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On the impossibility of core-selecting auctions

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When goods are substitutes, the Vickrey outcome is in the core and yields competitive seller revenue. In contrast, with complements, the Vickrey outcome is efficient but not necessarily in the core and revenue can be low. Non-core outcomes may be perceived as unfair since there are bidders willing to pay more than the winners' payments. Moreover, non-core outcomes render the auction vulnerable to defections, as the seller can attract better offers afterward. To avoid instabilities of this type, Day and Raghavan (2007), Day and Milgrom (2008), and Day and Cramton (2012) have suggested adapting the Vickrey pricing rule so that outcomes are in the core with respect to bidders' *reported* values.

If truthful bidding were an equilibrium of the resulting auction, then the outcome would also be in the core with respect to bidders' *true* values. We show, however, that when the equilibrium outcome of any auction is in the core, it is equivalent to the Vickrey outcome. In other words, if the Vickrey outcome is not in the core, no core-selecting auction exists. Our results further imply that the competitive equilibrium outcome, which always exists when goods are substitutes, can only be implemented when it coincides with the Vickrey outcome. Finally, for a simple environment, we show that compared to Vickrey prices, the adapted pricing rule yields lower expected efficiency and revenue as well as outcomes that are on average further from the core.

Keywords. Core outcomes, Vickrey auction, substitutes, complements, competitive equilibrium, Bayesian implementability.

JEL classification. D44.

1. Introduction

Practical auction design is often complicated by institutional details and legal or political constraints. For example, using bidder-specific bidding credits or reserve prices may be considered discriminatory and unlawful in some countries, making it impossible to implement an optimal auction design. More generally, the use of sizeable reserve prices

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may cause political stress due to the fear that it slows down technological progress when licenses remain unsold. While constraints of this nature are common and important in practice, mechanism design theory has typically treated them as secondary to incentive constraints.

Recent work by Day and Raghavan (2007), Day and Milgrom (2008), and Day and Cramton (2012) breaks with this tradition and asks how close incentive constraints can be approximated if other (institutional or political) constraints are put first. In particular, these authors have proposed an alternative payment rule to fix some drawbacks of the well known Vickrey–Clarke–Groves mechanism, or "Vickrey auction" for short. When goods are substitutes, the Vickrey auction produces an outcome, i.e., an allocation and payoffs for the seller and bidders, that is in the core. However, when goods are complements, the Vickrey outcome, while efficient, is not necessarily in the core and seller revenue can be very low as a result. Furthermore, non-core outcomes are "unfair" in that there are bidders willing to pay more than the winners' payments, which makes the auction vulnerable to defections as the seller can attract better offers afterward. The low revenue, perceived unfairness, and instability of Vickrey outcomes can create legal and political problems, which the alternative payment rule seeks to avoid.

The types of auctions proposed by Day et al. employ a payment rule that insures that outcomes are in the core with respect to *reported* values, i.e., the final allocation maximizes the total reported value and no coalition of bidders can block the outcome with respect to bidders' reports. Unless reported values are always truthful, however, it does *not* imply that core outcomes are produced with respect to bidders' *true* preferences. To distinguish these two cases, we use core to mean "core with respect to true values" and core* to mean "core with respect to reported values," and we refer to the auctions proposed by Day et al. as core*-selecting auctions.

Ideally, a core*-selecting auction produces outcomes that are in the core so that they are stable. The well known Green and Laffont (1977) and Holmstrom (1979) theorem states that the Vickrey auction is the only efficient auction with a dominant-strategy equilibrium. Thus, the Vickrey auction is the only candidate for core implementation in dominant strategies. Under the weaker notion of Bayesian Nash equilibrium, the Vickrey auction is no longer the unique efficient mechanism (e.g., in the single-unit, symmetric case, all standard auction formats are efficient in equilibrium). But we show that *any* core-selecting auction must be equivalent to the Vickrey auction in the sense that, for every possible valuation profile, the seller's revenue and bidders' profits are ex post identical. In other words, imposing dominant strategies comes at no cost for core implementation.

Our approach builds on previous work by Krishna and Perry (1998), Williams (1999), and Krishna and Maenner (2001), who generalize Myerson's "payoff equivalence" result and establish equivalence between the expected revenue of the Vickrey auction and the *expected* revenue of any efficient mechanism. Moreover, Ausubel and Milgrom (2002) show that the core constraints imply that the *ex post* revenue of any efficient mechanism can be no less than that of the Vickrey auction. Combining these results enables us to show that, in equilibrium, any efficient core-selecting auction yields the same ex post profits for the seller and bidders as the Vickrey auction. In other words, if the Vickrey outcome is in the core, then it is the unique Bayesian Nash (and unique dominant-strategy) implementable outcome and otherwise no core outcome can be implemented.

For the case of substitutes, it is well known that the Vickrey outcome is in the core. Our results imply that other mechanisms that result in different outcomes, including core*-selecting auctions, yield outcomes outside the core. When goods are not substitutes and the Vickrey outcome is not in the core, our results imply that *no* auction produces equilibrium outcomes that are in the core. In addition, as a competitive equilibrium with linear prices is a core outcome that may differ from the Vickrey outcome, our results imply that the competitive equilibrium outcome cannot always be implemented.

We also illustrate that core*-selecting auctions may perform worse than the Vickrey auction. We consider the BCV mechanism[1] proposed by Day and Cramton (2012), which has the following properties: (i) the outcome is in the core*, (ii) bidders' profits are maximized, and (iii) payments are as close as possible to the original Vickrey payments. For complete-information environments it has been shown that the BCV auction yields the Vickrey outcome when it is in the core and results in higher seller revenues when it is not. Moreover, with complete information, the BCV auction minimizes the maximal gain from deviating from truthful bidding (e.g., Day and Milgrom 2008).[2]

These positive results, however, rely crucially on the assumption that bidders' values and, hence, their bids are commonly known. In most practical applications, bidders' values constitute proprietary information. We therefore consider a simple incomplete-information environment where two local bidders interested in single items compete with a global bidder interested in the package. Bidders' values are privately known and uniformly distributed. We show that the BCV auction results in revenues on average lower than Vickrey revenues and outcomes that are inefficient and on average *further* from the core than Vickrey outcomes.[3] The reason for these negative results is that bidders no longer have a dominant strategy to bid truthfully.

To summarize, recent literature on "practical auction design" relaxes incentive constraints and focuses on the stability and fairness of outcomes. In particular, core*-selecting auctions yield stable and fair outcomes with respect to the bids, which are treated as exogenous parameters without making any assumptions about bidders' behavior, i.e., how their bids relate to their private information. In contrast, a core-selecting auction produces stable and fair outcomes with respect to bidders' true values assuming Bayesian Nash equilibrium behavior. Our results show that if a core-selecting auction exists, it is identical to the Vickrey auction. Hence, no core-selecting auction exists if the Vickrey auction is not core-selecting, in which case, the Vickrey auction may outperform the recently proposed core*-selecting auctions.

[1] BCV stands for bidder optimal, core* selecting, and Vickrey nearest.

[2] It is important to point out that bidders' maximal incentives for (possibly large) deviations are minimized. Erdil and Klemperer (2010) show that bidders always have marginal incentives to deviate from truthful bidding, and they propose the introduction of reference prices to reduce bidders' incentives for such marginal deviations.

[3] Our results do not rule out that the BCV auction may outperform the Vickrey auction in other environments. For example, Ausubel and Baranov (2013) show that when local bidders' values are correlated, the BCV auction performs better than the Vickrey auction if the degree of correlation is sufficiently high.

This paper is organized as follows. In the next section, we introduce the bidding environment and prove that core-selecting auctions are generally not possible. In Section 3, we explain the construction of BCV prices, analyze the Bayesian Nash equilibrium of the BCV auction, and evaluate its performance in terms of revenue and efficiency. Section 4 concludes.

2. Model and main result

The model follows Ausubel and Milgrom (2002) and Krishna and Maenner (2001). The set of auction participants, L, consists of the seller, labeled $l = 0$, and the buyers, labeled $l = 1, \ldots, |L| - 1$. Buyer l's type is K-dimensional with $t_l \in [0,1]^K$ and buyers' types are independently distributed according to a probability measure with full support. Let $t = (t_l \in [0,1]^K; l \in L \setminus 0)$ denote the profile of types. There are N indivisible items for sale, with Ω being the set of items. Let 2^Ω denote the power set of Ω, i.e., the set of all subsets of Ω. An allocation is a set $x = \{x_l \mid x_l \in 2^\Omega; l \in L \setminus 0\}$, where x_l denotes the package or bundle assigned to buyer l. An allocation is feasible if for any $\omega \in \Omega$ and $l, l' \in L \setminus 0$ with $l \neq l'$, $\omega \in x_l$ implies $\omega \notin x_{l'}$. The set of all feasible allocations is denoted by X. Buyer l values allocation $x \in X$ at $v_l(x,t) \in \mathbb{R}$, while the seller has no value for the items: $v_0(\cdot,\cdot) = 0$. A buyer with type 0 has zero value, $v_l(x,0) = 0$, as does a buyer who receives no items, $v_l(\{\varnothing, x_{-l}\}, t) = 0$. As in Krishna and Maenner (2001, Hypothesis I) we assume that buyer l's valuation function $v_l(x, \{t_l, t_{-l}\})$ is convex in t_l.[4] Finally, utilities are quasilinear, i.e., when the allocation is x and buyer l pays a price $h \in \mathbb{R}_+$ her utility is $u_l(h,x,t) = v_l(x,t) - h$.

An (indirect) auction mechanism consists of a feasible allocation rule $\hat{x}_l(b)$ and a payment rule $\hat{h}_l(b)$ based on bids, or reported values, $b = (b_l(x_l) \in \mathbb{R}_+; x_l \in 2^\Omega, l \in L \setminus 0)$ with $b_0(\cdot) = 0$.[5] The coalitional value for $S \subset L$ with respect to reported values is given by $B(S) = \max_{x \in X} \sum_{l \in S} b_l(x_l)$ if $0 \in S$ and $B(S) = 0$ otherwise. With this definition of the coalitional value, the core* can be defined as

$$\text{core}^*(L,B) = \left\{ \pi \;\middle|\; \sum_{l \in L} \pi_l = B(L), (\forall S \subset L)\; B(S) \leq \sum_{l \in S} \pi_l \right\}.$$

Definition 1. A core*-selecting auction $\{\hat{x}_l(b), \hat{h}_l(b)\}$ is such that $\hat{\pi}_l(b) \equiv b_l(\hat{x}_l(b)) - \hat{h}_l(b)$ for $l = 1, \ldots, |L| - 1$ and $\hat{\pi}_0(b) \equiv \sum_{l \in L \setminus 0} \hat{h}_l(b)$ satisfy $(\hat{\pi}_l(b); l \in L) \in \text{core}^*(L,B)$.

This definition takes bids as exogenous parameters. To define core-selecting auctions that yield allocations in the core with respect to bidders' true valuations, an assumption is needed about how bidders' private information translates into bids. We

[4]Together with convexity of the type space, this assumption guarantees payoff equivalence; see the proof of Proposition 1 in Krishna and Maenner (2001). For alternative sufficient conditions for payoff equivalence, see Hypothesis II in Krishna and Maenner (2001) or Williams (1999), who shows that equivalence requires that the interim expected valuation $V_l(t_l^* \mid t_l) \equiv E_{t_{-l}}[v_l(x(t_l^*, t_{-l}), t_l)]$ of each agent is continuously differentiable in (t_l^*, t_l) at points that satisfy $t_l^* = t_l$. Note that $v_l(x,\cdot)$ depends only on buyer l's own type in Williams' paper.

[5]Note that in core*-selecting auctions, a bidder's reported values only depend on her own allocation and not that of others, i.e., reported values do not reflect allocative externalities (even if actual values do). If a bidder does not report a value for a certain package, then this package is assumed to have zero value.

assume equilibrium behavior. Let $b^*(t)$ denote the Bayesian Nash equilibrium bids of an auction $\{\hat{x}_l(b), \hat{h}_l(b)\}$ and let $x_l(t) = \hat{x}_l(b^*(t))$, $\pi_l(t) = v_l(\hat{x}(b^*(t)), t) - \hat{h}_l(b^*(t))$ for $l = 1, \ldots, |L| - 1$, and $\pi_0(t) = \sum_{l \in L \setminus 0} \hat{h}_l(b^*(t))$ be the Bayesian Nash equilibrium allocations and payoffs, respectively. We refer to $\{x_l(t), \pi_l(t)\}$ as the (equilibrium) *outcome* of the auction mechanism.

The set of possible allocations is compact, so $\arg\max_{x \in X} \sum_{l \in S} v_l(x, t)$ is nonempty for all t. Given the type profile, t, the coalitional value is given by $w(S, t) = \max_{x \in X} \sum_{l \in S} v_l(x, t)$ if $0 \in S$ and $w(S, t) = 0$ otherwise. With this definition of the coalitional value, the core can be defined as

$$\text{core}(L, w, t) = \left\{ \pi \;\Big|\; \sum_{l \in L} \pi_l = w(L, t), (\forall S \subset L)\; w(S, t) \le \sum_{l \in S} \pi_l \right\}.$$

Definition 2. An auction $\{\hat{x}_l(b), \hat{h}_l(b)\}$ is core-selecting if the outcome $\{x_l(t), \pi_l(t)\}$ satisfies $(\pi_l(t); l \in L) \in \text{core}(L, w, t)$ for all t.

A direct corollary of this definition is that a core-selecting auction is efficient, losing bidders pay nothing, and winning bidders' payoffs are nonnegative.

Recall that two auctions, $\{\hat{x}_l(b), \hat{h}_l(b)\}$ and $\{\hat{x}'_l(b), \hat{h}'_l(b)\}$, are *interim* payoff equivalent if, for all $t_l \in [0, 1]^K$ and $l \in L$, the associated equilibrium payoffs satisfy $E_{t_{-l}}[\pi_l(t)] = E_{t_{-l}}[\pi'_l(t)]$. The stronger notion of ex post payoff equivalence requires that, for all t, $\pi_l(t) = \pi'_l(t)$.

Proposition 1. *Any core-selecting auction is ex post payoff equivalent to the Vickrey auction.*[6] *Hence, if the Vickrey auction is not core-selecting, no core-selecting auction exists.*

Proof. Bidders' types are revealed ex post and Ausubel and Milgrom (2002, Theorem 5) show that bidder l's Vickrey payoff $\pi_l^V(t) = w(L, t) - w(L \setminus l, t)$ is her highest payoff in the core. Thus, the payoff from any core-selecting auction satisfies $\pi_l(t) \le \pi_l^V(t)$ for all l and t. The model assumptions detailed at the start of the section match those in Krishna and Maenner (2001), which implies that we can apply their interim payoff equivalence result (see their Proposition 1): the expected payoff function of a Bayesian incentive compatible mechanism is, up to an additive constant, determined by the allocation rule. The assumption that a buyer with the lowest type 0 has zero value implies that the additive constant is 0.[7] In addition, any core-selecting auction employs the same efficient allocation rule.[8] Interim payoff equivalence thus implies that bidder l's

[6]Recall that the Vickrey auction is a direct auction mechanism $\{x_l^V(t), h_l^V(t)\}$ that truthfully implements an efficient allocation $x(t)$ in dominant strategies. Specifically, $x_l^V(\cdot) = x_l(\cdot)$ and $h_l^V(t) = v_l(x(t), t) - w(L, t) + w(L \setminus l, t)$.

[7]When bidder l's type is 0, the Vickrey payoff, $\pi_l^V(t) = w(L, t) - w(L \setminus l, t)$, is zero since no value results from assigning items to bidder l. Since the Vickrey payoff is the highest payoff in the core, $\pi_l(t) \le \pi_l^V(t)$, this implies that the core payoff of a bidder with type 0 is necessarily zero.

[8]When there are multiple efficient allocations, e.g., in case of a tie, we assume that the Vickrey and core-selecting auction select the same allocation.

interim expected payoff in any Bayesian incentive compatible, efficient mechanism is the same as in the Vickrey auction: $E_{t_{-l}}[\pi_l(t)] = E_{t_{-l}}[\pi_l^V(t)]$. Therefore, $\pi_l(t) = \pi_l^V(t)$ for almost all types. □

Remark 1. Only a small subset of core constraints is required for the above result, namely the ones that involve L, $L \setminus l$, and l for all $l \in L$.

Remark 2. Our results differ from the Green and Laffont (1977) and Holmstrom (1979) theorem, which establishes conditions under which the Vickrey auction is the unique efficient auction with a dominant-strategy equilibrium. We relax dominant-strategy equilibrium to Bayesian Nash equilibrium and replace the condition that losing bidders pay nothing with the requirement that the outcome must be in the core: we show that the only possibility is still the Vickrey auction.[9]

Remark 3. Auction papers that employ incomplete-information environments like the one studied here typically focus on only a single core constraint: ex post efficiency. The rationale is that the auction "should put the items in the hands of those that value them the most," i.e., no further gains from trade are possible and no aftermarket is needed. The main idea behind imposing additional ex post core constrains is that sales prices should be such that the seller cannot benefit from forming a coalition with a set of bidders different from the winning bidders. Recall that, by the revelation principle, the seller is able to infer bidders' values ex post and will want to sell to a different set of bidders if the outcome is not in the core. To summarize, the notion of ex post core is useful in incomplete-information environments because it guarantees that the seller does not renege on the auction outcome and no aftermarket is needed.

When does the Vickrey auction result in a core outcome? Ausubel and Milgrom (2002, Theorem 12) show that the sufficient and necessary condition is that goods are substitutes (see also Gul and Stacchetti 1999).[10] Specifically, following Ausubel and Milgrom (2002), let V be a collection of valuation profiles, where a valuation profile $\{v_l(\cdot)\}_{l\in L}$ is a set containing the seller's and bidders' valuation functions. Denote by V_{Sub} the set that contains all valuation profiles such that goods are substitutes for all bidders and types. Similarly, denote by V_{Add} the set of all valuation profiles containing solely additive valuations.[11]

Corollary 1. *For $V \supseteq V_{\text{Add}}$, a core-selecting auction exists if and only if $V \subseteq V_{\text{Sub}}$.*

[9] We also assume independence of types, which is obviously not required with ex post incentive compatibility.

[10] Here we assume that buyers' valuations do not depend on packages received by others. Recall that given t, goods are substitutes to a buyer if whenever under certain prices, the buyer's demand includes an item, he never drops the item if the prices for other items increase. Specifically, given a vector of item prices $p \in \mathbb{R}_+^{|\Omega|}$, the buyer's demand $x_l^*(p, t)$ solves $\max_{x_l \subset \Omega}(v_l(x_l, t) - \sum_{k\in x_l} p_k)$. Goods are substitutes to buyer l if for any t and p, item $m \in x_l^*(p, t)$ implies $m \in x_l^*(p', t)$ for any p' with $p'_m = p_m$ and $p'_j \geq p_j$ for $j \neq m$.

[11] With additive valuations, the value of a package is the sum of the values for individual items in that package.

This result follows from Theorem 12 in Ausubel and Milgrom (2002), which shows that for $V \supseteq V_{\text{Add}}$, the Vickrey auction is core-selecting if and only if $V \subseteq V_{\text{Sub}}$.[12] This fact, together with Proposition 1, yields Corollary 1.

Our results also have implications for the existence of mechanisms that lead to Walrasian, or equivalently, competitive equilibrium (CE), outcomes. The definition below follows the notation in Definition 2.

Definition 3. An auction $\{\hat{x}_l(b), \hat{h}_l(b)\}$ is CE-selecting if there exist item prices $\{\hat{p}_j(b)\}_{j\in\Omega}$ such that $\hat{h}_l(b) = \sum_{j\in\hat{x}_l(b)} \hat{p}_j(b)$ and the outcome $\{x_l(t), \pi_l(t)\}$ satisfies $x_l(t) \in \arg\max_{y\subseteq\Omega}\{v_l(y,t) - \sum_{j\in y} p_j(t)\}$ and $\pi_l(t) = v_l(x_l(t), t) - \sum_{j\in x_l(t)} p_j(t)$ for all $l \in L \setminus 0$ and all t, where $p_j(t) \equiv \hat{p}_j(b^*(t))$.

Since a CE-selecting auction is necessarily core-selecting, Proposition 1 implies the following corollary.

Corollary 2. *Any CE-selecting auction is ex post payoff equivalent to the Vickrey auction. Hence, if the Vickrey auction is not CE-selecting, no CE-selecting auction exists.*

To illustrate, suppose there are $2M$ identical items (with $M \geq 2$) and two bidders with valuation functions $v_i(m,t) = v(m)t_i$ for $i = 1, 2$, where the t_i are independently distributed and $v(m)$ is a strictly concave function for $0 \leq m \leq 2M$. Consider type profiles for which the types are "close," e.g., $t_1 = t(1+\epsilon)$ and $t_2 = t(1-\epsilon)$ with ϵ small. An efficient outcome dictates that both bidders get M units and the resulting Vickrey payments are approximately $(v(2M) - v(M))t$ for both bidders. In other words, the Vickrey per-unit price is

$$p^V = \frac{v(2M) - v(M)}{M} t.$$

A lower bound for the competitive equilibrium price, p, follows from the requirement that at price p, neither bidder desires an additional unit: $v(M)t - Mp \geq v(M+1)t - (M+1)p$ or

$$p \geq \frac{v(M+1) - v(M)}{1} t,$$

which exceeds the Vickrey per-unit price by strict concavity of $v(\cdot)$. Thus, in any CE-selecting auction, if it exists, the competitive equilibrium prices are higher than the Vickrey prices for a positive measure of types, resulting in lower bidder payoffs than those in the Vickrey auction. Therefore, Corollary 2 implies that a CE-selecting auction does not exist.

This result has important implications for equilibrium behavior in commonly used auction formats. For example, it implies that straightforward bidding cannot be an equilibrium of the simultaneous ascending auction for all possible valuation profiles, since straightforward bidding results in competitive equilibrium prices (e.g., Milgrom 2004).

[12] If $V \nsubseteq V_{\text{Sub}}$, then goods are not substitutes for some bidder in some valuation profile in V. Given such a bidder's valuation function, Ausubel and Milgrom (2002, Theorem 12) show how to construct additive valuations (or choose valuations from V_{Add}) for three additional bidders such that under the valuation profile consisting of these four bidders, the Vickrey auction is not core-selecting.

3. BCV auction: An example

Consider an environment with two local bidders and a single global bidder who compete for two items labeled **A** and **B**. Local bidder 1 (2) is interested only in acquiring item **A** (**B**), for which she has value $v_1 \in \mathbb{R}$ ($v_2 \in \mathbb{R}$). The global bidder 3 is interested only in the package **AB** consisting of both items, for which she has value $V \in \mathbb{R}$. For simplicity, we assume bidders can only bid on packages they are interested in. Let b_i be local bidder i's bid (or reported value) for the item she is interested in and let B be the global bidder's bid for package **AB**.

In the Vickrey auction, bidders simply bid their values for the object they are interested in: $b_i(v_i) = v_i$ and $B(V) = V$. This yields a fully efficient outcome, i.e., the global bidder wins all items if and only if (iff) $B > b_1 + b_2$ or, equivalently, iff $V > v_1 + v_2$. While the Vickrey auction generates full efficiency, it is well known that it may result in low revenues when the outcome is not in the core (e.g., Ausubel and Milgrom 2006).

The BCV auction $\{\hat{x}_l(b), \hat{h}_l(b)\}$ selects bidder-optimal points[13] in the core* and minimizes the distance, $(\sum_{l \subset L\setminus 0}(\hat{h}_l(b) - \hat{h}_l^V(b))^2)^{1/2}$, between the Vickrey payment $\hat{h}_l^V(b)$ and the BCV auction payment $\hat{h}_l(b)$, where $b \equiv \{b_1, b_2, B\}$.[14]

Definition 4. For the simple environment studied here, the BCV auction $\{\hat{x}_l(b), \hat{h}_l(b)\}$ is characterized by payments $\hat{h}_3(b) = (b_1 + b_2)\mathbf{1}_{B > b_1 + b_2}$ for the global bidder and

$$\hat{h}_i(b) = \Big(\max(0, B - b_{-i}) + \tfrac{1}{2}(B - \max(0, B - b_i) - \max(0, B - b_{-i}))\Big)\mathbf{1}_{b_1 + b_2 \geq B}$$

for local bidders $i \in \{1, 2\}$. The allocation rule $\hat{x}_l(b)$ is the same as in the Vickrey auction: the global bidder wins the package **AB** if $B > b_1 + b_2$; otherwise each local bidder wins the item she is interested in.

When the global bidder wins, the outcome is always in the core* with respect to submitted bids, so there are no adjustments to the global bidder's payment. As a result, the global bidder's strategy is unaffected, i.e., truthful bidding remains optimal. For the local bidders, truthful bidding is no longer optimal, however, since their own bids affect their payments.

The next result shows the degree to which local bidders "shade" their bids in response to the change in payment rule. We assume that local bidders' values are uniformly distributed on $[0, 1]$ and the global bidder's value for the package is uniformly distributed on $[0, 2]$.

Result 1. A Bayesian Nash equilibrium of the BCV auction is given by $B(V) = V$ and

$$b(v) = \max(0, v - \alpha),$$

where $\alpha = \frac{1}{2}E(b(v)) = 3 - 2\sqrt{2}$.

[13] The outcome $(\hat{\pi}_l; l \in L)$ in the core* is bidder-optimal in the core* if no other outcome $(\hat{\pi}'_l; l \in L)$ in the core* exists such that $\hat{\pi}'_l \geq \hat{\pi}_l$ for all $l \in L \setminus 0$ and $\hat{\pi}'_l > \hat{\pi}_l$ for some l.

[14] See Day and Cramton (2012) for more details.

Result 1 (see the Appendix for a proof) shows that the introduction of BCV prices creates incentives for local bidders to "free ride," i.e., each local bidder wants to win but prefers other local bidders to bid high. The consequences for the auction's performance are easy to calculate. A direct computation shows that the relative average efficiency of the core*-selecting auction is $E_{\mathrm{BCV}}/E_V = 98\%$, relative average revenue is $R_{\mathrm{BCV}}/R_V = 91\%$, and the relative average Euclidean distance to the core is $d_{\mathrm{BCV}}/d_V = 126\%$.[15] We can thus come to the following conclusion.

Result 2. Compared to the Vickrey auction, the BCV auction has lower expected efficiency and revenue and it produces outcomes that are on average further away from the core.

The main insight of Result 1, i.e., that truthful bidding is not an equilibrium in the BCV auction, holds under more general conditions.[16]

4. Conclusion

The BCV auction has some remarkable properties in complete-information environments where bidders' values and, hence, their bids are commonly known (e.g., Day and Milgrom 2008). In particular, when bids are completely predictable, the introduction of BCV prices ensures outcomes that are "fair" and seller revenues that are not embarrassingly low.

This paper considers the performance of the BCV auction for the realistic case when bidders' values are privately known and, hence, their bids are not perfectly predictable. In such incomplete-information environments, if truthful bidding were optimal, then the BCV auction would reliably outperform the Vickrey auction. However, our analysis shows that truthful bidding is *not* an equilibrium. We show that the BCV auction may result in lower expected revenue and efficiency as well as outcomes that are on average *further* from the core than Vickrey outcomes.

We study whether core allocations can be achieved by any mechanism in an environment where goods can be substitutes and/or complements. We prove that the ex post surplus, seller's revenue, and bidders' profits in any core-selecting auction are identical to their Vickrey counterparts, i.e., any core-selecting auction is equivalent to the Vickrey auction. A fortiori, if the Vickrey outcome is outside the core, no core-selecting auction exists.

Our impossibility result is akin to Myerson and Satterthwaite's (1983) finding that efficient bilateral trade is not generally possible. In both cases the intuition is that incentive compatibility requires that market participants have information rents (reflecting

[15] In this example, the average distance from the core is $\sqrt{E_t[\sum_{S\subset L} \max\{0, w(S,t) - \sum_{l\in S} \pi_l(t)\}^2]}$, where $\pi_l(t)$ is agent l's profit from the mechanism in consideration, specifically, $\hat{\pi}_l(t)$ or $\pi_l^V(t)$.

[16] For general distributions of the local bidders' values, $F(v)$, the optimal bid function is given by $b(v) = \max(0, v - \alpha)$, where now $\alpha = \frac{1}{2}\int_\alpha^1 (v-\alpha)\, dF(v)$. For some distributions, the resulting performance measures of the BCV auction are *worse* than those of Result 2. Likewise, the assumption that the global bidder's value is uniformly distributed can be relaxed. The resulting bidding function for the local bidders is no longer linear but free riding still occurs.

their private information). In the two-sided setting studied by Myerson and Satterthwaite, traders' expected information rents may exceed the surplus generated by trade. In the one-sided setting studied here, bidders' information rents imply an upper bound on the seller's revenue—an upper bound that generally conflicts with some of the core constraints.

Core allocations are possible when all goods are substitutes. The interest in core*-selecting auctions, however, derives from environments in which the substitutes assumption is relaxed. In this case, competitive equilibrium does not necessarily exist and "... the conception of auctions as mechanisms to identify market clearing prices is fundamentally misguided" (Milgrom 2004, p. 296). The core, which always exist in these one-sided applications, seems the natural and relevant solution concept since "... competitive equilibrium outcomes are always core outcomes, so an outcome outside the core can be labeled uncompetitive" (Milgrom 2004, p. 303). Our results, however, demonstrate that with incomplete information, core assignments are not generally possible unless all goods are substitutes.

Appendix: Proof

Proof of Result 1. As under the VCG mechanism, the global bidder's payment under BCV does not depend on his bid. So the dominant strategy is to bid $B(V)=V$. Suppose local bidder 1 with value v_1 bids $b_1 \geq 0$. Consider a deviation to $b_1+\varepsilon$ with $\varepsilon>0$. The expected gain of this deviation takes place when it turns the local bidder from a losing bidder to a winner, i.e., when the global bidder's value lies between $b_1+b_2(v_2)$ and $b_1+\varepsilon+b_2(v_2)$, where $b_2(\cdot)$ is bidder 2's bidding function. Since the global bidder's bid and the sum of the local bidders' bids are equal (up to order of ε) in this case, bidder 1's BCV payment is simply $B-b_2(v_2)=b_1$, i.e., her own bid, and the profit is v_1-b_1. (As a result, if $v_1-b_1<0$, a reduction of bid decreases the chance of winning and thus reduces the loss. A reduction of bid also reduces the expected payment as shown in the following. So $b_1>v_1$ cannot be optimal.) Since the distribution of the global bidder's value is uniform on $[0,2]$ and we have $b_i \leq v_i \leq 1$, the probability that B lies between $b_1+b_2(v_2)$ and $b_1+\varepsilon+b_2(v_2)$ is equal to $\frac{1}{2}\varepsilon$. Hence, the expected gain from deviation is

$$\tfrac{1}{2}\varepsilon(v_1-b_1).$$

To determine the expected cost of deviation, note that the only term in local bidder 1's payment affected by an increase in bidder 1's bid is the $\frac{1}{2}\max(0,B-b_1)$ term. Hence, when bidder 1 raises her bid by ε, her payment goes up by $\frac{1}{2}\varepsilon$ if and only if the global bidder's value is greater than b_1 and less than $b_1+b_2(v_2)$ (since otherwise the local bidders do not win). So the expected cost of local bidder 1's deviation is simply

$$\tfrac{1}{2}\varepsilon\int_0^1\int_{b_1}^{b_1+b_2(v_2)}\tfrac{1}{2}\,dV\,dv_2=\tfrac{1}{4}\varepsilon E(b_2(v_2)).$$

Thus, for the bid $b_1 \geq 0$ to be optimal, the gain from deviation to $b_1 + \varepsilon$ with $\varepsilon > 0$ must not be greater than the loss. We have

$$v_1 - b_1 \leq \frac{1}{2}E(b_2(v_2)).$$

Similarly, for $b_1 > 0$, a deviation to $b_1 + \varepsilon$ with $\varepsilon < 0$ leads to the same result except with opposite signs of changes,

$$v_1 - b_1 \geq \frac{1}{2}E(b_2(v_2)).$$

Combining the above two conditions, we have when $v_1 > \frac{1}{2}E(b_2(v_2))$, the optimal bid $b_1^* = v_1 - \frac{1}{2}E(b_2(v_2))$. When $v_1 \leq \frac{1}{2}E(b_2(v_2))$, b_1^* cannot be positive; otherwise, the second inequality is violated and deviation to $b_1 + \varepsilon$ is profitable. Thus, $b_1^* = 0$ for $v_1 \leq \frac{1}{2}E(b_2(v_2))$. Note that the above conditions also guarantee the global optimality of b_1^* because at any $b_1 \neq b_1^*$, locally moving toward b_1^* yields higher profit. We conclude that local bidder i's unique optimal response is $b_i(v_i) = \max(0, v_i - \frac{1}{2}E(b_{-i}(v_{-i})))$. Let $\alpha_1 = \frac{1}{2}E(b_2(v_2))$. We have $E(b_1(v_1)) = \frac{1}{2}(1-\alpha_1)^2 \equiv 2\alpha_2$. Similarly, $\frac{1}{2}(1-\alpha_2)^2 = 2\alpha_1$. Therefore, the equilibrium with $\alpha_1 = \alpha_2 = 3 - 2\sqrt{2}$ can be derived. □

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