycle is  $h_2 \rightarrow i_2 \rightarrow h_2$  and it is not simple. Because again there were no simple cycles in Step 2(a), we again enter Step 2(b). Broker  $i_2$  and another broker point to house  $h_2$  brokered by  $i_2$ , and thus broker  $i_2$  is forced to point to his second choice  $h_1$ , and we again return to Step 2.

**Step 2 (3rd iteration).** The only cycle is  $h_1 \rightarrow i_1 \rightarrow h_2 \rightarrow i_2 \rightarrow h_1$  and it is not simple. Because again there were no simple cycles in Step 2(a), we again enter Step 2(b). Brokers  $i_2$  and  $i_3$  point to  $h_1$ ; among them  $i_2$  is in the cycle of  $h_1$ , and thus  $i_2$  is forced to point to his third choice  $h_3$ , and we again return to Step 2.

**Step 2 (4th iteration).** The only cycle is  $h_1 \rightarrow i_1 \rightarrow h_2 \rightarrow i_2 \rightarrow h_3 \rightarrow i_3 \rightarrow h_1$ , and it is not simple. We enter Step 2(b), and we clear this cycle because no broker in this cycle is pointing to a house that is pointed to by another broker or owner.

**Step 3.** The algorithm terminates with outcome  $\sigma^1 = \{(i_1, h_2), (i_2, h_3), (i_3, h_1)\}$ .

## Appendix G Proof of Theorem 8

The proof incorporates the possibility of three brokers in one round of TC into the proof of Theorem 1. An inspection of Step 2 of the TC algorithm tells us that if there are three brokers in a round, then this is the last round. Because our arguments are by recursion on the rounds of TC, we only need to adjust the proof of Theorem 1 in instances with exactly three relevant agents and houses, and the straightforward verification of cases is sufficient. To highlight the trading-cycles structure of the problem, we proceed in a more analytical way.

**Lemma 19.** *No broker is forced to downgrade his pointing more than twice in a round. If two or more brokers point to the same house  $h$  at some step of a round, then exactly one of these*



*brokers continues to point to  $h$  till the end of this round and is assigned  $h$ . If agent  $i$  pushes  $j$  to downgrade his pointing, then  $j$  does not push  $i$  to downgrade his pointing in the same round; and if  $i$  pushes  $j$  to downgrade and  $j$  pushes  $k$  to downgrade, then  $k$  does not push  $i$  in the same round.*

**Proof.** Suppose there are three unmatched houses and three brokers, as otherwise the lemma is immediate. Consider the round from its beginning. If no two brokers point to the same house, then the round ends and the lemma obtains. If all three brokers point to  $h$ , then the broker of  $h$  is pushed to downgrade his pointing, and then there remain two brokers who point to  $h$ . Suppose, thus, that exactly two brokers point to  $h$  in a step 2(b) of the algorithm.

Let  $j$  be the broker who is forced to stop pointing to house  $h$ , let  $i$  be the other broker who points to  $h$ , and let  $k$  be the third broker. We may assume that, having downgraded his pointing,  $j$  points to a house  $h'$  that broker  $k$  also points to; indeed, otherwise the round ends and the lemma is true. Since  $j$  was forced to downgrade his pointing, either (i)  $k$  is the broker of  $h$  and points to the object brokered by  $j$ , or (ii)  $j$  is the broker of  $h$ . In the former case,  $j$  is then forced to downgrade his pointing, there is no conflict after this downgrade, and the lemma obtains. In the latter case: If  $h'$  is brokered by  $i$ , then  $k$  is outside the cycle, and again  $j$  downgrades his pointing, there is no conflict after the downgrade, and the lemma obtains. If  $h'$  is brokered by  $k$ , then  $k$  downgrades her pointing, and either there is no conflict after the downgrade and the lemma obtains, or  $k$  points to  $h$  after the downgrade. In the latter case,  $k$  is in the cycle and hence is forced to downgrade her pointing again; after this downgrade there is no further conflict, and the lemma obtains. **QED**

**Lemma 20.** *If broker  $i_1$  points to house  $h_2$  brokered by  $i_2$ , and  $i_2$  points to house  $h_3$  brokered by the third broker, then TC assigns  $h_2$  to  $i_1$  and  $h_3$  to  $i_2$ .*

**Proof.** Either the third broker  $j$  points to the house brokered by  $i_1$ , and the three brokers swap, or the third broker  $j$  points to one of  $h_2$  or  $h_3$ . Broker  $j$  is then in cycle and is forced to point to a lower ranked house. If  $j$  then points to another house in  $\{h_2, h_3\}$ , he is again in cycle and forced to downgrade his pointing. Eventually, there is a three-agent swap. **QED**

**Proof of Theorem 8. TC is Pareto efficient and group strategy-proof.** Our argument for Pareto efficiency goes through once we notice that a broker is only forced to downgrade his pointing when another agent points to the brokered object.

To adapt the argument for individual strategy-proofness, notice that R4, R5, and R6 imply that if there are no three brokers at  $\min\{s, s'\}$ , then this control-rights assignment cannot happen as long as  $i$  is unmatched. Thus, we only need to consider the case when there are three brokers in round  $\min\{s, s'\}$  and the algorithm terminates in this round under both  $\succ$  and  $\succ'$ . By way of contradiction, suppose  $i$  obtains house  $h$  under  $\succ'$  and  $i$  prefers  $h$  over his outcome under  $\succ$ . This implies that under  $\succ$ ,  $i$  was a broker who pointed to  $h$  and was

downgraded: some other agent  $j$  pointed to  $h$  and  $i$  was in the cycle of  $h$ . By Lemma 19, under  $\succ$  broker  $i$  neither pushed  $j$  to downgrade his pointing, nor pushed the third broker who then pushed  $j$ ; hence under  $\succ'$ , agent  $j$  still points to  $h$  at some step. Being pushed to downgrade below  $h$  under  $\succ$ , agent  $i$  either brokers  $h$  or the broker of  $h$  points to the house brokered by  $i$ , and, by Lemma 3, the same control rights obtain under  $\succ'$ . In the former case,  $i$  would still be pushed to downgrade below  $h$ , contrary to him getting it. In the latter case, let  $k$  be the broker of  $h$  and note that  $k \neq j$ . Because  $j$  points to  $h$  at some step, for  $i$  to obtain  $h$  he needs to push  $j$  to downgrade because  $k$  cannot do it as the broker of  $h$ . Thus, at the step  $i$  is pushing  $j$  to downgrade,  $k$  points to the house brokered by  $j$ . In effect, either under  $\succ'$ , or under  $\succ$ ,  $k$  is pushed by another broker to downgrade her pointing below the house brokered by  $i$ , or  $j$ , respectively. Such a push under  $\succ'$  would mean that  $k$  and  $j$  point to the house brokered by  $i$ , and  $i$  points to  $h$ ; but then  $j$  would never point to  $h$ , a contradiction. Similarly, such a push under  $\succ$  would mean that  $k$  and  $i$  point to the house brokered by  $j$ , and  $j$  points to  $h$ ; but then again  $i$  would never point to  $h$ . This contradiction concludes the proof of individual strategy-proofness.

Similarly, to adapt the argument for non-bossiness, we only need to consider the case with three brokers in round  $s = \min\{s, s'\}$  and the case with three brokers in the inductive round. In the inductive assumption, we modify the three possibilities by adding the claim that each of the cycles involves at least one owner, and we add a fourth possibility that three brokers are matched in the same round under  $\succ$  (not necessarily in the same cycle) if and only if these three agents are matched in the same round under  $\succ'$  and obtain the same houses.

If there are three brokers in round  $s$ , then  $i_*$  is one of them,  $s = s'$ , and Lemma 19 implies that  $i_*$  pushed no one or the same single agent in round  $s = s'$  (under both rankings), and hence all other agents obtain the same outcomes.

In the inductive step, suppose there are three brokers  $i^1, i^2, i^3$  and three houses  $h_1, h_2$ , and  $h_3$  in round  $r$ . As before,  $t$  is the round in which the first  $\sigma^{r-1}[\succ]$  broker or brokered house are matched under  $\succ'$ , and  $\nu = \sigma^{r-1}[\succ] \cup \sigma^{t-1}[\succ']$  is a submatching. Because  $\sigma^{r-1}[\succ]$ -brokers  $i^1, i^2, i^3$  are unmatched at  $\sigma^{t-1}[\succ']$ , we conclude that  $\nu = \sigma^{r-1}[\succ]$ . At  $\sigma^{t-1}[\succ']$ , one of these agents or one of the houses is matched in a cycle  $C$ . If there are three brokers at this submatching, then they have the same control rights at  $\nu$ , and, hence, at  $\sigma^{r-1}[\succ]$ , and the inductive step is proven. Otherwise, the cycle  $C$  contains an owner. By the inductive assumption, this owner is unmatched at  $\sigma^{r-1}[\succ]$ . By R4, this owner remains an owner at  $\nu$ , a contradiction that concludes the proof of non-bossiness. **QED**

**Every group strategy-proof and Pareto efficient mechanism is TC.** The lemmas and proofs from Appendices C.1 and C.2 go through unchanged, except that the statement of Lemma 12 and its proof need to be adjusted to reflect the relaxed R1.

In Appendix C.3, we only need to modify the proof of Claim 3. In particular, the inductive assumption and the statement of Claim 3 are unchanged. If there is a single broker removed

in round  $r$ , then the proof from Appendix C.3 applies. There is now the additional possibility that three brokers are removed in round  $r$  and matched with the only three houses left. Let  $i^1, i^2, i^3$  be these brokers,  $h^1, h^2, h^3$  be their brokered houses, and  $g^1, g^2, g^3$  be the houses they are assigned under  $\psi^{c,b}[\succ]$ , respectively.

Claim: It is not possible that  $h^1 \rightarrow i^1 \rightarrow h^1$  is a removed cycle and  $i^1$  ranks some  $g^\ell$  for  $\ell \neq 1$  higher than  $g^1$ . By way of contradiction, suppose this happens. Then  $i^1$  points to  $g^2$  or  $g^3$  before pointing to  $g^1 = h^1$ . By symmetry, suppose  $i^1$  first points to  $g^2$ . At the step he is pushed to point to a lower choice,  $i^1$  must be in a two-agent cycle. But then the other broker in this cycle points to  $h^1$ , and Lemma 19 ensures that  $i^1$  is forced not to point to  $h^1$ . Since  $i^1$  was assigned  $h^1$  under  $\psi^{c,b}[\succ]$ , this is a contradiction.

We now define  $\succ *$ . Note that the above claim implies that, subject to a renaming of agents, one of the following cases obtains:

Case 1. Each  $i^\ell$  ranks  $g^\ell$  higher than any other house  $\{h^1, h^2, h^3\} - \{g^\ell\}$  at  $\succ$ . Then, let  $\succ^* = \succ$ .

Case 2.  $h^1 \rightarrow i^1 \rightarrow h^2 \rightarrow i^2 \rightarrow h^1$  and  $h^3 \rightarrow i^3 \rightarrow h^3$  are the removed cycles, and  $i^1$  ranked some  $g^\ell$  for  $\ell \neq 1$  higher than  $g^1$ . Then,  $g^3 = h^3$  cannot be ranked higher than  $g^1 = h^2$  under  $\succ_{i^1}^*$ , as then  $i^3$  would never get  $g^3$  under  $\psi^{c,b}[\succ]$ . Thus,  $g^2 = h^1$  is the only unmatched house at  $\sigma^{r-1}$  that is ranked higher than  $g^1 = h^2$  in  $\succ_{i^1}$ :  $h^1 \succ_{i^1} h^2 \succ_{i^1} h^3$ . We also have  $h^3 \succ_{i^3} h^1, h^2$ , by the above claim. Moreover, by the symmetric argument for  $i^1$  applied to  $i^2$ , we have  $h^1 \succ_{i^2} h^3$ . If  $h^2 \succ_{i^2} h^1$ , then  $i^2$  and  $i^1$  could swap their assignments  $h^1$  and  $h^2$ , respectively, under  $\psi^{c,b}[\succ]$  and improve, contradicting the Pareto efficiency of  $\psi^{c,b}[\succ]$ . Thus, we also have  $h^1 \succ_{i^2} h^2, h^3$ . Let  $\succ^*$  be a monotonic transformation of  $\succ$  under  $\psi^{c,b}$  such that  $\succ_i^* = \succ_i$  for  $i \in I - \{i^1, i^2, i^3\}$  and

$$\begin{aligned} h^1 \succ_{i^1}^* h^2 &= g^1 \succ_{i^1}^* h^3 \\ h^1 &= g^2 \succ_{i^2}^* h^3 \succ_{i^2}^* h^2 \\ h^3 &= g^3 \succ_{i^3}^* h^1 \succ_{i^3}^* h^2 \end{aligned} \tag{3}$$

and otherwise  $\succ_i^*$  ranks houses the same way as  $\succ_i$  for  $i = i^1, i^2, i^3$ .

Case 3. The three brokered houses are removed all in one cycle. Then, let  $\succ^* = \succ$ .

By Maskin monotonicity,  $\psi^{c,b}[\succ^*](i^\ell) = g^\ell$  for  $\ell = 1, 2, 3$  in all three cases. The inductive assumption implies that  $\varphi[\succ^*](i) = \psi^{c,b}[\succ^*](i)$  for all  $i \in I^{r'}$  for all  $r' < r$ . Hence, by Maskin monotonicity, to prove that  $\varphi[\succ](i^\ell) = g^\ell$ , it is enough to show that  $\varphi[\succ^*](i^\ell) = g^\ell$  for  $\ell = 1, 2, 3$ . In Case 1, the inductive assumption and Maskin monotonicity give  $\varphi[\succ^*](i^\ell) = g^\ell$ . It remains to consider Cases 2 and 3.

Case 2 (continued). Let  $e^\ell \in \{h^1, h^2, h^3\}$  refer to  $i^\ell$ 's assignment under  $\varphi[\succ^*]$ . By way of contradiction, suppose that at least one  $e^\ell \neq g^\ell$ . Then, (3) implies that  $e^3 \neq h^1$ , as otherwise Pareto efficiency would be violated (any of the other two agents can swap with  $i^3$  and both

improve). We have two subcases:  $e^3 = h^2$  or  $e^3 = h^3$ :

Subcase  $e^3 = h^2$ : Then  $e^1 \neq h^3$  by Pareto efficiency, as otherwise  $i^1$  and  $i^3$  could swap to improve. We have  $e^1 = h^1$  and  $e^2 = h^3$ . Then, at submatching  $\sigma^{r-1}$ , a profile  $\succ^{h^1, h^3, h^2}$  is a monotonic transformation of  $\succ^*$  under  $\varphi$ . By Maskin monotonicity,  $\varphi[\succ^{h^1, h^3, h^2}] = \varphi[\succ^*]$ . On the other hand, by construction of control rights  $(c, b)$  and Lemma 11,  $\varphi[\succ^{h^1, h^3, h^2}](i^1) = h^3$ ,  $\varphi[\succ^{h^1, h^3, h^2}](i^2) = h^1$ , and  $\varphi[\succ^{h^1, h^3, h^2}](i^3) = h^2$ , contradicting the previous statement.

Subcase  $e^3 = h^3$ : Then  $e^1 = h^1$  and  $e^2 = h^2$  because  $e^\ell \neq g^\ell$  for at least one  $\ell$ . By construction of control rights  $(c, b)$  and Lemma 11, for any  $\succ^{h^3, h^1, h^2}$ ,  $\varphi[\succ^{h^3, h^1, h^2}](i^1) = h^2$ ,  $\varphi[\succ^{h^3, h^1, h^2}](i^2) = h^3$ , and  $\varphi[\succ^{h^3, h^1, h^2}](i^3) = h^1$ . Then, the inductive assumption and Maskin monotonicity imply that  $\varphi[\succ^{h^3, h^1, h^2}_{i^2}, \succ^*_{-i^2}] = \varphi[\succ^{h^3, h^1, h^2}]$ . This contradicts strategy-proofness of  $\varphi$ , as  $i^2$  improves by reporting  $\succ^{h^3, h^1, h^2}_{i^2}$  at  $\succ^*$ .

Case 3 (continued). At  $\succ^*$ , for at least one agent  $i^\ell$ , house  $g^\ell$  is his top choice among houses  $h^1, h^2, h^3$  as otherwise the three agents or two of them can swap houses and improve their outcome, contradicting the Pareto efficiency of  $\varphi$ . Because at  $\succ^*$ ,  $g^1$  is not  $i^1$ 's top choice among these three houses, we may assume that  $i^3$  prefers  $g^3$  over other houses  $h^1, h^2, h^3$ . Because we have a single cycle matching all three agents,  $g^3 \neq h^3$ , and there are two subcases.

Subcase  $g^3 = h^2$ : Then  $g^1 = h^3$  and  $g^2 = h^1$  because there is a single cycle. To show that  $h^1 \succ^*_{i^2} h^3$ , suppose, to the contrary, that  $h^3 \succ^*_{i^2} h^1$ . By Pareto efficiency of TC and the assumption that the best choice of  $i^1$  among the remaining houses is not  $g^1$ , we would have  $h^2 \succ^*_{i^1} h^3 \succ^*_{i^1} h^1$ . By Lemma 20,  $g^1 = h^2$  and  $g^2 = h^3$ , a contradiction. By construction of control rights  $(c, b)$  and Lemma 11, for any  $\succ^{h^2, h^1, h^3}$  we have  $\varphi[\succ^{h^2, h^1, h^3}](i^1) = h^3 (= g^1)$ ,  $\varphi[\succ^{h^2, h^1, h^3}](i^2) = h^1 (= g^2)$ , and  $\varphi[\succ^{h^2, h^1, h^3}](i^3) = h^2 (= g^3)$ . The inductive assumption and Maskin monotonicity imply that  $\varphi[\succ^*] = \varphi[\succ^{h^2, h^1, h^3}]$ , and, hence,  $\varphi[\succ^*] = \psi^{c, b}[\succ^*]$ , as required.

Subcase  $g^3 = h^1$ : Then  $g^1 = h^2$  and  $g^2 = h^3$  because there is a single cycle. To show that  $h^3 \succ^*_{i^2} h^1$ , suppose to the contrary that  $h^1 \succ^*_{i^2} h^3$ . If  $i^1$  ranked  $g^1 = h^2$  first among the three remaining houses, the TC outcome would not change by Maskin monotonicity. By Lemma 20,  $i^1$  and  $i^3$  could at least secure  $h^2$  and  $h^1$ , respectively, improving  $i^3$  and keeping  $i^1$  indifferent; contradicting the group-strategy-proofness of TC, and establishing that  $h^3 \succ^*_{i^2} h^1$ . Two subcases are possible:

- $h^3 \succ^*_{i^2} h^2$ : By the construction of control rights  $(c, b)$  and Lemma 11, for any  $\succ^{h^3, h^1, h^2}$  at  $\sigma^{r-1}$ ,  $\varphi[\succ^{h^3, h^1, h^2}](i^1) = h^2 (= g^1)$ ,  $\varphi[\succ^{h^3, h^1, h^2}](i^2) = h^3 (= g^2)$ , and  $\varphi[\succ^{h^3, h^1, h^2}](i^3) = h^2 (= g^3)$ . The inductive assumption and Maskin monotonicity together imply that  $\varphi[\succ^*] = \varphi[\succ^{h^3, h^1, h^2}]$ , and, hence,  $\varphi[\succ^*] = \psi^{c, b}[\succ^*]$  as required.
- $h^2 \succ^*_{i^2} h^3$ : We have  $h^2 \succ^*_{i^2} h^3 = g^2 \succ^*_{i^2} h^1$ . By Pareto efficiency of TC,  $h^2 = g^1 \succ^*_{i^1} h^3$ . Because the first choice of  $i^1$  is not  $g^1$  among the three houses (by assumption), we have  $h^1 \succ^*_{i^1} h^2 = g^1 \succ^*_{i^1} h^3$ . By Claim 7 of Lemma 11's proof, we have  $\varphi[\succ^{h^1, h^2}_{\{i^1, i^3\}}, \succ^*_{-\{i^1, i^3\}}](i^1) = h^2$  and  $\varphi[\succ^{h^1, h^2}_{\{i^1, i^3\}}, \succ^*_{-\{i^1, i^3\}}](i^3) = h^1$ . Moreover, if the three agents  $i^1, i^2, i^3$  had ranked

$h^2, h^3$ , and  $h^1$  as their first choices among the three houses, respectively, then they would get those respective houses under  $\varphi$  by Pareto efficiency. Therefore, by group strategy-proofness, we also have  $\varphi[\succ_{\{i^1, i^3\}}^{h^1, h^2}, \succ_{-\{i^1, i^3\}}^*](i^2) = h^3$ . Maskin monotonicity and the inductive assumption imply that  $\varphi[\succ^*] = \varphi[\succ_{\{i^1, i^3\}}^{h^1, h^2}, \succ_{-\{i^1, i^3\}}^*]$ , and, hence,  $\varphi[\succ^*] = \psi^{c,b}[\succ^*]$  as required.

This completes the proof of Claim 3 and the proof of the theorem. QED

## Appendix H Properties of Strategy-Proof and Efficient Mechanisms

Knowing that all group strategy-proof and Pareto efficient mechanisms are trading-cycles mechanisms allows us to derive properties common to all such mechanisms.

We start by noticing that in any trading-cycles mechanism, and for any preference profile, there is a group of agents—the decisive group—each of whom can get one of their three top choices irrespective of the preferences submitted by agents not in the group.

**Corollary 3.** (*Decisive Group*) *Fix a group strategy-proof and Pareto efficient mechanism  $\phi$ . For any preference profile  $\succ$ , there is a group of agents  $I_1 \subseteq I$  such that: (i) all agents from  $I_1$  get one of their three top choices, and (ii) the allocation of agents from  $I_1$  does not depend on preferences of agents not in  $I_1$ , that is for all  $\succ'$  we have  $\phi[\succ]_{I_1} = \phi[\succ_{I_1}, \succ'_{I-I_1}]_{I_1}$ .*

We further observe that all strategy-proof and efficient mechanisms have a recursive structure: the agents in the decisive group determine their allocation; given their preferences there is another group of agents who obtain one of their top three choices and who can determine their allocation irrespective of the preferences of others, etc. For instance, in a serial dictatorship (Satterthwaite and Sonnenschein, 1981; Svensson, 1994, 1999; Ergin, 2000), which is a special case of trading cycles, the first dictator chooses his most preferred object, then a second dictator chooses his most preferred object among the objects which were not chosen by the prior dictator, and so on until all agents have objects.

**Corollary 4.** (*Recursive Structure*) *Fix a group strategy-proof and Pareto efficient mechanism  $\phi$ . For every preference profile  $\succ$ , there is a partition  $I_1, \dots, I_k$  of the set of agents such that: (i) all agents from  $I_\ell$  get one of their three top choices among objects unmatched at  $\phi[\succ](I_1 \cup \dots \cup I_{\ell-1})$ , and (ii) the allocation of agents from  $I_\ell$  does not depend on preferences of agents not in  $I_1 \cup \dots \cup I_{\ell-1} \cup I_\ell$ .*