Supplement to: Party Platforms in Electoral Competition with Heterogeneous Constituencies

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In this supplement, we analyse the case where the two candidates have different valuations for winning the election. Candidate A_δ's valuation is given by $2V_A$ and B_δ's by $2V_B$. Note that this model with different valuations is equivalent to a model where candidates pay different (constant) marginal costs for locating away from their party platforms. More generally, if parties' platforms are given by a and b and candidate A_δ's utility if she wins at position α is $2V_A - c_A|\alpha - a|$, and B_δ's if she wins at β is $2V_B - c_B|\beta - a|$ then the candidates' (and parties') maximisation problems have the same solutions as in a model with utilities for winning of $2\frac{V_A}{c_A} - |\alpha - a|$ and $2\frac{V_B}{c_B} - |\beta - a|$, respectively.

Proposition 1 *Let $a \leq b \leq \delta$. For any pair of candidates' equilibrium strategies $\left(F^*_{A_\delta}, F^*_{B_\delta}\right)$ the following characterises $F^*_{A_\delta}(a, b; \cdot)$ and $F^*_{B_\delta}(a, b; \cdot)$:*

- *If $b - a \geq 2V_A$, then both candidates retain party platform.*

  Suppose $b - a < 2V_A$.

- *If $V_A - V_B \leq \frac{b-a}{2}$ and*
  - *$a + 2V_A \leq \delta$, then $F^*_{A_\delta}(a, b; a) = \frac{2(V_B - V_A) + b - a}{2V_B}$, $F^*_{B_\delta}(a, b; b) = \frac{b-a}{2V_A}$, and candidates randomise over $(b, a + 2V_A)$ with densities $f^*_{A_\delta}(a, b; t) = \frac{1}{2V_B}$ and $f^*_{B_\delta}(a, b; t) = \frac{1}{2V_A}$.*
  - *$\frac{a+b}{2} + V_A \leq \delta \leq a + 2V_A$ then $F^*_{A_\delta}(a, b; a) = \frac{2(V_B - V_A) + b - a}{2V_B}$ and $F^*_{B_\delta}(a, b; b) = \frac{b-a}{2V_A}$, candidates take the median voter's position with probabilities $\Pr_A(\delta) = \frac{2V_A - \delta + a}{V_B}$ and $\Pr_B(\delta) = \frac{2V_A + a - \delta}{V_A}$, respectively, and randomise continuously on $(b, 2\delta - a - 2V_A)$ with densities $f^*_{A_\delta}(a, b; t) = \frac{1}{2V_B}$ and $f^*_{B_\delta}(a, b; t) = \frac{1}{2V_A}$.*
  - *$a + V_A \leq \delta \leq \frac{a+b}{2} + V_A$, then $F^*_{A_\delta}(a, b; t) = \frac{V_B - \delta + b}{V_B}$ for each $t \in [a, \delta)$ and $F^*_{B_\delta}(a, b; t) = \frac{\delta - a - V_A}{V_A}$ for each $t \in [b, \delta)$.*

- *If $V_A - V_B > \frac{b-a}{2}$ and*
  - *$b + 2V_B < \delta$ then $F^*_{A_\delta}(a, b; a) = 0$, $F^*_{B_\delta}(a, b; b) = \frac{V_A - V_B}{V_A}$, and candidates randomise over $(b, b + 2V_B)$ with densities $f^*_{A_\delta}(a, b; t) = \frac{1}{2V_B}$ and $f^*_{B_\delta}(a, b; t) = \frac{1}{2V_A}$.*

- $b + 2V_B \geq \delta > b + V_B$ *then* $F^*_{A_\delta}(a, b; t) = \frac{t-b}{2V_B}$ *for* $t \in (b, 2\delta - b - 2V_B)$ *and* $F^*_{A_\delta}(a, b; t) = \frac{\delta - b - V_B}{V_B}$ *for* $t \in [2\delta - b - 2V_B, \delta)$*; also,* $F^*_{B_\delta}(a, b; t) = \frac{V_A - V_B}{V_A} + \frac{t-b}{2V_A}$ *for* $t \in [b, 2\delta - b - 2V_B)$*, and* $F^*_{B_\delta}(a, b; t) = \frac{V_A - V_B}{V_A} + \frac{\delta - b - V_B}{V_A}$ *for* $t \in [2\delta - b - 2V_B, \delta)$.
- $a + V_A < \delta \leq b + V_B$*, then* $F^*_{A_\delta}(a, b; t) = \frac{V_B - \delta + b}{V_B}$ *for each* $t \in [a, \delta)$ *and* $F^*_{B_\delta}(a, b; b) = \frac{\delta - a + V_A}{V_A}$ *for each* $t \in [b, \delta)$.

- *If* $\delta \leq \min\{a + V_A, b + V_B\}$*, then both candidates take the median voter's position.*[1]

Proof: Lemmas 1 and 2 in the text continue to hold for asymmetric valuations. It is straightforward to show that when candidate A_δ mixes with density f_{A_δ} on some interval (u, v), $b \leq u < v \leq \delta$ that $f_{A_\delta} = \frac{1}{2V_B}$, and similarly that B_δ mixes with density $f_{B_\delta} = \frac{1}{2V_A}$. The remainder of the proof consists of checking which of the configurations outlined in the text are equilibria and is omitted.

Q.E.D.

From the previous Proposition we can calculate the candidates' expected payoffs for any subgame. We here just state the result.

Corollary 1 *Assume* $a \leq b \leq \delta$*. If* $b - a < 2V_A$ *Candidate* A_δ*'s equilibrium expected payoff in any subgame is as follows:*

$$U_{A_\delta}(a, b, \delta) = \begin{cases} 0 & \text{if} \quad V_A - V_B \leq \frac{b-a}{2} \text{ and } \delta \geq V_A + a \\ V_A - \delta + a & \text{if} \quad \delta < \min\{a + V_A, b + V_B\} \\ 0 & \text{if} \quad V_A - V_B > \frac{b-a}{2} \text{ and } V_A + a \leq \delta < V_B + b \\ 2(V_A - V_B) + a - b & \text{if} \quad V_A - V_B > \frac{b-a}{2} \text{ and } \delta \geq V_B + b \end{cases}$$

Candidate B_δ*'s equilibrium expected payoff in any subgame is as follows:*

$$U_{B_\delta}(a, b, \delta) = \begin{cases} 2(V_B - V_A) - a + b & \text{if} \quad V_A - V_B \leq \frac{b-a}{2} \text{ and } \delta \geq V_A + \frac{a+b}{2} \\ 2(V_B - \delta + b) & \text{if} \quad V_A + a \leq \delta < \min\{V_A + \frac{a+b}{2}, V_B + b\} \\ V_B - \delta + b & \text{if} \quad \delta < \min\{a + V_A, b + V_B\} \\ 0 & \text{if} \quad V_A - V_B > \frac{b-a}{2} \text{ and } \delta \geq V_B + b \end{cases}$$

Theorem 1 *If* $\frac{1}{2} \geq V_A \geq V_B$ *and* $V_A \leq \min\left(7V_B - 1, \frac{3}{5}V_B + \frac{1}{5}\right)$ *the parties' equilibrium strategies* (a^*, b^*) *are given by*

$$\begin{aligned} a^* &= \frac{11}{8}V_A - \frac{13}{8}V_B + \frac{3}{8}, \\ b^* &= \frac{13}{8}V_A - \frac{11}{8}V_B + \frac{5}{8}. \end{aligned}$$

[1] Equilibrium is unique up to redefining cases by changing strict inequalities to weak ones. Doing so would not affect parties' expected continuation payoffs as the boundaries of these different regions occur with probability zero.

Proof of Theorem: Assume party A takes platform a and party B takes b and w.l.o.g. $a \leq b$. Similar to the proof of Theorem 1 in the paper candidate A's payoff for any realisation of δ can be derived from Proposition 1 by a relabeling of variables: A's payoff is $U_A(x, y, \delta)$, where

$$(x, y) = \begin{cases} (2\delta - b, 2\delta - a) & \text{if} \quad \delta < a \\ (2\delta - b, a) & \text{if} \quad a \leq \delta \leq \frac{a+b}{2} \\ (a, 2\delta - b) & \text{if} \quad \frac{a+b}{2} < \delta \leq b \\ (a, b) & \text{if} \quad b < \delta. \end{cases}$$

Using this, we can calculate A's expected payoff $\overline{U}_A(a, b)$ from platforms (a, b) from Proposition 1. This can be used to establish the following claim.

Claim 1: Party A will never choose a platform a with $b - a > 2V_B$.

Proof:

If $b - a \geq \max(2V_B, 2V_A)$ we have that

$$\overline{U}_A(a, b) = 2V_A\left(\frac{a+b}{2} - V_B\right) + \int_{\frac{a+b}{2}-V_B}^{\frac{a+b}{2}+V_A-V_B} (2(V_A - V_B) + a + b - 2\delta)\, d\delta$$

and thus $\frac{\partial \overline{U}_A(a,b)}{\partial a} = V_A > 0$ and party A prefers to take a position such that $b - a < \max(2V_B, 2V_A)$. Thus if $V_B > V_A$ the claim is established. If $2V_A > b - a > 2V_B$. We have that

$$\begin{aligned} \overline{U}_A(a, b) &= 2V_A\left(\frac{a+b}{2} - V_B\right) + \int_{\frac{a+b}{2}-V_B}^{b-V_B} (2(V_A - V_B) + a + b - 2\delta)\, d\delta \\ &+ \int_{b-V_B}^{\min\{V_A+a, V_B+b, 1\}} (V_A - \delta + a)\, d\delta \\ &+ \int_{\min\{V_B+b,1\}}^{1} \max\{2(V_A - V_B) + a - b, 0\} d\delta \end{aligned}$$

and thus $\frac{\partial \overline{U}_A(a,b)}{\partial a} > 0$ and party A prefers to take a position such that $b - a < 2V_B$ as well, which proves the claim.

Claim 1 implies that only platforms a with $b - a \leq 2V_B$ can be optimal for A. When

$0 \leq b - a \leq 2V_B$, we get

$$\begin{aligned}
\overline{U}_A(a,b) &= \max\{0, 2(V_A - V_B) + b - a\} \max\left\{\frac{a+b}{2} - V_B, 0\right\} \\
&+2\int_{\max\left\{\frac{a+b}{2}-V_B,0\right\}}^{\max\{\min\{b-V_B,a\},0\}} \max\{0, V_A + \delta - a\} d\delta \\
&+\int_{\max\{\min\{b-V_B,a\},0\}}^{a} \max\{0, V_A + \delta - a\} d\delta \\
&+2\int_{a}^{\max\{\min(b-V_B,a+V_A),a\}} (V_A - \delta + a)\, d\delta \\
&+\int_{\max\{\min(b-V_B,a+V_A),a\}}^{\min\{V_A+a,V_B+b,1\}} (V_A - \delta + a)\, d\delta \\
&+\max\{0, 1 - b - V_B\} \max\{0, 2(V_A - V_B) + a - b\}.
\end{aligned} \tag{1}$$

We use this representation to establish party A's optimal position a in the interval $[0, b]$ given b.

Assume first that $V_A \geq V_B$. We distinguish the following cases: (i) $b \leq V_B$; (ii) $b - V_B > a \wedge \frac{a+b}{2} - V_B < 0$ (iii) $0 < b - V_B < a \wedge \frac{a+b}{2} - V_B < 0$, (iv) $b - V_B \geq a \wedge \frac{a+b}{2} - V_B \geq 0$ and (v) $b - V_B < a \wedge \frac{a+b}{2} - V_B \geq 0$.

(i) As $b \leq V_B \wedge a \leq b \Longrightarrow \frac{a+b}{2} - V_B \leq 0$ we have that

$$\begin{aligned}
\overline{U}_A(a,b) &= \int_0^a (V_A + \delta - a)\, d\delta + \int_a^{\min\{V_A+a,V_B+b,1\}} (V_A - \delta + a)\, d\delta \\
&+\max\{0, 1 - b - V_B\} \max\{0, 2(V_A - V_B) + a - b\}.
\end{aligned}$$

If $V_A + a < 1$ we have $\frac{\partial \overline{U}_A(a,b)}{\partial a} \geq V_A - a > 0$ (as $V_A \geq V_B \geq b \geq a$). If $V_B + b \leq 1 \leq V_A + a$ we have $\frac{\partial \overline{U}_A(a,b)}{\partial a} \geq V_B + b - 2a \geq 0$. If $\min(V_A + a, V_B + b) > 1$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} \geq 1 - 2a$.

(ii) We have that

$$\begin{aligned}
\overline{U}_A(a,b) &= 2\int_0^a (V_A + \delta - a)\, d\delta + 2\int_a^{b-V_B} (V_A - \delta + a)\, d\delta \\
&+\int_{b-V_B}^{\min(V_A+a,V_B+b,1)} (V_A - \delta + a)\, d\delta \\
&+\max(0, 1 - b - V_B) \max(0, 2(V_A - V_B) + a - b).
\end{aligned}$$

and $\frac{\partial \overline{U}_A(a,b)}{\partial a} \geq \min(V_A - V_B + b - 3a, 2b - 4a, 1 + b - V_B - 4a) > 0$, where the inequality follows from $V_A > V_B, b - V_B > a$ and $\frac{a+b}{2} < b - a$.

(iii) We have that

$$\begin{aligned}\overline{U}_A(a,b) &= 2\int_0^{b-V_B}(V_A+\delta-a)\,d\delta+\int_{b-V_B}^{a}(V_A+\delta-a)\,d\delta\\ &\quad+\int_a^{\min\{V_A+a,V_B+b,1\}}(V_A-\delta+a)\,d\delta\\ &\quad+\max\{0,1-b-V_B\}\max\{0,2(V_A-V_B)+a-b\}\end{aligned}$$

and $\frac{\partial\overline{U}_A(a,b)}{\partial a}\geq V_B+V_A-b-a>0$ if $V_A+a<1$ or $V_B+b\leq 1$. We have $\frac{\partial\overline{U}_A(a,b)}{\partial a}=V_B-2a-b+1$ if $V_A+a\geq 1$ and $V_B+b>1$.

(iv) We have that

$$\begin{aligned}\overline{U}_A(a,b) &= (2(V_A-V_B)+b-a)(\frac{a+b}{2}-V_B)+2\int_{\frac{a+b}{2}-V_B}^{a}(V_A+\delta-a)\,d\delta\\ &\quad+2\int_a^{b-V_B}(V_A-\delta+a)\,d\delta+\int_{b-V_B}^{\min\{V_A+a,V_B+b,1\}}(V_A-\delta+a)\,d\delta\\ &\quad+\max\{0,1-b-V_B\}\max\{0,2(V_A-V_B)+a-b\}\end{aligned}$$

Assume that $V_B+b<1$. If $V_A+a\leq V_B+b$ and $2(V_A-V_B)+a-b<0$ we have that $\frac{\partial\overline{U}_A(a,b)}{\partial a}=\frac{3}{2}b-\frac{5}{2}a+V_A-2V_B$ and if $V_A+a\leq V_B+b$ and $2(V_A-V_B)+a-b\geq 0$ we have that $\frac{\partial\overline{U}_A(a,b)}{\partial a}=\frac{1}{2}b-\frac{5}{2}a+V_A-3V_B+1$. If $V_B+b<V_A+a$ we have $\frac{\partial\overline{U}_A(a,b)}{\partial a}=\frac{5}{2}b-\frac{7}{2}a-V_B$. Assume now that $V_B+b\geq 1$. If $1\leq V_A+a$ we have $\frac{\partial\overline{U}_A(a,b)}{\partial a}=\frac{3}{2}b-\frac{7}{2}a-2V_B+1$. If $V_A+a<1$ we have $\frac{\partial\overline{U}_A(a,b)}{\partial a}=\frac{3}{2}b-\frac{5}{2}a+V_A-2V_B$.

(v) We have that

$$\begin{aligned}\overline{U}_A(a,b) &= (2(V_A-V_B)+b-a)(\frac{a+b}{2}-V_B)+2\int_{\frac{a+b}{2}-V_B}^{b-V_B}(V_A+\delta-a)\,d\delta\\ &\quad+\int_{b-V_B}^{a}(V_A+\delta-a)\,d\delta+\int_a^{\min\{V_A+a,V_B+b,1\}}(V_A-\delta+a)\,d\delta\\ &\quad+\max\{0,1-b-V_B\}\max\{0,2(V_A-V_B)+a-b\}.\end{aligned}$$

Assume that $V_B+b<1$. If $V_A+a\leq V_B+b$ we have that $\frac{\partial\overline{U}_A(a,b)}{\partial a}=V_A-\frac{a+b}{2}$ If $V_B+b<V_A+a$ we have $\frac{\partial\overline{U}_A(a,b)}{\partial a}=V_B-\frac{a+b}{2}+b-a$. Assume now that $V_B+b\geq 1$. If $1<V_A+a$ we have $\frac{\partial\overline{U}_A(a,b)}{\partial a}=1-\frac{a+b}{2}-a$. If $V_A+a\leq 1$ we have $\frac{\partial\overline{U}_A(a,b)}{\partial a}=V_A-V_B+1-\frac{a}{2}-\frac{3}{2}b$.

Assume now that $V_A<V_B$. We again distinguish the following cases: (i) $b\leq V_B$; (ii) $b-V_B>a\wedge\frac{a+b}{2}-V_B<0$ (iii) $0<b-V_B<a\wedge\frac{a+b}{2}-V_B<0$, (iv) $b-V_B\geq a\wedge\frac{a+b}{2}-V_B\geq 0$ and (v) $b-V_B<a\wedge\frac{a+b}{2}-V_B\geq 0$.

(i) As $b\leq V_B\wedge a\leq b\Longrightarrow\frac{a+b}{2}-V_B\leq 0$ we have that

$$\overline{U}_A(a,b)=\int_{\max\{0,a-V_A\}}^{a}(V_A+\delta-a)\,d\delta+\int_a^{\min\{V_A+a,1\}}(V_A-\delta+a)\,d\delta.$$

If $a > V_A$ and $V_A + a < 1$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = 0$ and if $a > V_A$ and $V_A + a \geq 1$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = 1 - V_A - a < 0$. If $a < V_A$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = V_A - a > 0$ if $V_A + a < 1$ and $\frac{\partial \overline{U}_A(a,b)}{\partial a} = 1 - 2a$ if $V_A + a > 1$.

(ii) We have that (as $a + V_B < b \Rightarrow a + V_A < 1$) :

$$\begin{aligned}\overline{U}_A(a,b) &= 2\int_{\max\{0,a-V_A\}}^{a} (V_A + \delta - a)\, d\delta + 2\int_{a}^{\min\{b-V_B,a+V_A\}} (V_A - \delta + a)\, d\delta \\ &\quad + \int_{\min\{b-V_B,a+V_A\}}^{V_A+a} (V_A - \delta + a)\, d\delta.\end{aligned}$$

If $a < V_A$ and $b - V_B \leq a + V_A$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = V_A - V_B - 3a + b$.If $a < V_A$ and $b - V_B > a + V_A$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = 2V_A - 2a > 0$. If $a > V_A$ and $b - V_B \leq a + V_A$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = b - a - V_A - V_B$. If $a > V_A$ and $b - V_B > a + V_A$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = 0$.

(iii) We have that

$$\begin{aligned}\overline{U}_A(a,b) &= 2\int_{\max\{0,a-V_A\}}^{\max\{a-V_A,b-V_B\}} (V_A + \delta - a)\, d\delta + \int_{\max\{a-V_A,b-V_B\}}^{a} (V_A + \delta - a)\, d\delta \\ &\quad + \int_{a}^{\min\{V_A+a,1\}} (V_A - \delta + a)\, d\delta.\end{aligned}$$

If $a < V_A$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = V_A + V_B - a - b$ if $V_A + a < 1$ and that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = V_B - b - 2a + 1$ if $V_A + a > 1$. If $0 < a - V_A < b - V_B$ and $V_A + a < 1$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = a - b + V_B - V_A < 0$ and if $0 < a - V_A < b - V_B$ and $V_A + a \geq 1$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = V_B - b - 2V_A + 1$. If $a - V_A > b - V_B$ and $V_A + a_a < 1$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = 0$ and if $a - V_A > b - V_B$ and $V_A + a \geq 1$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = 1 - a - V_A < 0$.

(iv) We have that

$$\begin{aligned}\overline{U}_A(a,b) &= \max(0, 2(V_A - V_B) + b - a)\left(\frac{a+b}{2} - V_B\right) \\ &\quad + 2\int_{\max\{\frac{a+b}{2}-V_B,a-V_A\}}^{a} (V_A + \delta - a)\, d\delta \\ &\quad + 2\int_{a}^{\min\{b-V_B,a+V_A\}} (V_A - \delta + a)\, d\delta + \int_{\min\{b-V_B,a+V_A\}}^{\min\{V_A+a,1\}} (V_A - \delta + a)\, d\delta.\end{aligned}$$

If $2(V_A - V_B) + b - a < 0$ we have that $a - V_A > \frac{a+b}{2} - V_B$ and thus if $b - V_B < V_A + a < 1$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = b - a - V_A - V_B$ and if $b - V_B > V_A + a$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = 0$ and if $V_A + a \geq 1$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = b - 2a - 2V_A - V_B$. If $2(V_A - V_B) + b - a > 0$ and $b - V_B \leq V_A + a < 1$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = V_A - 2V_B - \frac{5}{2}a + \frac{3}{2}b$. If $2(V_A - V_B) + b - a > 0$

and $V_A + a \geq 1$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = 1 - 2V_B + \frac{3}{2}b - \frac{7}{2}a$. If $2(V_A - V_B) + b - a > 0$ and $V_A + a < b - V_B$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = 2V_A - V_B - \frac{3}{2}a + \frac{1}{2}b$.

(v) We have that

$$\overline{U}_A(a,b) = \max\{0, 2(V_A - V_B) + b - a\}(\frac{a+b}{2} - V_B)$$
$$+2\int_{\frac{a+b}{2}-V_B}^{b-V_B} \max\{0, V_A + \delta - a\}d\delta$$
$$+\int_{b-V_B}^{a} \max\{0, V_A + \delta - a\}d\delta + \int_{a}^{\min\{V_A+a,1\}} (V_A - \delta + a)\, d\delta.$$

If $2(V_A - V_B)+b-a < 0$ and $a-V_A > b-V_B$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = 0$. If $2(V_A - V_B)+b-a < 0$ and $a-V_A < b-V_B$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = a-b+V_B-V_A < 0$. If $2(V_A - V_B)+b-a > 0$ and $V_A + a < 1$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = V_A - \frac{1}{2}a - \frac{1}{2}b < 0$. If $2(V_A - V_B) + b - a > 0$ and $V_A + a > 1$ we have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} = 1 - \frac{1}{2}b - \frac{3}{2}a$.

One important property of $\overline{U}_A(a,b)$ that can be deduced from this analysis (by simply going through all possible cases) is that whenever $\frac{\partial^2 \overline{U}_A(a,b)}{\partial a} > 0$ we must have that $\frac{\partial \overline{U}_A(a,b)}{\partial a} < 0$.Therefore $\overline{U}_A(a,b)$ is concave or decreasing and (as$\overline{U}_A(a,b)$ is continuous) the first order condition $\frac{\partial \overline{U}_A(a,b)}{\partial a} = 0$ is sufficient for a to be a best response to b iff we can show that A cannot improve by taking a location in $(b,1]$. The following claim helps to verify whether this last condition is fulfilled.

Claim 2: Assume that $b > \frac{1}{2}$. Then we have that for party A platform $a > b$ cannot be better than platform $2b - a$

Proof: Party A can weakly improve by taking platform $2b - a$ as, because of symmetry, party A's aggregate payoff from median voters on the interval $[2b-1, 1]$ will remain the same whereas that from median voters in $[0, 2b-1)$ cannot be less (as party A is closer to these voters and keeps the same distance to b.

To verify that (a^*, b^*) is an equilibrium we first show that a^* is a best response to b^*. First note that it suffices to show that a^* is optimal among the alternatives $[0, b^*]$ as according to claim 2 a platform $a \in (b^*, 1]$ is never (strictly) better than platform $2b - a$.

We have that $b^* > V_B$ (as $\frac{1}{2} \geq V_A \geq V_B$ implies $V_A \geq \frac{19}{13}V_B - \frac{5}{13}$), $b^* - a^* \leq 2V_B$ (as $V_A \leq 7V_B - 1$) , $b^* - V_B \geq a^*$ (as $\frac{1}{2} \geq V_A \geq V_B$), $\frac{a^*+b^*}{2} - V_B \geq 0$ (as $\frac{1}{2} \geq V_A \geq V_B$). If we have that $V_A < \frac{3}{13} + \frac{3}{13}V_B$ then we must have that $b^* < 1 - V_B$ and that $V_A < \frac{9}{7}V_B + \frac{1}{7}$ (as we have that $0 \leq V_A \leq 7V_B - 1$). The latter implies that $2(V_A - V_B) + a^* - b^* < 0$ and case (iv) for $V_A > V_B$ above shows that $\left.\frac{\partial \overline{U}_A(a,b^*)}{\partial a}\right|_{a=a^*} = 0$ such that a^* is a best response to b^*. If we have that $V_A \geq \frac{3}{13} + \frac{3}{13}V_B$ it must be that $b^* \geq 1 - V_B$ and as $V_A < \frac{13}{19}V_B + \frac{5}{19} < \frac{1}{5} + \frac{3}{5}V_A$ implies that $a^* < 1 - V_A$ we must have that case (iv) for $V_A > V_B$ applies and as above shows that a^* is a best response to b^*.

Showing that b^* is a best response to a^* is equivalent to showing that a party $\widetilde{A}$ with $V_{\widetilde{A}} := V_B$ and position $\widetilde{a} := 1 - b^*$ cannot improve if a party $\widetilde{B}$ with $V_{\widetilde{B}} := V_A$ takes platform $\widetilde{b} := 1 - a^*$. As above claim 2 shows that $\widetilde{A}$ cannot improve by locating in $(\widetilde{b}, 1)$

(as $\widetilde{b} = 1 - a^* > \frac{1}{2}$). Thus we must show that $\widetilde{a} = \frac{11}{8}V_{\widetilde{A}} - \frac{13}{8}V_{\widetilde{B}} + \frac{3}{8}$ is a best response to $\widetilde{b} = \frac{13}{8}V_{\widetilde{A}} - \frac{11}{8}V_{\widetilde{B}} + \frac{5}{8}$ for $V_{\widetilde{A}} < V_{\widetilde{B}}$. Note that we have that $\widetilde{b} > V_{\widetilde{B}}$ (as $V_A < \frac{13}{19}V_B + \frac{5}{19}$), $\widetilde{b} - \widetilde{a} \leq 2V_{\widetilde{A}} \leq V_{\widetilde{A}} + V_{\widetilde{B}}$ (as $V_A < 7V_B - 1$), $2\left(V_{\widetilde{A}} - V_{\widetilde{B}}\right) + \widetilde{b} - \widetilde{a} > 0$ (as $V_A < \frac{1}{5} + \frac{3}{5}V_B < \frac{9}{7}V_B + \frac{1}{7}$ for $V_B > \frac{1}{6}$), $V_{\widetilde{A}} + \widetilde{a} < 1$ (as $\frac{1}{2} \geq V_A \geq V_B$), $\widetilde{b} - \widetilde{a} > V_{\widetilde{B}}$ (as $V_A \leq \frac{3}{5}V_B + \frac{1}{5} < \frac{1}{3}V_B + \frac{1}{3}$) and $\widetilde{b} + \widetilde{a} \geq 2V_{\widetilde{B}}$ (as $V_A \leq \frac{3}{5}V_B + \frac{1}{5}$). Thus case (iv) for $V_{\widetilde{A}} < V_{\widetilde{B}}$ applies and we have that $\left.\frac{\partial \overline{U}_A\left(a,\widetilde{b}\right)}{\partial a}\right|_{a=\widetilde{a}} = 0$ and b^* is a best response to a^*. Q.E.D.

Concluding remark: To check the possibility of other equilibria where parties play pure strategies involves a large number of cases because A and B's best response functions are both defined by many subcases; checking whether they are mutually consistent involves sorting through a myriad of cases. As some cases are clearly not consistent with mutual best responses we were able to reduce the problem to roughly checking 60 cases. Each could be done similarly to the the case where $V_A \leq \min\left\{7V_B - 1, \frac{3}{5}V_B + \frac{1}{5}\right\}$.