Supplementary Appendix: A General Framework for Rational Learning in Social Networks

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This supplementary appendix presents the proofs of statements made in the paper, an extension of the framework, and examples omitted in the paper. The structure of the supplementary appendix is the following. In the first section I provide proofs of statements made in the paper. First I consider the difference in inferences on one's neighbors actions based on common and private observables, and establish equivalence of the respective learning processes. Second, I prove that in the expected utility setting each set of positive probability is a dominant set. In section two I provide an extension of local indifference to the case where the network structure or strategies are not common knowledge but are captured in an extended state space. Section three presents examples omitted in the paper. First, I present an example for failure of global indifference once learning ends. Second, I provide background on the learning process in the example of optimal communication networks. Third, I present an example of a union consistent choice correspondence that cannot be represented in the expected utility setting.

1 Proofs

1.1 The Learning Process

Consider the alternative learning process where agents make inferences on their neighbors actions based on the set of possible information sets they assign to them given their private observables. In the first period the information set of agent i is given by

$$\check{I}_i^1(\omega) = \mathcal{P}_i(\omega)$$

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and the set of information sets of his neighbor j privately considered possible by i is given by

$$\breve{\mathcal{I}}_j^1(\mathcal{P}_i(\omega)) = \{P_j \in \mathcal{P}_j : P_j \cap \breve{I}_i^1(\omega) \neq \emptyset\}.$$

The inference agent i makes based on his neighbor j's first period action is given by

$$\breve{\mathcal{D}}_j^1(a_j^1; \mathcal{P}_i(\omega)) = \{P_j \in \breve{\mathcal{I}}_j^1(\mathcal{P}_i(\omega)) : a_j^1 = s_j(P_j)\}.$$

The information set of agent i in period $t = 2$ is given by

$$\breve{I}_i^2(\mathcal{P}_i(\omega), h_i^2(\omega)) = \mathcal{P}_i(\omega) \cap \bigcap_{j \in N_i} \cup \breve{\mathcal{D}}_j^1(a_j^1; \mathcal{P}_i(\omega)).$$

In period t the information set of agent i is given by

$$\breve{I}_i^t(\mathcal{P}_i(\omega), h_i^t(\omega)) = \mathcal{P}_i(\omega) \cap \bigcap_{j \in N_i} \cup \breve{\mathcal{D}}_j^{t-1}(a_j^{t-1}; \mathcal{P}_i(\omega), h_i^{t-1}(\omega)).$$

The set of information sets of neighbor j considered possible by his agent i based on i's private observables is given by

$$\breve{\mathcal{I}}_j^t(\mathcal{P}_i(\omega), h_i^t(\omega)) = \left\{ \breve{I}_j^t(P_j, \hat{h}_j^t) : \begin{array}{l} \hat{h}_{ij}^t = h_{ij}^t(\omega) \\ \exists \breve{I}_j^{t-1} \in \breve{\mathcal{D}}_j^{t-1}(a_j^{t-1}; \mathcal{P}_i(\omega), h_i^{t-1}(\omega)) \text{ s.t. } \breve{I}_j^t(P_j, \hat{h}_j^t) \subset \breve{I}_j^{t-1} \\ \breve{I}_j^t(P_j, \hat{h}_j^t) \cap \breve{I}_i^t(\mathcal{P}_i(\omega), h_i^t(\omega)) \neq \emptyset \end{array} \right\}.$$

Based on the action a_j^t selected by agent j in period t, agent j makes the following inference regarding j's realized information set

$$\breve{\mathcal{D}}_j^t(a_j^t; \mathcal{P}_i(\omega), h_i^t(\omega)) = \{\breve{I}_j^t \in \breve{\mathcal{I}}_j^t(\mathcal{P}_i(\omega), h_i^t(\omega)) : a_j^t = s_j(\breve{I}_j^t)\}.$$

The information set of agent i in period $t+1$ is then given by

$$\breve{I}_i^{t+1}(\mathcal{P}_i(\omega), h_i^{t+1}(\omega)) = \mathcal{P}_i(\omega) \cap \bigcap_{j \in N_i} \cup \breve{\mathcal{D}}_j^t(a_j^t; \mathcal{P}_i(\omega), h_i^t(\omega)).$$

The following proposition formalizes the claim made in section four of the paper.[1]

Proposition 1 *For every player i, for every period t and every state of the world ω,*

$$I_i^t(\mathcal{P}_i(\omega), h_i^t(\omega)) = \breve{I}_i^t(\mathcal{P}_i(\omega), h_i^t(\omega)).$$

[1] Please note that numbering of propositions in the supplementary appendix is different from the paper.

Proof. I use an induction argument for the proof. Following the definition of the respective sets we have in period $t = 1$

$$\breve{\mathcal{D}}_j^1(a_j^1; \mathcal{P}_i(\omega)) \subset \mathcal{D}_j^1(a_j^1; \omega; i)$$

which implies

$$\mathcal{P}_i(\omega) \cap \bigcap_{j \in N_i} \cup \breve{\mathcal{D}}_j^1(a_j^1; \mathcal{P}_i(\omega)) \subset \mathcal{P}_i(\omega) \cap \bigcap_{j \in N_i} \cup \mathcal{D}_j^1(a_j^1; \omega; i).$$

For each agent $j \in N_i$ consider the set $\mathcal{D}_j^{1C}(a_j^1; \omega; i)$ of partition cells that is commonly considered possible among i and j, but privately excluded by i,

$$\mathcal{D}_j^{1C}(a_j^1; \omega; i) = \mathcal{D}_j^1(a_j^1; \omega; i) \setminus \breve{\mathcal{D}}_j^1(a_j^1; \mathcal{P}_i(\omega))$$

which implies

$$\mathcal{D}_j^1(a_j^1; \omega; i) = \breve{\mathcal{D}}_j^1(a_j^1; \mathcal{P}_i(\omega)) \cup \mathcal{D}_j^{1C}(a_j^1; \omega; i). \tag{1.1}$$

Consider the information set $I_i^2(\mathcal{P}_i(\omega), h_i^2(\omega))$

$$I_i^2(\mathcal{P}_i(\omega), h_i^2(\omega)) = \mathcal{P}_i(\omega) \cap \bigcap_{j \in N_i} \cup \mathcal{D}_j^1(a_j^1; \omega; i).$$

By equation (1.1) we have

$$I_i^2(\mathcal{P}_i(\omega), h_i^2(\omega)) = \mathcal{P}_i(\omega) \cap \bigcap_{j \in N_i} \cup \left(\breve{\mathcal{D}}_j^1(a_j^1; \mathcal{P}_i(\omega)) \cup \mathcal{D}_j^{1C}(a_j^1; \omega) \right)$$

which is equivalent to

$$I_i^2(\mathcal{P}_i(\omega), h_i^2(\omega)) = \bigcap_{j \in N_i} \left(\cup \breve{\mathcal{D}}_j^1(a_j^1; \mathcal{P}_i(\omega)) \cap \mathcal{P}_i(\omega) \right) \cup \left(\cup \mathcal{D}_j^{1C}(a_j^1; \omega) \cap \mathcal{P}_i(\omega) \right).$$

By definition of $\breve{\mathcal{D}}_j^1(a_j^1; \mathcal{P}_i(\omega))$ and $\mathcal{D}_j^{1C}(a_j^1; \omega)$ we have

$$\mathcal{D}_j^{1C}(a_j^1; \omega) \cap \mathcal{P}_i(\omega) = \emptyset$$

for all $j \in N_i$ yielding

$$I_i^2(\mathcal{P}_i(\omega), h_i^2(\omega)) = \bigcap_{j \in N_i} \left(\cup \breve{\mathcal{D}}_j^1(a_j^1; \mathcal{P}_i(\omega)) \cap \mathcal{P}_i(\omega) \right)$$

or equivalently

$$I_i^2(\mathcal{P}_i(\omega), h_i^2(\omega)) = \mathcal{P}_i(\omega) \cap \bigcap_{j \in N_i} \cup \breve{\mathcal{D}}_j^1(a_j^1; \mathcal{P}_i(\omega)) = \breve{I}_i^2(\mathcal{P}_i(\omega), h_i^2(\omega))$$

establishing the base case for $t = 2$. For the inductive step let us assume that in period t the respective information sets are equal for all ω, i.e.

$$\mathcal{P}_i(\omega) \cap \bigcap_{j \in N_i} \cup \mathcal{D}_j^{t-1}(a_j^{t-1}, h_{ij}^{t-1}(\omega); \omega) = \mathcal{P}_i(\omega) \cap \bigcap_{j \in N_i} \cup \breve{\mathcal{D}}_j^{t-1}(a_j^{t-1}; \mathcal{P}_i(\omega), h_i^{t-1}(\omega))$$

which implies

$$\breve{\mathcal{I}}_j^t(\mathcal{P}_i(\omega), h_i^t(\omega)) \subset \mathcal{I}_j^t(h_{ij}^t(\omega); \omega)$$

for all $j \in N_i$ and thus

$$\breve{\mathcal{D}}_j^t(a_j^t; \mathcal{P}_i(\omega), h_i^t(\omega)) \subset \mathcal{D}_j^t(a_j^t, h_{ij}^t(\omega); \omega).$$

As above, $\mathcal{D}_j^{tC}(a_j^t, h_{ij}^t(\omega); \omega)$ denotes the set of information sets of agent j in period t that are considered possible among j and i based on common observables, but excluded based on private observables of i,

$$\mathcal{D}_j^{tC}(a_j^t, h_{ij}^t(\omega); \omega) = \mathcal{D}_j^t(a_j^t, h_{ij}^t(\omega); \omega) \setminus \breve{\mathcal{D}}_j^t(a_j^t; \mathcal{P}_i(\omega), h_i^t(\omega)).$$

I need to establish that the respective information sets in period $t+1$ are equal as well. Let me do so by contradiction. Suppose that

$$\breve{I}_i^{t+1}(\mathcal{P}_i(\omega), h_i^{t+1}(\omega)) \neq I_i^{t+1}(\mathcal{P}_i(\omega), h_i^{t+1}(\omega))$$

which is equivalent to

$$\bigcap_{j \in N_i} \cup \mathcal{D}_j^t(a_j^t, h_{ij}^t(\omega); \omega) \neq \bigcap_{j \in N_i} \cup \breve{\mathcal{D}}_j^t(a_j^t; \mathcal{P}_i(\omega), h_i^t(\omega)).$$

Consider a state $\omega' \in I_i^{t+1} \setminus \breve{I}_i^{t+1}$. For such a state we have

$$\omega' \in \mathcal{D}_j^{tC}(a_j^t, h_{ij}^t(\omega); \omega)$$

for at least one $j \in N_i$ implying by definition of $\breve{\mathcal{I}}_j^t(\mathcal{P}_i(\omega), h_i^t(\omega))$

$$\omega' \cap \breve{I}_i^t(\mathcal{P}_i(\omega), h_i^t(\omega)) = \emptyset.$$

By the induction hypothesis we have

$$\breve{I}_i^t(\mathcal{P}_i(\omega), h_i^t(\omega)) = I_i^t(\mathcal{P}_i(\omega), h_i^t(\omega))$$

which then implies

$$\omega' \cap I_i^t(\mathcal{P}_i(\omega), h_i^t(\omega)) = \emptyset.$$

The shrinking property of information sets $I_i^{t+1} \subset I_i^t$ yields

$$\omega' \cap I_i^{t+1}(\mathcal{P}_i(\omega), h_i^{t+1}(\omega)) = \emptyset.$$

thereby establishing $I_i^{t+1} \setminus \hat{I}_i^{t+1} = \emptyset$ and thus proving the inductive step. ■

1.2 Dominant Set in Expected Utility Setting

In this section I demonstrate that the expected utility setting is captured by the general framework considered in the paper. Let me first define the expected utility framework in the context of my analysis.

Uncertainty is represented by a probability space $(\Omega, \mathcal{F}, p)$ where Ω represents the state space, $\mathcal{F}$ is a σ-algebra of subsets of Ω, and p is a common prior.[2] Let all elements of the join have positive probability. All agents share a common utility function $u : A \times \Omega \to \mathbb{R}$ which is bounded and measurable for each $a \in A$. In every period t agents select an action out of A that maximizes their expected utility given their information, i.e.

$$c(I) = \arg\max_{a \in A} E[u(a, \omega) \,|I] \,.$$

In order to assure the existence of a maximum in the case of infinite A, the action set A has to be a compact subset of a topological space $\mathbf{A}$, and u has to be continuous in $\mathbf{A}$ for every ω. The Bounded Convergence Theorem then gives existence.

Next I prove the claim that in the expected utility setting a set B with positive probability is a dominant set whenever $u(a, \omega)$ is bounded and measurable for all a.

Proposition 2 *If the utility function $u : A \times \Omega \to \mathbb{R}$ is bounded and measurable for each action a, then every set $B \in \mathcal{F}$ with $p(B) > 0$ is a dominant set.*

Proof. Take any set $B \in \mathcal{F}$ with $p(B) > 0$ and let a^* denote an optimal action given set B

$$E[u(a^*, \omega) \,|B] \geq E[u(a, \omega) \,|B]$$

for all $a \in A$. Consider a subsequence $\{B^{t_k}\}_{k=1}^{\infty}$ and an action a' such that

$$E[u(a', \omega) \left| B^{t_k} \right] \geq E[u(a, \omega) \left| B^{t_k} \right]$$

[2] $\mathcal{F}$ is generated by the join of partitions.

for all $k \in \mathbb{N}$ and $a \in A$. I establish the result via contradiction. The expected utility of choosing action a given information set B^t can be written as

$$E[u(a,\omega)\,|B^t] = \frac{p\left(B^t \smallsetminus B\right)}{p(B^t)} E[u(a,\omega)\,|B^t \smallsetminus B] + \frac{p(B)}{p(B^t)} E[u(a,\omega)\,|B]$$

As the utility function $u(a,\omega)$ is bounded so is its expected value. Together with the fact that $\{B^t\}_{t=1}^{\infty}$ converges to B and $p(B) > 0$ we have

$$\lim_{t\to\infty} E[u(a,\omega)\,|B^t] = E[u(a,\omega)\,|B]\,.$$

Suppose that a' is not optimal in the limit set,

$$E[u(a^*,\omega)\,|B] > E[u(a',\omega)\,|B]\ .$$

Let $d_{a'}$ denote the difference between limit utilities

$$d_{a'} = E[u(a^*,\omega)\,|B] - E[u(a',\omega)\,|B] > 0.$$

As the expected utility terms converge there exists a finite period $t^*(a')$ such that for all $t > t^*(a')$

$$\left|E[u(a',\omega)\,|B^t] - E[u(a',\omega)\,|B]\right| < \frac{d_{a'}}{2}$$

and

$$\left|E[u(a^*,\omega)\,|B^t] - E[u(a^*,\omega)\,|B]\right| < \frac{d_{a'}}{2}$$

which implies

$$E[u(a^*,\omega)\,|B^t] - E[u(a',\omega)\,|B^t] > 0$$

for all $t > t^*(a')$ contradicting the fact that a' is optimal for all sets in the infinite subsequence $\{B^{t_k}\}_{k=1}^{\infty}$ ■

2 Extension: Lack of Common Knowledge and Extension of the State Space

So far I assumed that the network structure is common knowledge. From a practical perspective this seems reasonable for small networks, but less so for large networks. In this section, I address the question of whether common knowledge of the network structure is necessary to achieve the local indifference result.

Another component that initially was assumed to be common knowledge is the set of strategies of all agents. In section six I have shown that under the absence of common knowledge of strategies and assuming only common knowledge of the choice correspondence, local indifference can fail in incomplete networks. However, the information sets failed to account for the different possible strategies agents could follow. Thus learning failed to capture all dimensions of uncertainty.

The purpose of this section is to establish that common knowledge of strategies or network structure is by no means necessary for local indifference to hold. The condition for validity of local indifference is that 1.) agents learn in an extended state space, capturing the initial states extended by possible strategies and/or network structures, 2.) decisions being based on information sets in the extended state space and 3.) the choice correspondence being union consistent in the extended state space.

Let me present an example to provide the reader with the necessary intuition. The example falls within the expected utility setting where agents share a common prior over the state space. Suppose that the network structure is not common knowledge. The notation is as above, Ω is the set of states of the world, $\mathcal{F}$ the corresponding σ-algebra and p the common prior probability measure. Suppose that all elements of the join of partitions have positive probability. The set of agents is denoted by V and they share a common utility function $u : A \times \Omega \to \mathbb{R}$.

To capture the lack of common knowledge of the network, assume that there exists a set of possible network structures Γ, which is commonly considered possible by all agents,

$$\Gamma \subset \{(V,E) : \; E \in V \times V\}$$

and let $\mathcal{G}$ denote the power set of Γ. Note that Γ and $\mathcal{G}$ are finite since the set of agents is finite. I assume that agents share a common prior p^Γ over the set of possible networks Γ, where each network in Γ has positive probability. Let $\mathcal{P}_i^\Gamma$ denote the partition of agent i over Γ,

$$\mathcal{P}_i^\Gamma(G) = \{G' \in \Gamma : N_i(G') = N_i(G)\}.$$

Thus agent i's knowledge about the realized network is limited to his neighbors. The extended state space $\tilde{\Omega}$ equals the Cartesian product of Ω and Γ with typical state $\tilde{\omega} = (\omega, G)$. Its σ-algebra is the product of the respective σ-algebras,

$$\begin{aligned} \tilde{\Omega} &= \Omega \times \Gamma \\ \tilde{\mathcal{F}} &= \mathcal{F} \times \mathcal{G}. \end{aligned}$$

The partition of agent i over $\tilde{\Omega}$ is given by $\tilde{\mathcal{P}}_i$,

$$\tilde{\mathcal{P}}_i(\omega, G) = \{(\omega', G') : \omega' \in \mathcal{P}_i(\omega),\ G' \in \mathcal{P}_i^\Gamma(G)\}.$$

Let the probability measures p and p^Γ be independent. The probability measure over $(\tilde{\Omega}, \tilde{\mathcal{F}})$ is then given by

$$\tilde{p}(I, X) = p(I)p^\Gamma(X) \text{ for all } I \in \mathcal{F},\ X \in \mathcal{G}.$$

As by assumption all elements of the join of partitions of Ω have positive probability, so do all elements of the join of partitions of $\tilde{\Omega}$. Finally, the common utility function $\tilde{u} : \tilde{\Omega} \to \mathbb{R}$ is given by

$$\tilde{u}(a; (\omega, G)) = u(a; \omega)$$

which translates into the following common behavioral rule

$$\tilde{c}(\tilde{I}) = \arg\max_{a \in A} E_{\tilde{p}} \left[\tilde{u}(a, \tilde{\omega}) \middle| \tilde{I} \right] \text{ for all } \tilde{I} \in \tilde{\mathcal{F}}.$$

The thus defined choice correspondence is union consistent in the extended state space.[3] The analysis of repeated interaction in the framework $\left(\left(\tilde{\Omega}, \tilde{\mathcal{F}}\right), \tilde{c}, \{\tilde{\mathcal{P}}_i\}_{i \in V}\right)$ differs only slightly from my previous analysis. The difference lies in the learning process as the common observables based upon agents assign sets of possible information sets to their neighbors in a given period is based on the history of agents that are commonly known to be common neighbors in that period. The set of possible information sets of an agent i, from perspective of his neighbor j, is based on the history of choices of agents that are commonly known to be neighbors of both i and j. Gradually the set of commonly known common neighbors can increase over time as agents commonly learn about the network structure. The reader can easily confirm that the proofs of Lemmas 1 and 2 follow in exactly the same way for this adapted learning process. Proposition 1 of the paper then gives the local indifference result once learning ends.

3 Examples

3.1 Failure of Global Indifference

Consider a network consisting of three agents, $1, 2$ and 3. They are organized in a star network with agent 2 as the center agent, $N_1 = \{2\}$, $N_2 = \{1, 3\}$, and $N_3 = \{2\}$. The state space consists

[3] Please see Proposition 2 in the supplementary appendix.

of nine states $\Omega = \bigcup_{i=1}^{9} \omega_i$ and the probability measure p over Ω is uniform. Let the state space be represented by the rectangle in figure 1.

	c^1	c^2	c^3
r^1	ω_1	ω_2	ω_3
r^2	ω_4	ω_5	ω_6
r^3	ω_7	ω_8	ω_9

Figure 1: State space of example 1.

Agents 1 and 3 have private information while agent 2 has not. Agent 1 learns the rows of the matrix while agent 3 learns the columns. The partitions are given by

$$\begin{aligned}\mathcal{P}_1 &= \{\omega_1\omega_2\omega_3, \omega_4\omega_5\omega_6, \omega_7\omega_8\omega_9\}\\ \mathcal{P}_2 &= \{\omega_1\omega_2\omega_3\omega_4\omega_5\omega_6\omega_7\omega_8\omega_9\}\\ \mathcal{P}_3 &= \{\omega_1\omega_4\omega_7, \omega_2\omega_5\omega_8, \omega_3\omega_6\omega_9\}.\end{aligned}$$

In every period agents have to choose one of three actions, $A = \{a, b, c\}$. The utility function is given by

$$\begin{aligned}u(a,\omega) &= \begin{cases} 1 \text{ if } \omega = 1,2,4,5,6,8 \\ 0 \text{ otherwise}\end{cases} \quad u(b,\omega) = \begin{cases} 1 \text{ if } \omega = 1,2,4,5,6,8,9 \\ 0 \text{ otherwise}\end{cases}\\ u(c,\omega) &= \begin{cases} 1 \text{ if } \omega = 2,4,5,6,8,9 \\ 0 \text{ otherwise}\end{cases}\end{aligned}$$

Let the strategies be such that agents 1 and 3 select a whenever a is optimal and action c whenever c is optimal and a is not, while agent 2 selects action b whenever it is optimal.[4] Therefor agent 1 selects action a when observing row one or row two and action c when observing row three. Agent 2 selects action b and agent 3 selects a when observing column one or column two and action c when observing column 3. Agent 2 has no private information to reveal through his first period action which implies that the information sets of the other two agents do not change after observing the first period choice of agent 2. Let us consider the set of possible information sets of agent 2 at the beginning of the second round and the corresponding action prescribed by agent 2's strategy.

[4] This does not provide a complete description of the strategy of agent β but is all we require in the given example.

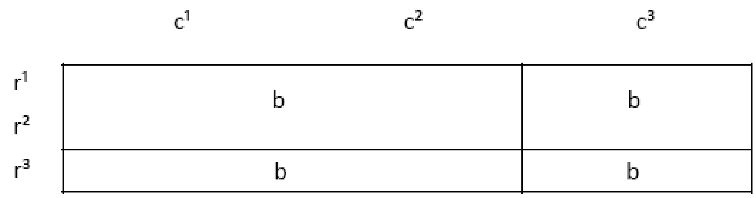

Figure 2: Information sets of agent 2 in round two and the corresponding action.

Therefor no inference can be made of agent 2's second period choice and learning ends for all agents. Suppose that state ω_3 is realized. Then agent 1 selects action a, which is not optimal for agent 3 conditioning on his information, and agent 3 selects action c which is not optimal for agent 1 conditioning on his information. Therefore global indifference fails in state ω_3.

3.2 Optimal Information Aggregation in Networks

In this subsection I provide the steps of the learning process in the incomplete network, exemplary for event Q_1 being realized. In the first stage, the information sets of agents are given by their partition cell, $I_1^1(\omega) = r^1$, $I_2^1(\omega) = d^1$, and $I_3^1(\omega) = c^1$, leading to first period announcements of $q_1^1(\omega) = \frac{1}{3}$, $q_2^1(\omega) = \frac{7}{18}$, and $q_3^1(\omega) = \frac{5}{12}$.

The second stage information sets are given by

$$\begin{aligned} I_1^2(r^1, h_1^2(\omega)) &= r^1 \cap d^1 \\ I_2^2(d^1, h^2(\omega)) &= \left(r^1 \cup r^2\right) \cap d^1 \cap \left(c^1 \cup c^2\right) \\ I_3^2(c^1, h_3^2(\omega)) &= d^1 \cap c^1. \end{aligned}$$

There is common knowledge among player 1 and 2 that player 2 faces an information set within $\mathcal{I}_2^2(h_{12}^2(\omega); \omega)$ where

$$\mathcal{I}_2^2(h_{12}^2(\omega); \omega) = \left\{ \left(r^1 \cup r^2\right) \cap d^1 \cap \left(c^1 \cup c^2\right), \left(r^1 \cup r^2\right) \cap d^1 \cap c^3 \right\}.$$

The posterior probabilities depending on the information set of agent 2 are

$$\begin{aligned} q_2^2\left(\left(r^1 \cup r^2\right) \cap d^1 \cap \left(c^1 \cup c^2\right)\right) &= \frac{1}{2} \\ q_2^2\left(\left(r^1 \cup r^2\right) \cap d^1 \cap c^3\right) &= 0. \end{aligned}$$

Thus at the beginning of stage three agent 1 learned the second stage information set of agent 2. Based on the common observables of agents 1 and 2 the information set of agent 1 is an element of

$\mathcal{I}_1^2(h_{12}^2(\omega);\omega)$ where

$$\mathcal{I}_1^2(h_{12}^2(\omega);\omega) = \left\{r^1 \cap d^1, r^2 \cap d^1\right\},$$

leading to the following probability announcements

$$\begin{aligned} q_1^2(r^1 \cap d^1) &= \frac{2}{5} \\ q_1^2(r^2 \cap d^1) &= \frac{1}{2}. \end{aligned}$$

Thus player 2 learns the true cell of agent 1 through his second stage announcements. Based on the common observables of agent 2 and 3 the set of possible information sets of agent 1 from perspective of agent 2 is

$$\mathcal{I}_1^2(h_{12}^2(\omega);\omega) = \left\{r^1 \cap d^1, r^2 \cap d^1\right\}$$

if agent 1 announced a probability of $\frac{1}{3}$ in the first stage. Otherwise the information set of agent 1 is commonly known among 1 and 2 to be equal to

$$I_1^2(r^3, h_1^2(\omega)) = r^3 \cap d^1.$$

For all three possible information sets agent 1 announces a different probability as

$$q_1^2(r^3 \cap d^1) = 0.$$

Thus it is common knowledge among all agents that player 2 learned the true cell of agent 1 at the outset of stage three. Based on the common observables of agents 2 and 3 it is common knowledge that agent 2's second stage information set is contained in

$$\mathcal{I}_2^2(h_{23}^2(\omega);\omega) = \left\{\left(r^1 \cup r^2\right) \cap d^1 \cap \left(c^1 \cup c^2\right), r^3 \cap d^1 \cap \left(c^1 \cup c^2\right)\right\},$$

leading to the following second stage announcements of agent 2

$$\begin{aligned} q_2^2\left(\left(r^1 \cup r^2\right) \cap d^1 \cap \left(c^1 \cup c^2\right)\right) &= \frac{1}{2} \\ q_2^2\left(r^3 \cap d^1 \cap \left(c^1 \cup c^2\right)\right) &= 0. \end{aligned}$$

Thus agent 3 learns the second stage information set of agent 2 through his second stage announcement. Based on the common observables of player 2 and player 3 it is common knowledge that the second stage information set of agent 3 is contained in

$$\mathcal{I}_3^2(h_{23}^2(\omega);\omega) = \left\{d^1 \cap c^1, d^1 \cap c^2\right\},$$

leading to the following second stage announcements of agent 2

$$\begin{aligned} q_2^2(d^1 \cap c^1) &= \frac{2}{5} \\ q_1^2(d^1 \cap c^2) &= \frac{1}{2}. \end{aligned}$$

Thus player 2 learns the true cell of agent 3 through his second stage announcement. Based on the common observables of agent 1 and 2 it is common knowledge that the set of possible information sets of agent 3 from perspective of agent 2 is equal to

$$\mathcal{I}_3^2(h_{23}^2(\omega);\omega) = \left\{d^1 \cap c^1, d^1 \cap c^2\right\}$$

if agent 3 announces a posterior of $q_3^1 = \frac{5}{12}$ in the first period. If agent 3 announced a posterior of $q_3^1 = 0$ in the first stage it is common knowledge among 2 and 3 that the true information set of agent 3 equals

$$I_3^2(c^3, h_3^2(\omega)) = d^1 \cap c^3$$

leading to

$$q_2^2(d^1 \cap c^3) = 0.$$

All three possible second stage information sets of agent 3 lead to different posterior announcements. Thus it is common knowledge among all agents that agent 2 knows the true cell of agent 3 at the beginning of stage three, if the true cell of agent 2 is equal to d^1.

Based on the considerations above it is common knowledge among all agents that agent 2 knows the true cell of the partition of agent 1 as well as of agent 3 at the beginning of stage three. Agents announce their second stage posteriors $q_1^2 = \frac{2}{5}$, $q_2^2 = \frac{1}{2}$ and $q_3^2 = \frac{2}{5}$ leading to the following third stage information sets

$$\begin{aligned} I_1^3(r^1, h_1^3(\omega)) &= r^1 \cap d^1 \cap \left(c^1 \cup c^2\right) \\ I_2^3(d^1, h^3(\omega)) &= r^1 \cap c^1 \\ I_3^3(c^1, h_3^3(\omega)) &= \left(r^1 \cup r^2\right) \cap d^1 \cap c^1 \end{aligned}$$

and announcements $q^3 = (\frac{1}{2}, \frac{1}{4}, \frac{1}{2})$. It is common knowledge among all players that player 2 knows the true cell of the join at the beginning of stage three. Thus after observing the third period announcement of agent 2 the other two agents will announce the third period posterior of agent 2 from period $t = 4$ on forward. For all periods $t \geq 4$ we have

$$q_1^t(r^1 \cap c^1) = q_2^t(r^1 \cap c^1) = q_3^t(r^1 \cap c^1) = \frac{1}{4}.$$

3.3 A Union Consistent Choice Correspondence without EU Representation

Let the state space contain three possible states $\Omega = \{1, 2, 3\}$ and the action space be given by $A = \{\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}\}$. The choice correspondence is given by

$$\begin{aligned} c(123) &= c(1) = \mathbf{a} \\ c(2) &= c(23) = \mathbf{b} \\ c(12) &= c(13) = \mathbf{d} \\ c(3) &= \mathbf{c}. \end{aligned}$$

Union consistency is not violated as it does not apply. No subset of Ω can be partitioned in sets that have an optimal action in common. For this state space, action set and choice correspondence there exist no pair of probability measure over Ω and utility function $u : A \times \Omega \to \mathbb{R}$ that is consistent with the above choice correspondence.

The choice made when the state is known imply

$$\begin{aligned} u(\mathbf{a}, 1) &> u(x, 1) \; \forall x \in A \setminus \mathbf{a} \\ u(\mathbf{b}, 2) &> u(x, 2) \; \forall x \in A \setminus \mathbf{b} \\ u(\mathbf{c}, 3) &> u(x, 3) \; \forall x \in A \setminus \mathbf{c}. \end{aligned}$$

The optimal choices when the information set contains two states imply that the probability of each state has to be positive, in particular $c(12), c(1), c(2)$ imply $p_1, p_2 > 0$ and $c(13), c(1), c(3)$ imply $p_1, p_3 > 0$. Furthermore $c(12) = \mathbf{d}$ and $c(13) = \mathbf{d}$ imply

$$\begin{aligned} u(\mathbf{d}, 2) &> u(\mathbf{a}, 2) \\ u(\mathbf{d}, 3) &> u(\mathbf{a}, 3) \end{aligned}$$

and

$$p(1) \times u(\mathbf{d}, 1) + p(2) \times u(\mathbf{d}, 2) > p(1) \times u(\mathbf{a}, 1) + p(2) \times u(\mathbf{a}, 2)$$

$$p\,(1) \times u(\mathbf{d}, 1) + p\,(3) \times u(\mathbf{d}, 3) > p\,(1) \times u(\mathbf{a}, 1) + p\,(3) \times u(\mathbf{a}, 3).$$

Summing up these two inequalities yields

$$2p(1) \times u(\mathbf{d}, 1) + p(2) \times u(\mathbf{d}, 2) + p(3) \times u(\mathbf{d}, 3) > 2p(1) \times u(\mathbf{a}, 1) + p(2) \times u(\mathbf{a}, 2) + p(3) \times u(\mathbf{a}, 3).$$

As $u(\mathbf{a}, 1)$ is greater than $u(\mathbf{d}, 1)$ we have

$$p(1) \times u(\mathbf{d}, 1) + p(2) \times u(\mathbf{d}, 2) + p(3) \times u(\mathbf{d}, 3) > p(1) \times u(\mathbf{a}, 1) + p(2) \times u(\mathbf{a}, 2) + p(3) \times u(\mathbf{a}, 3)$$

which contradicts $c(123) = \mathbf{a}$.