Supplementary Material

Supplement to "Rationalizable partition-confirmed equilibrium"

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Appendix D: Partition-confirmed equilibrium

RPCE is an analog of RSCE in games with terminal node partitions and it reduces to RSCE if the partitions are discrete. Here we define and analyze an analog of SCE that reduces to SCE in games with discrete terminal node partitions.

For $\pi \in \Pi$, let $H(\pi)$ denote the information sets reached with positive probability given π.

Definition D.1. Strategy profile π^* is a *partition-confirmed equilibrium* (*PCE*) if there exist a belief model V and an actual version profile v^* such that the following three conditions hold:

(i) Strategy profile π^* is generated by v^*.

(ii′) For each i and $v_i = (\pi_i, p_i)$, there exists μ_i such that (b) π_i is a best response to μ_i at $H(\pi_i, \pi_{-i})$ for all π_{-i} in the support of $b(\mu_i)$.

(iii) For all i, v_i^* is self-confirming with respect to π^*.

Remark D.1.

(a) Condition (i) says that the equilibrium strategy profile is generated by the specified belief model.

(b) Condition (ii′) ensures that players optimize against their beliefs at the "on-path" information sets. This is one of the conditions that we strengthened in our main solution concept.

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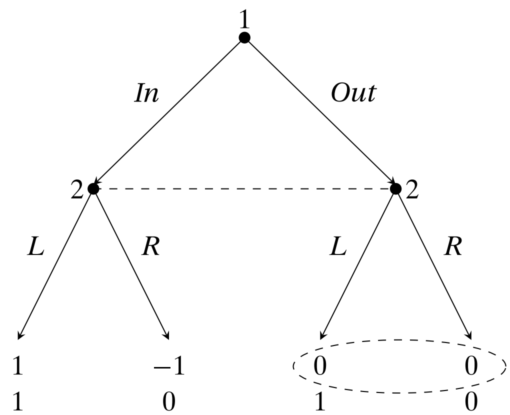

FIGURE 12. Example 12.

(c) Note that the above definition requires neither observational consistency nor coherent beliefs.

(d) If we define conjectural equilibrium (CE) as an $m^* \in M$ such that $(m_i^*, g_i(m^*))$ is g-rationalized by some $\mu \in \Delta(M_{-i})$ for all i, then the relationship between PCE and CE is analogous to that between RPCE and RCE.

The next example shows that adding the coherent belief condition to the PCE concept may rule out some outcomes even though adding it to SCE has no effect.

EXAMPLE 12. In Figure 12, the terminal node partition is that both players observe the exact terminal node reached except that player 1's partition does not reveal player 2's play if she plays *Out*.

First we argue that (Out, L) is a sensible outcome in this game if players do not know their opponents' payoff functions. To see this, note that L is a best response against *Out*, which player 2 indeed observes. *Out* is not a best response against L, but player 1 does not observe player 2's play when she plays *Out*, so she may well believe that player 2 is playing R. In this case, the expected payoff from playing *In* is -1, so playing *Out* is indeed a best response against such a belief.

Indeed, the following belief model and actual versions support this outcome as a PCE:

$$V_1 = \{v_1'\}, \quad v_1' = (Out, v_2')$$
$$V_2 = \{v_2'\}, \quad v_2' = (L, v_1')$$

The actual version profile is (v_1', v_2').

However, if we add the coherent belief condition by replacing condition (ii′) with condition (ii) (i.e., requiring that the μ_i to which π_i is a best response is coherent with p_i), this outcome is no longer supported in PCE. To see this, notice that the best response condition ensures that all versions of player 2 play L, as it is the dominant strategy. If we impose coherent beliefs, player 1's belief has to be a convex combination of player 2's strategies specified in player 2's possible versions. Hence, player 1 must believe that player 2 will play L with probability 1. But then the best response against this belief is *In*, invalidating the candidate outcome (Out, L).

This is in contrast to DFL's Theorem 2.1, which shows that adding the belief-closed condition (which fills the role of our coherent belief condition) to the SCE concept does not restrict the set of possible outcomes. In their context, players know opponents' play on the equilibrium path. Thus if a player's belief about an opponent's play at an information set h corresponds to a dominated strategy, then h must lie off the path of play. This conclusion fails if players do not observe all on-path play, which is why adding the coherent belief condition matters for PCE but not SCE.[1]

We note that if the terminal node partitions were discrete, player 1 could not play *In* in any PCE. So terminal node partitions allow extra actions not only under RPCE but also under PCE. ◊

Condition (iii) is the "self-confirming" part of the equilibrium concept. Notice that this condition is imposed only for actual versions. However, imposing the self-confirming condition for all versions does not restrict the set of equilibria.

Theorem D.1. *The set of PCE does not change if we replace condition (iii) with the following condition:*

For all i and v_i, v_i is self-confirming with respect to π^*.

Proof. Fix a PCE π^* generated by the actual version profile v^* and fix a belief model V. Construct the pair of actual version profile v^* and $\hat{V}$ in the same way as in the proof of part (i) of Theorem 3, where we replace d with D_i in condition (iii′). By definition, the new actual version v^* generates π^*. For each i and $v_i = (\pi_i, p_i)$ that still exists in $\hat{V}_i$, we did not change π_i, so the best response condition holds under the belief μ_i, under which the best response condition holds in the original belief model.

Finally, all remaining versions satisfy the self-confirming condition by the construction of $\hat{V}_i$ and the (extended notion of) perfect recall. □

The intuition for this result is simple: Since PCE does not require coherent beliefs, eliminating the hypothetical versions (who may not satisfy the self-confirming condition) does not invalidate the belief model. As stated in the main text, the distinction between conditions (iii) and (iii′) described in Example 11 relies on the fact that RSCE requires common knowledge of rationality (at reachable nodes). Theorem D.1 implies that this type of example indeed does not exist if we consider (nonrationalizable) SCE.

Appendix E: An epistemic interpretation of observational consistency

In this section, we use an epistemic model to make our interpretation of observational consistency precise.

[1] Note that the belief model we presented satisfies observational consistency.

Dekel and Siniscalchi (2014) model interactive knowledge with an *epistemic type structure*. This is a tuple $T = (I, (X_{-i}, T_i, \beta_i)_{i\in I})$, where X_{-i} is the space over which i has uncertainty and T_i is the set of i's types. Each $\beta_i : T_i \to \Delta(X_{-i} \times T_{-i})$ specifies i's beliefs. For our purpose, it is natural to define $X_i = \Pi_i \times \mathcal{P}_i$, where $\mathcal{P}_i$ is the set of i's possible terminal node partitions, and let $X_{-i} = \mathbf{\Pi}_{-i} \times \mathcal{P}_{-i}$. Type t_i is said to believe $E_{-i} \subseteq X_{-i} \times T_{-i}$ if $\beta_i(t_i)(E_{-i}) = 1$.

Whether a player's belief is self-confirming depends on the actual play of the others, so to define the event "player i believes that her opponents are self-confirming," we will define a belief operator on events in $\Omega := X \times T$.[2] Let $\Omega_i = X_i \times T_i$, so $\Omega = \times_{i\in I} \Omega_i$. Typical elements in Ω_i and Ω are denoted ω_i and ω, respectively. To do so, for $E \subseteq \Omega$, let $Y_{-i}(E; \omega_i) = \{\omega_{-i} \mid (\omega_i, \omega_{-i}) \subseteq E\}$ be the projection of $(\{\omega_i\} \times \Omega_{-i}) \cap E$ on Ω_{-i}. Then we define $B_i(E) = \{\omega_i \in \Omega_i \mid t_i(\omega_i)$ believes $Y_{-i}(E; \omega_i)\}$, where $t_i(\omega_i)$ is ω_i's type.

For any $E \subseteq \Omega$, let $B(E) = \times_{i\in I} B_i(E)$: this is the set of states where all players believe E. Notice that it may be the case that $B(E) \not\subseteq E$. This is essential, as we wish to allow players to have incorrect beliefs about each other's strategies.

Let $B^n(E) = B(B^{n-1}(E))$ with $B^0(E) = E$ and let $CB(E) = \bigcap_{n=1}^{\infty} B^n(E)$: this is the set of states where E is "common belief." We let $CK(E) = E \cap CB(E)$: this is the set of states where E is true and is a common belief; that is, it is "common knowledge."[3] Define also $K_i(E) = (B_i(E) \times \Omega_{-i}) \cap E \subseteq \Omega$.

Consider any finite product set $\bar{\Omega} = \times_{j\in I} \bar{\Omega}_j \subseteq \Omega$ such that $\bar{\Omega}$ is common knowledge at each $\omega \in \bar{\Omega}$, that is, $CK(\bar{\Omega}) = \bar{\Omega}$. Each $\omega \in \bar{\Omega}$ is called a state.

For the following discussion, it is convenient to introduce the notion of information sets $h_i(\omega_i)$, the set of states that i thinks possible. That is,

$$h_i(\omega_i) = \{(\omega_i, \omega'_{-i}) \mid \beta_i(t_i(\omega_i)) \text{ assigns positive probability to } \omega'_{-i}\}.$$

Note that given the restriction to the finite set $\bar{\Omega}$, for any given $E \subseteq \bar{\Omega}$, we have that $B_i(E) = \{\omega_i \in \Omega_i \mid h_i(\omega_i) \subseteq E\}$.

Let a generic element of $\Omega_i = X_i \times T_i$ be $\omega_i = ((\pi_i(\omega_i), P_i(\omega_i)), t_i(\omega_i))$, where $\pi_i(\omega_i) \in \Pi_i$ and $P_i(\omega_i) \in \mathcal{P}_i$ are i's strategy and partition at ω_i, respectively.[4] Denote by $\pi(\omega) = (\pi_i(\omega_i))_{i\in I}$ the strategy profile for all players at ω; $P_{-i}(\omega_{-i})$ and $P(\omega)$ are partition profiles for i's opponents and for all players, respectively.

Given $\bar{\Omega}$, construct a belief model $V^{\bar{\Omega}} = (V_i^{\bar{\Omega}})_{i\in I}$ such that $\#\bar{\Omega}_i = \#V_i^{\bar{\Omega}}$ for each i and for each ω_i in $\bar{\Omega}_i$, there is a $v_i(\omega_i) = (\pi_i(\omega_i), p_i(\omega_i))$, where $p_i(\omega_i)$ is a probability distribution over $V_{-i}^{\bar{\Omega}}$ that corresponds to $\beta_i(t_i(\omega_i))$. From here on, we only consider states in $\bar{\Omega}$ and we simply write V for $V^{\bar{\Omega}}$.

We define the sets

$$E_i^{\mathrm{SC}} = \{\omega \mid (\pi_i(\omega_i), p_i(\omega_i)) \text{ is self-confirming with respect to } \pi(\omega) \text{ under partition } P_i(\omega_i) \text{ under belief model } V\}$$

[2]Here we extend the framework of Dekel and Siniscalchi (2014), in which i's belief operator B_i depends only on Ω_{-i}.

[3]Notice that since we do not require $B(E) \subseteq E$, $CK(E)$ may be different from $CB(E)$.

[4]Recall that $t_i(\omega_i)$ is ω_i's type.

$$E_{-i}^{\mathrm{SC}} = \bigcap_{j \neq i} E_j^{\mathrm{SC}} \quad \text{and} \quad E^{\mathrm{SC}} = \bigcap_{j \in I} E_j^{\mathrm{SC}}$$

$$E_i^{\mathrm{OC}} = \{\omega \mid (\pi_i(\omega_i), p_i(\omega_i)) \text{ is observationally consistent}$$
$$\text{under partition } P_{-i}(\omega_{-i}) \text{ under belief model } V\}$$

$$E_i(\mathbf{P}_i) = \{\omega_i \mid P(\omega_i) = \mathbf{P}_i\}, \qquad E_{-i}(\mathbf{P}_{-i}) = \times_{j \neq i} E_j(\mathbf{P}_j), \quad \text{where } \mathbf{P}_{-i} = \times_{j \neq i} \mathbf{P}_j.$$

The next theorem states that the set of states where player i has correct beliefs about the partitions and believes that other players satisfy the self-confirming condition is the same as the set of states where player i has correct beliefs about the partitions and are observationally consistent.

Theorem E.1. *For each $i \in I$,*

$$\left(\bigcup_{\mathbf{P}_{-i} \in \mathcal{P}_{-i}} K_i(\bar{\Omega}_i \times E_{-i}(\mathbf{P}_{-i}))\right) \cap (B_i(E_{-i}^{\mathrm{SC}}) \times \bar{\Omega}_{-i}) = \left(\bigcup_{\mathbf{P}_{-i} \in \mathcal{P}_{-i}} K_i(\bar{\Omega}_i \times E_{-i}(\mathbf{P}_{-i}))\right) \cap E_i^{\mathrm{OC}}.$$

Proof. Fix $\omega \in \bigcup_{\mathbf{P}_{-i} \in \mathcal{P}_{-i}} K_i(\bar{\Omega}_i \times E_{-i}(\mathbf{P}_{-i}))$. We will show that $\omega \in B_i(E_{-i}^{\mathrm{SC}}) \times \bar{\Omega}_{-i}$ if and only if $\omega \in E_i^{\mathrm{OC}}$.

First, $\omega \in B_i(E_{-i}^{\mathrm{SC}}) \times \bar{\Omega}_{-i}$ is equivalent to the condition that for every ω'_{-i} that $\beta_i(t_i(\omega_i))$ assigns positive probability for any $j \neq i$, a version $(\pi_j(\omega'_j), p_j(\omega'_j))$ is self-confirming with respect to $\pi(\omega_i, \omega'_{-i})$ under partition $P_j(\omega'_j)$ under belief model V. Second, $\omega \in E_i^{\mathrm{OC}}$ is equivalent to the condition that if $\beta_i(t_i(\omega_i))$ assigns positive probability to ω'_{-i}, then for any $j \neq i$, version $(\pi_j(\omega'_j), p_j(\omega'_j))$ is self-confirming with respect to $\pi(\omega_i, \omega'_{-i})$ under partition $P_j(\omega_j)$ under belief model V.

The two conditions are different only in the partition that they consider, so the result holds if we prove $P_j(\omega_j) = P_j(\omega'_j)$, and this follows from $K_i(E) \subseteq E$ so that for any $\mathbf{P}_{-i}$, i's belief about the other players' partitions is correct on $K_i(\bar{\Omega}_i \times E_{-i}(\mathbf{P}_{-i}))$. □

We note that if we did not suppose that player i has a correct belief about the opponent's partitions, then $\omega \in B_i(E_{-i}^{\mathrm{SC}}) \times \bar{\Omega}_{-i}$ and $\omega \in E_i^{\mathrm{OC}}$ would not be equivalent. To see this, consider the following example.

Example 13 (Incorrect belief about the opponent's partition). Consider the two-player game in Figure 13. Here, only player 1 has a move, and chooses between L and R. Formally, we let player 2 play the action a in his singleton action set and let player 1's partition be $\mathbf{P}'_1$. There are two possible terminal node partitions for player 2 over the two terminal nodes: $\mathbf{P}'_2$ and $\mathbf{P}''_2$. Suppose the state space is[5]

$$\bar{\Omega}_1 = \{\omega'_1, \omega''_1\} \quad \text{with } \omega'_1 = ((L, \mathbf{P}'_1), \omega'_2), \omega''_1 = ((R, \mathbf{P}'_1), \omega'_2)$$
$$\bar{\Omega}_2 = \{\omega'_2, \omega''_2\} \quad \text{with } \omega'_2 = ((a, \mathbf{P}'_2), \omega''_1), \omega''_2 = ((a, \mathbf{P}''_2), \omega''_1).$$

[5]We abuse notation and denote a point belief in a particular state of the opponent by that state, e.g., $\omega'_1 = ((L, P'_1), \omega'_2)$ means $\omega'_1 = ((L, P'_1), \delta_{\omega'_2})$, where δ_x is the Dirac measure concentrated on x.

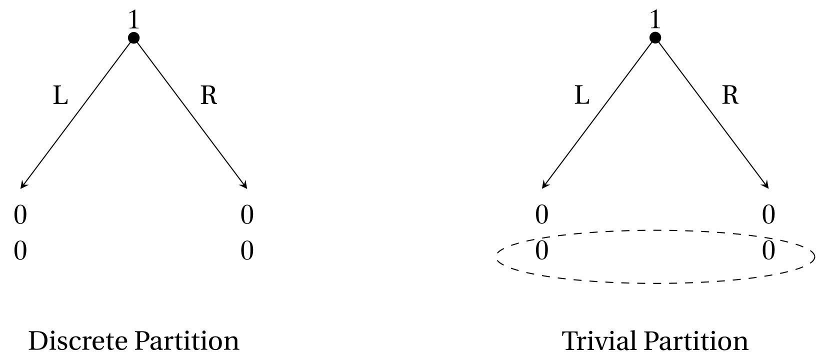

FIGURE 13. Example 13.

Note that at states (ω_1', ω_2'') and (ω_1'', ω_2''), player 1 has an incorrect belief about player 2's partition if $\mathbf{P}_2' \neq \mathbf{P}_2'$.

First suppose that $\mathbf{P}_2'$ is the discrete partition and $\mathbf{P}_2''$ is the trivial partition. Consider E_2^{SC}. Since player 2 has the trivial partition at ω_2'', any belief is self-confirming, so all states involving ω_2'' are in E_2^{SC}. At ω_2', player 2 has the discrete partition and believes ω_1'' is present. Thus $E_2^{\text{SC}} = \{(\omega_1', \omega_2''), (\omega_1'', \omega_2'), (\omega_1'', \omega_2'')\}$. Next consider $B_1(E_2^{\text{SC}})$. To see what is in this set, we consider $h_1(\omega_1)$ for each $\omega_1 \in \bar{\Omega}_1$. First, $h_1(\omega_1') = \{(\omega_1', \omega_2')\} \nsubseteq E_2^{\text{SC}}$ because at ω_1', player 1 thinks (only) ω_2' is present. Second, $h_1(\omega_1'') = \{(\omega_1'', \omega_2')\} \subseteq E_2^{\text{SC}}$ because ω_1'' thinks (only) ω_2' is present. So $B_1(E_2^{\text{SC}}) = \{\omega_1''\}$, hence, $B_1(E_2^{\text{SC}}) \times \bar{\Omega}_2 = \{(\omega_1'', \omega_2'), (\omega_1'', \omega_2'')\}$. Finally, consider E_1^{OC}. Again, since ω_2'' has the trivial partition, all the states involving ω_2'' are in E_1^{OC}. Since ω_2' thinks (only) ω_1'' is present, (ω_1'', ω_2') is in E_1^{OC} but (ω_1', ω_2') is not, so $E_1^{\text{OC}} = \{(\omega_1', \omega_2''), (\omega_1'', \omega_2'), (\omega_1'', \omega_2'')\}$ and, hence, $B_1(E_2^{\text{SC}}) \times \bar{\Omega}_2 \supsetneq E_1^{\text{OC}}$.

Next suppose that $\mathbf{P}_2'$ is the trivial partition and $\mathbf{P}_2''$ is the discrete partition. Proceeding as above, we can compute that $E_2^{\text{SC}} = \{(\omega_1', \omega_2'), (\omega_1'', \omega_2'), (\omega_1'', \omega_2'')\}$, $B_1(E_2^{\text{SC}}) = \{\omega_1', \omega_1''\}$, and $E_1^{\text{OC}} = \{(\omega_1', \omega_2'), (\omega_1'', \omega_2'), (\omega_1'', \omega_2'')\}$. Hence, $B_1(E_2^{\text{SC}}) \times \bar{\Omega}_2 \subsetneq E_1^{\text{OC}}$. $\diamondsuit$

The theorem and counterexample show that the interpretation of observational consistency as implying that *i* believes the opponents are self-confirming implicitly assumes that *i* has the correct belief about the opponents' partitions.

Now we consider higher order belief. The next theorem states that RPCE implies common belief of the partition structure and the self-confirming condition.

THEOREM E.2. *If π^* is a RPCE of an extensive-form game with partition P^*, then there exists a state space $\bar{\Omega}$ and a state $\omega \in CK(E^{\text{SC}}) \cap CK(E(\mathbf{P}^*)) \subseteq \bar{\Omega}$ such that $\pi(\omega) = \pi^*$.*

PROOF. Fix an extensive-form game with terminal node partition $\mathbf{P}^*$, and consider a belief model $\tilde{V}$ and actual versions profile v^* that supports π^* as a RPCE. For each player *i*, let $\hat{V}_i$ be the set of hypothetical versions in $\tilde{V}$ such that, for each $v_i \in \hat{V}_i$, there is no version v_j whose conjecture assigns positive probability to v_i. Then it must be the case that the belief model $\bar{V} = (\tilde{V}_i \setminus \hat{V}_i)_{i\in I}$ also supports π^* a RPCE under $\mathbf{P}^*$.

Construct $\bar{\Omega}$ such that $V^{\bar{\Omega}} = \bar{V}$, with a restriction that $P_i(\omega_i) = \mathbf{P}_i^*$ for all $\omega_i \in \bar{\Omega}_i$ for all player i.[6] Denote by $\omega_i^{v_i}$ the state for player i that corresponds to $v_i \in \bar{V}_i$.

First, since by construction $P(\omega) = \mathbf{P}^*$ for all $\omega \in \Omega$, it is immediate that $CK(E(\mathbf{P}^*)) = \bar{\Omega}$.

Second, we prove that $(\omega_i^{v_i^*})_{i\in I} \in CK(E^{\mathrm{SC}})$. To see this, note that $P(\omega) = \mathbf{P}^*$ for all $\omega \in \bar{\Omega}$ implies that $\bigcup_{\mathbf{P}_{-i}\in\mathcal{P}_{-i}} K_i(\bar{\Omega}_i \times E_{-i}(\mathbf{P}_{-i})) = \bar{\Omega}$. Also, as in RPCE, all versions satisfy the observational consistency condition under $\mathbf{P}^*$, $E_i^{\mathrm{OC}} = \bar{\Omega}_i$ for each i. By Theorem E.1, these facts imply that $(B_i(E_{-i}^{\mathrm{SC}}) \times \bar{\Omega}_{-i}) = \bar{\Omega}$ for all i, that is, $B_i(E_{-i}^{\mathrm{SC}}) = \bar{\Omega}_i$.

Now, let us show that $B_i(E_i^{\mathrm{SC}}) = \bar{\Omega}_i$. For this to be the case, we must have that for each $\omega_i \in \bar{\Omega}_i$, if ω_i assigns positive probability to *some* ω'_{-i}, then $D_i(\pi_i(\omega_i), \pi_{-i}) = D_i(\pi_i(\omega_i), \pi_{-i}(\omega'_{-i}))$ for *all* π_{-i} such that $\pi_{-i} = \pi_{-i}(\omega_{-i})$ for some ω_{-i} in the support of $p_i(\omega_i)$. This is immediate if $p_i(\omega_i)$'s support is a singleton, namely, $\{\omega'_{-i}\}$. If the support is not a singleton, then it suffices if $D_i(\pi_i(\omega_i), \pi_{-i})$ is constant across all π_{-i} such that $\pi_{-i} = \pi_{-i}(\omega_{-i})$ for some ω_{-i} in the support of $p_i(\omega_i)$. But this holds because by the construction of $\bar{V}$, for $v_i \in \bar{V}_i$ such that $\omega_i = \omega_i^{v_i}$, either $v_i \in E_i^{\mathrm{SC}}$ or there is some $v_j \in \bar{V}_j$ whose conjecture assigns positive probability to v_i. Hence, the claim holds regardless of whether the support of $p_i(\omega_i)$ is a singleton or not.

Since $B_i(E_{-i}^{\mathrm{SC}}) = B_i(E_i^{\mathrm{SC}}) = \bar{\Omega}_i$ for each i, it follows that $B_i(E^{\mathrm{SC}}) = \bar{\Omega}_i$ for each i. Hence, $CB(E^{\mathrm{SC}}) = \bigcap_{n=1}^{\infty} B^n(E^{\mathrm{SC}}) = \bar{\Omega}$. Thus it remains to show that $\omega \in E^{\mathrm{SC}}$ for some ω. But because the actual version v_i^* for each i satisfies the self-confirming condition, $(\omega_j^{v_j^*})_{j\in I} \in E_i^{\mathrm{SC}}$ for each i. Therefore, we have $(\omega_i^{v_i^*})_{i\in I} \in E^{\mathrm{SC}}$.

As we have already concluded that $CK(E(\mathbf{P}^*)) = \bar{\Omega}$, we have that $(\omega_i^{v_i^*})_{i\in I} \in CK(E^{\mathrm{SC}}) \cap CK(E(\mathbf{P}^*))$, completing the proof. □

To sum up this subsection, Theorem E.1 shows that the observational consistency condition corresponds to players having correct beliefs about the terminal node partitions and believing that the other players' beliefs are self-confirming, and Theorem E.2 shows that RPCE implies that there is common knowledge of the partition structure and the terminal node partitions. Thus the RPCE definition captures the idea that a player can make predictions about other players' actions based on her knowledge of things she does not directly observe but can infer from her observations and her beliefs about other players' payoffs and observation structures.

Appendix F: The effect of changes in terminal node partitions

In this section, we discuss the effect of changing the terminal node partitions. In Section F.1, we briefly discuss how the RPCE strategy profiles depend on the terminal node

[6] It is straightforward that such $\bar{\Omega}$ exists and is unique.

partitions. In Section F.2, we identify four ways that the set of an individual player's RPCE strategies is affected by terminal node partitions.[7]

F.1 *The effect of terminal node partitions on RPCE strategy profiles*

Consider how the set of RPCE strategy *profiles* (not an individual player's strategies) changes with the terminal node partitions. If the terminal node partitions **P** are coarser than **P**′, then any strategy profile that is a RPCE under **P**′ is also a RPCE under **P**: if a belief model supports a strategy profile under **P**′, then it can also be used to support the same strategy profile under **P**.[8]

However, versions in the belief model that support a strategy profile under **P** may not support it under a finer partition **P**′. Perhaps the most obvious reason is that a player may not want to play a particular action once she learns the unobserved play by the opponents. For example, the strategy profile discussed in Example 7 ((Out, L_2, L_3)) would not be a RPCE if player 1's terminal node partition were discrete: If she observes that the equilibrium that the opponents are coordinating on is different from the one that she was expecting, she wants to play *In*.

These examples show that not only the set of RPCE strategies, but also the RPCE outcomes of these games (the distributions over terminal nodes) can depend on the terminal node partitions.

F.2 *The effect of terminal node partitions on an individual player's RPCE strategies*

Now we ask how the set of an individual player's RPCE strategies depends on the terminal node partitions. As we explained above, coarsening the partitions cannot rule out a RPCE strategy profile. Obviously, this also means that if a strategy of player i is used in a RPCE under a particular partition **P**′, then it can also be used in a RPCE under a coarser partition **P**.

Example 5 illustrates one way by which terminal node partitions affect player i's RPCE strategies. In that example, when i's opponents' terminal node partitions are different in two games, she expects them to play differently and so change her own play.

Another effect of the changes of terminal node partitions is illustrated in Example 8: Since i plays an action because of a correlated belief about the opponents' unobserved on-path play, she cannot play that action when her terminal node partition is discrete, because the discrete partition reveals the actual on-path play and actual play is not correlated.

Next we present two more examples to show other ways in which the terminal node partitions affect RPCE strategies. Specifically, terminal node partitions affect RPCE strategies when player i's belief is only coherent with a conjecture that assigns strictly positive probabilities to multiple versions of the opponents (Example 14) and when some player j believes player i has an incorrect belief (Example 15 and Example 6).

[7] One motivation is that the analyst may only know the terminal node partitions of some of the players and/or may only observe some players' moves.

[8] This result is stated also in Battigalli and Friedenberg (2012).

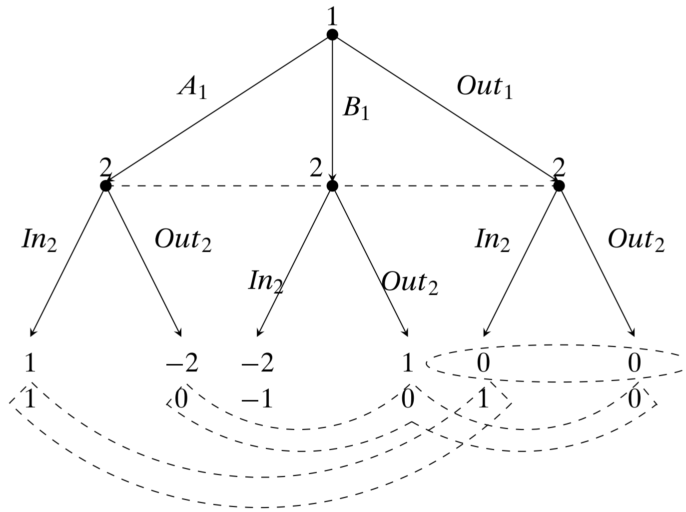

FIGURE 14. Example 14.

EXAMPLE 14. The game in Figure 14 has only two players, so beliefs are trivially independent. The terminal node partitions are that both players observe the exact terminal node reached except that player 1's partition does not reveal player 2's action if she plays Out_1, and player 2's partition is $\{(A_1, In_2), (Out_1, In_2)\}$, $\{(B_1, In_2)\}$, $\{(A_1, Out_2), (B_1, Out_2), (Out_1, Out_2)\}$. We will show that player 1 can play Out_1 under the original terminal node partition but not under a discrete terminal node partition.

First we show that player 1 can play Out_1. To see this, consider the belief model and actual versions

$$V_1 = \{v_1', v_1'', v_1'''\}, \quad v_1' = \left(Out_1, \left(\tfrac{1}{2}v_2', \tfrac{1}{2}v_2''\right)\right), v_1'' = (B_1, v_2''); v_1''' = (A_1, v_2')$$

$$V_2 = \{v_2', v_2''\}, \quad v_2' = (In_2, v_1'''), v_2'' = (Out_2, v_1'')$$

The actual version profile is (v_1', v_2').

Notice that although player 1's action is rationalized by a belief that corresponds to player 2's mixed strategies, she is sure that player 2 is playing a pure strategy: Both versions v_2' and v_2'' play pure strategies. If player 1's conjecture assigns probability 1 to either of these versions, player 1 cannot play Out_1: If player 1 expects In_2 with probability 1, then she wants to play A_1; if she expects Out_2 with probability 1, then she wants play B_1. Thus, the action Out_1 is possible only when player 1's belief corresponds to player 2's mixed strategy.

Player 1 can be unsure which of v_2' and v_2'' is present, because she plays Out_1 and does not observe the exact terminal node reached.

Now we argue that if player 1's terminal node partition is discrete, she can never play a strategy that assigns probability 1 to Out_1. To see this, we first note that no version of player 2 can play a mixed strategy if player 1 plays Out_1. This is because if player 1 plays Out_1 with probability 1 and player 2 assigns a positive probability to In_2, then player 2 expects payoff 1 from playing In_2 and 0 from playing Out_2. This means he is not indifferent, so he cannot mix.

Thus, whenever player 1 plays Out_1 with probability 1, player 2 should not play a mixed strategy. But this implies that player 1 is observing either (a) player 2 is playing In_2 with probability 1 or (b) player 2 is playing Out_2 with probability 1. However, as we have explained above, player 1 would be strictly better off by playing A_1 than Out_1 in case (a) and playing B_1 than Out_1 in case (b). Hence, she cannot play a strategy that assigns probability 1 to Out_1 if her terminal node partition is discrete, although this action could be played if the partition were not discrete.

The key here is that player 1's belief is coherent with the conjecture that assigns strictly positive probabilities to multiple versions of player 2, but the corresponding "mixed strategy" by player 2 cannot be played in RPCE. $\diamondsuit$

A remark on Example 14. Fudenberg and Levine (1993) and Kamada (2010) identify the conditions that guarantee that the outcome of a SCE is identical to a Nash outcome. To prove this theorem, they explicitly construct a Nash equilibrium from a SCE that satisfies these conditions: For an off-path information set h_j that player i can deviate to reach, they set player j to play as in i's belief, while strategies at other information sets are unchanged. Their conditions ensure that this modification is well defined. In particular, the independent-beliefs condition guarantees that the modification can be done information set by information set.

Given this, it might seem natural to conjecture that if π^* is a RPCE with independent beliefs under partitions $(\mathbf{P}_i, \mathbf{P}_{-i})$, then under $(\bar{\mathbf{P}}_i, \mathbf{P}_{-i})$ with $\bar{\mathbf{P}}_i$ being the discrete partition, we can let i's opponents play "as in i's belief" (while we do not change i's strategy) and the modified strategy profile constitutes a RPCE under $(\bar{\mathbf{P}}_i, \mathbf{P}_{-i})$, because of common knowledge of rationality. Example 14 above shows why this argument fails: The problem is that we cannot replace i's opponents' strategies "as in i's belief," even if we impose independent beliefs. This is what happens in Example 14. In Example 14, two versions of player 2 that player 1 assigns positive probabilities play different strategies that are rationalized by different beliefs, and it is not necessarily the case that we can rationalize a convex combination of these pure strategies by some single belief. The intuition is similar to the idea behind the need for unitary beliefs to establish the outcome equivalence between SCE and Nash: If heterogeneous beliefs are allowed, a single belief may not rationalize all of the pure strategies in the support of a player's mixed strategy, so the mixed action may not be played in a Nash equilibrium.

The next example illustrates the following situation: if i's terminal node partition is coarse, some player j may believe that i has an incorrect belief, while if i's terminal node partition is discrete, j knows that i sees the true distribution on terminal nodes.

Example 15. In the game in Figure 15, player 2 is indifferent between In_2 and Out_2 when player 1 plays Out_1. As usual, the terminal node partition is such that player i's partition reveals the opponent's action when she plays In_i, while it does not when she plays Out_i.

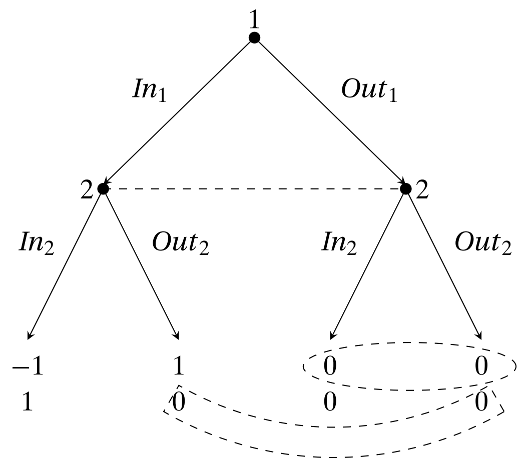

FIGURE 15. Example 15.

We first show that player 1 can play In_1 given these terminal node partitions. To see this, consider the belief model and actual versions

$$V_1 = \{v_1', v_1''\}, \quad v_1' = (In_1, v_2'), v_1'' = (Out_1, v_2'')$$

$$V_2 = \{v_2', v_2''\}, \quad v_2' = (Out_2, v_1''), v_2'' = (In_2, v_1'')$$

The actual version profile is (v_1', v_2').

Notice that player 2 plays Out_2 because he believes player 1 is playing Out_1. Such a belief is justified because given Out_1, player 1 does not observe player 2's play, so player 1 can *incorrectly* believe that player 2 is playing In_2. However, such an "incorrect belief" is not possible if player 1's terminal node partition is discrete, so player 2 cannot believe that player 1 plays Out_1 when he plays Out_2. This in turn rules out the possibility of the strategy that assigns probability 1 to In_1.

To see this formally, suppose that player 1's terminal node partition is discrete and she plays In_1 with probability 1. Then, for the best response condition for player 2 to be satisfied, player 2 must be playing In_2 with probability 1 or Out_2 with probability 1. However, for the best response condition for player 1 to hold, it must be the case that Out_2 is played with probability 1. For Out_2 to be a best response for player 2, his belief must assign probability 1 to Out_1. But then the observational consistency condition and the assumption that player 1's terminal node partition is discrete imply that there exists a version of player 1 who plays Out_1 with a belief that assigns probability 1 to Out_2. However, such a version violates the best response condition, as In_1 gives a strictly higher payoff than Out_1 against Out_2.[9]

[9]Note that the example hinges on the assumption that player 2 is indifferent between In_2 and Out_2 when player 1 plays Out_1, as otherwise either (In_2, Out_1) or (Out_2, Out_1) will not satisfy the best response condition. However, the argument does not require ties. An example that shows this is available on request.

We note that player 1 cannot play In_1 if player 2's terminal node partition becomes discrete. This is easy to check: If it were discrete, player 2 must play In_2 with probability 1 if player 1 plays In_1. However, then player 1 would be better off by playing Out_1 than In_1. ◊

Remark F.1. Example 15 also shows that RPCE can Pareto-dominate all Nash equilibria, even in two-player games. The RPCE discussed in the example has the payoff $(1, 0)$, while the unique Nash equilibrium, (Out_1, In_2) has the payoff $(0, 0)$.

Note that in Examples 14 and 15, it is important that an opponent's observation about other players' strategies depends on that opponent's action. In these examples, this dependence is captured by the terminal node partitions. To formalize this dependence, we introduce a notion of "nonmanipulability":

Let $\zeta : S \to Z$ be the map that assigns to each pure strategy profile the terminal node induced by that profile.

Definition F.1. A game with player i's terminal node partition $\mathbf{P}_i$ is *nonmanipulable for player i* if $\zeta(s_i, s_{-i})$ and $\zeta(s_i, s'_{-i})$ are in the same cell of $\mathbf{P}_i$ if and only if $\zeta(s'_i, s_{-i})$ and $\zeta(s'_i, s'_{-i})$ are in the same cell of $\mathbf{P}_i$.[10]

That is, the game is nonmanipulable for player i if i's action does not affect what she observes. The condition is satisfied, for example, in simultaneous-move games with discrete partitions, but it is more general. For example, game A is nonmanipulable for players 2 and 3.

Imposing nonmanipulability for players other than i rules out some, but not all, examples such as Example 15 in which j believes i has an incorrect belief, as shown in Example 6 of the main text. In that example, it is important that with nondiscrete partitions, some player believes another player has an incorrect belief. The difference from the logic in Examples 14 and 15 is that in these examples with a nondiscrete partition, i's opponent j believes that i is not best responding to j's play, yet if j knows i's partition reveals j's play, then j should expect i is best responding to j, so j should play differently. In Example 6, however, when the partition is discrete, j learns a third player k's strategy from the fact that i is observing k's play and best responding to it, and this information changes how j should act. This learning from player i's play was not an issue in Examples 14 and 15.

To sum up, the set of strategies a player can use in equilibrium is typically sensitive to the details of her terminal node partition.

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[10] Battigalli et al. (1992) defined this property to hold for all players.

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