Supplement to "Dynamic Project Selection"

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A Appendix: Auxiliary Technical Lemmas

Lemma A.1. (i) When $c < \underline{c}$, $\Phi^C(\cdot, c)$ on $[\underline{x}, \bar{x}]$ is positive at first, then intersects zero at a point, then is negative, then intersects zero at a point, and then is positive again.
(ii) $\Phi^C(\cdot, c)$ is nonnegative on $[\underline{x}, \bar{x}]$ if and only if $c \geq \underline{c}$.
(iii) $\Phi^A(\cdot, \cdot, c)$ is nonnegative on $[\underline{x}, \bar{x}]^2$ if and only if $c \geq \underline{c}$.
(iv) When $c < \bar{c}$, $\Phi^A(x_1, \cdot, c)$ is quasi-convex on $\left[x_1, b^{-1}(x_1)\right]$.

Proof. The proof proceeds in steps.

1. Claim: $1 - c - a > 0$ and $1 - c - a^2 > 0$ for any $a \in [\underline{x}, \bar{x}]$.

   Proof:

$$1 - c - a > 1 - c - \bar{x} = \frac{1 - 2c - \sqrt{1-4c}}{2} = \frac{4c^2}{2\left(1 - 2c + \sqrt{1-4c}\right)} > 0,$$

   which, coupled with $a^2 < a$, also implies that $1 - c - a^2 > 0$.

2. Claim: Define $\hat{a} = 1 - \sqrt{c}$. Then, $\hat{a} \in (\underline{x}, \bar{x})$, and $a \in [\underline{x}, \hat{a}) \cup (\hat{a}, \bar{x}]$ implies that $(a - \hat{a})\left(c - (1-a)^2\right) > 0$.

   Proof: The inequality follows by the definition of $\hat{a}$ and by inspection. It remains to verify that $\hat{a} \in (\underline{x}, \bar{x})$. Indeed,

$$\begin{aligned} \hat{a} - \underline{x} &= \frac{1 + \sqrt{1-4c} - 2\sqrt{c}}{2} > 0 \\ \bar{x} - \hat{a} &= \frac{2\sqrt{c} + \sqrt{1-4c} - 1}{2} > 0, \end{aligned}$$

where $c < \bar{c}$.

3. Claim: If $\Phi^C(a,c) < 0$ for some $a \in [\underline{x}, \bar{x}]$, then $\Phi^C(\cdot, c)$ is at first positive, then intersects zero at a single point to the left of $\hat{a}$, then is negative, then intersects zero at a single point to the right of $\hat{a}$, and then is again positive.

   Proof: Because $\Phi^C(\underline{x}, c) = 1/(1-\underline{x}) > 0$ and $\Phi^C(\bar{x}, c) = 1/(1-\bar{x}) > 0$, $\Phi(a,c) \leq 0 \implies a \in (\underline{x}, \bar{x})$. Differentiating,

$$\begin{aligned}\Phi_1^C(a,c) &= \frac{c\left(1-c-a^2\right)}{(1-a)(1-c-a)^2} + \ln\frac{c(1-a)}{a(1-c-a)} \\ &= \frac{\Phi^C(a,c)}{a} + \frac{\left(c-(1-a)^2\right)\left(1-c-a^2\right)}{a(1-a)(1-a-c)^2}.\end{aligned}$$

   If $\Phi^C(a,c) = 0$ for some $a \in (\underline{x}, \bar{x})$, then

$$\Phi_1^C(a,c) = \frac{\left(c-(1-a)^2\right)\left(1-c-a^2\right)}{a(1-a)(1-c-a)^2}.$$

   By Step 1, the sign of $\Phi_1^C(a,c)$ is the sign of $c-(1-a)^2$, which, by Step 2, switches the sign from negative to positive at $\hat{a} \in (\underline{x}, \bar{x})$. Hence, if a with $\Phi^C(a,c) < 0$ exists, then $\Phi^C(\cdot, c)$ intersects zero twice: once from above and to the left of $\hat{a}$, and once from below and to the right of $\hat{a}$.

4. Claim: If $c < \underline{c}$, then $\Phi^C(\hat{a}, c) < 0$. If $c \geq \underline{c}$, then $\Phi^C(\cdot, c)$ is nonnegative on $[\underline{x}, \bar{x}]$.

   Proof: Note that, at $a \in (\underline{x}, \bar{x})$,

$$\Phi_2^C(a,c) = \frac{a}{c}\left(\frac{1-a}{1-c-a}\right)^2 > 0.$$

   Furthermore,

$$\Phi^C(\hat{a}, \underline{c}) = 2 - (1-\sqrt{\underline{c}})\ln\frac{(1-\sqrt{\underline{c}})^2}{\underline{c}} = 0,$$

   where the first equality is by $\hat{a} = 1-\sqrt{\underline{c}}$, and the second one is by (11). Combining the two displays above delivers $\Phi^C(\hat{a}, c) < 0$ for any $c < \underline{c}$ and $\Phi^C(\hat{a}, c) \geq 0$ for any $c \geq \underline{c}$.

For $c < \underline{c}$, $\Phi^C(\hat{a}, c) < 0$ and Step 3 imply part (i).

For $c \geq \underline{c}$, $\Phi^C(\hat{a}, c) \geq 0$ and Step 3 imply part (ii).

5. Claim: Subject to $x_2 \geq x_1$, $\Phi^A(\cdot, \cdot, c)$ is minimized at $x_2 = x_1$.

   Proof: The claim follows from

$$\Phi_2^A(x_1, x_2, c) = \frac{(1 - x_1)}{(1 - x_2)^2}\left(\ln \frac{c(1 - x_1)}{x_1(1 - c - x_2)} + \frac{(1 - x_2)\left(1 - c - x_2^2\right)}{(1 - c - x_2)^2}\right) > 0,$$

   where the inequality follows because $1 - c - x_2^2 > 0$ and $1 - c - x_2 > 0$ by Step 1. Because $\Phi^A(z, z, c) = \Phi^C(z, c)$, Step 5 implies that $\Phi^A(\cdot, \cdot, c)$ has the same minimized value as $\Phi^C(\cdot, c)$ does. Hence, part (iii) is implied by part (ii).

6. Claim: Define $\kappa(x_1, x_2, c) \equiv (1 - x_2)\left(2c - 1 + x_2 + x_2^2 - x_1 x_2^2\right) - c^2$. Then, for some $y^A \in (\underline{x}, \bar{x})$ and for any $x_2 \in [\underline{x}, y^*) \cup (y^*, \bar{x}]$, $\left(x_2 - y^A\right)\kappa(x_1, x_2, c) > 0$.

   Proof: First, it will be shown that $\kappa(x_1, \bar{x}, c) > 0$. Indeed,

$$\kappa(x_1, \bar{x}, c) \geq \kappa(\bar{x}, \bar{x}, c) = c - \left(1 - \sqrt{1 - 4c}\right)\left(\frac{1}{2} - c\right) > 0,$$

   where the first inequality is by $\partial\kappa/\partial x_1 < 0$, and the last inequality follows because $\kappa(\bar{x}, \bar{x}, \cdot)$ is zero at $c \in \{0, \bar{c}\}$ and is positive at the only critical point ($c = 2/9$) in $(0, \bar{c})$.

   Next, it will be shown that $\kappa(x_1, \underline{x}, c) < 0$. Indeed,

$$\kappa(x_1, \underline{x}, c) \leq \kappa(\underline{x}, \underline{x}, c) = c - \left(1 + \sqrt{1 - 4c}\right)\left(\frac{1}{2} - c\right) < 0,$$

   where the first inequality is by $\partial\kappa/\partial x_1 < 0$, and the last inequality follows because $\kappa(\underline{x}, \underline{x}, \bar{c}) = 0$ and because $\partial\kappa(\underline{x}, \underline{x}, c)/\partial c > 0$.

Finally, $\partial^2 \kappa(x_1, x_2, c) / x_2^2 = -6(1-x_1)x_2 - 2x_1 < 0$. Hence, $\kappa(x_1, \underline{x}, c) < 0$ and $\kappa(x_1, \bar{x}, c) > 0$ imply that, on $(\underline{x}, \bar{x})$, $\kappa(x_1, \cdot, c)$ crosses zero and—by $\partial^2 \kappa(x_1, x_2, c) / x_2^2 < 0$—just once, from below, at some $y^A \in (\underline{x}, \bar{x})$.

7. Claim: $\Phi^A(x_1, \cdot, c)$ can be negative on, and only on, an interval.

   Proof: At any (x_1, x_2, c) with $\Phi^A(x_1, x_2, c) = 0$, by differentiation and substitution,

$$\Phi_2^A(x_1, x_2, x) = \frac{\kappa(x_1, x_2, c)}{(1-x_2)\, x_2\, (1-c-x_2)^2}.$$

   The sign of $\Phi_2^A(x_1, x_2, c)$ is the sign of $\kappa(x_1, x_2, c)$, which, by Step 6, switches from negative to positive at $y^A \in (\underline{x}, \bar{x})$ as x_2 rises; $\Phi^A(x_1, x_2, c)$ is quasi-convex. Part (iv) follows.

□

Lemma A.2. *The function M^C is uniquely maximized on $[\underline{x}, \bar{x}]$ at $\bar{x}$.*

Proof. Recall from the proof of Lemma 3 that M^C has two local maxima, at $\underline{a}$ and at $\bar{x}$. It remains to verify that $M^C(\bar{x}) > M^C(\underline{a})$.

Then,

$$M^C(\bar{x}) = 2c\sigma(\bar{x}) - \frac{1 - V(\bar{x}, \bar{x})}{(1-\bar{x})^2} = 2c\sigma(\bar{x}) - \frac{1}{1-\bar{x}},$$

where the last equality uses $V(\bar{x}, \bar{x}) = \bar{x}$, by direct substitution. Furthermore,

$$M^C(\underline{a}) = 2c\sigma(\underline{a}) - \frac{1 - V(\underline{a}, \underline{a})}{(1-\underline{a})^2} = 2c\sigma(\underline{a}) - \frac{c}{\underline{a}(1-\underline{a})^2} - \frac{1-\underline{a}}{1-\underline{a}-c},$$

where the last equality follows by substituting $\Phi(\underline{a}, c) = 0$ into the expression for $\frac{1-V(\underline{a},\underline{a})}{(1-\underline{a})^2}$.

As a result,

$$\begin{aligned} M^C(\bar{x}) - M^C(\underline{a}) &= 2c\sigma(\bar{x}) - 2c\sigma(\underline{a}) + \frac{c}{\underline{a}(1-\underline{a})^2} + \frac{1-\underline{a}}{1-\underline{a}-c} - \frac{1}{1-\bar{x}} \\ &= c\left[2\sigma(\bar{x}) - \frac{1}{\bar{x}(1-\bar{x})^2}\right] - c\left[2\sigma(\underline{a}) - \frac{1}{\underline{a}(1-\underline{a})^2}\right] + \frac{1-\underline{a}}{1-\underline{a}-c} > 0, \end{aligned}$$

where the last equality follows from $\bar{x}(1-\bar{x}) = c$ and by rearranging, and the inequality follows because the first bracket exceeds the second bracket, and the fraction $(1-\underline{a})/(1-\underline{a}-c)$ is positive (by $\underline{a} < \bar{x}$). The ordering of the brackets follows from $\bar{x} > \underline{a}$ and the observation

$$\frac{\mathrm{d}}{\mathrm{d}a}\left(2\sigma(a) - \frac{1}{a(1-a)}\right) = \frac{1}{a^2(1-a)^2} > 0, \qquad \forall a \in (0,1).$$

To summarize, $M^C(\bar{x}) > M^C(\underline{a})$, and, so, M^C has a unique maximand, $\bar{x}$, on $[\underline{x}, \bar{x}]$. □

Lemma A.3. *For M^A defined in (25),* $\arg\max_{a \in [x_1, b^{-1}(x_1)]} M^A(x_1, a) = \{b^{-1}(x_1)\}$, *where b^{-1} is the inverse of b defined in (14). As a result, on $\hat{\mathcal{A}}$, $u(x_1) < b^{-1}(x_1)$ and $\mathcal{F} \subset \mathcal{A}$.*

Proof. By Lemma 4, the only two local maxima of $M^A(x_1, \cdot)$ are $d(x_1)$ and $b^{-1}(x_1)$, so it suffices to show that $M^A(x_1, b^{-1}(x_1)) > M^A(x_1, d(x_1))$. Write

$$M^A\left(x_1, b^{-1}(x_1)\right) = c\eta\left(b^{-1}(x_1)\right) - \frac{1 - V\left(x_1, b^{-1}(x_1)\right)}{1 - b^{-1}(x_1)} = c\eta\left(b^{-1}(x_1)\right) - 1,$$

where the first equality is definitional, and the second one is by $V\left(x_1, b^{-1}(x_1)\right) = b^{-1}(x_1)$.

Evaluating V in (13) at $(x_1, d(x_1))$ and using $\Phi^A(x_1, d(x_1), c) = 0$ (by (28)), one can write

$$V(x_1, d(x_1)) = (1-x_1)\,d(x_1) - \frac{(1-d(x_1))\,c\,(1-c-x_1 d(x_1))}{d(x_1)\,(1-c-d(x_1))} - c + x_1.$$

Then,

$$M^A(x_1, d(x_1)) = c\eta(d(x_1)) - \frac{1 - V(x_1, d(x_1))}{1 - d(x_1)}$$
$$= c\left[\eta(d(x_1)) - \frac{1}{d(x_1)(1 - d(x_1))}\right] + x_1 - \frac{c(1 - x_1)}{1 - c - d(x_1)} - 1,$$

where the first equality is definitional, and the second one follows by substituting $V(x_1, d(x_1))$ and rearranging.

Then, suppressing the argument x_1 in $b^{-1}(x_1)$ and in $d(x_1)$, for compactness,

$$M^A\left(x_1, b^{-1}\right) - M^A(x_1, d) = c\left[\eta\left(b^{-1}\right) - \frac{1}{b^{-1}(1 - b^{-1})} - \left(\eta(d) - \frac{1}{d(1-d)}\right)\right]$$
$$+ \frac{b(d)\left(b^{-1} - x_1\right) + x_1\left(1 - b^{-1}\right)}{b^{-1}(1 - b(d))} > 0,$$

where the first equality follows by using the definitions of b and b^{-1} and rearranging, and the inequality uses $b^{-1}(x_1) > d(x_1)$ and

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\eta(y) - \frac{1}{y(1-y)}\right) = \frac{1}{y^2(1-y)} > 0$$

to conclude that the bracket in the first line is positive; and uses $x_1 < b^{-1}(x_1) < 1$ to conclude that the fraction in the second line is positive, too. That is, $M^A\left(x_1, b^{-1}(x_1)\right) > M^A(x_1, d(x_1))$, as desired.

The conclusion that, on $\hat{\mathcal{A}}$, $u(x_1) < b^{-1}(x_1)$ and $\mathcal{F} \subset \mathcal{A}$ follows by inspection of Lemma 4's Figure 7 (just validated by showing that $M^A\left(x_1, b^{-1}(x_1)\right) > M^A(x_1, d(x_1))$).

□

Lemma A.4. *On $\hat{\mathcal{B}}$, $w(x_1) < b^{-1}(x_1)$ and $\mathcal{F} \subset \mathcal{B}$.*

Proof. To conclude that $w(x_1) < b^{-1}(x_1)$, we shall show that $V\left(x_1, b^{-1}(x_1)\right) > B\left(x_1, b^{-1}(x_1)\right)$.

Note that $V\left(x_1, b^{-1}(x_1)\right) = b^{-1}(x_1)$ and, from the definition of B in (31),

$$B\left(x_1, b^{-1}(x_1)\right) = 1 - \left(1 - b^{-1}(x_1)\right)\left(\frac{1 - C(x_1)}{1 - x_1} + c\eta\left(b^{-1}(x_1)\right) - c\eta(x_1)\right).$$

Then,

$$V\left(x_1, b^{-1}(x_1)\right) - B\left(x_1, b^{-1}(x_1)\right) = \left(1 - b^{-1}(x_1)\right)\left(\frac{1 - C(x_1)}{1 - x_1} - 1 + c\eta\left(b^{-1}(x_1)\right) - c\eta(x_1)\right),$$

where

$$\begin{aligned}\frac{1 - C(x_1)}{1 - x_1} &= (1 - x_1)\left(\frac{1 - V(\underline{a}, \underline{a})}{(1 - \underline{a})^2} + 2c\left[\sigma(x_1) - \sigma(\underline{a})\right]\right) \\ &= (1 - x_1)\left(\frac{1}{1 - b(\underline{a})} + \frac{c}{\underline{a}(1 - \underline{a})^2} + 2c\left[\sigma(x_1) - \sigma(\underline{a})\right]\right).\end{aligned}$$

The first equality in the display above uses the definition of C in (19). The second equality uses the definitions of V in (13) and b in (14), and the condition $\Phi^C(\underline{a}, c) = 0$ in (21), which characterizes $\underline{a}$.

Then, substituting the display above into its precursor display gives

$$\begin{aligned}\frac{V\left(x_1, b^{-1}(x_1)\right) - B\left(x_1, b^{-1}(x_1)\right)}{1 - b^{-1}(x_1)} &= (1 - x_1)\left(\frac{1}{1 - b(\underline{a})} + \frac{c}{\underline{a}(1 - \underline{a})^2} + 2c\left[\sigma(x_1) - \sigma(\underline{a})\right]\right) \\ &\quad + c\left[\eta\left(b^{-1}(x_1)\right) - \eta(x_1)\right] - 1 \\ &= \left(\frac{(1 - x_1)}{1 - b(\underline{a})} + \frac{c}{b^{-1}(x_1)\left(1 - b^{-1}(x_1)\right)} - 1\right. \\ &\quad + (1 - x_1)\, c\left[2\sigma(x_1) - \frac{1}{x_1(1 - x_1)^2} - \left(2\sigma(\underline{a}) - \frac{1}{\underline{a}(1 - \underline{a})^2}\right)\right] \\ &\quad \left. + c\left[\eta\left(b^{-1}(x_1)\right) - \frac{1}{b^{-1}(x_1)\left(1 - b^{-1}(x_1)\right)} - \left(\eta(x_1) - \frac{1}{x_1(1 - x_1)}\right)\right]\right).\end{aligned}$$

Note that, using the definition of b in (14),

$$\begin{aligned}\frac{(1-x_1)}{1-b(\underline{a})}+\frac{c}{b^{-1}(x_1)(1-b^{-1}(x_1))}-1 &= \frac{(1-x_1)}{1-b(\underline{a})}+\frac{x_1}{b^{-1}(x_1)}-1\\ &= \frac{x_1\left[1-b^{-1}(x_1)\right]+b(\underline{a})\left[b^{-1}(x_1)-x_1\right]}{b^{-1}(x_1)(1-b(\underline{a}))}>0,\end{aligned}$$

where the inequality follows from $x_1 < b^{-1}(x_1) < 1$. Moreover,

$$2\sigma(x_1)-\frac{1}{x_1(1-x_1)^2}-\left(2\sigma(\underline{a})-\frac{1}{\underline{a}(1-\underline{a})^2}\right)>0$$

by $x_1 > \underline{a}$ and by

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(2\sigma(y)-\frac{1}{y(1-y)^2}\right)=\frac{1}{y^2(1-y)^2}>0$$

for any $y \in (0,1)$. Finally,

$$\eta\left(b^{-1}(x_1)\right)-\frac{1}{b^{-1}(x_1)(1-b^{-1}(x_1))}-\left(\eta(x_1)-\frac{1}{x_1(1-x_1)}\right)>0$$

by $b^{-1}(x_1) > x_1$ and by

$$\frac{\mathrm{d}}{\mathrm{d}y}\left(\eta(y)-\frac{1}{y(1-y)}\right)=\frac{1}{y^2(1-y)}>0$$

for $y \in (0,1)$. Thus, $V\left(x_1, b^{-1}(x_1)\right)-B\left(x_1, b^{-1}(x_1)\right)>0$, as required.

To show that $\left(\mathcal{F}\cap\hat{\mathcal{B}}\right)\subset\mathcal{B}$, from (31) and (13), write

$$\frac{B(x)-V(x)}{1-x_2}=(1-x_1)\left(\frac{1}{1-b(x_2)}+\frac{c}{1-x_2}\left[\eta(x_1)-\eta(b(x_2))\right]\right)-\frac{1-C(x_1)}{1-x_1}-c\left[\eta(x_2)-\eta(x_1)\right].$$

Differentiating, then simplifying, gives

$$\frac{\mathrm{d}}{\mathrm{d}x_2}\left(\frac{B(x)-V(x)}{1-x_2}\right)=-\frac{c\Phi^A(x,c)}{x_2(1-x_2)}.$$

As a result, because $\Phi^{A}(x,c) < 0$ implies that $B(x) > V(x)$, $\mathcal{B}$ covers $\mathcal{F}$, the failure region, on $\hat{\mathcal{B}}$. That is, $\mathcal{F} \cap \hat{\mathcal{B}} \subset \mathcal{B}$. □