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Which networks permit stable allocations? A theory of network-based comparisons

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Economic agents care about their relative well-being, and the comparisons are usually local. We capture this using a network model, in which an agent's payoff depends on the ranking of their allocation among their network neighbors. Given a network, an allocation is called α-stable if no blocking coalition whose size is an α fraction of the population can strictly improve their payoffs. We find a sufficient and necessary condition for a network to permit an α-stable allocation: the network has an independent set whose size is at least $1-\alpha$ of the network population. The characterization of permissive networks holds not only for our baseline ranking preference but also for a range of preferences under which the sets of stable allocations are expanded. We also provide a sufficient condition for an allocation to be stable. Extensions of the model concern directed networks and the case where agents have limited enforcement power.

Keywords. Network, social ranking, relative comparison, independent set, stable allocations.

JEL classification. D85, D91, D72, C71.

1. Introduction

Social comparisons influence our subjective well-being. Compelling evidence shows that people's perceptions of happiness can deteriorate when their *relative* well-being declines, even if their absolute income increases (Easterlin (1995), Tideman, Frijters, and Shields (2008)). Recent evidence further shows that individuals' comparisons are *network-based*. That is, they do not compare themselves with the entire population but only with their neighbors in certain networks, for example, friends, coworkers, and classmates (e.g., Luttmer (2005)). Therefore, it matters with whom people are connected.

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This paper asks the following question: When agents care about their rankings within a social network, what network structures permit stable allocations? Specifically, we consider a model in which agents care about their rankings among their neighbors within a given social network. An allocation profile assigns each agent resources, which could be wealth, income, seniority, office space, or anything these agents may care about. Agents compare their allocations to those of their neighbors within their network. An allocation profile is α-*stable* if no alternative allocations exist under which at least an α fraction of the population could strictly improve their rankings. We say a network *permits* α-stable allocations, or the network is α-*permissive*, if an α-stable allocation exists within that network.[1]

We find that the size of the largest independent set is the key factor that determines whether a network is α-permissive. An independent set of a network is a set of vertices such that no pairs of vertices in the set are connected to each other. Our main result shows that a network permits an α-stable allocation *if and only if* it has an independent set whose size is at least a $1-\alpha$ fraction of the population. In other words, the size of the maximal independent set allows us to fully determine whether a network can permit stable allocations.

Our characterization of permissive networks holds for a range of preferences, from the above mentioned baseline ranking preference to a strong ranking preference. These preferences are stronger than the baseline version in the sense that the sets of stable allocations expand under these preferences, resulting in weaker stability notions. The characterization of permissive networks, however, remains the same. We also examine preferences that fall outside of the range and show that our previous characterization of permissive networks holds for those preferences. One such example is the case in which agents care about both their absolute allocations and their rankings. Another example is the case where agents may prefer having moderate advantages over their neighbors but not the level of inequality that occurs when they are leading by too much. We show that our previous characterization of permissive networks also applies to these cases.

We then turn to a characterization of stable allocations. The key idea is to examine the rankings induced by an allocation, instead of the allocation itself. We provide a sufficient condition for an allocation to be stable by comparing the sum of the rankings to the number of links for each potential blocking coalition. This makes it easy to construct and verify stable allocations in any network. We provide several examples of both stable and unstable allocations to illustrate the above ideas.

We extend our model in the following directions. We first examine directed networks where links are not necessarily reciprocal. They include important applications in social media networks such as Twitter. We derive a necessary and sufficient condition for permissive networks in the above case as in the baseline model. We also study the situation in which agents have limited enforcement power, that is, they can only redistribute the resources of their own coalitions. We show that when agents care about their rankings and have limited enforcement power, *all* networks are permissive. This stands as an interesting benchmark for our previous analysis.

[1]We omit α in front of *stable* or *permissive* when there is no confusion of their meaning.

Related literature Economists have long recognized the importance of relative comparison. Classical discussion goes back to Veblen's study on conspicuous consumption (Veblen (1899)). Postlewaite (1998) discusses the socioeconomic and evolutionary basis for why people care about rankings. Rayo and Becker (2007) show that evolution favors agents caring about their relative rather than absolute success under certain physical constraints. Tideman, Frijters, and Shields (2008) use relative income concerns to explain the Easterlin puzzle: the observation that average happiness has remained constant over time, despite sharp increases in GNP per capita. Genicot and Ray (2020) provide an excellent survey on recent developments in research on social aspirations and their impacts on social behavior.

Further evidence implies that people's comparisons are usually local. That is, they compare themselves to certain reference groups: their friends, neighbors, colleagues, and so on, rather than to an entire population (See, for instance, Festinger (1954), Frank (1985a, 1985b), Van de Stadt, Kapteyn, and Van de Geer (1985), Luttmer (2005), and Kuhn, Kooreman, Soetevent, and Kapteyn (2011)).

Motivated by such findings, a few theoretical papers study the impact of local comparisons. Ghiglino and Goyal (2010), for example, examine what happens to equilibrium prices, resource allocations, and welfare when individuals' utilities depend negatively on their neighbors' consumption. Immorlica, Kranton, Manea, and Stoddard (2017) also study local comparisons, with a focus on agents who are "upward-looking," that is, they assume that individuals are only affected by neighbors whose consumption is higher than theirs and explore how such comparisons affect the agents' costly actions. Baumann (2018) and Bloch and Olckers (2018) study what mechanisms can induce agents to truthfully report their neighbors' rankings. Although the above papers and ours all study local comparisons based on social network settings, our focus is different: they are interested in either market equilibrium or specific strategic decisions such as choosing levels of consumption, whereas we focus on stable allocations within social networks.

Our paper is closely related to the literature that studies stable allocations within social networks. Among these studies, Demange (2004) and Kets, Iyengar, Sethi, and Bowles (2011) are the most related. In these papers, networks capture the structure of communication and/or coordination, whereas in ours, networks shape preferences via defining agents' reference groups. Specifically, Demange (2004) finds that hierarchical structures avoid instability by optimally allocating blocking power to certain subgroups, where blocking power allows a subgroup to jointly block default allocations. Kets et al. (2011) study the relationship between network structures and sustainable levels of inequality.

Finally, our α-stability notion is different from the stability concept introduced in the network-formation literature (see Demange and Wooders (2005), Jackson (2005, 2008b), Goyal (2007), and Mauleon and Vannetelbosch (2015) for useful surveys). Rather, our α-stability notion is related to the core concept of cooperative game theory (e.g., Aumann and Dreze (1974) and Myerson (1977)). The key differences between our paper and the previous literature are in how we define blocking coalitions and that we study comparison-based preferences.

The rest of this paper is organized as follows. Section 2 sets up the model. Section 3 presents the main results: we first provide a necessary and sufficient condition for permissive networks, and then a sufficient condition for allocations to be stable. Section 4 extends the model to directed networks and the case where agents have limited enforcement power. Section 5 concludes.

2. Model

2.1 *Baseline setup*

Consider a finite set $N = \{1, \dots, n\}$ of agents i and j, who are members of an undirected network $g \in \{0, 1\}^{n\times n}$, such that $g_{ij} = g_{ji}$, and $g_{ii} = 0$.[2] Agents i and j are neighbors if $g_{ij} = g_{ji} = 1$, and $N_i(g) = \{j \in N \mid g_{ij} = 1\}$ is the set of i's neighbors.[3]

An allocation $w \in \mathbb{R}^n$ allocates each agent i a level of resources w_i, which could represent wealth, income, seniority, opportunity, or anything of prestige or value these agents care about. To fix our ideas, throughout our paper we interpret w as income and let all possible income allocations be nonnegative and sum up to a fixed amount, that is, $w_i \geq 0$ and $\sum_i w_i = W \in \mathbb{W}$, in which W is the total allocation, which is normalized to $W = 1$ unless stated otherwise.[4]

Each agent i cares about their local ranking, that is, how w_i ranks among i's neighbors' income allocations $\{w_j \mid j \in N_i(g)\}$. Formally, we have the following definitions.

Definition 1 (Local ranking). Given a network g and an allocation w, i's local ranking is as follows:

$$r_i^g(w) \equiv \#\{j \in N_i(g) | w_j \geq w_i\}.$$

That is, r_i is the number of i's neighbors who have a (weakly) higher income than agent i. In particular, $r_i = 0$ implies that i receives an income that is the highest among its neighbors.

Agents' preferences regarding their allocations are defined by their local rankings: an agent i prefers allocation w' to w if i has a better local ranking under w'. In particular, we define our baseline ranking preference as follows.

Definition 2 (Baseline ranking preference). Given a network g, i's ranking preference is defined by

$$w' \succ_i^g w \quad \text{if and only if} \quad r_i^g(w') < r_i^g(w).$$

[2] We generalize the results to directed networks in Section 4.1.

[3] Comparisons with "neighbors" can be interpreted in various ways. For instance, we could have networks in which agents compare themselves only with their acquaintances, such as friends, colleagues, neighbors, and so on. Our model could also capture information networks in which each agent observes the allocations belonging to a subset of others and only compares themselves with those whom they can observe. More generally, different agents can compare with different reference groups and a network can simply be a modeling device that captures who compares with which group.

[4] The assumptions on the allocations are not needed for most parts of this paper because agents' preferences are defined in relative terms. However, we will need these assumptions when we discuss alternative preferences, for example, when agents care about both their rankings and their absolute allocations.

In a later part of this section, we will introduce a range of preferences to which our main results hold.[5] Below, we define the notion of stable allocations.

Definition 3 (α-stable allocation). An income allocation w is α-*stable* under preference $\succ^g$ if no alternative w' exists, such that $w' \succ_i^g w$ for strictly more than $\alpha \times n$ agents.

That is, an allocation, w, is α-stable if other agents in the network cannot block it for an alternative allocation under α-majority voting.

Here, we note that we put no restrictions on the alternative allocations. However, there could be other interesting cases: for instance, when agents have limited enforcement power, coalitions can only redistribute their own resources, resulting in a restriction on the set of feasible alternative allocations. We discuss such a case in Section 4.2.

The following example helps illustrate the above definitions.

Example 1 ((Non)stable allocations). Consider a network g as depicted in Figure 1. The left-hand panel represents the agents' initial allocations (numbers outside the brackets) and their corresponding local rankings (numbers in brackets).

Consider an alternative allocation that swaps the bottom two agents' allocations while the other three agents' allocations are kept unchanged.[6]

How are agents' local rankings changed under the alternative? The three agents that are represented by the shaded circles strictly improve their local rankings under the alternative, and hence, strictly prefer w' to w. Therefore, in the case where $\alpha = \frac{1}{2}$, w is not stable on g, that is, when simple majority voting is employed to replace the default allocation. ◊

We see in the above example that the initial allocation is not stable because enough agents' rankings can be improved under the alternative. Obviously, the network structure matters: it shapes agents' comparison groups, and hence, the stability of an allocation. Some networks permit stable allocations whereas others may not. Our next definition concerns whether a network is "permissive."

Definition 4 (α-permissive networks). A network g permits an α-stable allocation, or is α-permissive, if there exists an α-stable allocation under $\succ^g$.

Example 2 (Stylized $\frac{1}{2}$-permissive networks). Consider $\alpha = \frac{1}{2}$, that is, simple majority voting is employed to block an initial allocation.

[5]We note two things here. First, in the aforementioned preference, agents care only about their local rankings. We allow for a range of alternative preferences, which we introduce later. Second, according to Definitions 1 and 2, agent i would prefer being the only one with income w_i to being tied with agent j. This applies to the cases where rankings imply the (re)distribution of important resources such as power, opportunity, market share, etc. We discuss the relaxing of this assumption in Section 3.1.

[6]Note that the alternative allocation need not necessarily be a direct redistribution of income. For instance, suppose a pathway is to be built and it would make the life of the bottom-right agent more convenient but its construction would harm the bottom-left agent.

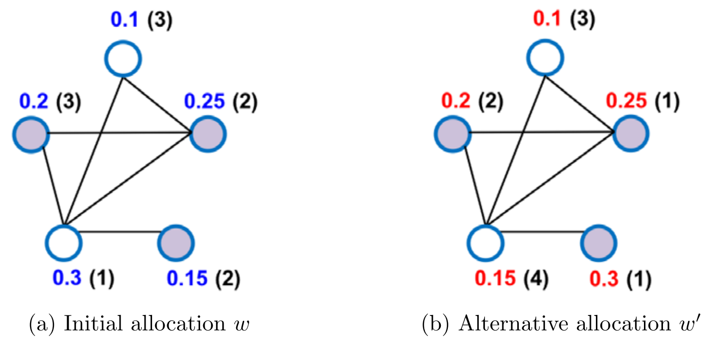

FIGURE 1. An example of a network in which the initial allocation is not α-stable when $\alpha = \frac{1}{2}$. The two graphs present the initial (left-hand panel) and alternative (right-hand panel) allocations, respectively, with local rankings shown in brackets.

- Bipartite networks are $\frac{1}{2}$-permissive.[7]
- Complete networks or rings with an odd-numbered population n are not $\frac{1}{2}$-permissive.

$\Diamond$

2.2 *A range of preferences*

Our baseline ranking preference assumes that one's preference solely depends on their ranking. One may view this as too extreme. Therefore, we relax this assumption and discuss a range of preferences. We show that our main results, which characterize permissive networks and stable allocations, hold for all of these preferences. The range of preferences to which our results hold is specified by two extreme preferences: the baseline ranking preference on one end and a strong ranking preference on the other, the latter of which we introduce below.

DEFINITION 5 (Strong ranking reference). Given a network g and sensitivity threshold $e \geq 0$, the *strong ranking preference* of i, denoted by $\succ_i^{S(e),g}$, is such that $w' \succ_i^{S(e),g} w$, if and only if:

(i) $\forall j \in N_i$: $w'_i - w'_j \geq w_i - w_j$; and

(ii) $\exists j \in N_i$: $w'_i - w'_j > e$, and $w_i - w_j \leq 0$.

That is, an agent i strictly prefers an alternative allocation if and only if (i) the income gaps between them and their neighbors improve, and (ii) there exists at least one

[7]A network g is a bipartite network if its nodes can be divided into two sets, X and Y, such that only connections between two nodes in different sets are allowed. A complete network is one where every pair of agents is linked. A ring network is one where all agents are connected and each agent connects to exactly two other agents.

neighbor behind whom agent i had fallen in the previous allocation but is now ahead of in the alternative allocation (by some sensitivity threshold $e \geq 0$).[8]

We will allow for any preference that is *between* $\succ$ and $\succ^S$. To do so, we formalize the relationship between the different preferences.

Definition 6 (Relationship between preferences). Consider two strict preference relations, $\succ_i^a$ and $\succ_i^b$, of some agent i on the allocation space $\mathbb{W}$. We say that $\succ_i^a$ *is stronger than* $\succ_i^b$ and $\succ_i^b$ *is weaker than* $\succ_i^a$ if

$$w' \succ_i^a w \quad \Longrightarrow \quad w' \succ_i^b w, \quad \forall w, w' \in \mathbb{W}.$$ [9]

For the preference profiles, we say that $\succ^a$ is stronger than $\succ^b$ and $\succ^b$ is weaker than $\succ^a$ if the relationship holds for every i.

Armed with Definition 6, we can define the preferences that are between the baseline preference $\succ$ and the strong ranking preference $\succ^{S(e)}$. Formally, we have the following.

Definition 7. A preference profile $\succ^c$ *is between preference profiles* $\succ^a$ *and* $\succ^b$ (suppose $\succ^b$ is stronger than $\succ^a$) if $\succ^c$ is stronger than $\succ^a$ and weaker than $\succ^b$.

For any network, g, it is easy to verify that the strong ranking preference $\succ^{S(e),g}$ is stronger than the baseline ranking preference $\succ^g$ (Definition 2).

As the above definitions imply, it is more demanding to block an allocation when the agents' preferences are stronger than the baseline version. As a result, these stronger preferences allow for expanded sets of stable allocations. And we summarize the above observation as the following lemma.

Lemma 1 (Expanded sets of stable allocations under stronger preferences). *Consider two preference profiles, $\succ^a$ and $\succ^b$, such that $\succ^a$ is stronger than $\succ^b$. Then for any allocation $w \in \mathbb{W}$ and $\alpha \in (0, 1)$, w is α-stable under $\succ^a \implies w$ is α-stable under $\succ^b$.*

Although the sets of stable allocations may differ under different preferences, we will show that the same characterization of permissive networks holds for any preferences that are between the baseline ranking preference and the strong ranking preference.

[8] We assume that the total allocation is large enough relative to the threshold e; in particular, $W > W(n, e) = n(n-1)e$. We remark that this is an upper bound of the required level of total wealth, W, and that this bound works uniformly for all α. This is not necessarily a tight bound and the required total wealth level could be further reduced. Actually, it will be clear from the proof of Theorem 1 that a tighter upper bound is in the order of $W(n, e) \sim O(ne)$ for any fixed $\alpha > 0$.

[9] In the literature, this relationship is seen as $\succ_i^b$ being an extension of $\succ_i^a$, emphasizing the fact that $\succ_i^b$ makes more comparisons than $\succ_i^a$. We thank the editor for pointing this out. We use the term "stronger" and "weaker" because it is more intuitive to refer to the comparative forms, such as "the strongest preference," and to discuss them according to which preferences result in more stable allocations, etc.

3. Main results

For our results, we first provide a sufficient and necessary condition that characterizes permissive networks. We show that having a large enough independent set is not only sufficient but also necessary for a network to be permissive. We provide several examples of preferences that are between $\succ$ and $\succ^S$ and also discuss some interesting exceptions. Lastly, we provide a sufficient condition for allocations to be stable.

3.1 *A sufficient and necessary condition for permissive networks*

We begin with a definition that is essential for characterizing permissive networks.

Definition 8 (Independent set). A subset of agents $M \subseteq N$ is an independent set of network g if $g_{ij} = 0, \forall i, j \in M$.

In our context, an independent set of agents is a subset wherein individual agents do not compare themselves with each other.

Theorem 1 (Permissive networks). *For any network g and any preference profile $\succsim$ that is between the baseline ranking preference, $\succ^g$, and the strong ranking preference, $\succ^{S(e),g}$, there exists an α-stable allocation if and only if g has an independent set M whose size is at least $(1-\alpha)n$.*

Proof of Theorem 1. It follows from Lemma 1 that if an allocation is α-stable under a weaker preference, then it is also α-stable under a stronger preference. We will prove this theorem in two steps. In step 1, we show the "if" part of the theorem for the baseline ranking preference, $\succ^g$, which is the weakest one in the range of preferences we specified in Section 2.2. In step 2, we show the "only if" part for the strong ranking preference, $\succ^{S(e),g}$, which is the strongest preference in the range.

The "if" part, for $\succ^g$: This part of our theorem follows from a simple intuition: a large enough independent set allows for a large group of agents who do not compare themselves with each other. Consequently, it is easy to find a stable allocation: one can give more income to the agents in the independent set, M, than to any other agents. One such example is $w_i^* = \frac{1}{|M|}$ if $i \in M$ and $w_i^* = 0$ otherwise (Figure 2).[10] Under such an allocation, all of the agents in the independent set receive strictly higher allocations than their neighbors, and hence, have no room to further improve their rankings. The other agents in the network do not have enough votes to block the current allocations.

The "only if" part for $\succ^{S(e),g}$: Suppose there exists some allocation w that is α-stable under $\succ^{S(e),g}$. We aim to find a large enough independent set of g. We begin by assigning labels $\{l_i\}$ to the agents:

0. Initialize $l_i = null$ for all i.

1. Pick $i \in \arg\max_{j \text{ s.t. } l_j = null} w_j$, and let $l_i = 1$.

[10] There can be stable allocations of other forms. We provide a formal discussion in Section 3.2.

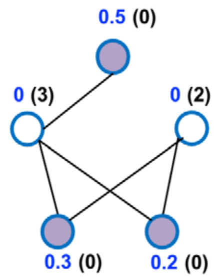

FIGURE 2. A $\frac{1}{2}$-stable allocation (with rankings in brackets). The network has a large enough independent set (whose size is $3 > n/2$, as represented by shaded circles).

2. Let $l_i = 0$ if $l_i = null$ and i is a neighbor of some j with $l_j = 1$.
3. Repeat steps 1 and 2 until all of the agents are labeled either 0 or 1.

Let $M_k = \{j \in N \mid l_j = k\}$, $k = 0, 1$, and we have $M_1 \cup M_0 = N$ and $M_1 \cap M_0 = \emptyset$. Consider an alternative allocation:

$$w'_i = \begin{cases} w_i + \dfrac{\displaystyle\sum_{i \in M_1} (w_i - \underline{w})}{|M_0|} & \text{if } i \in M_0, \\ \underline{w} & \text{if } i \in M_1. \end{cases} \tag{1}$$

That is, we collect the allocations from set M_1 and equally allocate these to the agents in M_0. We observe that the reallocated amount is large enough so that each agent in M_0 receives an extra amount that is larger than e.[11]

By construction, $w' \succ_i^{S(e),g} w, \forall i \in M_0$: under w', everyone in M_0 receives a strictly higher allocation, by more than e, than everyone in M_1; whereas under w, every $i \in M_0$ has a neighbor $j \in M_1$ such that $w_i \leq w_j$. In addition, the allocation differences among the agents in M_0 remain unchanged.

Therefore, the fact that w is α-stable on g implies $|M_0| \leq \alpha n$, or $|M_1| \geq (1 - \alpha)n$. Finally, M_1 is an independent set of g: for any $i \in M_1$, all i's neighbors are labeled 0. Hence, any pair of agents in M_1 is not linked to each other.

We have constructed M_1, an independent set of g, and $|M_1| \geq (1 - \alpha)n$. This concludes the proof of the "only if" part of the theorem. □

Intuition Theorem 1 provides a clean characterization of permissive networks. The sufficient part follows from the observation that when a network of n agents has a large enough independent set (of size at least $(1 - \alpha)n$), an allocation is stable if everyone in this independent set has the highest local ranking. Such an allocation is possible, for example, when we divide all of the resources equally among the agents in the independent set and leave nothing for the remaining agents. In this case, the agents in the

[11] To see this, recall from the first step of the algorithm that one agent with the highest w_i is in M_1; let that agent be agent 1. Then $\sum_{j \in M_1} w_j \geq w_1 \geq W/n > (n-1)e$, so $w_i \geq \frac{\sum_{j \in M_1} w_j}{|M_0|} > e, \forall i \in M_0$.

independent set have no incentive to support any alternative allocations because they have already achieved their best ranking and the rest of the agents do not have enough votes to block the current allocation.[12]

The other direction is more surprising: any permissive network must have a large enough independent set. The construction of the proof for this part of the theorem is in line with an observation that when people are thinking about disrupting a status quo or staging a revolution, they often start by redistributing the resources of the wealthiest. In particular, the intuition of the algorithm is to start from the wealthiest, then identify whom among their neighbors includes those who could benefit (improve their rankings) from a redistribution of the allocations of the the wealthiest agents in this set. Then we pick the wealthiest from the remaining agents and redistribute their resources, and so on. In doing so, the wealthiest agents identified in each step will not be connected and, as a result, they form an independent set. Lastly, this set of "wealthiest agents" must be large enough to ensure that other people do not have enough votes to block the current allocation.

Examples of preferences in the range As we showed above, Theorem 1 holds for a range of preferences that are between the baseline ranking preference $\succ$ and the strong ranking preference $\succ^{S(e)}$. This range covers many interesting cases and we present two examples below.

The first example captures a situation where the agents are insensitive to small income differences. That is, agent i ranks themself above a neighbor only when their income is higher than that neighbor's, beyond some threshold e. This example is meant to capture people's inability to perceive small differences, which is well documented in the literature.[13]

Example 3 (Inability to perceive small differences). Consider agent i's preference, $\succ_i^{I(e,c),g}$, such that $w' \succ_i^{I(e,c),g} w$ if and only if $r_i^{I(e,c),g}(w') < r_i^{I(e,c),g}(w)$, in which the ranking score is given by $r_i^{I(e,c),g}(w) = \sum_{j \in N_i} r_{ij}(w)$, such that for some $c \in (0, 1)$,

$$r_{ij}(w) \equiv \begin{cases} 0 & \text{if } w_i - w_j > e, \\ c & \text{if } w_i - w_j \in (0, e], \\ 1 & \text{if } w_i - w_j \leq 0. \end{cases}$$

For every $c \in (0, 1)$ and $e > 0$, the preference $\succ_i^{I(e,c),g}$ is stronger than the baseline ranking preference $\succ_i$, and weaker than the strong ranking preference $\succ_i^{S(e)}$. $\diamondsuit$

For the second example, we note that Theorem 1 also allows the agents to have heterogenous preferences.

[12] We note here that the stable allocation mentioned above is only an extreme case. Having agents in the independent set rank the highest is not a necessary consequence of stable allocations. We discuss other examples in Section 3.2 where a characterization of stable allocations/rankings is provided.

[13] For instance, Rayo and Becker (2007) attribute the insensitivity to small differences as one of the key constraints that gives the local comparison utility function its evolutionary advantage.

Example 4 (Heterogenous preferences). Some agents may have weaker preferences while others hold stronger preferences. For instance, in Example 3, we could allow each agent to have a different e_i, and the larger the e_i, the stronger the preference. As long as every agent's preference is between the baseline ranking preference $\succ^g$ and the strong ranking preference $\succ^{S(e),g}$, then the condition for network g to permit stable allocations remains the same. $\Diamond$

Although the range of preferences between $\succ$ and $\succ^{S(e)}$ covers many important applications, there could be interesting cases in which agents' preferences are not in the range. We introduce two additional preferences that fall outside of the range between $\succ$ and $\succ^{S(e)}$ and show that our previous characterization of permissive networks applies to them as well.

Examples of preferences that are not in the range

Preference for both absolute allocation and ranking In our previous analysis, agents only care about their local rankings. Here, we explore a situation in which agents care about both their rankings and their absolute allocations.

Definition 9 (Preference with both absolute and relative considerations). Given a network, g, the (strict) preference relation of i is defined by

$$w' \succ_i^{A,g} w \quad \text{if and only if} \quad r_i^g(w') < r_i^g(w) \quad \& \quad w_i' > w_i,$$

in which r_i is i's local ranking as defined in Definition 1.

Therefore, an agent strictly prefers an alternative allocation to the initial one when their local ranking and absolute allocation would both strictly improve under the alternative. Recall that we already required that all of the allocations considered satisfy the same total amount, $\sum_i w_i = W$, and consequently the additional requirement of the absolute terms cannot be trivially met by adding some constant to everyone's allocation.

We now show that the characterization of permissive networks remains the same under the new preferences $\succ^A$ as introduced in Definition 9. The notion of permissiveness is defined according to these new definitions of the preferences and stable allocations.

Corollary 1 (Permissive networks with both absolute and relative considerations). *For any network, g, consider the corresponding preference $\succ^{A,g}$. There exists an α-stable allocation under $\succ^{A,g}$ if and only if g has an independent set whose size is at least $(1-\alpha)n$.*

The proof of the above result is implied by the proof of Theorem 1 and is presented in the Appendix.

Preference for moderate advantages Another example of a preference that is not between $\succ$ and $\succ^{S(e)}$ is when agents prefer only *moderate* advantages. In our baseline preference, agents dislike ties and their rankings improve if their allocations improve. One may call such a preference "antiegalitarian." In this subsection, we consider a modification in which agents may prefer a higher ranking when their income is below or close to their neighbors' but dislike the inequality that develops when they are ahead of their neighbors by a lot. We formalize the above in the following definition.

Definition 10 (Preference for moderate advantages). Given a network g, a threshold $e \geq 0$, and some $c \geq 0$, the (strict) preference relation of agent i, denoted by $w' \succ_i^{M(e,c),g} w$, is defined as follows:

$w' \succ_i^{M(e,c),g} w$ if and only if the ranking score $r_i^{M(e,c),g}(w') < r_i^{M(e,c),g}(w)$, where $r_i^{M(e,c),g}(w) \equiv \sum_{j \in N_i(g)} r_{ij}(w)$, such that

$$r_{ij}(w) \equiv \begin{cases} c & \text{if } w_i - w_j > e, \\ 0 & \text{if } w_i - w_j \in (0, e], \\ 1 & \text{if } w_i - w_j \leq 0. \end{cases}$$

That is, the agents still enjoy being ahead, yet they dislike the inequality that occurs when they are ahead by too much (higher than e); this latter part is reflected by the penalty $c \geq 0$ in the ranking score.

We show below that our characterization of the permissive network remains in this case.

Corollary 2 (Permissive networks when agents prefer moderate advantages). *For any network g and preference $\succ_i^{M(e,c),g}$ (with any $e, c \geq 0$), there exists an α-stable allocation under $\succ_i^{M(e,c),g}$ if and only if g has an independent set whose size is at least $(1-\alpha)n$.*

The proof of the above result follows from a simple modification of the proof of Theorem 1 and is presented in the Appendix.

3.2 *A sufficient condition for an allocation to be stable*

Theorem 1 answers our main question about which *networks* permit stable allocations. One may still wonder about the conditions for *allocations* to be stable. In this subsection, we address this issue and provide a sufficient condition for an allocation to be stable.

The characterization of stable allocations is based on the ranking profile induced by an allocation. Formally, recall that an agent's ranking induced by an allocation $w \in \mathbb{W}$, given the network, g, is as follows:

$$r_i^g(w) \equiv \#\{j \in N_i(g) | w_j \geq w_i\}.$$

In addition, for any ranking profile, r, let

$$N^{0,r} \equiv \{i \in N \mid r_i = 0\}$$

be the set of agents who are already ranked the highest among their neighbors.

Finally, given any network, g, and any subset of this network's agents, $M \subseteq N$, let $g|_M$ be the induced network that only contains links among M, that is, $(g|_M)_{ij} = 1$ if and only if $g_{ij} = 1$ and $i, j \in M$. Let

$$L(g|M) \equiv \left|\{i, j \in M \mid g_{ij} = 1, i < j\}\right|$$

be the total number of links among the agents in M.

Now we are ready to present a sufficient condition for stable allocations.

Proposition 1 (A sufficient condition for allocations to be α-stable). *Consider any network, g, a preference profile $\succsim$ stronger than $\succ^g$, and $\alpha \in (0, 1)$. An allocation $w \in \mathbb{W}$ is α-stable under $\succsim$ if the ranking profile induced by this allocation, $r^g(w)$, satisfies the following condition:*

$$\sum_{i \in M} r_i^g(w) - |M| < L(g|_M) \quad \forall M \subseteq N \setminus N^{0,r} \text{s.t.} |M| > \alpha n. \tag{2}$$

The above proposition says that if an allocation is unstable, the rankings induced by the allocation must violate the above inequality for some blocking coalition, M. Intuitively, the rankings under an unstable allocation must be high enough, relative to the total number of links in the blocking coalition, so that there is room for improvement.

Let us take a closer look at the above sufficient condition. $M \subseteq N \setminus N^{0,r}$ such that $|M| > \alpha n$ is the condition for a subset of agents, M, to be a potential blocking coalition: it contains enough agents, all of whom still have room to improve their rankings.

Then the sufficient condition follows from a key observation: each link ij induces a comparison between agents i and j and adds (at least) one ranking level to either r_i or r_j.[14] Therefore, for any ranking profile, $r^g(w')$, induced by any alternative allocation, w', we must have $\sum_{i \in M} r_i^g(w') \geq L(g|_M)$; which states that the sum of the rankings among M shall be at least the number of links (comparisons) among these agents.

At the same time, for all of the members in M to strictly improve their rankings (by at least one) under some alternative allocation, w', we must have $\sum_{i \in M} r_i^g(w') \leq \sum_{i \in M}(r_i^g(w) - 1) = \sum_{i \in M} r_i^g(w) - |M|$.

Putting all of this together, for M to block the current allocation, it must be that $\sum_{i \in M} r_i^g(w) - |M| \geq \sum_{i \in M} r_i^g(w') \geq L(g|_M)$, which violates the above inequality (2). In other words, the condition stated in the above proposition is a sufficient condition for an allocation to be stable. We note that Proposition 1 applies to all preferences considered in Theorem 1, because they are all stronger than the baseline preference $\succ^g$.

Examples of stable and unstable allocations (rankings) Proposition 1 provides an easy method to verify whether an allocation is stable, based on the ranking profile it induces. We discuss several examples below for illustration. We revisit the network in Figure 2, which has an independent set of size 3 (e.g., nodes a, c, and d form an independent set). Figure 3 presents three allocations (with induced rankings in brackets) that are $\frac{1}{2}$-stable. Subfigure 3(a) replicates the previous example shown in Figure 2. The allocations

[14] If $w_i = w_j$, then link ij adds one ranking level to both $r_i(w)$ and $r_j(w)$.

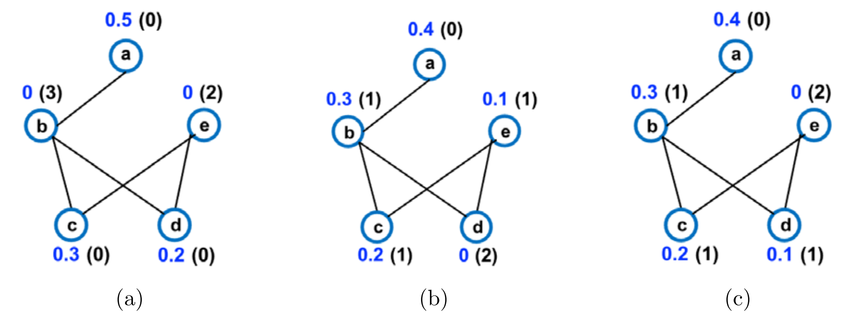

FIGURE 3. Examples of $\frac{1}{2}$-stable allocations (with rankings in brackets) in a network.

(rankings) shown in Subfigures 3(b) and 3(c) are also stable but are quite different from that in Subfigure 3(a). In particular, the agents (e.g., c and d) in the independent set need not rank the highest under stable allocations. This shows that the extreme nature of the allocations in Figure 3(a) is not a necessary consequence of α-stability.[15] Using Proposition 1, one can easily verify that these rankings are indeed $\frac{1}{2}$-stable.

Proposition 1 also helps identify unstable allocations (rankings) and potential blocking coalitions. Figure 4 presents three examples that are not $\frac{1}{2}$-stable under the baseline ranking preference $\succ$. In each example, there are at least three agents (represented by shaded circles) who can jointly improve their rankings. For instance, in Subfigure 4(a), the coalition $M = \{a, c, e\}$ violates Condition (2) and can jointly improve their rankings from $(1, 2, 1)$ to $(0, 1, 0)$.[16] The same exercise can be applied to the other two examples.

4. Extensions

In this section, we first extend the model to directed networks. We provide a sufficient and necessary condition for networks to be permissive that is similar to that of the case

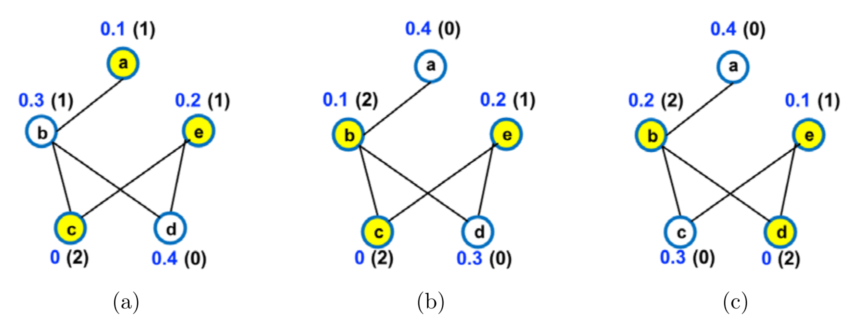

FIGURE 4. Examples of allocations (with rankings in brackets) that are not $\frac{1}{2}$-stable under the baseline ranking preference $\succ$. In each example, the three agents that are represented by shaded circles can jointly improve their rankings.

[15]We thank the referee for suggesting this comment.

[16]That is, to have agents a and e rank the highest locally, and agent c rank just below agent e. For instance, consider an alternative allocation $(w_a, w_b, w_c, w_d, w_e) = (0.4, 0.1, 0.2, 0, 0.3)$.

of undirected networks. We then discuss what happens when agents have limited enforcement power, that is, where coalitions can only reallocate their own resources. We show that in this case all networks are permissive.

4.1 *Directed networks*

Thus far, we have studied undirected networks. However, some situations are better captured by directed networks. For instance, people need not follow their followers on Twitter. Also, an agent's wealth is not observable to everyone, and the relationship where one observes the other's wealth need not be reciprocal. In such cases, when an agent, a, compares themself with another agent, b, agent b's comparison group need not include agent a.

In this section, we extend our model to directed networks and characterize a necessary and sufficient condition for permissive networks. Specifically, let $g_{ij} = 1$ if i compares their wealth to j's wealth, and 0 otherwise. A network, g, is directed such that $g_{ij} = 1$ does not imply $g_{ji} = 1$. Each agent i ranks themself among their (outgoing) neighbors $N_i^{\text{out}}(g) \equiv \{j \mid g_{ij} = 1\}$, that is, neighbors whose wealth i pays attention to; whereas $N_i^{\text{in}}(g) \equiv \{j \mid g_{ji} = 1\}$ represents the set of i's incoming neighbors, who compare themselves with i. The stability and permissiveness notions are defined as before, with the modified concept of ranking (based on outgoing neighbors).

Definition 11 (Path). A (directed) path in a network, g, is a sequence of links $i_1 i_2, \dots, i_{K-1} i_K$, such that $g_{i_k i_{k+1}}$ for each $k \in \{1, \dots, K-1\}$.

Definition 12 (Cycle). A (directed) cycle in a network, g, is a path, $i_1 i_2, \dots, i_{K-1} i_K$, such that $i_1 = i_K$. A network is acyclic if it contains no cycles.

Proposition 2 (α-permissiveness for directed networks). *A network, g, is α-permissive if and only if there exists a subset of agents, $M \subseteq N$, such that $|M| \geq (1-\alpha)n$ and $g|_M$ is acyclic.*

That $g|_M$ is acyclic implies that it has at least one "topological ordering"—an ordering of M into a sequence such that for every link, the starting agent of the link appears earlier in the sequence than the ending agent of the link.[17] To see the intuition behind the above result, consider an allocation that gives more income to agents in M than to those outside M, and higher income to the agents in M who were earlier in the sequence. This allocation is α-stable when M is large enough, as everyone in M ranks at the top of their (outgoing) neighborhood and there is no room to improve. This is an extreme example of a permissive network that is similar to Figure 2, which presents the undirected case.

[17] See Section 22.4 in Cormen, Leiserson, Rivest, and Stein (2009) for an example.

4.2 *Limited enforcement power*

So far, we have assumed that the "blocking coalition," that is, the set of agents that supports an alternative allocation, has unlimited enforcement power in the sense that no restriction is imposed on the alternative allocation, $w' \in \mathbb{W}$. While this assumption suits many situations, there are cases in which agents' enforcement power is limited. Here, we discuss what happens in one such case: a blocking coalition can only reallocate their own resources. Formally, our previous definition of stability (Definition 3) is replaced by the following.

Definition 13 (α-stable allocation with limited enforcement power). An allocation, w, is *α-stable* under some preference $\tilde{\succsim}$ if there exists no blocking coalition, $M \subseteq N$ s.t. $|M| > \alpha n$, and an alternative w' such that:

(i) $w' \tilde{\succsim}_i w$, $\forall i \in M$.

(ii) $w'_i = w_i$, $\forall i \notin M$, and $\sum_{i \in M} w'_i = \sum_{i \in M} w_i$.

The new requirement of limited enforcement power is captured by Condition (ii) in the above definition. We then have the following result: when agents have relative consideration and are with limited enforcement power, every network is α-permissive.

Proposition 3 (Limited enforcement power: every network is permissive). *Consider the new stability definition in which agents have limited enforcement power (Condition (ii)), then every network is permissive.*

In particular, for every $\alpha \in (0, 1)$ and network, g, there exists an allocation that is α-stable under $\succ^g$ (and any preference $\tilde{\succsim}$ stronger than $\succ^g$).

To prove this proposition, we construct an allocation that is stable regardless of the network structure.

Example 5 (An allocation that is α-stable in every network). Normalize the total endowment to $W = 1$. Label the agents $i = 1, 2, \ldots, n$, and consider the following allocation:

$$w_i = \frac{1 - 2^n}{2^i}, \quad i = 1, \ldots, n.$$

That is, agent 1 receives half of the total endowment, agent 2 receives one-quarter, and so on.

Claim: under the condition of limited enforcement power (Condition (ii)), no nontrivial set of agents, $M \subseteq \{1, \ldots, n\}$, can strictly improve the rankings of all of the members. ◊

Proof of Example 5. By construction, $w_i = \sum_{j>i} w_j$, $\forall i$, that is, even by assigning the resources of all agents in $\{i+1, \ldots, n\}$ to one single agent in that set, that agent is still unable to (strictly) catch up with agent i (or anyone labeled prior to i).

Now we prove the claim by contradiction. Suppose there exists a nonempty set of agents (i.e., a blocking coalition), $M \subseteq N$, each of whom can strictly improve their ranking. Let $i_0 = \min i \in M$. By the above observation, for any $k < i_0$, $w_k \geq w_{i_0-1} \geq \sum_{j \geq i_0} w_j \geq \sum_{j \in M} w_j$. That is, regardless of how the resources are reallocated in M, agent i_0 is unable to improve their ranking (to strictly catch up with any $k < i_0$). This leads to a contradiction. □

Proposition 3 then immediately follows from the above example: since no agent can strictly improve their ranking, they would not prefer any alternative over the current allocation, w_i, under the baseline ranking preference $\succ$ (or any stronger preferences). Intuitively, the resource that any blocking coalition is able to redistribute can be insufficient to allow those agents to catch up with others outside the coalition. This result shows that when the agents' main concern is their ranking, then limited enforcement is a strong assumption under which all networks are permissive.

5. Discussion

In this paper, we provide a sufficient and necessary condition for networks to permit at least one stable allocation. This condition allows us to discuss comparative statics, and specifically, to examine which classes of networks are more permissive. For instance, working with Erdős–Rényi random networks, we show that as networks become larger, more connected, or more homophilous, they permit stable allocations for a smaller set of α, that is, networks become less permissive. These comparative statics results have interesting implications. For instance, if we interpret the development of the internet, social media, and the increase in the transparency of information in general as making the network more connected, then we can predict that such changes may make networks less permissive. Also, the homophily in our setting hurts permissiveness and this fact may provide another important factor when we consider group formation.[18] In addition, we discuss in the paper what happens if agents have limited enforcement power. We could further extend our model to general sets of blocking coalitions. We provide a necessary and sufficient condition for permissive networks in this case as well: a network is permissive if and only if an expansive enough independent set exists whose intersection with any blocking coalition is nonempty.[19]

Appendix

A.1 *Omitted proofs*

Proof of Corollary 1

"If": By definition, given any network g, allocation w, and alternative allocation w', $w' \succ_i^A w$ implies $w' \succ_i w$, $\forall i$. So replacing the initial allocation with any alternative is uniformly (weakly) more difficult under $\succ^A$ than under $\succ$. Therefore, if w is α-stable on

[18] Interested readers could refer to Appendix A.2 for more details.

[19] We refer interested readers to the working paper version of this paper for more details.

g under $\succ$, then it must be stable under $\succ^{A}$. As a result, the "if" part of Corollary 1 is an immediate corollary to the "if" part of Theorem 1.

"Only if": We aim to show that this part (of Corollary 1) is implied by the "only if" part of proof of Theorem 1. The key is to observe that for the alternative allocation, w', constructed in equation (1), we have $w' \succ_i^{A} w, \forall i \in M_1$: first, the strict improvement in the local ranking is shown in the proof of Theorem 1; second, as for the absolute part, we have by construction $w_i' > w_i, \forall i \in M_1$.

Therefore, the current part of Corollary 1 can be shown by repeating the argument in the "only if" part of the proof of Theorem 1, with the preference profile replaced by $\succ^{A}$.

Proof of Corollary 2

This proof follows from the proof of Theorem 1, with simple modifications to guarantee that all of the differences are "moderate" under the constructed allocations. Here, we formally present the modifications while we skip the other steps.

For the "if" part, we modify the constructed allocation w^* as

$$\widetilde{w^*} = \frac{W}{n} + \frac{e}{\max\limits_j w_j' - \min\limits_j w_j'}\left(w_i' - \frac{W}{n}\right).$$

Recall that $\frac{W}{n}$ is the average allocation, and thus, the above applies a mean reversion modification such that the modified allocation, $\widetilde{w^*}$, keeps the ranking(s) in w^* and has the largest difference capped by e.

For the "only if" part, we modify the alternative allocation, w', constructed in (1): we construct a new alternative, $\widetilde{w}'$, such that

$$\widetilde{w}_i' = \frac{1}{n} + \frac{e}{\max\limits_j w_j' - \min\limits_j w_j'}\left(w_i' - \frac{1}{n}\right).$$

Again, $\widetilde{w}'$ keeps the same ranking(s) as in w' and, in the meantime, guarantees that all of the differences are "moderate," that is, $\max_j \widetilde{w_j'} - \min_j \widetilde{w_j'} = e$.

Proof of Proposition 1

We prove the sufficient condition under the baseline ranking preference, $\succ^{g}$. Then it follows from Lemma 1 that the statement also holds under other preferences that are specificed in Theorem 1, all of which are stronger than $\succ^{g}$.

We show the contrapositive. Suppose w is not α-stable under $\succ^{g}$, we aim to show that condition (2) must be violated.

By definition, there exists some blocking coalition, M, such that $|M| > \alpha n$ and an alternative allocation, w', such that $r_i^{g}(w') \leq r_i^{g}(w) - 1, \forall i \in M$.

The above implies that $r_i^{g}(w) \neq 0$, that is, $M \in N \setminus N^{0,r}$. In addition,

$$\sum_{i \in M} r_i^{g}(w') \leq \sum_{i \in M} \left(r_i^{g}(w) - 1\right) = \sum_{i \in M} r_i^{g}(w) - |M|$$

At the same time, we have:

$$\begin{aligned}\sum_{i\in M} r_i^g(w') &\geq \sum_{i\in M}\sum_{j\in N} g_{ij}1_{\{w'_j\geq w'_i\}}\\ &\geq \sum_{i,j\in M} g_{ij}1_{\{w'_j\geq w'_i\}}\\ &= \sum_{i,j\in M,i<j} g_{ij}(1_{\{w'_j\geq w'_i\}}+1_{\{w'_i\geq w'_j\}})\\ &\geq \sum_{i,j\in M,i<j} g_{ij}\\ &= L(g|_M)\end{aligned}$$

Putting this together, we find some $M\subseteq N\setminus N^{0,r}$, $|M|>\alpha n$ such that

$$\sum_{i\in M} r_i^g(w)-|M|\geq L(g|_M).$$

Proof of Proposition 2

It is understood in the literature that $g|_M$ being acyclic implies that it has at least one "topological ordering"—an ordering of M in a sequence such that for every link the starting agent of the link appears earlier in the sequence than the ending agent of the link.[20] A large part of the proof (especially in the "if" part) is to explicitly construct an allocation that corresponds to such a sequence under the current context.

"If": Suppose there exists $M\subseteq N$ such that $g|_M$ is acyclic and $|M|\geq(1-\alpha)n$. We aim to find an allocation w^* such that $r_i(w^*)=1$, $\forall i\in M$.

Recall that $N_i^{\text{out}}(g)$ $[N_i^{\text{in}}(g)]$ consists of i's (immediate) outgoing [incoming] neighbors. Let $\widetilde{N}_i^{\text{out}}(g)$ $[\widetilde{N}_i^{\text{in}}(g)]$ consist of i's outgoing [incoming] neighbors, neighbors of neighbors, etc. Formally,

$$\widetilde{N}_i^{\text{out}}(g)\equiv\{j\mid \text{ there is a path from } i \text{ to } j \text{ on } g\};$$
$$\widetilde{N}_i^{\text{in}}(g)\equiv\{j\mid \text{ there is a path from } j \text{ to } i \text{ on } g\}.$$

The following observation is a key to the proof:

$$g\text{ is acyclic}\quad\Longrightarrow\quad \widetilde{N}_i^{\text{out}}(g)\cap\widetilde{N}_i^{\text{in}}(g)=\emptyset,\ \forall i.$$

Otherwise, if $j\in\widetilde{N}_i^{\text{out}}(g)\cap\widetilde{N}_i^{\text{in}}(g)$, there exists a path from i to j and a path from j to i, and thus, there is a path from i to j and then back to i, that is, a cycle.

Now we are ready to assign values (allocations) to the agents; first, to agents in M, and then to those outside the set.[21]

[20] See, for example, Section 22.4 in Cormen et al. (2009)

[21] We temporarily drop the restriction of the fixed total resource, $\sum_i w_i=W=1$, to simplify notation. To satisfy this assumption, we can simply define $w_i=w_i^*/(\sum_j w_j^*)$.

1. For every $i \in M$ that is isolated in $g|_M$, let $w^*(i) = 0.5$.
2. For every $i \in M$ such that $\widetilde{N}_i^{\text{out}}(g|_M) = \emptyset$ and $w^*(i) \neq 0.5$, let $w^*(i) = 1$.
3. For every $i \in M$ such that $\widetilde{N}_i^{\text{in}}(g|_M) = \emptyset$ and $w^*(i) \neq 0.5$, let $w^*(i) = 2$.
4. Pick any agent $i \in M$ that is not yet assigned a value, note that this implies that $\widetilde{N}_i^{\text{out}}(g|_M)$ and $\widetilde{N}_i^{\text{in}}(g|_M)$ are both nonempty. Let

$$B(i) \equiv \{ j \in M \mid w^*(j) \text{ is defined before } w^*(i) \},$$
$$w^*(i) \equiv 0.5\Big(\max_{j \in \widetilde{N}_i^{\text{out}}(g|_M) \cap B(i)} w^*(j) + \min_{j \in \widetilde{N}_i^{\text{in}}(g|_M) \cap B(i)} w^*(j) \Big),$$

in which the max and min are based on the js whose values are already assigned.[22]

5. Repeat Step 4 until all of the agents in M are assigned a value under w^*.
6. For every $i \notin M$, let $w^*(i) = 0$.

Now we aim to show a key property: for every $i \in M$ whose value is defined in the above step 4 (or 5)

$$\max_{j \in \widetilde{N}_i^{\text{out}}(g|_M) \cap B(i)} w^*(j) < \min_{j \in \widetilde{N}_i^{\text{in}}(g|_M) \cap B(i)} w^*(j). \tag{3}$$

And as a result,

$$w^*(i) \in \Big(\max_{j \in \widetilde{N}_i^{\text{out}}(g|_M) \cap B(i)} w^*(j), \min_{j \in \widetilde{N}_i^{\text{in}}(g|_M) \cap B(i)} w^*(j) \Big) \subseteq (1, 2) \tag{3a}$$

We prove (3) by using the mathematical induction with respect to the order in which i is defined.

For the first i defined in step 4, $\max_{j \in \widetilde{N}_i^{\text{out}}(g|_M) \cap B(i)} w^*(j) = 1$ because all of those $w^*(j)$s are defined in step 2, and $\max_{j \in \widetilde{N}_i^{\text{out}}(g|_M) \cap B(i)} w^*(j) = 2$ because all of those $w^*(j)$s are defined in step 3. Therefore, equation (3) holds.

In addition, pick any i, and suppose equation (3) holds for all js whose values are defined before i. We aim to show that (3) also holds for i. If not, then there would exist $j_1, j_2 \in B(i)$, such that $j_1 \in \widetilde{N}_i^{\text{out}}(g|_M)$, $j_2 \in \widetilde{N}_i^{\text{in}}(g|_M)$, and $w^*(j_1) \geq w^*(j_2)$. It follows from the definitions of $\widetilde{N}_i^{\text{out}}$ and $\widetilde{N}_i^{\text{in}}$ that there exists a path from j_2 to i and a path from i to j_1 which, combined together, form a path from j_2 to j_1 (all on $g|_M$). Hence, $j_1 \in \widetilde{N}_{j_2}^{\text{out}}(g|_M)$ and $j_2 \in \widetilde{N}_{j_1}^{\text{in}}(g|_M)$ and, therefore, $w^*(j_1) \geq w^*(j_2)$ implies that (3) is violated for either j_1 or j_2 (the one whose w^* was defined earlier), which is a contradiction.

Thus, we conclude the proof of equation (3).

[22] Notice that the max and min are well-defined, given $g|_M$ is acyclic: there exist $j_1 \in \widetilde{N}_i^{\text{out}}(g|_M)$ and $j_2 \in \widetilde{N}_i^{\text{in}}(g|_M)$, such that $w^*(j_1) = 1$ and $w^*(j_2) = 2$.

To see this, the nonemptiness of $\widetilde{N}_i^{\text{out}}(g|_M)$ and $\widetilde{N}_i^{\text{in}}(g|_M)$ implies the existence of i' and i'' such that $i'i$, ii'' is a path on $g|_M$. It follows from the acyclicity of $g|_M$ that extending the path $i'i$, ii'' forward [backward] will eventually reach some agent j_1 [j_2] with $\widetilde{N}_{j_1}^{\text{out}}(g|_M) = \emptyset$ [$\widetilde{N}_{j_1}^{\text{in}}(g|_M) = \emptyset$]. By construction, j_1 and j_2 are not isolated in $g|_M$, so their values are 1 and 2, respectively.

The last piece of the proof of the "if" part is to show that for every $i \in M$, $r_i(w^*) = 1$. Notice that if $i \in M$ and $j \notin M$, then $w^*(i) \geq 0.5 > 0 = w^*(j)$.

Therefore, it suffices to show for all $i, j \in M$ s.t. $j \in \widetilde{N}_i^{\text{out}}(g)$,

$$w^*(j) < w^*(i) \tag{4}$$

If $w^*(i)$ is defined in steps 1 or 2, then $\widetilde{N}_i^{\text{out}}(g|_M) = \emptyset$, so that (4) automatically holds.

If $w^*(i)$ is defined in step 3, then $j \in \widetilde{N}_i^{\text{out}}(g)$ is defined in steps 2, 4, or 5 and, therefore, (4) is implied by (3a).

Finally, consider the case where $w^*(i)$ is defined in step 4 (or 5). If $j \in B(i)$, that is, $w^*(j)$ is defined before $w^*(i)$, then it follows from equation (3a) (for i) that

$$w^*(j) \leq \max_{j' \in \widetilde{N}_i^{\text{out}}(g|_M) \cap B(i)} w^*(j') < w^*(i).$$

Otherwise, $w^*(j)$ is defined after $w^*(i)$; hence, $w^*(j)$ is also defined in step 4 (or 5). Then it follows from equation (3a) (for j) that

$$w^*(j) < \min_{j' \in \widetilde{N}_j^{\text{in}}(g|_M) \cap B(j)} w^*(j') \leq w^*(i),$$

This concludes the proof of the "if" part.

"Only if": Suppose g is α-permissive so that there exists some allocation, w, that is α-stable on g. Now we assign labels to the agents:

0. Initialize $l_i = null$ for all i.
1. Pick $i \in \arg\max_{j \text{ s.t. } l_j = null} w_j$, let $l_i = 1$.
2. Let $l_i = 0$ if $l_i = null$ and $i \in N_j^{\text{out}}$ for some $l_j = 1$.
3. Repeat steps 2 and 3 until all of the agents are labeled, 0 or 1.

Let $M_k = \{j \in N \mid l_j = k\}$, $k = 0, 1$, then $M_1 \cup M_0 = N$ and $M_1 \cap M_0 = \emptyset$.

We use the same method to construct an alternative allocation, according to equation (1). By construction, all of the agents in M_0 strictly prefer w' to w: under w, every $i \in M_0$ has a neighbor $j \in M_1$ such that $w_i \leq w_j$; whereas under w', the rankings among the agents in M_0 remain the same and everyone in M_0 has a strictly higher allocation than everyone in M_1.

Thus, the fact that w is α-stable on g implies that $|M_0| \leq \alpha n$, or $|M_1| \geq (1-\alpha)n$.

Finally, we show that $g|M_1$ is acyclic. By construction, all of the nodes in M_1 received a label of 1, one at a time in step 2. Therefore, we can uniquely rank the nodes in M_1 as $i_1, i_2, \ldots, i_{|M_1|}$, in the order in which they are labeled. In addition, for any $i_j, i_k \in M_1$, $j < k$ (i.e., i_j is labeled before i_k) implies that $i_k \notin N_{i_j}^{\text{out}}$, that is, $g_{i_j i_k} = 0$. That is to say, in $g|M_1$, it is only possible to have links from nodes with lower subscripts to nodes with higher subscripts. Hence, $g|M_1$ is acyclic.

We constructed a set M_1, whose size is at least $(1-\alpha)n$, and $g|M_1$ is acyclic.

A.2 *Comparative statics: Which networks are more permissive?*

Recall that Theorem 1 provides a necessary and sufficient condition for permissive networks. Armed with this clean characterization, we are able to compare across networks to see which one is more permissive.

Definition 14 (Level of permissiveness). The level of permissiveness $\Gamma(g)$ for a network, g, is defined as follows:

$$\gamma(g) \equiv \frac{\Gamma(g)}{n},$$

where $I(g)$ is the "independence number," or the size of the largest independent set, in network g.

It follows from Theorem 1 that a network is α-permissive for any $\alpha \geq 1 - \gamma(g)$, and hence, a larger $\gamma(g)$ implies the network is α-permissive for a larger set of α.

Lemma 2 (Monotonicity). *Consider two networks g, g' such that $g' \leq g$. Then $\gamma(g') \geq \gamma(g)$. Moreover, $\forall \alpha \in (0, 1)$:*

- *If g is α-permissive, then g' is also α-permissive;*
- *If g' is not α-permissive, then g is not α-permissive.*

For further comparative statics, one challenge is that finding the largest independent set in an arbitrary network is a classical NP-hard problem.[23] Fortunately, independence numbers can be approximated for random networks. We now present a series of comparative statics results that are based on these approximations.

Comparative statics in Erdős–Rényi random networks Consider a set of nodes $N = \{1, \dots, n\}$, and let a link between any two nodes, i and j, be formed with probability p, where $0 < p < 1$. The formation of links is independent, and each agent has an expected degree of $d = np$.[24] This is referred to as an Erdős–Rényi random graph/network model, denoted by $G(n, p)$. In this section, we study the permissiveness of the Erdős–Rényi random networks.

Let $\Gamma(n, p)$ be a random variable that captures the size of the largest independent set, that is, the independence number of a network that is randomly formed according to the above mentioned process $G(n, p)$. The literature provides approximations of the properties of large random networks, including $\Gamma(n, p)$. There are two commonly studied cases, depending on which term is fixed as $n \to \infty$:

(i) The "sparse" case keeps the expected degree $d = np$ fixed as n increases;

(ii) The "dense" case keeps the link probability p fixed as n increases.

[23] See, for instance, Bomze, Budinich, Pardalos, and Pelillo (1999).

[24] Throughout this section, we focus on large $n \gg 1$, and thus, we need not distinguish between np and $(n-1)p$.

For the following definition, we adopt the approximations of $S(n, p)$ provided by Frieze and Karoński (2016) and Grimmett and McDiarmid (1975).[25]

Definition 15 (Level of permissiveness of the Erdős–Rényi networks). The level of permissiveness of $G(n, p)$ is defined as follows:

(i) in the sparse case: $\tilde{\gamma}^{\text{sparse}}(n, p) \equiv \frac{2 \ln d}{d}$, in which $d = np$;

(ii) in the dense case: $\tilde{\gamma}^{\text{dense}}(n, p) \equiv \frac{2}{n} \ln_{1/(1-p)}(n)$.

Proposition 4 (Comparative statics). *The level of permissiveness $\tilde{\gamma}(n, p)$ decreases in n and p, in both the dense and sparse cases.*

That is, networks tend to be less permissive as they become more populated (larger n), or more connected (larger p). The above result follows directly from Definition 15 and we omit the proof here.

The comparative statics have interesting implications. For instance, people can interpret their networks as being informational. That is, they compare themselves with those for which they have information on income, wealth, social status, etc. Then, with the development of the internet, social media, and so on, people have more and more information about other people, potentially making their networks more connected. We predict that in this situation, networks will become less permissive.

Homophily Lastly, we examine whether more homophilous networks are more or less permissive. Homophily is a pattern that shows how individuals with similarities tend to interact more with each other. This pattern is endemic in social networks and is of great social and economic importance (see McPherson, Smith-Lovin, and Cook (2001) for an overview). How would homophily affect the permissiveness of networks?

In particular, consider a simple version of homophily following Jackson (2008a). Suppose the population $N = \{1, \dots, n\}$ is partitioned into two groups, N_1 and N_2. This partition captures the characteristics (types) of the agents so that agents of the same type are within the same group. Depending on the application, a specific type of agents might embody ethnicity, gender, education level, profession, etc.

Let p_{ij} be the probability of a link being formed between an agent in group i and an agent in group j. *Homophily* refers to the pattern in which the agents are more likely to link with those in their own group, that is, $p_{11}, p_{22} > p_{12}$.

To simplify the notation, suppose the two groups are of equal size $|N_1| = |N_2| = n/2$ (with even n), and $p_{11} = p_{22} = p + \Delta$, $p_{12} = p - \Delta$, for some $\Delta \in [0, p)$, so that everyone's expected degree is still $d = np$.

We denote such a model as $G(n, p, \Delta)$. Homophily increases as Δ increases: it is more likely to form a link within a group than across groups.

Proposition 5 (Homophily hurts permissiveness). *The permissiveness of random networks in the $G(n, p, \Delta)$ model decreases as Δ becomes larger. That is, networks are less permissive when they become more homophilous.*

[25] The detailed approximations are provided in A.2.1. From now on, we assume that the population size n (and that of any subgroup) is large enough.

Intuitively, homophily creates dense local structures, so that everyone compares themselves with more people in the same group. This effect dominates the decrease in the across-group linking probabilities. Consequently, as networks become more homophilous, it is more difficult to find a large enough independent set.

Proof of Proposition 5. We prove a stronger statement in the Appendix: for any (sub)set of agents, the probability that they form an independent set decreases in the homophily index, Δ. Consider a subset of agents $M \subseteq N$ that consists of k_1 agents from group 1 and k_2 agents from group 2. We aim to show the probability that M is an independent set that decreases in Δ. To simplify the notation, we focus on the case where $k_1, k_2 \gg 1$, so that we need not distinguish between k_i and $k_i - 1$:

$$\begin{aligned} P(M \text{ is an independent set}) &= P(g_{ij} = 0, \forall i, j \in M) \\ &= (1 - p - \Delta)^{0.5(k_1)^2 + 0.5(k_2)^2} \times (1 - p + \Delta)^{k_1 k_2} \\ &\equiv P(\Delta). \end{aligned}$$

Denote $c(\Delta) \equiv (1 - p - \Delta)^{0.5(k_1)^2 + 0.5(k_2)^2 - 1} \times (1 - p + \Delta)^{k_1 k_2 - 1} > 0$ $(\forall \Delta)$, we have

$$\begin{aligned} P'(\Delta) &= c(\Delta) \cdot \big[-\big(0.5(k_1)^2 + 0.5(k_2)^2\big)(1 - p + \Delta) + k_1 k_2 (1 - p - \Delta)\big] \\ &= c(\Delta) \cdot \big[-0.5(k_1 - k_2)^2 (1 - p) - 0.5(k_1 + k_2)^2 \Delta\big] \\ &< 0. \end{aligned}$$

This concludes the proof that $P(M \text{ is an independent set})$ decreases in Δ. □

A.2.1 *Approximations of the independence number* The following approximations for $\Gamma(n, p)$ are cited from Frieze and Karoński (2016)[26] and Grimmett and McDiarmid (1975), respectively. In addition, $\Gamma(n, p)$ is shown to have a very peaked density function.[27]

Lemma 3 (Independence number in Erdős–Rényi random networks). *For any $\varepsilon > 0$, we have the following approximations for the independence number $\Gamma(n, p)$:*

[26] We thank Ben Golub for this reference.

[27] For instance, for the sparse case, the literature has established exponential bounds for the tail probabilities (cf. Frieze and Karoński (2016) p. 119, equation (7.9)):

$$P\left(\left|\Gamma(n, p) - \mathbb{E}\big(\Gamma(n, p)\big)\right| \geq \frac{\varepsilon \ln d}{2d} n\right) \leq \exp\left\{-O\left(\frac{\varepsilon \ln d}{d}\right)^2 n\right\}.$$

For the dense case, Bollobás and Erdös (1976), Matula (1976) independently found a strong concentration result that the independence number for any E-R random graph is specified within two adjacent integers. In particular, fix $\forall p \in (0, 1)$, $\Gamma(n, p)$ is either $k(n)$ or $k(n) + 1$ with a probability close to 1, for some integer $k(n)$ that depends on n. See Frieze and Karoński (2016) for a recent review. We thank the editor for suggesting the above discussion.

(i) In the sparse case, there exists some $d_\epsilon > 0$ such that we fix any expected degree $d \geq d_\epsilon$, and there exists some $\bar{n}$ s.t. for any $n > \bar{n}$,

$$\left|\Gamma(n, p) - \frac{2 \ln d}{d} n\right| \leq \frac{\varepsilon \ln d}{d} n$$

holds with a probability of at least $1 - \varepsilon$.

(ii) In the dense case, if we fix any link probability $p \in (0, 1)$, then there exists some $\bar{n}$ such that for any $n > \bar{n}$,

$$\left|\Gamma(n, p) - 2 \ln_{1/(1-p)}(n)\right| \leq \varepsilon \ln n$$

holds with a probability of at least $1 - \varepsilon$.

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