Supplementary Material

Supplement to "Majority runoff elections: Strategic voting and Duverger's hypothesis": Technical appendices

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In Appendix C, for a large class of Poisson games, we define Bayes–Nash equilibrium and extend the definition of *strictly perfect* equilibrium (Okada 1981). We show some useful properties of Nash equilibria and we characterize the set of strictly perfect equilibria. We define a limit property of Nash equilibria as the number of players grows large: asymptotic strict perfection. Our characterization allows for a simple procedure to find all asymptotically strictly perfect equilibria in any Poisson voting game with infinite type set.

In Appendix D, we extend the model to include an endogenous second round where the probability of upset victory converges to zero as the number of voters grows large. We prove that the results of our initial model hold (at least qualitatively) regardless of the speed of convergence to zero. We discuss the differences.

Appendix C

A Poisson game $\Gamma \equiv (n, \mathcal{T}, F, \mathcal{C}, u)$ is defined by the expected number of voters $n \in \mathbb{N}$, the set of types (a metric space) $\mathcal{T}$ with t being a typical element of $\mathcal{T}$, a probability measure F defined over $\mathcal{T}$, a set of actions $\mathcal{C}$, and a payoff vector $u_t : \mathcal{C} \times Z(\mathcal{C}) \to \mathbb{R}$, for all $t \in \mathcal{T}$, where $Z(\mathcal{C})$ is the set of all action profiles. Without loss of generality, we restrict attention to $\mathcal{T} = \mathbb{R}^{|\mathcal{C}|-1}$. In this appendix, we provide a series of definitions and results concerning a class of Poisson games, namely the games satisfying the following assumption.

Call $\sigma \equiv ((\sigma_t(c))_{c\in\mathcal{C}})_{t\in\mathcal{T}} \in \Delta(\mathcal{C})^{\mathcal{T}}$ a profile of strategies.[1] Define $\tau : \Delta(\mathcal{C})^{\mathcal{T}} \times \mathcal{F} \to \Delta(\mathcal{C})$,

$$\tau(\sigma, F) \equiv \left(\int_T \sigma_t(c)\, dF(t)\right)_{c\in\mathcal{C}},$$

where $\mathcal{F}$ is a set of distributions over $\mathcal{T}$ that admit a density and have full support over $\mathcal{T}$.

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[1]More precisely, the strategy $\sigma_t(c)$ is the marginal distribution of Milgrom–Weber distributional strategies (Milgrom and Weber 1985).

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Assumption S1. *We have $F \in \mathcal{F}$.*

We define the expected gain from action $c \in \mathcal{C}$ for type $t \in \mathcal{T}$ when the expected profile of actions is $n\tau$ as $G_t(c, n\tau)$. As usual in the literature, this expected gain is defined as the difference between the expected utility for a voter of type t if she votes for c and if she abstains. This gain depends on the voter's type and on the strategy function for all voters, σ.

Notice that Assumption S1 implies that, given any $\tau \in \Delta(\mathcal{C})$, the set

$$\mathcal{T}^{n\tau}_{c,c'} \equiv \{t \in \mathcal{T} : G_t(c, n\tau) = G_t(c', n\tau)\}$$

has probability measure zero: $\int_{\mathcal{T}^{n\tau}_{c,c'}} dF(t) = 0$.

In the remainder of this appendix, we define (Bayes–Nash) equilibrium and extend the definition of *strictly perfect* equilibrium (Okada 1981) to Poisson games. We show a few useful properties of Nash equilibria and we characterize the set of strictly perfect equilibria. We define a new concept: asymptotic strict perfection. Our characterization allows for a simple procedure to find all asymptotically strictly perfect equilibria in any Poisson voting game with infinite types.

A best response for type t given $\tau \in \Delta(\mathcal{C})$ is

$$\mathcal{B}_t(\tau) = \arg \max_{\sigma_t \in \Delta(C)} \sum_{c \in C} \sigma_t(c) G_t(c, n\tau).$$

Definition S1. The strategy profile $\sigma^* \in \Delta(\mathcal{C})^{\mathcal{T}}$ is said to be a (Bayes–Nash) equilibrium of Γ if $\sigma^*_t \in \mathcal{B}_t(\tau(\sigma^*, F))$ for every type $t \in \mathcal{T}$.

The following proposition partially characterizes the set of equilibria. It shows that in every equilibrium (i) there are types for which the equilibrium strategy is not a strict best response and (ii) the measure of these types is zero.

Proposition S1. *Let σ^* be an equilibrium of Γ. This equilibrium σ^* is a (pure strategy) strict best response to itself for all but a measure zero of types. Moreover, there always exists a type $t \in \mathcal{T}$ for which σ^*_t is not a strict best response to itself.*

Proof. For any pair of actions $c, c' \in \mathcal{C}$, let $\mathcal{T}_{c,c'}$ be the set of types with $\sigma^*_t(c), \sigma^*_t(c') > 0$. Then it must be

$$\mathcal{T}_{c,c'} \subseteq \mathcal{T}^{n\tau(\sigma^*_t(c),F)}_{c,c'} \equiv \left\{t \in \mathcal{T} : G_t(c, n\tau(\sigma^*_t(c), F)) = G(c', n\tau(\sigma^*_t(c), F))\right\}.$$

That is, $\mathcal{T}_{c,c'}$ is a subset of the types that are indifferent between c and c'. By Assumption S1, the measure of this set is zero. Hence, the set of players playing a mixed strategy or being indifferent between two strategies is the sum of measure zero sets. Nonetheless, by Assumption S1, there is at least one t who is indifferent between every possible pair of actions $c, c' \in \mathcal{C}$. Hence, the equilibrium is not strict. □

DEFINITION S2. Let $\eta \equiv (\eta_{t,c})_{t\in\mathcal{T},c\in\mathcal{C}} \in \mathbb{R}^{\mathcal{C}^{\mathcal{T}}}$ be such that $\eta_{t,c} \geq 0$, $\sum_{c\in\mathcal{C}} \eta_{t,c} < 1$, and define

$$\Sigma_t(\eta) \equiv \{\sigma_t \in \Delta(\mathcal{C}) : \sigma_t(c) \geq \eta_{t,c}, \forall c \in \mathcal{C}\}.$$

The perturbed game (Γ, η) is the infinite strategies game $(n, \mathcal{T}, F, (\Sigma_t(\eta))_{t\in\mathcal{T}}, u)$. Let $\mathcal{B}_t^{\eta}(\tau)$ be a best response in (Γ, η).

DEFINITION S3. The strategy profile $\sigma^* \in (\Delta(\mathcal{C}))_{t\in\mathcal{T}}$ is said to be a strictly perfect equilibrium of Γ if for any arbitrary sequence $\{\eta^k = (\eta^k_{t,c})\}_{k=1}^{\infty}$ such that

$$\begin{aligned} \eta^k &> 0 \quad \forall k \\ \eta^k &\to 0 \quad (k \to \infty), \end{aligned}$$

there exists $\sigma^*(\eta^k) \in (\Sigma_t(\eta^k))_{t\in\mathcal{T}}$, $k = 1, \ldots$, such that

$$\begin{aligned} \sigma_t^*(\eta^k) &\in \mathcal{B}_t^{\eta^k}(\tau(\sigma^*(\eta^k), F)) \quad \forall t \in \mathcal{T} \\ \sigma^*(\eta^k) &\to \sigma^*. \end{aligned}$$

The following proposition identifies a necessary and sufficient condition for a strategy profile to be a strictly perfect equilibrium. This condition greatly simplifies the characterization of the set of strictly perfect equilibria.

PROPOSITION S2. *In a Poisson voting game satisfying Assumption S1, σ^* is a strictly perfect equilibrium if and only if $\exists \epsilon > 0$ such that if $\tau \in \Delta(\mathcal{C})$: $|\tau - \tau(\sigma^*, F)| < \epsilon$, then $\sigma_t^* \in \mathcal{B}_t(\tau)$ $\forall t \in \mathcal{T}$.*

PROOF. We begin by proving the *if* direction. Assume $\exists \epsilon > 0$ such that $\forall \tau \in \Delta(\mathcal{C})$: $|\tau - \tau(\sigma^*, F)| < \epsilon$, $\sigma_t^* \in \mathcal{B}_t(\tau)$ $\forall t \in \mathcal{T}$. Let $\{(\Gamma, \eta^k)\}_{k=1}^{\infty}$ be a sequence of perturbed games with $\eta^k \to 0$. Choose a subsequence $(\eta^k)_{k=K,\ldots}$ with $\eta^K < \epsilon$. Let

$$\bar{\sigma}(\eta^k) \equiv \Big(\arg \min_{\sigma_t(\eta^K)\in\Sigma_t(\eta^k)} |\sigma_t(\eta^k) - \sigma^*|\Big)_{t\in\mathcal{T}}$$

be the (perturbed game) strategy profile when all players minimize their tremble with respect to the (unperturbed) equilibrium strategy. Since with $\bar{\sigma}(\eta^k)$, $\eta^k < \epsilon$, each player deviates to a different action $c \in \mathcal{C}$ with probability at most ϵ, τ_c can increase by at most ϵ for all c. That is

$$\big|\tau(\bar{\sigma}(\eta^k), F) - \tau(\sigma^*, F)\big| < \epsilon.$$

By assumption it follows that $\sigma_t^* \in \mathcal{B}_t(\tau(\bar{\sigma}(\eta^k), F))$. That is, if players could choose to play the original equilibrium strategy, this would be a best response to $\bar{\sigma}(\eta^k)$. Since $\bar{\sigma}(\eta^k)$ minimizes the tremble from σ_t^*, it follows that $\bar{\sigma}_t(\eta^k) \in \mathcal{B}_t^{\eta}(\tau(\bar{\sigma}(\eta^k), F))$. By construction, it is also true that $\bar{\sigma}(\eta^k) \to \sigma^*$. Hence, σ^* is a strictly perfect equilibrium.

We turn now to prove the *only if* direction. Let $\{\sigma_t^*(\eta^k)\}_{k=1}^{\infty}$ be any sequence of equilibria of perturbed games converging to a strictly perfect equilibrium σ^*. There exists

$\epsilon > 0$ such that $\forall \eta^k < \epsilon$—that is, a limit subsequence of $\{\eta^k\}_{k=1}^{\infty}$—$\sigma_t^* \in \mathcal{B}_t(\tau(\sigma^*(\eta^k), F))$ $\forall t$ and $\sigma_t^* = \mathcal{B}_t(\tau(\sigma^*(\eta^k), F))$ for all positive measures of types. To see this, recall that by Proposition S1, σ^* is a strict best response to itself for all positive measures of types. Suppose that the claim is not true and even for large k, $\sigma_t^* \notin \mathcal{B}_t(\tau(\sigma^*(\eta^k), F))$ for all $t \in \bar{\mathcal{T}}$, where $\bar{\mathcal{T}}$ is some positive measure subset of $\mathcal{T}$. That is, type t players play some action $c \in \mathcal{C}$ in equilibrium and strictly prefer this action to all others. Nevertheless, they play it with zero probability in all perturbed games along the sequence. This implies $\sigma^*(\eta^k) \nrightarrow \sigma^*$, contradicting the assumption that σ^* is strictly perfect. Furthermore, notice that for k sufficiently large, if $\sigma^*(\eta^k) \to \sigma^*$, then the profile $\tau(\sigma^*(\eta^k), F)$ is arbitrarily close to $\tau(\sigma^*, F)$. Since σ^* is a strict equilibrium for all positive measures of players, it must be that for $\tau(\sigma^*(\eta^k), F)$ sufficiently close to $\tau(\sigma^*, F)$, for all $t \in \bar{\mathcal{T}}$, $\sigma_t^* = \mathcal{B}_t(\tau(\sigma^*(\eta^k), F))$.

Notice that since $\sigma_t^* = \mathcal{B}_t(\tau(\sigma^*(\eta^k), F))$ for all positive measures of types, then, for the same types, $\bar{\sigma}_t(\eta^k) = \mathcal{B}_t^{\eta^k}(\tau(\sigma^*(\eta^k), F))$. By construction $\bar{\sigma}(\eta^k) \to \sigma^*$. Call $\bar{\epsilon}$ the minimum of all the ϵ for each sequence $\eta^k \to 0$. For all such sequences, there is a limit sub-sequence with $\eta^k < \bar{\epsilon}$ such that $\bar{\sigma}(\eta^k)$ is a sequence of equilibria (for all positive measures of types) of the perturbed games $\bar{\sigma}(\eta^k) \to \sigma^*$ and $\sigma_t^* \in \mathcal{B}_t(\tau(\bar{\sigma}(\eta^k), F))$.

Notice that

$$\{\tau \in \Delta(\mathcal{C}) : \tau = \tau(\bar{\sigma}(\eta), F),\ \text{some } \eta < \bar{\epsilon}\} = \{\tau \in \Delta(\mathcal{C}) : |\tau - \tau(\sigma^*, F)| < \bar{\epsilon}\}.$$

To see this, recall from Proposition S1 that all but a measure zero of players must play a pure strategy in σ^*. Therefore, any $\tau(c)$: $|\tau(c) - \tau(\sigma^*, F)(c)| < \bar{\epsilon}$ can be generated by simply choosing $\eta_{t,c}$, all t and c such that

$$\tau(c) = \int_{\mathcal{T}} \bar{\sigma}_t(\eta)\, dF(t) \equiv \int_{\mathcal{T}} \min\{1, \sigma_t^*(c) + \eta_{t,c}\}\, dF(t).$$

Hence, since $\sigma_t^* \in \mathcal{B}_t(\tau(\bar{\sigma}(\eta), F))$ $\forall \eta < \bar{\epsilon}$, then $\sigma_t^* \in \mathcal{B}_t(\tau)$ $\forall \tau$: $|\tau - \tau(\sigma^*, F)| < \bar{\epsilon}$. □

A refinement concept with similar intuition to strict perfection has been employed in the analysis of voting games. Expectationally stable equilibria (Fey 1997, Palfrey and Rosenthal 1991) are equilibria that are robust to a small deviation in the expectation of the voters regarding the result of an election. That is, suppose that voters expect $\tau(\sigma^*, F)$ to be the share of votes and $\sigma^* \in \mathcal{B}(\tau(\sigma^*, F))$ so that σ^* is an equilibrium. This equilibrium is expectationally stable if for sufficiently small deviations of the strategy profiles of all the voters, if a single voter is allowed to reoptimize, he does not deviate excessively from the original equilibrium.[2]

[2]The exact definitions of Fey (1997) and Palfrey and Rosenthal (1991) cannot be applied to generic games, since it is valid only when the equilibrium strategy is a one-dimensional cutoff strategy. Both Fey (1997) and Palfrey and Rosenthal (1991) seem to suggest that their interpretation of the solution concept implies a more restrictive definition of expectational stability, where the reoptimized strategy is *closer* to the original equilibrium or is *in the direction* of it. This more stringent solution concept trivially implies our definition of expectational stability.

Definition S4. The strategy profile σ^* such that $\sigma_t^* \in \mathcal{B}_t(\tau(\sigma^*, F))$ $\forall t \in \mathcal{T}$ is an expectationally stable equilibrium if for all $\epsilon > 0$, there exists $\delta > 0$ such that, if σ is a strategy profile such that $|\sigma_t - \sigma_t^*| < \delta$ $\forall t \in \mathcal{T}$, then $|\mathcal{B}_t(\tau(\sigma, F)) - \sigma_t^*| < \epsilon$.

We now show that this concept is equivalent to strict perfection in large Poisson voting games with infinite types.

Proposition S3. *Let* Γ *be a Poisson game satisfying Assumption S1. Then* σ^* *is an expectationally stable equilibrium of* Γ *if and only if it is strictly perfect.*

Proof. We begin by considering the *if* direction. Let σ^* be a strictly perfect equilibrium. Hence, by Proposition S2, there exists $\eta > 0$ such that for any $\tau \in \Delta(\mathcal{C})$: $|\tau - \tau(\sigma^*, F)| < \eta$, $\sigma_t^* \in \mathcal{B}_t(\tau)$ for all $t \in \mathcal{T}$. Let each player play σ_t: $|\sigma_t - \sigma_t^*| < \eta$. Since each player deviates to a different action $c \in \mathcal{C}$ with probability at most η, τ_c can increase by at most η for all c. That is

$$|\tau(\sigma, F) - \tau(\sigma^*, F)| < \eta.$$

By assumption, it follows that $\sigma_t^* \in \mathcal{B}_t(\tau(\sigma, F))$. Hence, for any $\epsilon > 0$, there exists $\delta = \eta$ such that $\forall \sigma$: $|\sigma_t - \sigma_t^*| < \delta$, $\forall t \in \mathcal{T}$, $|\mathcal{B}_t(\tau(\sigma, F)) - \sigma_t^*| = 0 < \epsilon$.

To prove the *only if* direction, let σ^* be an expectationally stable equilibrium. By contradiction, let us consider the possibility that, for any $\delta > 0$, $\sigma_t^* \notin \mathcal{B}_t(\tau(\sigma, F))$ for some $t \in \mathcal{T}$ and σ: $|\sigma_t - \sigma_t^*| < \delta$. This implies that there is an action $c \in \mathcal{C}$ played with positive probability by type t players in σ_t^* that is not a best response to σ for the same players. That is, $\mathcal{B}_t(\tau(\sigma, F))$ does not contain strategy profiles where players of type t play c with positive probability. Let $\epsilon < \sigma_t^*(c)$. Then $|\mathcal{B}_t(\tau(\sigma)) - \sigma_t^*| > \epsilon$. Hence, it must be that $\sigma_t^* \in \mathcal{B}_t(\tau(\sigma, F))$ for all $t \in \mathcal{T}$ and all σ: $|\sigma_t - \sigma_t^*| < \delta$. Since σ: $|\sigma_t - \sigma_t^*| < \delta$ spans the entire set $\tau \in \Delta(\mathcal{C})$: $|\tau - \tau(\sigma^*, F)| < \delta$, we conclude that the equilibrium is strictly perfect. □

In Poisson voting games, the object of the analysis is usually the limit of the set of equilibria as $n \to \infty$. In the remainder of this appendix, we refer to a point in this set as an *aymptotic equilibrium*.

Definition S5. Let $\hat{\Gamma} \equiv \{\Gamma_n\}_{n\to\infty}$ be a sequence of games $\Gamma_n \equiv (n, \mathcal{T}, F, \mathcal{C}, u)$. A strategy profile σ_t^* $\forall t \in \mathcal{T}$ is an *asymptotic equilibrium* of $\hat{\Gamma}$ if there exists a sequence of Nash equilibria $\{\sigma_n^*\}_{n\to\infty}$ of Γ_n such that $\sigma_{n,t}^* \to \sigma_t^*$ for almost all $t \in \mathcal{T}$.

Let us clarify the meaning of this definition by means of an example. Take a Duverger's law equilibrium in a plurality voting game. In such an equilibrium, only two serious candidates receive a positive expected share of votes. For any finite n, there exists a positive measure of voters who are expected to vote for a third candidate—those that are almost indifferent between the two serious candidates, but like this third candidate very much. What we mean when we say that there is a Duverger's law equilibrium as $n \to \infty$ is that, as n grows large, the measure of voters expected to vote for a third candidate goes to zero.

We apply a similar limit concept to strict perfection. For n finite, the equilibrium is not strictly perfect, because there exist voters whose best response is not robust to some perturbation of the strategy space. We say that a sequence of Nash equilibria $\{\sigma_n^*\}_{n\to\infty}$ is *asymptotically strictly perfect* if (i) it admits a limit and (ii) as n grows large, $\Pr[t \in \mathcal{T} : \sigma_t^* \notin \mathcal{B}_t(\tau_n)] \to 0$ for any τ_n sufficiently close to $\tau(\sigma_n^*, F)$.

Definition S6. An asymptotic equilibrium σ^* is *asymptotically strictly perfect* (ASP) if there exists a sequence of Nash equilibria $\{\sigma_n^*\} \to \sigma^*$ for almost all $t \in \mathcal{T}$ such that, for any $\delta > 0$, there exist $N \in \mathbb{N}$ and $\epsilon > 0$ such that, for any $n > N$, if $\tau_n \in \Delta(\mathcal{C})$: $|\tau_n - \tau(\sigma_n^*, F)| < \epsilon$, then $\Pr[t \in \mathcal{T} : \sigma_t^* \notin \mathcal{B}_t(\tau_n)] < \delta$.

Proposition S4 (below) shows that if, as n grows large, a strategy profile is a best response to itself only if two pivotal events have identical magnitudes, then it is not an ASP equilibrium. Importantly, this does not imply that a strategy profile that is not associated with a unique largest magnitude (as n goes to infinity) cannot be an ASP equilibrium. To prove this proposition, we introduce two lemmata.

Myerson (2000) shows that, in a Poisson game, the probability of an exact profile of action shares is exponentially decreasing in the expected number of players, n, and converges to zero at a speed proportional to its *magnitude*. The probability of the action profile x depends on τ, which itself depends on σ and F. In particular, this probability is

$$\Pr(x|\tau) = \prod_{i\in C}\left(\frac{\exp(-n\tau_i)(n\tau_i)^{x_i}}{x_i!}\right).$$

To lighten notation, we omit the τ from the notation of the probability of any action profile or set of action profiles.

An event E is a set of action profiles that satisfy given constraints, i.e., a subset of $Z(\mathcal{C})$. As shown in Myerson (2000, Theorem 1), for a large population of size n, the probability of an event E is such that

$$\mu(E) \equiv \lim_{n\to\infty} \frac{\log[\Pr(E)]}{n} = \max_{x\in E} \sum_i \frac{x_i}{n}\left(1 - \log\left(\frac{x_i}{n\tau_i}\right)\right) - 1.$$

That is, the probability that event E occurs is exponentially decreasing in n; $\mu(E) \in [-1, 0]$ is called the magnitude of event E. Its absolute value represents the "speed" at which the probability decreases toward 0: the more negative is the magnitude, the faster the probability goes to 0.

Furthermore, Myerson (2000, Corollary 1) shows the following lemma.

Lemma S1. *Compare two events $E, E' \subseteq 2^{Z(\mathcal{C})}$ with different magnitudes under $\tau \in \Delta(\mathcal{C})$: $\mu_\tau(E) < \mu_\tau(E')$. Then the probability ratio of the former over the latter event goes to zero as n increases:*

$$\mu_\tau(E) < \mu_\tau(E') \quad \Rightarrow \quad \frac{\Pr(E|n\tau)}{\Pr(E'|n\tau)} \underset{n\to\infty}{\to} 0.$$

The intuition is that the probabilities of different events do not converge toward zero at the same speed. Hence, unless two events have the same magnitude, their likelihood ratio converges either to zero or to infinity when the number of players grows large. Myerson calls this result the *magnitude theorem*.

In a voting game, it is possible to find a subset of events where the action of a single player can change the payoff outcome of the game. These events are called *pivotal events*. The set of pivotal events is $\text{PIV} \subseteq 2^{Z(\mathcal{C})}$. Let $\bar{G}_t(c|E)$ be the gain of action c, conditional on some event E having occurred, for a voter of type $t \in \mathcal{T}$. By definition, $\bar{G}_t(c|E) \neq 0 \Rightarrow E \in \text{PIV}$. Furthermore, it is easy to see that

$$G_t(c, n\tau) = \sum_{E \in \text{PIV}} \Pr(E|n\tau)\bar{G}_t(c|E).$$

That is, when deciding what action to take, all the players only consider the relative probability of different pivotal events.

We say that two pivotal events E and E' are *generically independent* if $\mu_\tau(E) = \mu_\tau(E')$ only for a measure zero of expected voting shares.

Definition S7. Two pivotal events E and E' in PIV are *generically independent* if

$$\{\tau \in \Delta(\mathcal{C}) : \mu(E) = \mu(E')\}$$

has measure zero.

The following proposition allows us to rule out the possibility of strictly perfect equilibria relying on a specific ratio between the probabilities of two generically independent pivotal events.

Proposition S4. *Let σ^* be an asymptotic equilibrium only if two generically independent pivotal events have equal magnitudes under $\tau(\sigma^*, F)$. Then σ^* is not ASP.*

Proof. We prove this statement by contradiction. Let E and E' be two pivotal events. Let σ^* be an asymptotic equilibrium only if $\mu(E) = \mu(E')$ under $\tau^* \equiv \lim_{n\to\infty} \tau(\sigma^*, F)$. By definition, this means that, as $n \to \infty$, $\sigma_t^* \in \mathcal{B}_t(\tau_n)$ for almost all $t \in \mathcal{T}$ and $\sigma_t^* \notin \mathcal{B}_t(\tau_n)$ for a positive measure of voters' types for any $\{\tau_n\}_{n\to\infty}$ inducing $\mu(E) \neq \mu(E')$. Suppose that σ^* is asymptotically strictly perfect. This means that for any $\epsilon > 0$ sufficiently small, as $n \to \infty$, $\sigma_t^* \in \mathcal{B}_t(\tau_n)$ for all $\tau \in \Delta(\mathcal{C})$: $|\tau_n - \tau^*| < \epsilon$ and almost all $t \in \mathcal{T}$. Since, E and E' are generically independent, then there exists $\tau \in \Delta(\mathcal{C})$: $|\tau - \tau^*| < \epsilon$ such that $\mu(E) \neq \mu(E')$. (Proposition 2 in Myerson 2002 guarantees that $\mu(E)$ and $\mu(E')$ are continuous function of τ.) Then asymptotic strict perfection implies that, as $n \to \infty$, $\sigma_t^* \in \mathcal{B}_t(\tau_n)$ for some $\{\tau_n\}_{n\to\infty}$ inducing $\mu(E) \neq \mu(E')$, reaching a contradiction. □

Proposition S4 highlights that asymptotic strict perfection is equivalent to require stability with respect to perturbations in the distribution of preferences. Indeed, $\tau(\sigma^*, F)$ changes if F changes and σ^* remains the same.

Appendix D

The model presented in this paper assumes that for any given pair of candidates participating in the second round, the probability of victory is exogenous, positive, and constant (i.e., independent of the size of the electorate). As shown in Bouton (2013), generically, the chance of second-round victory for a large electorate is never constant in the size of the population. In this appendix, we show that our results do not hinge on the particular reduced-form way of modeling the second round that we have assumed. Indeed, all our results hold (at least qualitatively) in a model with an endogenous second round.

As shown in Bouton (2013), when there is uncertainty about the realized distribution of preferences in the electorate after the first round, when the expected number of voters grows large, the risk of an upset victory converges to zero. The speed of convergence to zero, i.e., the magnitude of the upset victory event, depends on the particular assumptions on the distribution of preferences made to obtain such uncertainty. We prove that the results of our initial model hold (at least qualitatively) no matter what is the speed of convergence to zero.[3] That is, in the spirit of the rest of the paper, we are able to prove general properties of the set of equilibria for any distribution of preferences.

In a model that includes the second round, we show that (i) there always exist at least *two* strictly perfect Duverger's law equilibria, and (ii) there may exist strictly perfect equilibria in which three candidates receive a positive share of the votes (i.e., a Duverger's hypothesis equilibrium). As in the model with the reduced-form second round, the characteristics of the Duverger's hypothesis equilibria are as follows: (i) they never support pushover, (ii) they never support sincere voting by all voters, and (iii) they can lead to the exclusion of the Condorcet winner from the second round.

Even if the results are qualitatively identical to those derived with the reduced-form second round, there are some small differences that deserve to be highlighted. First, the Duverger's law equilibrium in which W is the runner-up might not exist. In line with Bouton (2013), this equilibrium exists whenever the support for W against the front-runner is large enough. The exact condition for existence depends on the structure of the second round. For instance, if there is a complete new draw of voters (from the same distribution of preferences), as in Bouton (2013), then the support for W against the front-runner must be larger than 6.7% of the electorate. The intuition is the same as in Bouton (2013, pp. 1268–1274). Second, there may exist a Duverger's hypothesis equilibrium in which some voters who prefer R to the runner-up do not vote for the former. This can only happen when W is the runner-up. One direct implication is that in the model with an endogenous second round, the (expected) vote share of the front-runner may increase between the two rounds. This is so, even if all voters are strategic.

Proposition S9 below compares the set of equilibria with reduced-form and endogenous second round. We show that the set of equilibria with reduced-form second round

[3] Note that we could also just adopt the exact same structure as in Bouton (2013), i.e., there is a complete new draw of voters between the two rounds. This would imply that, for almost all F, $\forall i \in \{S, W\}$, either $\lim_{n\to\infty} \Pr(R|R, i) = 1$ or $\lim_{n\to\infty} \Pr(R|R, i) = 0$, and we would be able to pinpoint precisely the rate of convergence of these events.

converges to a subset of the set of equilibria with endogenous second round if we let $\Pr(i|Ri)$ converge to 0 or 1 for all candidates $i \neq R$. Not surprisingly, this subset is identified by those equilibria in which the difference between first-round pivotal events dominates the difference between second-round winning probabilities.

LEMMA S2. *In any strictly perfect equilibrium with endogenous second round,*

$$\mu(S|RS) \geq \mu(W|RW).$$

PROOF. By definition, for any finite n, the chances of S defeating R are higher than those of W defeating R. □

Before we begin the analysis, we define

$$\chi \equiv [\mu(piv_{R/RS}) - \mu(piv_{RS/RW})] - [\mu(S|RS) - \mu(W|RW)].$$

Notice that if $\chi > 0$, then the difference between first-round pivotal events dominates the difference between second-round winning probabilities.

PROPOSITION S5. *There always exist strictly perfect Duverger's law equilibria. If $\chi > 0$ or if the front-runner is the Condorcet loser, then any opponent can be the runner-up. Otherwise, only the strong opponent can be the runner-up.*

PROOF. In a Duverger's law equilibrium, there exists a candidate $j \in \mathcal{C}$ with $\tau_j = 0$. Hence, in equilibrium, we have $\mu(piv_{R/Ri}) = \mu(piv_{i/Ri}) \geq$ any other magnitude. Since magnitude formulae are continuous in τ, to show that we have a strictly perfect Duverger's law equilibrium, we need to show that, as $n \to \infty$, $\sigma_t(j) = 0$ is a best response for almost all $t \in \mathcal{T}$, whenever $\mu(piv_{R/Ri}) \geq \mu(piv_{i/Ri}) >$ any other magnitude.

We begin by showing the result for $j = W$. Let

$$\rho \equiv \frac{\Pr(piv_{S/RS})}{\Pr(piv_{R/RS})}$$

$$\phi \equiv \frac{\Pr(piv_{RS/RW})}{\Pr(piv_{R/RS})}.$$

Notice that $\mu(piv_{R/Ri}) = \mu(piv_{i/Ri}) \Rightarrow \rho \to_{n\to\infty} \bar{\rho} \in (0,1)$, $\phi \to 0$, and $\mu(piv_{R/Ri}) > \mu(piv_{i/Ri}) \Rightarrow \phi, \rho \to 0$. Then we have

$$\begin{aligned}
&\lim_{n\to\infty} \frac{G_t(R, n\tau)}{\Pr(piv_{R/RS})} - \lim_{n\to\infty} \frac{G_t(S, n\tau)}{\Pr(piv_{R/RS})} \\
&\quad = 2\Pr(S|RS)(U(R|t) - U(S|t)) \\
&\qquad\quad + 2\rho\big[U(R|t) - U(S|t) + \Pr(S|RS)(U(S|t) - U(R|t))\big] \\
&\qquad\quad - \phi\big[\Pr(S|RS)(U(S|t) - U(R|t)) + \Pr(W|RW)(U(R|t) - U(W|t))\big]
\end{aligned} \tag{S1}$$

and

$$\lim_{n\to\infty}\frac{G_t(R,n\tau)}{\Pr(piv_{R/RS})}-\lim_{n\to\infty}\frac{G_t(W,n\tau)}{\Pr(piv_{R/RS})}$$
$$=2\Pr(S|RS)(U(R|t)-U(S|t))$$
$$-\phi\big[\Pr(S|RS)(U(R|t)-U(S|t))+\Pr(W|RW)(U(W|t)-U(R|t))\big] \tag{S2}$$
$$\lim_{n\to\infty}\frac{G_t(S,n\tau)-G_t(W,n\tau)}{\Pr(piv_{R/RS})}$$
$$=2\rho[U(S|t)-U(R|t)+\Pr(S|RS)(U(R|t)-U(S|t))]$$
$$+2\phi\big[\Pr(S|RS)(U(S|t)-U(R|t))+\Pr(W|RW)(U(R|t)-U(W|t))\big].$$

We can now determine the behavior of voters. There are three cases to consider: (i) R is a Condorcet loser, i.e., $\lim_{n\to\infty}\Pr(R|RS)=0=\lim_{n\to\infty}\Pr(R|RW)$; (ii) R is a Condorcet winner, i.e., $\lim_{n\to\infty}\Pr(R|RS)=1=\lim_{n\to\infty}\Pr(R|RW)$; and (iii) R is neither of the two, i.e., $\lim_{n\to\infty}\Pr(R|RS)=0<\lim_{n\to\infty}\Pr(R|RW)=1$.

First, notice that in all three cases, (S1) and (S2) converge to a multiple of $U(R|t)-U(S|t)$. The last condition converges to

$$2\big[U(S|t)-U(R|t)+\Pr(S|RS)(U(R|t)-U(S|t))\big]$$

in cases (i) and (iii), and to $U(S|t)-U(R|t)$ in case (ii). It sufficient to notice that for all finite n, $\Pr(S|RS)<1$ for all finite n to conclude that all voters who prefer R to S vote for R and all those who prefer S to R vote for S. The measure of voters voting for W equals 0.

We now consider the case of $j=S$. Let

$$\rho\equiv\frac{\Pr(piv_{W/RW})}{\Pr(piv_{R/RW})}$$
$$\phi\equiv\frac{\Pr(piv_{RS/RW})}{\Pr(piv_{R/RW})}.$$

We have

$$\lim_{n\to\infty}\frac{G_t(R,n\tau)}{\Pr(piv_{R/RW})}-\lim_{n\to\infty}\frac{G_t(S,n\tau)}{\Pr(piv_{R/RW})}$$
$$=2\Pr(W|RW)(U(R|t)-U(W|t))$$
$$-\rho\big[\Pr(W|RW)(U(R|t)-U(W|t))+\Pr(S|RS)(U(S|t)-U(R|t))\big] \tag{S3}$$

and

$$\lim_{n\to\infty}\frac{G_t(R,n\tau)}{\Pr(piv_{R/RW})}-\lim_{n\to\infty}\frac{G_t(W,n\tau)}{\Pr(piv_{R/RW})}$$
$$=2\Pr(W|RW)(U(R|t)-U(W|t))$$
$$+2\phi\big[U(R|t)-U(W|t)+\Pr(W|RW)(U(W|t)-U(R|t))\big]$$
$$-\rho\big[\Pr(W|RW)(U(W|t)-U(R|t))+\Pr(S|RS)(U(R|t)-U(S|t))\big] \tag{S4}$$

$$
\begin{aligned}
&\lim_{n\to\infty} \frac{G(S,n\tau)}{\Pr(piv_{R/RW})} - \lim_{n\to\infty} \frac{G_t(W,n\tau)}{\Pr(piv_{R/RW})} \\
&\quad = 2\phi\big[U(R|t) - U(W|t) + \Pr(W|R,W)(U(W|t) - U(R|t))\big] \qquad \text{(S5)} \\
&\qquad\qquad + 2\rho\big[\Pr(W|RW)(U(R|t) - U(W|t)) + \Pr(S|RS)(U(S|t) - U(R|t))\big].
\end{aligned}
$$

We can now determine the behavior of voters. There are three cases to consider: (i) R is a Condorcet loser, i.e., $\lim_{n\to\infty}\Pr(R|R,S) = 0 = \lim_{n\to\infty}\Pr(R|R,W)$; (ii) R is a Condorcet winner, i.e., $\lim_{n\to\infty}\Pr(R|R,S) = 1 = \lim_{n\to\infty}\Pr(R|R,W)$; and (iii) R is neither of the two, i.e., $\lim_{n\to\infty}\Pr(R|R,S) = 0 < \lim_{n\to\infty}\Pr(R|R,W) = 1$.

Notice that if $\chi > 0$, then

$$
\frac{\Pr(W|RW)}{\phi\Pr(S|RS)} \to \infty.
$$

This is sufficient to show that (S3) and (S4) converge to a multiple of $U(R|t) - U(W|t)$ in all three cases. Condition (S5) converges to a multiple of $U(R|t) - U(W|t)$ in cases (ii) and (iii), and to

$$
2\big[U(R|t) - U(W|t) + \Pr(W|RW)(U(W|t) - U(R|t))\big]
$$

in case (i). It sufficient to notice that for all finite n, $\Pr(W|RW) < 1$ for all finite n to conclude that all voters who prefer R to W vote for R and all those who prefer W to R vote for W. The measure of voters voting for S equals 0.

Notice that if $\chi < 0$, then

$$
\frac{\Pr(W|RW)}{\phi\Pr(S|RS)} \to 0.
$$

This implies that in cases (ii) and (iii), (S3) converges to $U(R|t) - U(S|t)$, leading to a positive share of voters voting for S and contradicting the hypothesis.

Last, notice that case (i) corresponds to the R being the Condorcet loser to finish the proof. □

Proposition S6. *For some distribution of preferences, there exist strictly perfect Duverger's hypothesis equilibria in which all voters who prefer the front-runner to the runner-up vote for the front-runner. If the front-runner is not the Condorcet loser, $\chi > 0$, and the runner-up is the weak opponent, some of the supporters of the weak opponent vote for the strong opponent. Otherwise, all voters who prefer the runner-up to the front-runner vote for their preferred candidate.*

Proof. The equilibria in this proposition are all those with

$$
\mu(piv_{R/Ri}) \ge \mu(piv_{RS/RW}) \ge \text{any other magnitude},
$$

where $i \in \{S, W\}$ is the runner-up. Since magnitude formulae are continuous in τ, to show that σ^* is a strictly perfect equilibrium, we need to show that, as $n \to \infty$, σ^* is a best response for almost all $t \in \mathcal{T}$, whenever $\mu(piv_{R/Ri}) > \mu(piv_{RS/RW}) >$ any other magnitude.

We begin by showing the result for $i = S$. Let

$$\phi \equiv \frac{\Pr(piv_{RS/RW})}{\Pr(piv_{R/RS})}.$$

Crucially, notice that $\phi \to 0$ as n grows large.

We have

$$\lim_{n\to\infty} \frac{G_t(S, n\tau)}{\Pr(piv_{R/RS})} - \frac{G_t(W, n\tau)}{\Pr(piv_{R/RS})}$$
$$= 2\phi U(R|t)[\Pr(R|RS) - \Pr(R|RW)] + 2\phi[U(S|t)\Pr(S|RS) - U(W|t)\Pr(W|RW)]$$

$$\lim_{n\to\infty} \frac{G_t(R, n\tau)}{\Pr(piv_{R/RS})} - \frac{G_t(W, n\tau)}{\Pr(piv_{R/RS})}$$
$$= 2\Pr(S|RS)[U(R|t) - U(S|t)] \tag{S6}$$
$$- \phi\big[\Pr(W|RW)(U(W|t) - U(R|t)) + \Pr(S|RS)(U(R|t) - U(S|t))\big],$$

and

$$\lim_{n\to\infty} \frac{G_t(R, n\tau)}{\Pr(piv_{R/RS})} - \frac{G_t(S, n\tau)}{\Pr(piv_{R/RS})}$$
$$= 2\Pr(S|RS)(U(R|t) - U(S|t))$$
$$- \phi\big[\Pr(S|RS)(U(S|t) - U(R|t)) + \Pr(W|RW)(U(R|t) - U)(W|t)\big].$$

We can now determine the behavior of voters. There are three cases to consider: (i) R is a Condorcet loser, i.e., $\lim_{n\to\infty}\Pr(R|RS) = 0 = \lim_{n\to\infty}\Pr(R|RW)$; (ii) R is a Condorcet winner, i.e., $\lim_{n\to\infty}\Pr(R|RS) = 1 = \lim_{n\to\infty}\Pr(R|RW)$; and (iii) R is neither of the two, i.e., $\lim_{n\to\infty}\Pr(R|RS) = 0 < \lim_{n\to\infty}\Pr(R|RW) = 1$.

First, we show that all voters who prefer R to S vote for R. That is, in all three cases, (S6) and (S9) converge to a multiple of $U(R|t) - U(S|t)$. To see this point, it is sufficient to recall that $\phi \to 0$ and $\mu(S|RS) \geq \mu(W|RW)$.

Second, we can determine the behavior of voters who prefer S to R. Using condition (S8), we have that such a voter votes for S only if

$$U(S|t) - U(W|t) > \Pr(R|RW)[U(R|t) - U(W|t)] + \Pr(R|RS)[U(S|t) - U(R|t)]. \tag{S7}$$

Otherwise she votes for W.

For each of the three cases, we have to determine the behavior of the three types of voters who prefer S to R, i.e., t_{SRW}, t_{SWR}, and t_{WSR}.

Case (i): R is a Condorcet loser. In this case, the right-hand side of (S7) converges to zero. Therefore, voters who prefer S to W (i.e., t_{SWR}, and t_{SRW}) vote for S and voters who prefer W to S (i.e., t_{WSR}) vote for W.

Case (ii): R is a Condorcet winner. In this case, the right-hand side of (S7) converges to $U(S|t) - U(W|t)$ from below (because $\Pr(R|Ri) < 1$ for any finite n). Therefore, voters who prefer S to W (i.e., t_{SWR}, and t_{SRW}) vote for S and voters who prefer W to S (i.e., t_{WSR}) vote for W.

Case (iii): R is neither. In this case, there is no Duverger's hypothesis equilibrium because (S7) converges to $U(S|t) - U(R|t)$ (all voters who prefer S to R vote for S). Notice that this implies $\mu(piv_{R/Ri}) = \mu(piv_{i/Ri}) \geq$ any other magnitude.

A similar proof holds for $\mu(piv_{RS/RW}) = \mu(piv_{R/RW})$.
We now consider the case of $i = W$. Let

$$\phi \equiv \frac{\Pr(piv_{RS/RW})}{\Pr(piv_{R/RS})}.$$

Crucially, notice that $\phi \to 0$ as n grows large.
We have

$$\begin{aligned}\lim_{n\to\infty} \frac{G_t(S, n\tau)}{\Pr(piv_{R/RW})} - \frac{G_t(W, n\tau)}{\Pr(piv_{R/RW})} &= 2\phi \Pr(S|RS)[U(S|t) - U(R|t)] \\ &\quad + 2\phi \Pr(W|RW)[U(R|t) - U(W|t)]\end{aligned} \tag{S8}$$

$$\begin{aligned}\lim_{n\to\infty} \frac{G_t(R, n\tau)}{\Pr(piv_{R/RW})} - \frac{G_t(W, n\tau)}{\Pr(piv_{R/RW})} &= (2+\phi) \Pr(W|RW)[U(R|t) - U(W|t)] \\ &\quad + \phi \Pr(S|RS)[U(R|t) - U(S|t)]\end{aligned} \tag{S9}$$

and

$$\begin{aligned}\lim_{n\to\infty} \frac{G_t(R, n\tau)}{\Pr(piv_{R/RW})} - \frac{G_t(S, n\tau)}{\Pr(piv_{R/RW})} &= (2-\phi) \Pr(W|RW)[U(R|t) - U(W|t)] \\ &\quad + \phi \Pr(S|RS)[U(R|t) - U(S|t)].\end{aligned} \tag{S10}$$

We can now determine the behavior of voters. There are three cases to consider: (i) R is a Condorcet loser, i.e., $\lim_{n\to\infty} \Pr(R|RS) = 0 = \lim_{n\to\infty} \Pr(R|RW)$; (ii) R is a Condorcet winner, i.e., $\lim_{n\to\infty} \Pr(R|RS) = 1 = \lim_{n\to\infty} \Pr(R|RW)$; and (iii) R is neither of the two, i.e., $\lim_{n\to\infty} \Pr(R|RS) = 0 < \lim_{n\to\infty} \Pr(R|RW) = 1$.

First, we show that all voters who prefer R to W vote for R. To see this point, recall that $\phi \to 0$ and $\mu(S|RS) \geq \mu(W|RW)$. Also, let $\chi > 0$ and notice that this implies

$$\frac{\Pr(W|RW)}{\phi \Pr(S|RS)} \to \infty.$$

With this in mind, it is easy to see that (S9) and (S10) both converge to a multiple of $U(R|t) - U(W|t)$ in all three cases. Hence, a voter who prefers R to W votes for R.

On the contrary, if $\chi < 0$, (S9) and (S10) both converge to a multiple of $U(R|t) - U(W|t)$ only in case (i). Otherwise, they converge to a multiple of $U(R|t) - U(S|t)$.

Second, we can determine the behavior of voters who prefer W to R. From (S8), we have that such a voter will vote for W only if

$$\Pr(S|RS)[U(S|t) - U(R|t)] < \Pr(W|RW)[U(W|t) - U(R|t)].$$

Otherwise she votes for S. In case (i), this converges to $U(S|t) < U(W|t)$. Otherwise it converges to $U(S|t) < U(R|t)$. We can, therefore, conclude that in case (i), all voters who prefer R to W vote for R and the remaining voters vote for their most preferred candidate. In cases (ii) and (iii), if $\chi > 0$, then all voters who prefer R to W vote for R, and the remaining voters vote for W if and only if they prefer R to S. That is, some W supporter votes for S. In cases (ii) and (iii), if $\chi < 0$, there is no voter who votes for W. Notice that this implies a Duverger's law equilibrium with $\mu(piv_{R/Ri}) = \mu(piv_{i/Ri}) \geq$ any other magnitude. A similar proof holds for $\mu(piv_{RS/RW}) = \mu(piv_{R/RS})$. □

Proposition S7. *For some distribution of preferences, if $\chi < 0$ and the front-runner is not the Condorcet loser, then there exists a strictly perfect Duverger's hypothesis equilibrium where (i) the weak opponent is the runner-up, (ii) all voters who prefer the runner-up to the front-runner vote for the runner-up, and (iii) among the voters who prefer the front-runner to the runner-up, those who prefer the strong opponent to the front-runner vote for the strong opponent.*

Proof. Notice that if $\chi < 0$,

$$\frac{\Pr(W|RW)}{\phi \Pr(S|RS)} \to 0.$$

Then, when $\mu(piv_{R/RW}) > \mu(piv_{W/RW}) >$ any other magnitude, in cases (ii) and (iii), (S3) in the Proof of Proposition S5 converges to $U(R|t) - U(S|t)$. Also, (S4) and (S5) converge to a multiple of $U(R|t) - U(W|t)$. This says that all voters who prefer R to both S and W vote for R. All voters who prefer W to R vote for W. Yet, this is not a Duverger's law equilibrium because some of the voters who prefer R to the runner-up, W, vote for S. These are the voters who prefer S to R and R to W (i.e., t_{SRW} voters). □

Proposition S8. *There is no other strictly perfect equilibrium.*

From Lemma 3 and Proposition 3, we know that in any strictly perfect equilibrium, the order of first-round magnitudes must be either

$$\mu(piv_{R/Ri}) \geq \mu(piv_{i/Ri}) \geq \text{any other magnitude}$$

or

$$\mu(piv_{R/Ri}) \geq \mu(piv_{RS/RW}) \geq \text{any other magnitude.}$$

The first case is completely discussed in the proofs of Propositions S5 and S7, and leads only to equilibria described in those two propositions. The second case is completely discussed in the proof of Proposition S6. Finally, notice that, by Lemma 1, if a strategy profile is an equilibrium only if $\chi = 0$, then it is not strictly perfect.

Corollary S1. *There is no strictly perfect Duverger's hypothesis equilibria supporting pushover or sincere voting.*

Proof. The corollary follows from the description of the equilibria in Propositions S5, S6, and S7, and that these are all the strictly perfect equilibria. □

We now turn to the comparison of the model with reduced-form second round. For this exercise, we denote by $\varsigma \in (0,1)$ and $\omega \in (0,1)$ the exogenously given probabilities that candidate S and W defeat R in the second round. Let $\mathcal{E}(\varsigma, \omega; F)$ be the set of strictly perfect equilibria, as $n \to \infty$, when F is the distribution of voters, and second-round probabilities are given exogenously by ς and ω. We also define

$$s \equiv \lim_{n\to\infty} \Pr(S|RS)$$
$$w \equiv \lim_{n\to\infty} \Pr(W|RW)$$
$$\Gamma \equiv \lim_{n\to\infty} \frac{\Pr(W|RW)}{\Pr(S|RS)}.$$

Then we can denote the set of strictly perfect equilibria (as $n \to \infty$) with endogenous second-round probabilities as $\bar{\mathcal{E}}(s, w; F)$.

Let $\bar{\mathcal{E}}(s, w; F|\chi > 0)$ be the set of elements of $\bar{\mathcal{E}}(s, w; F)$ for which $\chi > 0$. Notice that this includes all equilibria where R is the Condorcet loser. The following proposition says that provided that $\chi > 0$, the model with reduced-form second round converges to the model with endogenous second round when the reduced-form second-round winning probabilities converge to the limit of their endogenous counterpart.

Proposition S9. *For all distributions of voters' preferences in the first round F, and for all endogenous second-round winning probabilities converging to s and w,*

$$\lim_{\substack{\varsigma\to s,\,\omega\to w\\ \frac{\omega}{\varsigma}\to\Gamma}} \mathcal{E}(\varsigma, \omega; F) = \bar{\mathcal{E}}(s, w; F|\chi > 0).$$

Proof. We show the result for W being the runner-up. The case for S being the runner-up is similar with the exception that χ plays no role.

If $\chi > 0$, there always exist three Duverger's law equilibria for any exogenous or endogenous second-round specification. Hence, the proposition is trivially true for Duverger's law equilibria.

We now turn to the Duverger's hypothesis equilibria. In all equilibria with an exogenous second round, all voters who prefer R to W vote for R. With an endogenous second round, this is true whenever $\chi > 0$. It remains to show that the choice between S and W is the same as $\varsigma \to s$ and $\omega \to w$.

Notice that this choice depends on

$$\text{vote for } S \text{ only if} \quad [U(S|t) - U(R|t)] > \frac{\omega}{\varsigma}[U(W|t) - U(R|t)]. \tag{S11}$$

There are three cases to consider: (i) when R is the Condorcet loser, that is, $s = w = 1$; (ii) when R is the Condorcet winner, that is, $s = w = 0$; and (iii) when R is neither, that is, $s = 1 > 0 = w$.

In the first case, $\omega/\varsigma \to 1^-$; hence, (S11) converges to $U(S|t) > U(W|t)$. This is the same as the case in Proposition S6 when R is the Condorcet loser.

In the second and third cases, $\omega/\varsigma \to 0$. (We know from the endogenous model that any strictly perfect equilibrium with $w = 0$ must be an equilibrium also when $\Gamma = 0$, that is, when $\mu(S|RS) > \mu(W|RW)$.) Then (S11) converges to $U(S|t) > U(R|t)$. This is the same as the case in Proposition S6 when R is not the Condorcet loser. □

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