Judicial precedent as a dynamic rationale for axiomatic bargaining theory: Appendix with proofs

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Appendix

Proof of Prop. 2. Let $D = \{i, j, 3j, k, 2k, l, 3l, m, 2m, n\}$, for

$$
\begin{aligned}
i &= co\{(0,0), (12,0), (0,12)\} \\
j &= co\{(0,0), (3,0), (2,2), (0,2)\} \\
k &= co\{(0,0), (3,0), (2,2), (0,2.5)\} \\
l &= co\{(0,0), (2,0), (2,2), (0,3.5)\} \\
m &= co\{(0,0), (2.5,0), (2,2), (0,3.5)\} \\
n &= co\{(0,0), (6,0), (4,4), (0,7)\}.
\end{aligned}
$$

The ten domains are illustrated in Figure 1.

1. The Nash theorem holds on D.

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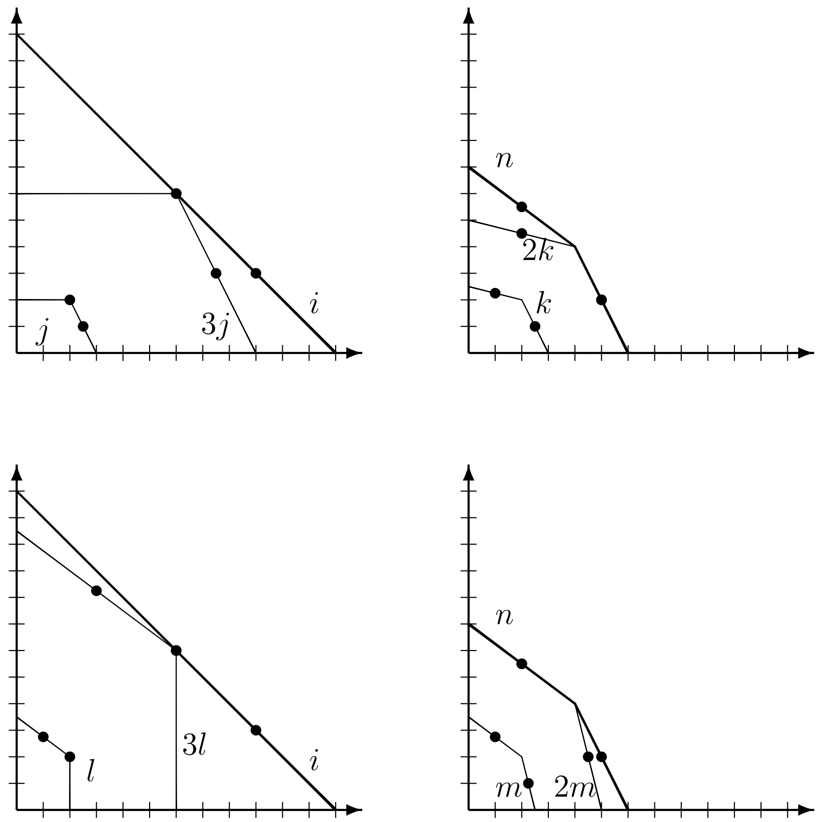

Figure 1: The domain used in the proof of Prop. 2

By WP and Sym, $\varphi(i) = N(i) = (6,6)$. By Ind, $\varphi(3j) = N(3j) = (6,6)$ and $\varphi(3l) = N(3l) = (6,6)$. By ScInv, $\varphi(j) = N(j) = (2,2)$ and $\varphi(l) = N(l) = (2,2)$.

By Ind, $\varphi(k)$ must be on the segment $(2,2)\,(0,2.5)$ and $\varphi(m)$ must be on the segment $(2.5,0)\,(2,2)$. By ScInv, $\varphi(2k)$ must be on the segment $(4,4)\,(0,5)$ and $\varphi(2m)$ must be on the segment $(5,0)\,(4,4)$. Therefore, by Ind, $\varphi(n)$ must be on the segment $(4,4)\,(0,7)$ and on the segment $(6,0)\,(4,4)$, which implies that $\varphi(n) = N(n) = (4,4)$.

Then, by Ind, $\varphi(2k) = N(2k) = (4,4)$ and $\varphi(2m) = N(2m) = (4,4)$. By ScInv, $\varphi(k) = N(k) = (2,2)$ and $\varphi(m) = N(m) = (2,2)$, which completes the proof.

2. Consider the following strategy for the Judge (it is illustrated in Fig. 1):

- for i, pick $N(i)$ or $(9,3)$, whichever has lower penalty (this clause applies to all choices below);

- for j, pick $N(j)$ or $(2.5, 1)$;

- for $3j$, pick $N(3j)$ or $(7.5, 3)$;

- for k, pick $(1, 2.25)$ or $(2.5, 1)$;

- for $2k$, pick $(2, 4.5)$ or $(5, 2)$;

- for l, pick $N(l)$ or $(1, 2.75)$;

- for $3l$, pick $N(3l)$ or $(3, 8.25)$;

- for m, pick $(2.25, 1)$ or $(1, 2.75)$;

- for $2m$, pick $(4.5, 2)$ or $(2, 5.5)$;

- for n, pick $(5, 2)$ or $(2, 5.5)$.

Note that for the sets k, $2k$, m, $2m$ and n, the Nash point is not one of the points possibly selected by this strategy. We now prove that whatever the previous sequence of problems $(i_t)_{t=1,...,T-1}$ (including the case in which there is no previous sequence and $T = 1$), if this strategy has been followed on $t = 1, ..., T-1$ the Judge can always minimize penalty by sticking to the strategy in T. We consider a general system of penalties: the penalty attached to WP, Sym, ScInv, Ind is, respectively, a, b, c, d (with discounting by δ for the last two).

In order to check that the strategy is optimal, we have to check that its recommendations minimize the penalty in every set of the domain. In every set, we will ignore the points which do not belong to the upper frontier, as every such point always entails a greater penalty (due to WP) than some point on the frontier.

- For i : the choice in ∂i is constrained only by Sym, and by choices made in i, $3j$ and $3l$.

Let

$$\begin{aligned} T_1 &= \{t < T \mid (i_t, x_t) = (3j, (7.5, 3)) \text{ or } (3l, (3, 8.25))\}, \\ T_1' &= \{t < T \mid (i_t, x_t) = (i, N(i))\}, \\ T_1'' &= \{t < T \mid (i_t, x_t) = (i, (9, 3))\}. \end{aligned}$$

The penalty for choosing $N(i)$ is $\sum_{t\in T_1}\delta^{T-t}d+\sum_{t\in T_1''}\delta^{T-t}(c+d)$. The penalty for choosing $(9,3)$ is $b+\sum_{t\in T_1'}\delta^{T-t}(c+d)$. The penalty for choosing any other point on ∂i is $b+\sum_{t\in T_1'\cup T_1''}\delta^{T-t}(c+d)$. The last value is never lower than the penalty for $(9,3)$, and therefore the Judge minimizes the penalty by choosing either $N(i)$ or $(9,3)$, as dictated by the strategy.

Observe that the reasoning can be simplified. The penalties due to previous choices made in i push toward making the same choices as previously and therefore reinforce the contemplated strategy. In the sequel we will ignore the penalties due to previous choices in the same set. If we can prove that the strategy minimizes the sum of the remaining penalties, a fortiori it minimizes the complete sum of penalties.

- For j : the choice in ∂j is constrained only by choices made in $3j$ and in k (ignoring j itself).

Let $T_2=\{t<T\mid(i_t,x_t)=(3j,(7.5,3))\}$, $T_3=\{t<T\mid(i_t,x_t)=(k,(2.5,1))\}$, $T_4=\{t<T\mid(i_t,x_t)=(3j,N(3j))\}$.

The penalty for choosing $N(j)$ is $\sum_{t\in T_2}\delta^{T-t}c+\sum_{t\in T_3}\delta^{T-t}d$. The penalty for choosing $(2.5,1)$ is $\sum_{t\in T_4}\delta^{T-t}c$. The penalty for choosing any other point on ∂j is the sum of these penalties. Whichever of the first two values is lower, it can be obtained by the strategy.

- For k : the choice in ∂k is constrained only by choices made in j and in $2k$ (ignoring k itself).

Let $T_5=\{t<T\mid(i_t,x_t)=(2k,(5,2))\}$, $T_6=\{t<T\mid(i_t,x_t)=(2k,(2,4.5))\}$, $T_7=\{t<T\mid(i_t,x_t)=(j,N(j))\}$.

The penalty for choosing $(1,2.25)$ is $\sum_{t\in T_5}\delta^{T-t}c$. The penalty for choosing $(2.5,1)$ is $\sum_{t\in T_6}\delta^{T-t}c+\sum_{t\in T_7}\delta^{T-t}d$. The penalty for choosing any other point is at least $\sum_{t\in T_5\cup T_6}\delta^{T-t}c$. This is necessarily as least as great as the lowest of the first two values because for non-negative numbers x,y,z one always has $x+y\geq\min\{x,y+z\}$.

- For $3j$: the choice in ∂j is constrained only by choices made in i and in j (ignoring $3j$ itself).

Let $T_8 = \{t < T \mid (i_t, x_t) = (j, (2.5, 1))\}$, $T_9 = \{t < T \mid (i_t, x_t) = (j, N(j))\}$, $T_{10} = \{t < T \mid (i_t, x_t) = (i, N(i))\}$.

The penalty for choosing $N(3j)$ is $\sum_{t \in T_8} \delta^{T-t} c$. The penalty for choosing $(7.5, 3)$ is $\sum_{t \in T_9} \delta^{T-t} c + \sum_{t \in T_{10}} \delta^{T-t} d$. The penalty for choosing any other point on $\partial 3j$ is the sum of the previous two values, $\sum_{t \in T_8 \cup T_9} \delta^{T-t} c + \sum_{t \in T_{10}} \delta^{T-t} d$.

- For $2k$: the choice in $\partial 2k$ is constrained only by choices made in k and in n (ignoring $2k$ itself).

Let $T_{12} = \{t < T \mid (i_t, x_t) = (k, (1, 2.25))\}$, $T_{13} = \{t < T \mid (i_t, x_t) = (k, (2.5, 1))\}$, $T_{14} = \{t < T \mid (i_t, x_t) = (n, (5, 2))\}$.

The penalty for choosing $(5, 2)$ is $\sum_{t \in T_{12}} \delta^{T-t} c$. The penalty for choosing $(2, 4.5)$ is $\sum_{t \in T_{13}} \delta^{T-t} c + \sum_{t \in T_{14}} \delta^{T-t} d$. Any other point on $\partial 2k$ incurs $\sum_{t \in T_{12} \cup T_{13}} \delta^{T-t} c$ at least, which is not less than the lowest of the first two values.

- The cases of $l, m, 3l, 2m$ are similar to, respectively, $j, k, 3j, 2k$.

- For n : the choice in ∂n is constrained only by choices made in $2k$ and in $2m$ (ignoring n itself).

Let $T_{15} = \{t < T \mid (i_t, x_t) = (2k, (2, 4.5))\}$, $T_{16} = \{t < T \mid (i_t, x_t) = (2m, (4.5, 2))\}$.

The penalty for choosing $(5, 2)$ is $\sum_{t \in T_{15}} \delta^{T-t} d$. The penalty for choosing $(2, 5.5)$ is $\sum_{t \in T_{16}} \delta^{T-t} d$. Any other point on ∂n incurs at least one of the two penalties. This concludes the proof. ■

Proof of Prop. 4. If: Let φ be any solution on D satisfying the axioms WP, Sym, ScInv and IMon. Let $i \in D$. By Condition ($\mathrm{C_{KS}}$) there is a special chain $j_1, \ldots j_n$ beginning at i. By Sym, $\varphi(j_n) = KS(j_n)$ and one can roll back along the special chain to i, and at each step $\varphi(j_k) = KS(j_k)$ either by IMon (case (i)) or by ScInv (case (ii)). For $k = 1$, we have $\varphi(i) = KS(i)$. It follows that $\varphi = KS$ on D.

Only if: 1. For all solutions φ and all problems $k, i \in D$ such that either $k \subseteq i$ or $k \supseteq i$, let $C_\varphi(k; i)$ denote the constraint imposed on $\varphi(k)$ by $\varphi(i)$, IMon and WP, i.e., $C_\varphi(k; i)$ is the subset of ∂k such that IMon is not violated if $\varphi(k) \in C_\varphi(k; i)$, given $\varphi(i)$.

Specifically, there are eight cases:

(i) $I(k) = I(i)$ and $\varphi(i) \in \partial k$. Then $C_\varphi(k;i) = \{\varphi(i)\}$.

(ii) $I(k) = I(i)$ and $\varphi(i) \in k \setminus \partial k$. Then $C_\varphi(k;i) = \{x \in \partial k \mid x \geq \varphi(i)\}$.

(iii) $I(k) = I(i)$ and $\varphi(i) \notin k$. Then $C_\varphi(k;i) = \{x \in \partial k \mid x \leq \varphi(i)\}$.

(iv) $I_1(k) = I_1(i)$ and $I_2(k) < I_2(i)$. Then $C_\varphi(k;i) = \{x \in \partial k \mid x_2 \leq \varphi_2(i)\}$.

(v) $I_1(k) = I_1(i)$ and $I_2(k) > I_2(i)$. Then $C_\varphi(k;i) = \{x \in \partial k \mid x_2 \geq \varphi_2(i)\}$.

(vi) $I_1(k) < I_1(i)$ and $I_2(k) = I_2(i)$. Then $C_\varphi(k;i) = \{x \in \partial k \mid x_1 \leq \varphi_1(i)\}$.

(vii) $I_1(k) > I_1(i)$ and $I_2(k) = I_2(i)$. Then $C_\varphi(k;i) = \{x \in \partial k \mid x_1 \geq \varphi_1(i)\}$.

(viii) $I_1(k) \neq I_1(i)$ and $I_2(k) \neq I_2(i)$. Then $C_\varphi(k;i) = \partial k$.

Note that if φ satisfies WP and $\varphi(k) \in C_\varphi(k;i)$, necessarily $\varphi(i) \in C_\varphi(i;k)$. This can be checked for each case:

(i) This also corresponds to case (i) for i : $\varphi(k) = \varphi(i)$ and $C_\varphi(i;k) = C_\varphi(k;i)$.

(ii) This corresponds to case (iii) for i : As $\varphi(i) \in k \setminus \partial k$, $\varphi(k) > \varphi(i)$, $\varphi(k) \notin i$ and $C_\varphi(i;k) = \{x \in \partial i \mid x \leq \varphi(k)\}$.

(iii) This corresponds to case (ii) for i : As $\varphi(i) \notin k$, $\varphi(k) < \varphi(i)$, $\varphi(k) \in i \setminus \partial i$ and $C_\varphi(i;k) = \{x \in \partial i \mid x \geq \varphi(k)\}$.

(iv) This corresponds to case (v) for i : $\varphi_2(k) \leq \varphi_2(i)$ and $C_\varphi(i;k) = \{x \in \partial i \mid x_2 \geq \varphi_2(k)\}$.

(v) This corresponds to case (iv) for i : $\varphi_2(k) \geq \varphi_2(i)$ and $C_\varphi(i;k) = \{x \in \partial i \mid x_2 \leq \varphi_2(k)\}$.

(vi),(vii) are treated similarly.

(viii): This also corresponds to case (viii) for i and $C_\varphi(i;k) = \partial i$.

2. Let D_+ denote the subset of D containing the problems i at which a special chain begins. We must show that $D_+ = D$ if the Kalai-Smorodinsky theorem holds on D. Suppose that $D \setminus D_+$ is not empty. For every $k \in D \setminus D_+$:

1) k is not symmetric because symmetric problems are in D_+;

2) k is not a rescaling of a problem $i \in D_+$ because otherwise $k, i, ...$ starts a special chain

beginning at k, implying that $k \in D_+$;

3) k can be related by IMon to a problem $i \in D_+$. But necessarily, either $I(k) \neq I(i)$ or $KS(i) \notin \partial k$; otherwise, a special chain beginning at k can be formed and thus $k \in D_+$.

The set $D \setminus D_+$ can be partitioned into the equivalence classes of the equivalence relation "is a rescaling of". Say that two equivalence classes E, E' are "linked" if there is $i \in E$, $i' \in E'$ such that $I(i) = I(i')$, $KS(i) = KS(i')$, and either $i \subseteq i'$ or $i \supseteq i'$. The relation "is linked to" is not transitive in general because it may happen that, for a particular triple E, E', E'', E and E' are linked via two problems i, i', E' and E'' are linked via j', j'', and E and E'' are not linked. We will also be interested in its transitive closure, called "directly or indirectly linked to".

3. Pick one particular equivalence class E^* (for the rescaling relation) and all the equivalence classes that are directly or indirectly linked to it. Call the union of these classes D^*. This is a subset of $D \setminus D_+$ (not necessarily a proper subset).

Pick a member of E^*, i^*. E^* and i^* will play a special role in the rest of the proof. If $D^* \neq E^*$, let E be any other equivalence class in D^*. Consider first the case in which the equivalence classes E and E^* are linked by two problems $i \in E$, $j^* \in E^*$. Therefore there exists (not necessarily in E) a rescaling of i, denoted i_E, such that $I(i^*) = I(i_E)$, $KS(i^*) = KS(i_E)$, and either $i^* \subseteq i_E$ or $i^* \supseteq i_E$. Consider now the case in which E and E^* are only indirectly linked (which means that they are not linked but are directly or indirectly linked). One can then pick an arbitrary i in E and construct a rescaling of i, denoted i_E, such that $I(i^*) = I(i_E)$. (In this case there is no guarantee that $KS(i^*) = KS(i_E)$, $i^* \subseteq i_E$ or $i^* \supseteq i_E$.)

For every equivalence class E in D^*, one can construct such a i_E following the approach described for each of the two cases in the previous paragraph. The problems i_E may or may not belong to D^*. Let D^{**} be the (possibly empty) subset of these problems i_E that do not belong to D^*. Note that $D^{**} \cap D_+ = \varnothing$, because if one had $i_E \in D_+$, any $k \in E$, being a rescaling of i_E, would then be the beginning of a special chain, contradicting the fact that $E \cap D_+ = \varnothing$.

4. Consider any two problems $k \in D^*$, $i \in D \setminus D^*$. Suppose that k is related to i by IMon, which implies that $k \subseteq i$ or $k \supseteq i$. Necessarily, either $I(k) \neq I(i)$ or $KS(i) \notin \partial k$, as we now show. First, suppose that $i \in D_+$. Then $I(k) = I(i)$ and $KS(i) \in \partial k$ would imply that $k \in D_+$, a contradiction. Second, suppose that $i \in D \setminus D_+$. In this case, $I(k) = I(i)$ and $KS(i) \in \partial k$ would imply that the equivalence classes of k and i are linked, contradicting the fact that $k \in D^*$ and $i \in D \setminus D^*$.

We now derive consequences from the fact that either $I(k) \neq I(i)$ or $KS(i) \notin \partial k$ whenever $k \in D^*$ and $i \in D \setminus D^*$ are related by IMon. Consider first the case $I(k) \neq I(i)$. Four subcases are possible: $C_{KS}(k;i) = \{x \in \partial k \mid x_2 \leq KS_2(i)\}$, $C_{KS}(k;i) = \{x \in \partial k \mid x_2 \geq KS_2(i)\}$, $C_{KS}(k;i) = \{x \in \partial k \mid x_1 \leq KS_1(i)\}$, $C_{KS}(k;i) = \{x \in \partial k \mid x_1 \geq KS_1(i)\}$. Focus on the first subcase, the other subcases being similar. Because this first subcase corresponds to $k \subseteq i$, $I_1(k) = I_1(i)$ and $I_2(k) < I_2(i)$, one then has $KS_2(k)/KS_1(k) < KS_2(i)/KS_1(i)$. As $k \subseteq i$, the point $\hat{x} \in \partial k$ such that $\hat{x}_2 = KS_2(i)$ (which belongs to $C_{KS}(k;i)$) satisfies $\hat{x}_1 \leq KS_1(i)$ and thus $\hat{x}_2/\hat{x}_1 \geq KS_2(i)/KS_1(i)$. Therefore $KS(k)$, which is obviously an element of $C_{KS}(k;i)$, satisfies $KS_2(k)/KS_1(k) < \hat{x}_2/\hat{x}_1$ and is not an extreme point of $C_{KS}(k;i)$. Consider the second case, $KS(i) \notin \partial k$. As the subcase in which $I(k) \neq I(i)$ has already been examined, we can focus on the subcase in which $I(k) = I(i)$. One then has either $KS(k) \gg KS(i)$ or $KS(k) \ll KS(i)$ and again $KS(k)$ is not an extreme point of $C_{KS}(k;i)$.

5. For every $k \in D^*$, let $C(k) = \bigcap_{i \in D \setminus D^*} C_{KS}(k;i)$. This set contains $KS(k)$ and $KS(k)$ is not an extreme point of it, because these two properties are satisfied by each of the $C_{KS}(k;i)$, of which there is a finite number.

Take any $k^* \in D^*$. Let E be its equivalence class. For every $k \in E$, there is $\alpha \in \mathbb{R}^2_{++}$ such that k^* is an α-rescaling of k. Let $C|_{k^*}(k)$ denote the α-rescaling of $C(k)$. This is a subset of ∂k^*. Note that $KS(k^*)$ is an α-rescaling of $KS(k)$. As $KS(k)$ is a non-extreme element of $C(k)$, then $KS(k^*)$ is a non-extreme element of $C|_{k^*}(k)$. Therefore the subset

$\bigcap_{k \in E} C|_{k^*}(k)$ contains $KS(k^*)$ and $KS(k^*)$ is not an extreme point of this subset. Let $C^*(k^*)$ denote this subset. Note that when k is a rescaling of k', $C^*(k)$ is a rescaling of $C^*(k')$.

Take any $i_E \in D^{**}$, where E is an equivalence class in D^*. Recall that $i_E \notin D^*$ but i_E is a rescaling of each member of E. The subset $\bigcap_{k \in E} C|_{i_E}(k)$ contains $KS(i_E)$ and $KS(i_E)$ is not an extreme point of this subset. Let $C^*(i_E)$ denote this subset. Note that for all $k \in E$, $C^*(i_E)$ is a rescaling of $C^*(k)$.

6. Now let us look again at the particular i^* and all the i_E (that may belong to D^* or D^{**}) that were introduced in step 3. One has $I(i^*) = I(i_E)$ and therefore $KS_2(i^*)/KS_1(i^*) = KS_2(i_E)/KS_1(i_E)$ for all E in D^*, while $KS(i^*)$ is not an extreme point of $C^*(i^*)$ just as $KS(i_E)$ is not an extreme point of $C^*(i_E)$. Therefore there is $\mu \in \mathbb{R}^2_{++}$, $\mu \neq KS_2(i^*)/KS_1(i^*)$ such that the point $x \in \partial i^*$ such that $x_2/x_1 = \mu$ belongs to $C^*(i^*)$ and for all E in D^*, the point $x \in \partial i_E$ such that $x_2/x_1 = \mu$ belongs to $C^*(i_E)$.

Now we are ready to define a solution φ as follows. On $D \setminus D^*$, it coincides with KS. For $k \in D^*$, there is $\alpha \in \mathbb{R}^2_{++}$ and E in D^* such that k is an α-rescaling of i_E (or i^*); then $\varphi(k)$ is the point $x \in \partial k$ such that $x_2/x_1 = (\alpha_2/\alpha_1)\mu$. Note that, as $C^*(k)$ is a α-rescaling of $C^*(i_E)$ (or of $C^*(i^*)$), this implies that $\varphi(k) \in C^*(k)$.

7. It is obvious that φ satisfies WP, Sym and ScInv. It also obviously satisfies IMon on $D \setminus D^*$.

Consider two problems $k \in D^*$, $i \in D \setminus D^*$ that are related by IMon. By construction, $\varphi(k) \in C^*(k) \subset C(k) \subset C_{KS}(k; i) = C_\varphi(k; i)$, where the last equality is due to $\varphi(i) = KS(i)$. And conversely this implies $\varphi(i) \in C_\varphi(i; k)$.

Consider two problems $i, k \in D^*$ that are related by IMon. First case: If they belong to the same equivalence class, IMon is satisfied because it is implied by ScInv in this case. Second case: Suppose i is a rescaling of i_E, k a rescaling of $i_{E'}$ (one problem among $i_E, i_{E'}$ may be i^*). One has $\varphi_2(i_E)/\varphi_1(i_E) = \varphi_2(i_{E'})/\varphi_1(i_{E'}) = \mu$. Without loss of generality, suppose that $i \subseteq k$ and $I_1(i) = I_1(k)$. If $I_2(i) = I_2(k)$, then i and k are

α-rescalings of i_E and $i_{E'}$, respectively, for the same α (recall that $I(i_E) = I(i_{E'}) = I(i^*)$). Then $\varphi_2(i)/\varphi_1(i) = \varphi_2(k)/\varphi_1(k) = (\alpha_2/\alpha_1)\mu$ and IMon is satisfied. If $I_2(i) < I_2(k)$, then $\varphi_2(i)/\varphi_1(i) = (\alpha_2/\alpha_1)\mu$ and $\varphi_2(k)/\varphi_1(k) = (\alpha_2'/\alpha_1)\mu$ for some $\alpha_1, \alpha_2, \alpha_2'$ such that $\alpha_2 < \alpha_2'$. IMon would be violated if one had $\varphi_2(k) < \varphi_2(i)$. This inequality is equivalent to

$$\varphi_2(k) = \varphi_1(k)(\alpha_2'/\alpha_1)\mu < \varphi_1(i)(\alpha_2/\alpha_1)\mu = \varphi_2(i),$$

$$\varphi_1(k)(\alpha_2'/\alpha_2) < \varphi_1(i).$$

One would then obtain $\varphi(i) \gg \varphi(k)$, contradicting the fact that $i \subseteq k$ and that φ satisfies WP. Therefore IMon is satisfied.

The solution φ coincides with KS only on $D \setminus D^*$, which shows that if $D \setminus D_+$ is not empty, the Kalai-Smorodinsky theorem does not hold on D. This achieves the proof of the "only if" part of the proposition. ■