Supplementary Material

Supplement to "Collusion constrained equilibrium"

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Appendix S1. A Dirichlet based family of random belief models

We show here that there are ϵ-random belief models for every positive value of ϵ. An obvious idea is to take a smooth family of probability distributions with mean equal to the truth and small variance. A good candidate for a smooth family is the Dirichlet since we can easily control the precision by increasing the "number of observations." However using an unbiased probability distribution will not work because it is ill-behaved on the boundary: if we try to keep the mean equal to the truth, then as we approach the boundary, the variance has to go to zero, and on the boundary there will be a spike. A simple alternative is to bias the mean slightly toward a fixed strictly positive probability vector with a small weight on that vector, and then let that weight go to zero as we take the overall variance to zero. Set $h(\epsilon) = (\epsilon/2)^3$. Fix a strictly positive probability vector over A^{-k} denoted by β^{-k} and call the *ϵ-Dirichlet belief model* the Dirichlet distribution with parameter vector (dimension cardinality of A^{-k})

$$\frac{1}{h(\epsilon)}\left[\left(1-\frac{\epsilon}{2\sqrt{2}}\right)\alpha^{-k}(a^{-k}) + \frac{\epsilon}{2\sqrt{2}}\beta^{-k}(a^{-k})\right].$$

Theorem S1. *The ϵ-Dirichlet belief model is an ϵ-random belief model.*

Proof. Since the parameters are away from the boundary by at least $\epsilon/2$, this has the requisite continuity property. The random variable $\tilde{\alpha}$ has mean $\overline{\alpha}^{-k} = (1-\frac{\epsilon}{2\sqrt{2}})\alpha^{-k} +$

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$\frac{\epsilon}{2\sqrt{2}}\beta^{-k}$. Since the covariances of the Dirichlet are negative, $E|\tilde{\alpha}^{-k} - \overline{\alpha}^{-k}|^2$ is bounded by the sum of the variances and we may apply Chebyshev's inequality to find

$$\Pr\left[\left|\tilde{\alpha}^{-k} - \overline{\alpha}^{-k}\right| > \epsilon/2\right] \leq E\left|\tilde{\alpha}^{-k} - \bar{\alpha}^{-k}\right|^2/(\epsilon/2)^2.$$

To evaluate the last expression, let $\delta_\epsilon(a^{-k}) \equiv \frac{1}{h(\epsilon)}[(1-\frac{\epsilon}{2\sqrt{2}})\alpha^{-k}(a^{-k}) + \frac{\epsilon}{2\sqrt{2}}\beta^{-k}(a^{-k})]$ and observe that $\sum_{a^{-k}} \delta_\epsilon(a^{-k}) = 1/h(\epsilon)$. Then by the standard Dirichlet variance formula we have

$$\frac{E|\tilde{\alpha}^{-k} - \bar{\alpha}^{-k}|^2}{(\epsilon/2)^2} = \frac{1}{(\epsilon/2)^2} \frac{\left(\sum_{a^{-k}} \delta_\epsilon(a^{-k})\right)^2 - \sum_{a^{-k}} \left(\delta_\epsilon(a^{-k})\right)^2}{\left(\sum_{a^{-k}} \delta_\epsilon(a^{-k})\right)^2 \left(\sum_{a^{-k}} \delta_\epsilon(a^{-k}) + 1\right)}$$

$$\leq \frac{1}{(\epsilon/2)^2} \frac{\left(1/h(\epsilon)\right)^2}{\left(1/h(\epsilon)\right)^2 \left(1/h(\epsilon)+1\right)} \leq \frac{h(\epsilon)}{(\epsilon/2)^2} = \frac{\epsilon}{2}.$$

We also have $|\bar{\alpha}^{-k} - \alpha^{-k}| = \frac{\epsilon}{2\sqrt{2}}|\alpha^{-k} - \beta^{-k}| \leq \frac{\epsilon}{2}$. Then $|\tilde{\alpha}^{-k} - \alpha^{-k}| > \epsilon$ implies $|\tilde{\alpha}^{-k} - \overline{\alpha}^{-k}| > \epsilon/2$; hence $\Pr(|\tilde{\alpha}^{-k} - \alpha^{-k}| > \epsilon) \leq \Pr[|\tilde{\alpha}^{-k} - \overline{\alpha}^{-k}| > \epsilon/2] \leq \epsilon/2 \leq \epsilon$, which shows that this is indeed an ϵ-random belief model. □

Appendix S2. Analysis of the leading example

The leading example consists of a three-player game in which each player chooses between C and D. The payoff matrices if player 3 plays C (left) or D (right) are

	C	D			C	D
C	6, 6, 5	0, 8, 0		C	10, 10, 0	0, 8, 5
D	8, 0, 0	2, 2, 0		D	8, 0, 5	2, 2, 5

Given α^3, the probability that player 3 plays C, the payoff matrix for players 1 and 2 is then

	C	D
C	$6+4(1-\alpha^3), 6+4(1-\alpha^3)$	0, 8
D	8, 0	2, 2

Note that if players 1 and 2 believe that $\alpha^3 < 1/2$, their best incentive compatible plan is CC, while if $\alpha^3 > 1/2$, their best incentive compatible plan is DD.

Nash equilibrium

We first examine Nash equilibrium in the ordinary sense. There is no Nash equilibrium where $\alpha^3 > 1/2$, for, if 1 and 2 play DD (as they have to in equilibrium), player 3 prefers D ($\alpha^3 = 0$). Similarly for $\alpha^3 = 1/2$: in equilibrium, 1 and 2 could only play either CC, in which case player 3 strictly prefers C, or DD, in which case she strictly prefers D.

Consider now $\alpha^3 < 1/2$. The CC within-group equilibrium for 1 and 2 cannot be part of a Nash equilibrium because then 3 prefers C ($\alpha^3 = 1$). Hence 1 and 2 must either play DD or mix.

If 1 and 2 play DD, then 3's best response is D, that is, $\alpha^3 = 0$ and therefore DDD is Nash.

Suppose then 1 and 2 mix, and denote by α^i the probability that $i = 1, 2$ plays C. In equilibrium $\alpha^1 = \alpha^2 = 1/2(1-\alpha^3)$, whence $\alpha^1 = \alpha^2 \geq 1/2$. Player 3 prefers D strictly if $\alpha^1 = \alpha^2 < 1/\sqrt{2} \approx 0.7$, so the only Nash in this range has $\alpha^1 = \alpha^2 = 1/2$ and $\alpha^3 = 0$.

For $\alpha^1 = \alpha^2 = 1/\sqrt{2}$ there is a fully mixed equilibrium with $\alpha^1 = \alpha^2 = 1/\sqrt{2}$ and α^3 given by $1/2(1-\alpha^3) = 1/\sqrt{2}$, that is, $\alpha^3 = 1 - 1/\sqrt{2}$.

Finally, there are no equilibria with $\alpha^1 = \alpha^2 > 1/\sqrt{2}$ because, for such values, 3 would play C for sure, which cannot happen in equilibrium.

In conclusion there are three Nash equilibria: DDD; one where player 3 plays D and players 1 and 2 mix 50–50 between C and D; a fully mixed one $\alpha^1 = \alpha^2 = 1/\sqrt{2}$ and $\alpha^3 = 1 - 1/\sqrt{2}$.

The payoffs in the Nash equilibria: in DDD payoffs are $(2, 2, 5)$. In the partially mixed equilibrium payoffs are $(5, 5, 3.75)$. In the fully mixed equilibrium payoffs are $(\varsigma, \varsigma, 2.5)$ where $\varsigma = 8/\sqrt{2} + 2(1 - 1/\sqrt{2}) \approx 6.24$.

Perturbations

We ease the group notation a bit. Group $\mathbf{2}$ consists of only one player, player 3, who has to choose between C and D. For ease of readability we identify the correlated strategy of this individual with the actual strategy: that is, we let $\alpha^3 = \rho^{\mathbf{2}}[C] = \rho^{-\mathbf{1}}[C]$. Since the only actual group is group $\mathbf{1}$, which is also group $-\mathbf{2}$ from the point of view of group $\mathbf{1}$, we drop the superscript from $\rho^{\mathbf{1}} = \rho^{-\mathbf{2}}$ and simply write ρ, with ρ_{CC} and ρ_{DD} the probabilities that group $\mathbf{1}$ plays CC or DD. For individual play we will also use α^i, as before, for the probability that $i = 1, 2$ plays C.

Player 3's payoff from C is $5\rho_{CC}$; from D it is $5(1-\rho_{CC})$. Consequently if player 3 is to be indifferent, it must be that $\rho_{CC} = 1/2$: if $\rho_{CC} > 1/2$, he plays C; if $\rho_{CC} < 1/2$, he plays D.

Belief equilibrium

Assume the Dirichlet belief model (defined in Section S1). What do the group response functions look like? Recall that σ indicates the beliefs are variable. For group $\mathbf{1}$, they play only CC and DD, and the probability $F^{\mathbf{1}}(\alpha^3)[CC]$ of playing CC is the probability that the belief $\sigma^{-\mathbf{1}}[C] < 1/2$; this is strictly between 0 and 1, symmetric around $\alpha^3 = 1/2$, where it is equal to $1/2$ and strictly decreasing in α^3.

For player 3, the probability $F^{\mathbf{2}}(\rho)[C]$ of playing C is the probability that the belief $\sigma^{-\mathbf{2}}[CC] > 1/2$; this is strictly between 0 and 1 and strictly increasing in ρ_{CC}.

Consider what happens at $\rho_{CC} = \rho_{DD} = 1/2$ and write $f^{\mathbf{2}}_{1/2}(\sigma^{-\mathbf{2}})$ for the density of $\mathbf{2}$'s beliefs. Then by symmetry,

$$\begin{aligned} &f^{\mathbf{2}}_{1/2}\big(\sigma^{-\mathbf{2}}[CC] = s \mid \sigma^{-\mathbf{2}}[CC] + \sigma^{-\mathbf{2}}[DD] = S\big) \\ &\quad = f^{\mathbf{2}}_{1/2}\big(\sigma^{-\mathbf{2}}[DD] = s \mid \sigma^{-\mathbf{2}}[CC] + \sigma^{-\mathbf{2}}[DD] = S\big) \end{aligned}$$

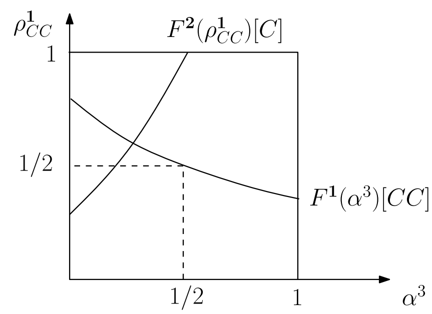

Figure S1. Random belief equilibrium.

so that

$$f^{\mathbf{2}}_{1/2}\big(\sigma^{-\mathbf{2}}[CC]=s|\sigma^{-\mathbf{2}}[CC]+\sigma^{-\mathbf{2}}[DD]=S\big)$$
$$=f^{\mathbf{2}}_{1/2}\big(\sigma^{-\mathbf{2}}[CC]=S-s|\sigma^{-\mathbf{2}}[CC]+\sigma^{-\mathbf{2}}[DD]=S\big).$$

In other words given $\sigma^{-\mathbf{2}}[CC]+\sigma^{-\mathbf{2}}[DD]=S$, then $\sigma^{-\mathbf{2}}[CC]$ is symmetric around $S/2$; hence $\sigma^{-\mathbf{2}}[CC]>1/2$ occurs less than 1/2 the time, so $F^{\mathbf{2}}(\rho_{CC})[C]<1/2$. Hence the intersection of $F^{\mathbf{1}}$ and $F^{\mathbf{2}}$ occurs for $\alpha^3<1/2$ and and $\rho_{CC}>1/2$, with $\rho_{CD}=\rho_{DC}=0$, as illustrated in Figure S1.

As beliefs converge to true values, the $F^{\mathbf{2}}$ function shifts to the right and the intersection occurs at $(1/2, 1/2)$.

Player 3 in leadership and costly enforcement equilibrium

Player 3's incentive constraint is the same as his objective function: he has the standard best response function: if $\rho^{\mathbf{1}}_{CC}>1/2$, he plays C; if $\rho^{\mathbf{1}}_{CC}<1/2$, he plays D; if $\rho^{\mathbf{1}}_{CC}=1/2$, he is indifferent. Because he is the only one in his group, he faces no incentive constraint and relaxing the incentive constraint either in leadership or costly enforcement equilibrium cannot matter; hence our focus on players 1 and 2 in group **1**, which we refer to simply as *the group*.

Costly enforcement equilibrium

As described in the text, the sequence

$$C^k_n\big(\alpha^k,\rho^{-k}\big)=\frac{\pi_n}{1-\pi_n}\sum_{k(i)=k}G^i\big(\alpha^k,\rho^{-k}\big)$$

with $\pi_n\to 1$ is a high cost sequence. To pin down the group's best response correspondence, note that for $\alpha^3\le 1/2$, it is simply CC. If the group chooses CC, the objective function takes a value of $2[6+4(1-\alpha^3)]-2\frac{\pi_n}{1-\pi_n}[2-4(1-\alpha^3)]$. This turns out to be higher than the value of 4 achieved by playing DD if and only if $\alpha^3<\frac{4-3\pi_n}{2}$. It turns out that no other mixed strategy profile is ever an element of the best response set. We establish this next. Consider any mixed strategy profile for the group. The group payof is then

$$\alpha^1\alpha^2 2\big[6+4(1-\alpha^3)\big]+\big[\alpha^1(1-\alpha^2)+\alpha^2(1-\alpha^1)\big]8$$
$$+(1-\alpha^1)(1-\alpha^2)4-\frac{\pi_n}{1-\pi_n}\big[2\alpha^1\alpha^2\big[2-4(1-\alpha^3)\big]+\big[\alpha^1(1-\alpha^2)+\alpha^2(1-\alpha^1)\big]2\big],$$

which can be rewritten as

$$(\alpha^1\alpha^2)\left\{2\big[6+4(1-\alpha^3)\big]-\frac{\pi_n}{1-\pi_n}2\big[2-4(1-\alpha^3)\big]\right\}$$
$$+\big[\alpha^1(1-\alpha^2)+\alpha^2(1-\alpha^1)\big]\left\{8-2\frac{\pi_n}{1-\pi_n}\right\}+\big[(1-\alpha^1)(1-\alpha^2)\big]4.$$

For $\pi_n > 4/5$, the term $8-2\frac{\pi_n}{1-\pi_n}$ must be negative. So the value to the group from such a mixed strategy profile is the convex combination of the group's value from playing CC: the negative quantity $8-2\frac{\pi_n}{1-\pi_n}$ and 4. When $\alpha^3 > \frac{4-3\pi_n}{2}$, then the group's value from playing CC is strictly less than 4. Consequently, every mixed strategy profile other than DD must give a value strictly less than 4. Hence the unique group best reply is DD. When $\alpha^3 < \frac{4-3\pi_n}{2}$, then the group's value from playing CC is strictly greater than 4. So every mixed strategy profile other than CC must have a value strictly less than that from playing CC. The unique group best reply is therefore CC. Similarly when $\alpha^3 = \frac{4-3\pi_n}{2}$, CC and DD are the only elements of the group best reply correspondence.

It follows immediately that the costly enforcement equilibrium consists of the group randomizing 50–50 between CC and DD while player 3 plays $\alpha^3 = \frac{4-3\pi_n}{2}$ for all $\pi_n > 4/5$. It is easy to see how this equilibrium converges to the CCE as $\pi_n \to 1$.

Leadership equilibrium

For $\alpha^3 < 1/2$, playing CC is incentive compatible for the group. The question is how much can they mix out of the unique bad within-group equilibrium DD when $\alpha^3 > 1/2$ given that they are willing to forgo gains no larger than ν.

From the payoff matrix of group **1**, we see that utility for player 1 is given by $u^1(\alpha^1,\alpha^2,\alpha^3) = 4\alpha^1\alpha^2(1-\alpha^3) - 2\alpha^1 + 6\alpha^2 + 2$. The group utility (with weights $\beta^1 = \beta^2 = 1$) is $v^{\mathbf{1}}(\alpha^1,\alpha^2,\alpha^3) = u^1 + u^2 = 8\alpha^1\alpha^2(1-\alpha^3) + 4\alpha^1 + 4\alpha^2 + 4$; notice that it is increasing in α^1 and α^2 for any α^3.

Consider the utility gained by player 1 upon deviating from $(\alpha^1,\alpha^2,\alpha^3)$ to $(0,\alpha^2,\alpha^3)$, namely $2\alpha^1[1-2\alpha^2(1-\alpha^3)]$. This is strictly positive when $\alpha^3 > 1/2$ for any positive value of α^1 and so the optimal deviation from such profiles is precisely to play D with utility $6\alpha^2+2$ and utility gain $2\alpha^1[1-2\alpha^2(1-\alpha^3)]$. Group **1** must play ν incentive compatible profiles, that is, profiles with gain no larger than ν.

When $\alpha^3 > 1/2$, increasing α^2 reduces the utility gain from player 1's optimal deviation and hence relaxes the incentive constraint for any ν. So in a strict ν equilibrium we should choose $\alpha^1 = \alpha^2$, and either the constraint binds in that $2\alpha^1[1-2\alpha^1(1-\alpha^3)] = \nu$ or $\alpha^1 = \alpha^2 = 1$ since group utility is increasing in both α^1 and α^2 for any α^3.

Notice that the utility gain $G(\alpha^1) = -4(\alpha^1)^2(1-\alpha^3) + 2\alpha^1$ is quadratic concave with $G(0) = 0$, $G' = 2[1 - 4\alpha^1(1-\alpha^3)]$ so that $G'(0) > 0$ and $G'(1) = 2[1 - 4(1-\alpha^3)]$, meaning $G'(1) < 0$ for $\alpha^3 < 3/4$.

Since group utility increases in α^1 and α^2, if the utility gain at $\alpha^1 = \alpha^2 = 1$, that is, $G(1) = 2[1 - 2(1-\hat{\alpha}^3)]$, turns out to be less than ν, group **1** plays CC and player 3 plays C, which is not an equilibrium. If this is greater than ν, then regardless of the sign of $G'(1)$, $G(\alpha^1)$ reaches ν while increasing, and group **1** plays $\hat{\alpha}^1 = \hat{\alpha}^2$ such that $G(\hat{\alpha}^1) = \nu$, that is, both players mix a little just until the incentive constraint is satisfied with equality. For small enough ν, the solution to $G(\hat{\alpha}^1) = \nu$ must be an $\hat{\alpha}^1$ so small that $\rho^{\mathbf{1}}_{CC} < 1/2$. This in turn would make player 3 play D—again not an equilibrium.

Finally consider the case of $G(1) = \nu$ so that group **1** shadow mixes between CC and the smaller solution of $-4(\hat{\alpha}^1)^2(1-\hat{\alpha}^3) + 2\hat{\alpha}^1 = \nu$. For this to be an equilibrium, since player 3 is mixing, player 1 must mix so that $\rho^{\mathbf{1}}_{CC} = 1/2$. Letting p be the probability of shadow mixing on CC, we may compute $p + (1-p)(\hat{\alpha}^1)^2 = \rho^{\mathbf{1}}_{CC} = 0.5$ from which we get

$$p = \frac{0.5 - (\hat{\alpha}^1)^2}{\left(1 - (\hat{\alpha}^1)^2\right)}.$$

So in this equilibrium player 3 has, a greater than 50% chance of playing C and the group has a less than 50% chance of playing DD, a 50% chance of playing CC and some small chance of playing CD, DC. Here the solution for player 3 is on the opposite side of 1/2 from the belief equilibrium case.

Thus equilibrium has $G(1) = G(\hat{\alpha}^1) = \nu$, that is, $2\hat{\alpha}^1[1 - 2\hat{\alpha}^1(1-\hat{\alpha}^3)] = 2[1 - 2(1-\hat{\alpha}^3)] = \nu$. As $\nu \to 0$, we get $\hat{\alpha}^3 \to 1/2$ and the smaller solution $\hat{\alpha}^1 \to 0$ so that in the limit, the group shadow mixes 50–50 between CC and DD.

Appendix S3. Analysis of the voting game

In the three-player voting game in the text, each player chooses between 0 (meaning do not vote) and 1 (meaning vote). Players 1 and 2 are assumed to form group 1, with player 3 the sole member of the remaining group.

The payoffs in the example can be written in bi-matrix form: if player 3 does not vote, the payoff matrix for the actions of players 1 and 2 is

$$\begin{array}{ccc} & 1 & 0 \\ 1 & \tau-1, \tau-1, -2\tau & \tau-1, \tau, -2\tau \\ 0 & \tau, \tau-1, -2\tau & 0, 0, 0 \end{array}$$

This game between players 1 and 2 has a unique dominant strategy equilibrium at which neither votes if $\tau < 1$. If player 3 does vote, the payoff matrix for the actions of players 1 and 2 becomes

$$\begin{array}{ccc} & 1 & 0 \\ 1 & \tau-1, \tau-1, -2\tau-1 & -\tau-1, -\tau, -2\tau-1 \\ 0 & -\tau, -\tau-1, -2\tau-1 & -\tau, -\tau, 2\tau-1 \end{array}$$

If $\tau > 1/2$, this is a coordination game for players 1 and 2 due to the fact that a tie is as bad as a loss: for a large party member not voting and having a tie results in $-\tau$, whereas voting and winning results in $\tau - 1 > -\tau$. Similarly, voting and having a tie is as bad as a loss and it would be better to not vote and lose, suffering the same loss but not paying the cost of voting.

Overview of results

We first summarize our conclusions about the structure of collusion constrained, Nash, and free enforcement equilibria in this model. There are a number of equilibria of different kinds in the various ranges of τ: (i) an equilibrium N where nobody votes (only for $\tau < 1/2$); (ii) an equilibrium L in which player 3 does not vote and the large group wins by casting a single vote. In the case of Nash equilibria, there is also (iii) an equilibrium S in which only player 3 votes (and wins); (iv) equilibria L_2 and L_3, where player 3 plays a pure strategy and the group members randomize with positive probability on both voting and not voting; (v) a fully mixed equilibrium M in which the large group members randomize as in the previous case; (vi) two asymmetric mixed equilibria A in which only one of the group members votes with positive probability. In the case of collusion constrained equilibrium (CCE) and free enforcement equilibria (FEE), there are two types of equilibria with player 3 mixing, which in the CCE case involve shadow mixing: (vii) m_1 and M_1 in which the large group either stays out or casts a single vote and (viii) m_2 and M_2 in which the large group either stays out or casts two votes. In all the equilibria where player 3 mixes, the probability that neither group member votes is always $\rho^{\mathbf{1}}[0,0] = 1/2\tau$.

We define $\tilde{\tau} \equiv 1/(3-\sqrt{5}) \approx 1.31$. Our results concerning the entire set of equilibria can be summarized in the following table.

Lower τ	Upper τ	CCE	Nash	FEE
0	1/2	N	N	N
1/2	3/4	m_2	S	L, M_1, M_2
3/4	1	m_2	S	L
1	$\tilde{\tau}$	m_2, m_1, L	S, L, A	L
$\tilde{\tau}$	3/2	m_2, m_1, L	S, L, M, A, L_3	L
3/2	2	L	S, L, L_3	L
2	∞	L	S, L, L_2, L_3	L

Discussion Before establishing these results there are several basic points to be made. If $\tau < 1/2$, then it is strictly dominant for player 3 not to vote: if the group casts no votes, not voting gives 0 rather than $\tau - 1$, and if the group does cast votes, then voting has no effect or results in an undesirable tie. Given that player 3 is not voting and $\tau < 1/2$, it is optimal both for player 1 and player 2 individually not to vote and, for the group as a whole, for neither player 1 or 2 to vote: there is no conflict here between individual incentives and group goals. Hence, in all types of equilibrium (CCE, Nash, and FEE), when $\tau < 1/2$ the unique equilibrium involves no voting and this is efficient.

The interesting case is what happens when the stakes increase to $\tau > 1/2$. Here it cannot be an equilibrium for nobody to vote because, in this case, player 3 would prefer to vote. Of particular interest are the S and L equilibria: these are always the best for the small group and large group, respectively. To see this, observe that the best that can happen if nobody in a group votes is to get 0. Alternatively, the best thing that can happen if a group casts at least one vote is that it casts only one vote and it wins, in which case the group gets $2\tau - 1$. When $\tau > 1/2$, this is better than not voting. In the equilibrium S and L in which exactly one player votes, total welfare is always -1, reflecting the cost of the single vote that is cast.

Additional observations in the light of the details presented below are the following. In the range $3/4 < \tau < 1$, equilibria of all types are unique, which allows for sharp equilibrium comparison. The Nash equilibrium is S and the FEE is L. The CCE is less efficient than either, but the large group does better than S and does worse than L. In this range, as the stakes τ increase the probability of both members of the large group voting, the probability of everyone voting and the probability of the large group winning all increase, while total welfare decreases.

There are a few parameter ranges where there are equilibria giving higher welfare than the S, L value of -1: for FEE, the M_1 when it exists gives higher welfare; for CCE, m_1 gives higher welfare in the range $1 \leq \tau \leq 9/8$. All remaining equilibria give welfare less than -1. In the Nash case, S is always an equilibrium and indeed for $\tau < 1$ is the only equilibrium. By contrast, in CCE and FEE the small player always gets a negative utility. Moreover, in both cases, when the stakes τ are high enough, the only equilibrium is L, although this occurs for a smaller value of τ for FEE than CCE.

In the range $1/2 < \tau < 3/2$, shadow mixing is a possibility for CCE and for $1/2 < \tau < 1$, there is a unique CCE with shadow mixing m_2. In the shadow mixing equilibria, the small group does better than at L while the large group does worse than at L.

It is interesting to compare m_2 and M_2 in the range $1/2 < \tau < 3/4$, the former for CCE and the latter for FEE. In both equilibria the group mixes the same way, but the third player must vote more frequently in CCE than in FEE. The reason is that if the third player votes too infrequently, then the incentive constraint fails when both members of the group vote.

Another observation of interest is that there are CCE and FEE that give the large group more utility but a lower probability of winning. Specifically in $1/2 < \tau < 3/4$ for FEE, we have that M_1 is better for the large group than M_2 but gives them a smaller probability of winning, and the same is true for CCE in the range $1 < \tau < \tilde{\tau}$ for the shadow mixing equilibria m_1 and m_2.

In a rough sense Nash is best for the small group, FEE is best for the large group, and CCE is in between. This rough "in between" picture also emerges in the sense that CCE changes more gradually in favor of the large group as τ increases than does FEE.

Remark. With respect to welfare of the large group, we have computed it in the obvious way as expected utility. For shadow mixing, whether or not this is correct depends on the underlying model: with random beliefs it is correct. However in costly enforcement equilibrium, shadow mixing appears as actual mixing, meaning that the group must be indifferent between the alternatives. In m_1 and m_2, staying out is strictly worse than casting either one or two votes. Hence in the costly enforcement equilibrium, the cost of overcoming the incentive constraints to allow the casting of votes must exactly equal the difference in utility between casting the votes and staying out: that is to say, all the gain from vote casting must be dissipated in enforcement cost. Hence, in the limit, we should evaluate the utility of the group as the least utility of profiles over which shadow mixing occurs, that is to say, the utility from staying out. As we describe in detail below, the expected utility to the large group in m_1 and m_2 is $3-2\tau-\frac{1}{2\tau}$ and $-3+2\tau+\frac{1}{2\tau}$, respectively, while the probability of player 3 not voting is $\frac{1}{\tau}$ and $1-\frac{1}{\tau}$, respectively. Hence the utility of staying out is $2(1-\tau)$ and -2, respectively, and this is the appropriate utility for the large group. In particular in the range $1 \le \tau \le 9/8$, it is no longer true that m_1 does better from an overall welfare perspective than L and S.

In the leadership case, the utility assigned to a group when shadow mixing is ambiguous. From the perspective of the followers, the correct calculation is expected utility. From the perspective of the leader, the correct calculation is the least utility of profiles over which shadow mixing occurs: from the leader's point of view, the punishment needed to make him indifferent dissipates the benefit of the better profiles. One may wonder why anyone would agree to be leader given that he or she get less utility than the followers. Although a discussion of who leaders are and why they are leaders is beyond the scope of this paper, it is natural to imagine they get some additional compensation from the group for agreeing to be leader. In this case, the follower utility seems the most relevant.

Detailed results

We now provide a more detailed summary of the different types of equilibria and payoffs. Table S1 summarizes the different types of equilibria using the notation of the text. The first column is the designation of the equilibrium. The second column gives the equilibrium strategies. The final three columns give the total payoff of the group, player 3, and the sum of all the payoffs, respectively. The probability of voting in the group's mixed strategy is denoted by p. We also write ρ_{ab} for $\rho^{\mathbf{1}}[a,b]$.

We repeat for convenience the table (see Table S2) from the overview that gives the ranges of τ for which these equilibria exist, where as above $\tilde{\tau} \approx 1.31$.

The next table, Table S3, contains payoff comparisons: we compare payoffs from the point of view of the whole set of players, represented by the total payoff, and from the point of view of the large group. We use $\succ_W$ and $\succ_{\mathbf{1}}$ to denote, respectively, welfare and large group preference. We neglect M, L_2, and L_3 (notice that A is a special case of m_1).

The last table, Table S4, contains information about the electoral outcome. We let $H=\rho_{11}(1-\alpha^3)$ denote the probability of all voting (High turnout); $D=(1-\alpha^3)(1-\rho_{00}-\rho_{11})$ denote the probability of deadlock, and $\Lambda=\alpha^3(1-\rho_{00})+(1-\alpha^3)\rho_{11}$ denote the probability that the large group wins. In this table, the rows denote different types of equilibria and the columns provide the relevant values of H, D, and Λ.

TABLE S1. Equilibrium table.

	Equilibrium strategies	Group payoff	Pl. 3 payoff	Total payoff (W)
N	$\alpha^3 = \rho_{00} = 1$	0	0	0
L	$\alpha^3 = 1$, $\rho_{10} + \rho_{01} = 1$	$2\tau - 1$	-2τ	-1
S	$\alpha^3 = 0$, $\rho_{00} = 1$	-2τ	$2\tau - 1$	-1
m_1	$\alpha^3 = \frac{1}{\tau}$, $\rho_{00} = \frac{1}{2\tau}$, $\rho_{10} + \rho_{01} = 1 - \frac{1}{2\tau}$	$3 - 2\tau - \frac{1}{2\tau}$	$1 - 2\tau$	$4 - 4\tau - \frac{1}{2\tau}$
M_1	$\alpha^3 = \frac{1}{2\tau}$, $\rho_{00} = \frac{1}{2\tau}$, $\rho_{10} + \rho_{01} = 1 - \frac{1}{2\tau}$	$1 - 2\tau$	$1 - 2\tau$	$2 - 4\tau$
m_2	$\alpha^3 = 1 - \frac{1}{2\tau}$, $\rho_{00} = \frac{1}{2\tau}$, $\rho_{11} = 1 - \frac{1}{2\tau}$	$-3 + 2\tau + \frac{1}{2\tau}$	$1 - 2\tau$	$-2 + \frac{1}{2\tau}$
M_2	$\alpha^3 = 2(1 - \frac{1}{2\tau})$, $\rho_{00} = \frac{1}{2\tau}$, $\rho_{11} = 1 - \frac{1}{2\tau}$	$2\tau - 2$	$1 - 2\tau$	-1
L_2	$\alpha^3 = 1$, $p = 1 - \frac{1}{\tau}$	$2\tau - 2$	$-2\tau + \frac{2}{\tau}$	$-2 + \frac{2}{\tau}$
L_3	$\alpha^3 = 0$, $p = \frac{1}{2\tau}$	-2τ	$2\tau - 5 + \frac{1}{\tau}$	$-5 + \frac{1}{\tau}$
M	$\alpha^3 = \frac{1}{\tau}\frac{2p\tau-1}{3p-1}$, $p = 1 - \frac{1}{\sqrt{2\tau}}$	$2\frac{\sqrt{2\tau}-\tau\sqrt{2\tau}-1+3\tau}{3-2\sqrt{2\tau}}$	$1 - 2\tau$	$1 - 2\tau + 2\frac{\sqrt{2\tau}-\sqrt{2}\tau^{\frac{3}{2}}-1+3\tau}{3-2\sqrt{2\tau}}$
A	$\alpha^3 = \frac{1}{\tau}$, $p_i = 1 - \frac{1}{2\tau}$, $p_j = 0$, $i \neq j = 1, 2$	$3 - 2\tau - \frac{1}{2\tau}$	$1 - 2\tau$	$4 - 4\tau - \frac{1}{2\tau}$

TABLE S2. Existence table.

Lower τ	Upper τ	CCE	Nash	FEE
0	$1/2$	N	N	N
$1/2$	$3/4$	m_2	S	L, M_1, M_2
$3/4$	1	m_2	S	L
1	$\tilde{\tau}$	m_2, m_1, L	S, L, A	L
$\tilde{\tau}$	$3/2$	m_2, m_1, L	S, L, M, A, L_3	L
$3/2$	2	L	S, L, L_3	L
2	∞	L	S, L, L_2, L_3	L

TABLE S3. Payoff comparisons.

τ	CCE	Nash	FEE	$\succ_W$	$\succ_{\mathbf{1}}$
$1/2 < \tau < 3/4$	m_2	S	L, M_1, M_2	$M_1 \succ_W L \sim_W S \sim_W M_2 \succ_W m_2$	$L \succ_{\mathbf{1}} M_1 \succ_{\mathbf{1}} M_2 \succ_{\mathbf{1}} m_2 \succ_{\mathbf{1}} S$
$3/4 < \tau < 1$	m_2	S	L	$L \sim_W S \succ_W m_2$	$L \succ_{\mathbf{1}} m_2 \succ_{\mathbf{1}} S$
$1 < \tau \leq 9/8$	m_1, m_2, L	S, L	L	$m_1 \succ_W L \sim_W S$	$L \succ_{\mathbf{1}} m_1 \succ_{\mathbf{1}} m_2 \succ_{\mathbf{1}} S$
$9/8 \leq \tau < \tilde{\tau}$	m_1, m_2, L	S, L	L	$L \sim_W S \succ_W m_1 \succ_W m_2$	$L \succ_{\mathbf{1}} m_1 \succ_{\mathbf{1}} m_2 \succ_{\mathbf{1}} S$
$\tilde{\tau} < \tau < 3/2$	m_1, m_2, L	S, L	L	$L \sim_W S \succ_W m_2 \succ_W m_1$	$L \succ_{\mathbf{1}} m_2 \succ_{\mathbf{1}} m_1 \succ_{\mathbf{1}} S$
$3/2 < \tau < 2$	L	S, L	L	$S \sim_W L$	$L \succ_{\mathbf{1}} S$
$\tau > 2$	L	S, L	L	$S \sim_W L$	$L \succ_{\mathbf{1}} S$

TABLE S4. Electoral outcome probabilities.

	ρ_{11}	H	D	Λ
S	0	0	0	0
L	0	0	0	1
m_1	0	0	$(1 - \frac{1}{2\tau})(1 - \frac{1}{\tau})$	$\frac{1}{\tau}(1 - \frac{1}{2\tau})$
m_2	$1 - \frac{1}{2\tau}$	$(1 - \frac{1}{2\tau})\frac{1}{2\tau}$	0	$1 - \frac{1}{2\tau}$
M_1	0	0	$(1 - \frac{1}{2\tau})^2$	$\frac{1}{2\tau}(1 - \frac{1}{2\tau})$
M_2	$1 - \frac{1}{2\tau}$	$(1 - \frac{1}{2\tau})(\frac{1}{\tau} - 1)$	0	$1 - \frac{1}{2\tau}$

Proofs of results

We first explain how the assertions in the discussion follow from the detailed Tables S1–S4. Then we derive those tables.

Assertions in the discussion From Tables S1 and S2, the total payoff W is negative except for the nonvoting equilibrium N.

From Table S3, M_1 gives welfare greater than -1 and m_1 gives welfare greater than -1 in the range $1 \leq \tau \leq 9/8$.

From Tables S1 and S2, all equilibria other than M_1, m_1, and N give welfare no more than -1.

From Tables S1 and S2, in CCE and FEE the small player always gets a negative utility.

In the range $3/4 < \tau < 1$ from Table S3, m_2 is less efficient than S or L but the large group does better than S and does worse than L.

In the range $3/4 < \tau < 1$ from Table S4, as the stakes τ increase at m_2, the probability of both members of the large group voting, the probability of everyone voting, and the probability of the large group winning all increase, while from Table S1, total welfare decreases.

In the range $1/2 < \tau < 3/2$ in m_1 and m_2 from Table S1, the small group does better than at L with utility of $1 - 2\tau$ versus -2τ, while from Table S3, the large group does worse than L.

In the range $1/2 < \tau < 3/4$ from Table S1, in m_2 and M_2, the group mixes the same way, but the third player must vote more frequently in m_2 and M_2.

In the range $1/2 < \tau < 3/4$ for FEE, we have from Table S3 that $M_1 \succ_{\mathbf{1}} M_2$, but from Table S4 it gives them a smaller probability of winning Λ.

In the range $1 < \tau < \tilde{\tau}$ for CCE, we have from Table S3 that $m_1 \succ_{\mathbf{1}} m_2$, but from Table S4 it gives them a smaller probability of winning Λ.

Equilibria: All cases It is convenient in the analysis of equilibria to create a single group **1** payoff matrix as a function of α^3 by averaging together the two matrices corresponding to 3 not voting and voting:

$$
\begin{array}{ccc}
 & 1 & 0 \\
1 & \tau - 1, \tau - 1 & (2\alpha^3 - 1)\tau - 1, (2\alpha^3 - 1)\tau \\
0 & (2\alpha^3 - 1)\tau, (2\alpha^3 - 1)\tau - 1 & (\alpha^3 - 1)\tau, (\alpha^3 - 1)\tau
\end{array}
$$

To the matrix above we add the constant $1 + \tau(1 - 2\alpha^3)$ since this is independent of group **1** play; this gives the following payoff matrix for group **1**:

$$
\begin{array}{ccc}
 & 1 & 0 \\
1 & 2\tau(1 - \alpha^3), 2\tau(1 - \alpha^3) & 0, 1 \\
0 & 1, 0 & 1 - \alpha^3\tau, 1 - \alpha^3\tau
\end{array}
$$

We also make the observation that optimality of the small group (player 3) depends only on ρ_{00} and that if $\rho_{00} < 1/(2\tau) \equiv \Upsilon$, then $\alpha^3 = 1$, if $\rho_{00} > \Upsilon$, then $\alpha^3 = 0$, and if $\rho_{00} = \Upsilon$, then player 3 is indifferent. Notice also that $\Upsilon \leq 1$ if and only if $\tau \geq 1/2$. Hence if $\tau < 1/2$, then $\alpha^3 = 1$ in any equilibrium.

We now derive Tables S1 and S2 concerning the types and nature of equilibrium.

Collusion constrained equilibria

Case 1. $\tau < 1/2$. *Nobody votes; equilibrium N. It is easy to check that this is the only group correlated equilibrium.*

Case 2. $1/2 < \tau < 1$. *There is a unique CCE where $\alpha^3 = 1 - \Upsilon$, $\rho_{00} = 1/(2\tau) = \Upsilon$, and $\rho_{11} = 1 - \Upsilon$. This is m_2. This CCE has shadow mixing. The remaining group correlated equilibria are $\rho_{00} = \Upsilon$, $\rho_{11} = 1 - \rho_{00}$, and $0 < \alpha^3 < 1 - 1/(2\tau)$; and $\alpha^3 = 0$, $\rho_{00} \geq \Upsilon$, and $\rho_{11} = 1 - \rho_{00}$.*

Proof. If $2\tau(1 - \alpha^3) < 1$, that is, $\alpha^3 > 1 - 1/2\tau$, the only within-group equilibrium for **1** is 00 and then 3 would prefer to vote, whence $\alpha^3 = 0$. It must then be $2\tau(1 - \alpha^3) \geq 1$, that is, $\alpha^3 \leq 1 - 1/2\tau$ in any group correlated equilibrium. In this case the group faces a coordination game with three within-group Nash equilibria: both vote, neither votes, and symmetric mixed equilibrium.

Let p be the probability of voting in the symmetric mixed equilibrium. The indifference is $2\tau(1 - \alpha^3)p = p + (1 - p)(1 - \alpha^3\tau)$ whence $p = (1 - \alpha^3\tau)/[\tau(2 - 3\alpha^3)]$. This increases in α^3 from $p(0) = 1/2\tau > 1/2$ to $p(1 - 1/2\tau) = 1$. Since $\alpha^3 < 1$, for this to be part of an equilibrium, player 3 should weakly prefer voting (otherwise $\alpha^3 = 1$) and this means $-[1 - (1 - p)^2] + (4\tau - 1)(1 - p)^2 \geq 2\tau(1 - p)^2$, which is equivalent to $p \leq 1 - 1/\sqrt{2\tau} < 1 - 1/2\tau < 1/2$; this is not in the range of equilibrium p's for group **1**. Hence **1** playing their mixed Nash in any group correlated equilibrium is ruled out.

Next, in any group correlated equilibrium, the probability that **1** plays $(0, 0)$ must be positive; otherwise 3 prefers not voting ($\alpha^3 = 1$) and **1** would play $(0, 0)$ for sure. Also the probability that **1** plays $(1, 1)$ must be positive; otherwise when 1 is told to vote, he knows 2 is not voting and would deviate. So $\rho_{00}, \rho_{11} > 0$. For the possible values of ρ_{10} and ρ_{01} we are left to consider that there are the two cases where correlated equilibrium probability is concentrated on $(1, 1)$, $(1, 0)$, $(0, 0)$ or on $(1, 1)$, $(0, 1)$, $(0, 0)$, which are essentially the same. We consider the first. Player 1 indifference gives $\rho_{11} \cdot 2\tau(1 - \alpha^3) = \rho_{11} + \rho_{10}(1 - \alpha^3\tau)$, that is, $\rho_{11}[2\tau(1 - \alpha^3) - 1] = \rho_{10}(1 - \alpha^3\tau)$ and, analogously, from player 2 we get $\rho_{10}[2\tau(1 - \alpha^3) - 1] = \rho_{00}(1 - \alpha^3\tau)$; from $\rho_{11} + \rho_{10} + \rho_{00} = 1$, letting $A = [2\tau(1 - \alpha^3) - 1]/(1 - \alpha^3\tau)$, we get in particular $\rho_{00} = A^2/(1 + A + A^2)$. Again player 3 should weakly prefer voting, which in this case gives $-(\rho_{11} + \rho_{10}) + (4\tau - 1)\rho_{00} \geq 2\tau\rho_{00}$, that is, $\rho_{00} \geq 1/2\tau$. Thus for **1**'s correlated equilibrium to be part of an equilibrium, it must be $2\tau \geq (1 + A + A^2)/A^2$. Now the right hand side decreases in A and A reaches its maximum for $\alpha^3 = 0$, where its value is $A_0 = 2\tau - 1$. So $(1 + A + A^2)/A^2 \geq 1 + 2\tau/(2\tau - 1)^2$. But since $0 < 2\tau - 1 < 1$, we have $(2\tau - 1)^3 < 2\tau$, which is equivalent to $2\tau < 1 + 2\tau/(2\tau - 1)^2$, whence $2\tau \geq (1 + A + A^2)/A^2$ is false for all admissible values of A. This shows that $\rho_{01} = \rho_{10} = 0$ in any group correlated equilibrium.

Summing up, group correlated equilibria have $\alpha^3 \leq 1 - 1/2\tau$ and $\rho_{00} + \rho_{11} = 1$ with $\rho_{00}, \rho_{11} > 0$. That player 3 should weakly prefer voting gives $\rho_{00} \geq \Upsilon$, with equality if $\alpha^3 > 0$. This yields the equilibrium set in the statement.

For CCE, the threshold between dominant strategy and coordination game occurs when given that one party member votes, the other is indifferent to voting: the condition is $2\tau(1-\alpha^3)=1$ so that $\alpha^3 = 1-1/(2\tau)$. This is strictly positive, so $\rho_{00} = \Upsilon$. The equilibria with smaller α^3 are not CCE because collusion would lead the group to play the voting equilibrium for sure. □

CASE 3. $1 < \tau < 3/2$. *There are three sets of CCEs: (a) a continuum of CCEs where player 3 does not vote and the group mixes with any probability over* $(1,0)$ *and* $(0,1)$*, which is L; (b) a CCE where* $\alpha^3 = 1-1/2\tau$ *and the group plays* $(1,1)$ *with probability* $1-1/2\tau$ *and* $(0,0)$ *with probability* $1/2\tau$*, which is* m_2*; and (c) a CCE with* $\alpha^3 = 1/\tau$ *where, with probability* $1-1/2\tau$*, the group mixes over* $(1,0)$ *and* $(0,1)$ *while with probability* $1/2\tau$ *they play* $(0,0)$*, which is* m_1.

PROOF. For $\alpha^3 \le 1-1/2\tau$, $(1,1)$ and $(0,0)$ are within-group Nash equilibria along with a mixed strategy equilibrium. The highest payoff for the group comes from $(1,1)$. For $1-1/2\tau < \alpha^3 < 1/\tau$, the game becomes dominance solvable with the unique equilibrium $(0,0)$. For all higher values of α^3, the within-group Nash equilibria are $(1,0)$ and $(0,1)$ along with the mixed equilibrium. The highest payoff for the group in this case turns out to be any of the group correlated equilibria with mixing over $(1,0)$ and $(0,1)$. For these higher values of α^3 where $1/\tau < \alpha^3$ and $1<\tau<3/2$, the expected payoff to each player from the mixed Nash is always strictly less than that from the group correlated equilibrium average payoff. Indeed, the inequality is $2p\tau(1-\alpha^3) < 1/2$, which since $\alpha^3 > 1/\tau > 2/3$ reads $4(\alpha^3\tau-1)(1-\alpha^3) < 3\alpha^3-2$, that is $4\alpha^3\tau(1-\alpha^3) < 2-\alpha^3$, the left member is decreasing in α^3, and using this and $\tau < 3/2$, we get $4\alpha^3\tau(1-\alpha^3) < \frac{4}{3} < 2-\alpha^3$, the last inequality from $\alpha^3 > 2/3$.

Thus in this case the group best response correspondence is

$$\begin{aligned} (1,1) \quad & \textit{if}\; \alpha^3 \le 1-1/2\tau, \\ (0,0) \quad & \textit{if}\; 1-1/2\tau \le \alpha^3 \le 1/\tau, \\ \textit{correlated} \quad & \textit{if}\; 1/\tau \le \alpha^3. \end{aligned}$$

So for any $1<\tau<3/2$, we get three sets of CCEs: (a) $\alpha^3 = 1$ and the group mixes over $(1,0)$ and $(0,1)$, (b) $\alpha^3 = 1-1/2\tau$ and the group plays $(1,1)$ with probability $1-1/2\tau$ and $(0,0)$ with probability $1/2\tau$, and (c) $\alpha^3 = 1/\tau$ and with probability $1-1/2\tau$, the group mixes over $(1,0)$ and $(0,1)$ while with probability $1/2\tau$ they play $(0,0)$, as asserted. □

CASE 4. $\tau > 3/2$. *There is a continuum of CCEs, where player 3 does not vote and the group mixes with any probability over* $(1,0)$ *and* $(0,1)$.

PROOF. It is seen from group **1** payoff matrix that for $\alpha^3\tau \le 1$, $(1,1)$ and $(0,0)$ are within-group Nash equilibria along with a mixed strategy symmetric equilibrium where the probability say p that a player votes is given by

$$p = \frac{1-\alpha^3\tau}{\tau(2-3\alpha^3)}.$$

The highest payoff for the group comes from $(1,1)$. For $1/\tau < \alpha^3 < 1 - 1/2\tau$, the game becomes dominance solvable with the unique within-group equilibrium $(1,1)$. For $\alpha^3 = 1 - 1/2\tau$, there are three within-group equilibria—$(1,1)$, $(1,0)$ and $(0,1)$—and again the best within-group equilibrium for the group is $(1,1)$.

For $\alpha^3 > 1 - 1/2\tau$, the within-group equilibria are $(1,0)$ and $(0,1)$ and the mixed equilibrium is as above. Turning to the group payoff, the two pure Nash equilibria (NE) give the same payoff; hence so does any mixture of the two. The alternative to consider is the mixed equilibrium. In the latter, the expected payoff to each player (say when player 1 plays 1) is $2p\tau(1-\alpha^3)$; in the former the per-player payoff is $1/2$. Recalling that in the range under consideration $\alpha^3\tau > 1$, the condition for the mixed to be better than the correlated mixtures becomes

$$\frac{2-\alpha^3}{4\alpha^3(1-\alpha^3)} \le \tau.$$

In the relevant range, $\tau > 3/2$ and $\alpha^3 > 1 - 1/2\tau$ imply $\alpha^3 > 2/3$, the left hand side is increasing, so letting $\hat{\alpha}^3(\tau)$ solve the above inequality as an equality, we get that the mixed Nash is better for $\alpha^3 \le \hat{\alpha}^3(\tau)$, while the mixture over the two pure Nash is better for $\alpha^3 > \hat{\alpha}^3(\tau)$. So the group best response correspondence is

$$\begin{aligned} (1,1) \quad & \textit{if}\, \alpha^3 \le 1 - 1/2\tau, \\ \textit{mixed} \quad & \textit{if}\, 1 - 1/2\tau < \alpha^3 \le \hat{\alpha}^3(\tau), \\ \textit{correlated} \quad & \textit{if}\, \alpha^3 > \hat{\alpha}^3(\tau). \end{aligned}$$

Now we can search for collusion constrained equilibria. Player 3's best response to the group playing $(1,1)$ is to set $\alpha^3 = 1$. So there cannot be a CCE with $\alpha^3 \le 1 - 1/2\tau$. Since player 3's best response to the group mixing over $(1,0)$ and $(0,1)$ is to again play $\alpha^3 = 1$, we must also rule out CCE where $\hat{\alpha}^3(\tau) < \alpha^3 < 1$. The group mixing over $(1,0)$ and $(0,1)$ with some probability and player 3 choosing $\alpha^3 = 1$ is indeed a CCE. Consider the possibility of a CCE that involves the group playing the mixed Nash equilibrium and player 3 mixing too. For player 3 to be indifferent (so as to mix) it must be that $p = 1 - 1/\sqrt{2\tau}$. Now the equilibrium p in the mixed Nash is decreasing in α^3 over the relevant region: it takes values from 1 when $\alpha^3 = 1 - 1/2\tau$ to $(\tau-1)/\tau$ when $\alpha^3 = 1$. Since $(\tau - 1)/\tau > 1 - 1/\sqrt{2\tau}$ for $\tau > 2$, for such values of τ we cannot have such a CCE. For $3/2 < \tau \le 2$ there does exist an α^3 that solves

$$\frac{1-\alpha^3\tau}{\tau(2-3\alpha^3)} = 1 - \frac{1}{\sqrt{2\tau}},$$

but the solution has $\alpha^3 > \hat{\alpha}^3(\tau)$, whence there is no CCE in the range $1 - 1/2\tau < \alpha^3 \le \hat{\alpha}^3(\tau)$ either.[1] □

[1] *Proof of this statement.* The displayed equality can be rewritten as $3\sqrt{\frac{\tau}{2}}(\alpha^3 - \frac{2}{3}) = 1 - 2\tau(1-\alpha^3)$, while $\alpha^3 \le \hat{\alpha}^3(\tau)$ reads as $\alpha^3[1+4\tau(1-\alpha^3)] \ge 2$. Since $\tau > 3/2$, we have $3\sqrt{\frac{\tau}{2}}(\alpha^3 - \frac{2}{3}) > \frac{3}{2}\sqrt{3}(\alpha^3 - \frac{2}{3}) = \sqrt{3}(\frac{3}{2}\alpha^3 - 1)$,

Nash Recall that $\tilde{\tau} \equiv 1/(3-\sqrt{5}) \approx 1.31$. We reiterate the payoff matrix for the group for visibility:

$$\begin{array}{ccc} & 1 & 0 \\ 1 & 2\tau(1-\alpha^3), 2\tau(1-\alpha^3) & 0,1 \\ 0 & 1,0 & 1-\alpha^3\tau, 1-\alpha^3\tau \end{array}$$

CASE 1. $\rho_{00} < \Upsilon$ *and* $\alpha^3 = 1$. *The payoff matrix for the group is*

$$\begin{array}{ccc} & 1 & 0 \\ 1 & 0,0 & 0,1 \\ 0 & 1,0 & 1-\tau, 1-\tau \end{array}$$

If $\tau < 1$, *then it is dominant to play 0 and this is not an equilibrium. If* $\tau > 1$, *then there are two pure equilibria where one voter in the group votes and these imply* $\rho_{00} < \Upsilon$, *so this corresponds to the equilibrium L. The other equilibrium is symmetric and mixed, and continuing to use p for the probability of voting, the indifference condition is* $p + (1-p)(1-\tau) = 0$ *or*

$$p = \frac{\tau - 1}{\tau} = 1 - 2\Upsilon.$$

Here $p > 0$ *requires* $\Upsilon \leq 1/2$. *The probability that neither player votes is* $4\Upsilon^2$, *which must satisfy* $4\Upsilon^2 < \Upsilon$ *or* $\Upsilon < 1/4$. *Hence we have an equilibrium of this type (it is* L_2*) if* $1/(2\tau) < 1/4$ *or* $\tau > 2$. *Notice that in this equilibrium the probability that the group wins* $1 - 4\Upsilon^2$ *is larger than* $3/4$.

CASE 2. $\rho_{00} > \Upsilon$ *and* $\alpha^3 = 0$. *Recall that this requires* $\tau \geq 1/2$ *(otherwise* $\alpha^3 = 1$*). The payoff matrix for the group is*

$$\begin{array}{ccc} & 1 & 0 \\ 1 & 2\tau, 2\tau & 0,1 \\ 0 & 1,0 & 1,1 \end{array}$$

This coordination game has one pure strategy equilibrium where both vote, which contradicts $\rho_{00} > \Upsilon$, *and one where neither votes, corresponding to the equilibrium S, which therefore exists for all values of* $\tau \geq 1/2$. *It also has a unique symmetric mixed equilibrium where the indifference condition is* $p2\tau = 1$ *or* $p = \Upsilon$. *The probability that neither votes is then* $(1-\Upsilon)^2$ *and the condition is* $(1-\Upsilon)^2 > \Upsilon$. *This is* $1 - 3\Upsilon + \Upsilon^2 > 0$, *which has roots at* $(3 \pm \sqrt{5})/2$ *and is positive only for* Υ *smaller than the lesser root* $(3-\sqrt{5})/2 \approx 0.38$. *That is to say, we have an equilibrium of this type when* $\tau > 1/(3-\sqrt{5}) = \tilde{\tau}$. *This is* L_3.

so the equality implies $1 - 2\tau(1-\alpha^3) > \sqrt{3}(\frac{3}{2}\alpha^3 - 1)$, that is, $2\tau(1-\alpha^3) < 1 - \sqrt{3}(\frac{3}{2}\alpha^3 - 1)$, whence

$$\alpha^3[1 + 4\tau(1-\alpha^3)] < \alpha^3\left[3 - 2\sqrt{3}\left(\frac{3}{2}\alpha^3 - 1\right)\right] = \alpha^3\sqrt{3}[\sqrt{3} - (3\alpha^3 - 2)] < \frac{2}{3}\sqrt{3}\left[\sqrt{3} - \left(3\frac{2}{3} - 2\right)\right] = 2,$$

where the last inequality follows from the fact that in the relevant range $\alpha^3 \geq 2/3$, the function $\alpha^3\sqrt{3}[\sqrt{3} - (3\alpha^3 - 2)]$ is decreasing.

Case 3. $\rho_{00} = \Upsilon$. *Indifferences give the same values of p and α^3 as in the case of* $1/2 < \tau < 1$, *that is,*

$$p = 1 - 1/\sqrt{2\tau} = 1 - \sqrt{\Upsilon} \quad \alpha^3 = \frac{1}{\tau}\frac{2p\tau - 1}{3p - 1}.$$

This equilibrium, labeled M, exists for

$$\tilde{\tau} < \tau < 3/2.$$

In addition, for $1 < \tau < \frac{3}{2}$, there is an asymmetric partially mixed equilibrium where one of the players in the group does not vote and the other votes with probability $1 - \Upsilon$ while $\alpha^3 = 2\Upsilon$. This is equilibrium A. Notice that this is a special case of m_1.

Proof. If both group members mix, we must have symmetry and this gives $(1-p)^2 = \Upsilon$ or $p = 1 - \sqrt{\Upsilon} > 0$. From the group payoff matrix, we see that if $\tau < 1/2$, then 0 is strictly dominant, so this is impossible. Assume $\tau > 1/2$. For $\tau > 1/2$, the indifference condition of player 1 between voting and not voting when pkayer 2 votes with probability p gives $p2\tau(1-\alpha^3) = p + (1-p)(1-\alpha^3\tau)$, which yields

$$\alpha^3 = \frac{1}{\tau}\frac{2p\tau - 1}{3p - 1}.$$

We then plug in $p = 1 - \sqrt{\Upsilon}$ and look at the signs of the numerator and denominator of this expression. The numerator is $2\tau(1 - 1/\sqrt{2\tau}) - 1 = 2\tau - \sqrt{2\tau} - 1$. This is positive if and only if $2\tau - 1 > \sqrt{2\tau}$, which, since $\tau > 1/2$ is equivalent to $(2\tau - 1)^2 > 2\tau$, is $4\tau^2 - 6\tau + 1 > 0$. This has roots $(3 \pm \sqrt{5})/4$ and is negative in between. Note that the lesser root is $< 1/2$. The denominator is positive for $3(1 - 1/\sqrt{2\tau}) - 1 > 0$, that is, for $\tau > 9/8$. Note that $(3 + \sqrt{5})/4 = 1/(3 - \sqrt{5}) > 9/8$; hence for $\tau > 1/2$, the numerator and denominator have the same sign if and only if $1/2 < \tau < 9/8$ (both negative) or $\tau > 1/(3 - \sqrt{5})$ (both positive). In the latter case, $\alpha^3 < 1$ requires $2p\tau - 1 < 3p - 1$, which is to say $2\tau < 3$ or $\tau < 3/2$, and in this range this equilibrium exists. In the former case, $\alpha^3 \leq 1$ would require $2p\tau - 1 \geq 3p - 1$, which is true only for $\tau \geq 2$, so this range is ruled out.

Now consider the possibility of only one group member mixing. Say player 1 mixes while player 2 plays 0 with certainty. It must be that $1 - p_1 = \rho_{00} = \Upsilon$. For player 1 to be so indifferent, we need $\alpha^3 = 2\Upsilon$. For player 2 to prefer not voting to voting, we need $(1 - \frac{1}{2\tau})(3 - 2\tau) \geq 0$. Satisfying this inequality along with $\alpha^3 \leq 1$ gives the range $1 < \tau < \frac{3}{2}$. So for each $1 < \tau < \frac{3}{2}$, we get two more mixed equilibria, in each of which one group member plays 0 for sure while the other does so with probability Υ and $\alpha^3 = 2\Upsilon$. □

Free enforcement equilibrium Assuming uniform weights in the group utility, group **1** payoffs are $1 - \alpha^3\tau$ if neither votes, $1/2$ if one votes, and $2\tau(1-\alpha^3)$ if both vote. Recalling that if $\rho_{00} < 1/(2\tau) \equiv \Upsilon$, then $\alpha^3 = 1$, if $\rho_{00} > \Upsilon$, then $\alpha^3 = 0$, and if $\rho_{00} = \Upsilon$, then player 3 is indifferent, equilibrium analysis goes as follows.

Case 1. *$\rho_{00} < \Upsilon$ and $\alpha^3 = 1$. Group payoffs are $1-\tau$, $1/2$, and 0. If $1-\tau > 1/2$, that is, $\tau < 1/2$, the optimum is not to vote and this is an equilibrium since $\Upsilon > 1$ for $\tau < 1/2$. If $\tau > 1/2$, the optimum is for exactly one to vote, leading to the equilibrium L; hence this is the equilibrium for $\tau > 1/2$.*

Case 2. *$\rho_{00} > \Upsilon$ and $\alpha^3 = 0$. Group payoffs are 1, $1/2$, and 2τ. If $\tau > 1/2$, the optimum is vote, which is not an equilibrium given $\rho_{00} > 0$. For $\tau < 1/2$, notice that $\alpha^3 = 0$ cannot be optimal. So, no equilibrium corresponds to this case.*

Case 3. *$\rho_{00} = \Upsilon$. This case requires that $1-\alpha^3\tau \geq 1/2, 2\tau(1-\alpha^3)$ with at least one equality.*

Case 3a. *$1-\alpha^3\tau = 1/2$ and $1/2 \geq 2\tau(1-\alpha^3)$. The first equation solves as $\alpha^3 = \Upsilon$, which we know requires $\tau \geq 1/2$. The inequality becomes $1/2 \geq 2\tau(2\tau-1)/(2\tau) = 2\tau - 1$, that is, $\tau \leq 3/4$. Hence for $1/2 < \tau < 3/4$, there is an equilibrium with $\rho_{11} = 0$ and $\alpha^3 = \Upsilon$. This is M_1.*

Case 3b. *$2\tau(1-\alpha^3) = 1-\alpha^3\tau$ and $1-\alpha^3\tau \geq 1/2$. The first equation gives $\alpha^3 = 2-1/\tau = 2(1-\Upsilon)$. For Υ we need as usual $\tau \geq 1/2$. We also need $2-1/\tau \leq 1$ or $1 \leq 1/\tau$ or $\tau \leq 1$. Plugging into the inequality, we get $1-(2-1/\tau)\tau \geq 1/2$, which gives $\tau \leq 3/4$. Hence if $1/2 < \tau < 3/4$, there is another equilibrium with $\rho_{11} = 1-\Upsilon$ and $\alpha^3 = 2(1-\Upsilon)$. This is M_2.*

Payoff comparisons

We next derive Table S3. For the welfare of all three players combined we have

$$\begin{aligned} L,S \succ_W m_2 &\iff \tau > 1/2, & L,S \succ_W m_1 &\iff \tau \gtrapprox 1.14,\\ m_1 \succ_W m_2 &\iff \tau < \tilde{\tau}, & M_1 \succ_W L,S &\iff \tau < 3/4. \end{aligned}$$

For the large group the inequalities are

$$\begin{aligned} L \succ_{\mathbf{1}} S &\iff \tau > 1/4, & L \succ_{\mathbf{1}} m_1 &\impliedby \tau > 1,\\ m_1 \succ_{\mathbf{1}} m_2 &\iff 0.2 < \tau < \tilde{\tau}, & M_1 \succ_{\mathbf{1}} M_2 &\iff \tau < 3/4,\\ L \succ_{\mathbf{1}} M_1 &\iff \tau > 1/2, & M_1 \succ_{\mathbf{1}} m_2 &\implies \tau \lessapprox 0.85,\\ M_2 \succ_{\mathbf{1}} m_2 &\iff \tau > 1/2, & m_2 \succ_{\mathbf{1}} S &\impliedby \tau > 1/2,\\ m_1 \succ_{\mathbf{1}} S &\iff \tau > 1/6. \end{aligned}$$

Going in the order of the last display, for the three players we have

$$\begin{aligned} L,S \succ_W m_2 &\iff -1 > -2 + \frac{1}{2\tau} \iff \frac{1}{2\tau} < 1 \iff \tau > 1/2,\\ L,S \succ_W m_1 &\iff -1 > 4 - 4\tau - \frac{1}{2\tau} \iff 8\tau^2 - 10\tau + 1 > 0 \end{aligned}$$

$$
\begin{aligned}
&\iff\quad 0.11 \precapprox \tau \precapprox 1.14,\\
m_1 \succ_W m_2 \quad &\iff\quad 4-4\tau-\frac{1}{2\tau} > -2+\frac{1}{2\tau} \quad\iff\quad 6-4\tau-\frac{1}{\tau} > 0\\
&\iff\quad 4\tau^2-6\tau+1>0 \quad\iff\quad 0.19 \precapprox \tau \le \tilde{\tau},\\
M_1 \succ_W L, S \quad &\iff\quad 2-4\tau > -1 \quad\iff\quad 3 > 4\tau \quad\iff\quad \tau < 3/4.
\end{aligned}
$$

For the large group,

$$
\begin{aligned}
L \succ_{\mathbf{1}} S \quad &\iff\quad 2\tau-1 > -2\tau \quad\iff\quad 4\tau > 1 \quad\iff\quad \tau > 1/4,\\
L \succ_{\mathbf{1}} m_1 \quad &\iff\quad 2\tau-1 > 3-2\tau-\frac{1}{2\tau} \quad\iff\quad 4\tau-4+\frac{1}{2\tau} > 0\\
&\iff\quad 8\tau^2-8\tau+1>0 \quad\Longleftarrow\quad \tau > 0.85,\\
m_1 \succ_{\mathbf{1}} m_2 \quad &\iff\quad 3-2\tau-\frac{1}{2\tau} > -3+2\tau+\frac{1}{2\tau} \quad\iff\quad 6-4\tau-\frac{1}{\tau} > 0\\
&\iff\quad 4\tau^2-6\tau+1<0 \quad\iff\quad 0.2 < \tau < \tilde{\tau},\\
M_1 \succ_{\mathbf{1}} M_2 \quad &\iff\quad 1-2\tau > 2\tau-2 \quad\iff\quad 3 > 4\tau,\\
L \succ_{\mathbf{1}} M_1 \quad &\iff\quad 2\tau-1 > 1-2\tau \quad\iff\quad 4\tau > 2,\\
M_1 \succ_{\mathbf{1}} m_2 \quad &\iff\quad 1-2\tau > -3+2\tau+\frac{1}{2\tau} \quad\iff\quad 8\tau^2-8\tau+1<0\\
&\iff\quad 0.15 \precapprox \tau \precapprox 0.85,\\
M_2 \succ_{\mathbf{1}} m_2 \quad &\iff\quad 2\tau-2 > -3+2\tau+\frac{1}{2\tau} \quad\iff\quad 1 > \frac{1}{2\tau} \quad\iff\quad \tau > 1/2,\\
m_2 \succ_{\mathbf{1}} S \quad &\iff\quad -3+2\tau+\frac{1}{2\tau} > -2\tau \quad\iff\quad 8\tau^2-6\tau+1>0 \quad\Longleftarrow\quad \tau > 1/2,\\
m_1 \succ_{\mathbf{1}} S \quad &\iff\quad 3-2\tau-\frac{1}{2\tau} > -2\tau \quad\iff\quad 3 > \frac{1}{2\tau} \quad\iff\quad \tau > 1/6.
\end{aligned}
$$

We check that it is always the case that $M \prec_W S, L$. Indeed this is equivalent to $1-2\tau+2\frac{\sqrt{2\tau}-\sqrt{2}\tau^{\frac{3}{2}}-1+3\tau}{3-2\sqrt{2\tau}} < -1$, that is, $2-2\tau+2\frac{\sqrt{2\tau}-\sqrt{2}\tau^{\frac{3}{2}}-1+3\tau}{3-2\sqrt{2\tau}} < 0$. In the relevant range, the denominator in the fraction is always negative, so after multiplying, we get $(2-2\tau)(3-2\sqrt{2\tau})+2(\sqrt{2\tau}-\tau\sqrt{2\tau}-1+3\tau) > 0$, which simplifies to $2\sqrt{2}[\sqrt{2}-\sqrt{\tau}+\tau\sqrt{\tau}] > 0$, which is true for every $\tau > 0$.

Electoral outcome probabilities

Finally we find the electoral outcome probabilities. Electoral outcome probabilities are also elementarily obtained. Recall that $H = \rho_{11}(1-\alpha^3)$, $D = (1-\alpha^3)(1-\rho_{00}-\rho_{11})$, and $\Lambda = \alpha^3(1-\rho_{00}) + (1-\alpha^3)\rho_{11}$; we just have to apply these formulas.

We follow the order of the table. In S, we have $H = D = \Lambda = 0$. In L, the only difference is $\Lambda = 1$.

In m_1, we have $\alpha^3 = \frac{1}{\tau}$, $\rho_{00} = \frac{1}{2\tau}$, and $\rho_{10}+\rho_{01} = 1-\frac{1}{2\tau}$, so $H = 0$, $D = (1-\frac{1}{\tau})(1-\frac{1}{2\tau})$, and $\Lambda = \frac{1}{\tau}(1-\frac{1}{2\tau})$.

In m_2, it is $\alpha^3 = 1 - \frac{1}{2\tau}$, $\rho_{00} = \frac{1}{2\tau}$, and $\rho_{11} = 1 - \frac{1}{2\tau}$, so $H = (1 - \frac{1}{2\tau})\frac{1}{2\tau}$, $D = 0$, and $\Lambda = (1 - \frac{1}{2\tau})^2 + (1 - \frac{1}{2\tau})\frac{1}{2\tau} = 1 - \frac{1}{2\tau}$.

In M_1, we have $\alpha^3 = \frac{1}{2\tau}$, $\rho_{00} = \frac{1}{2\tau}$, and $\rho_{10} + \rho_{01} = 1 - \frac{1}{2\tau}$, so $H = 0$, $D = (1 - \frac{1}{2\tau})^2$, and $\Lambda = \frac{1}{2\tau}(1 - \frac{1}{2\tau})$.

Finally, in M_2, we have $\alpha^3 = 2(1 - \frac{1}{2\tau})$, $\rho_{00} = \frac{1}{2\tau}$, and $\rho_{11} = 1 - \frac{1}{2\tau}$, so $H = (1 - \frac{1}{2\tau})[1 - 2(1 - \frac{1}{2\tau})] = (1 - \frac{1}{2\tau})(\frac{1}{\tau} - 1)$, $D = 0$, and $\Lambda = 2(1 - \frac{1}{2\tau})(1 - \frac{1}{2\tau}) + [1 - 2(1 - \frac{1}{2\tau})](1 - \frac{1}{2\tau}) = 1 - \frac{1}{2\tau}$.

For the ranges of H in m_2 and M_2, and of D in m_1 and M_1, we have the following equalities.

Range H in m_2: Up from 0 for $\tau = 1/2$ to $2/9$ for $\tau = 3/4$, still up to $1/4$ for $\tau = 1$, and then down to $2/9$ again for $\tau = 3/2$.

Range H in M_2: Up from 0 for $\tau = 1/2$ to $1/8$ for $\tau = 2/3$, then down to $1/9$ for $\tau = 3/4$.

Range D in m_1: Up from 0 for $\tau = 1$ to $2/9$ for $\tau = 3/2$.

Range D in M_1: Up from 0 for $\tau = 1/2$ to $1/9$ for $\tau = 3/4$.

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