Supplementary Material

Supplement to “On asymmetric reserve prices”

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A. Proofs for Section 2

A.1 *Proof of Theorem 2a (the semi-separating equilibrium)*

The proof of Theorem 2a is divided into a series of lemmas and remarks. We begin by outlining the general argument.

Outline

Related literature Elements of our argument draw on reasoning proposed by Lebrun (1997, 1999, 2004, 2006). Though our notation is generally different, we also employ η to denote the maximal equilibrium bid. We further draw on Lebrun's conclusions regarding the monotonicity of the equilibrium bidding strategy as a function of η, and we follow the common practice of examining inverse bid functions. As previously noted, Maskin and Riley (2003) and Lizzeri and Persico (2000) propose similar arguments in their analyses. The presence of a jump discontinuity in the equilibrium strategy of some bidders differentiates our argument from prior studies. To keep the discussion manageable and to reduce the number of cases, we suppose $N_1 \geq 2$ and $N_2 \geq 2$. If $N_k = 1$, then the equilibrium will feature the bidder in group k bidding r_k.

Preliminaries To prove Theorem 2a, we first complete the definition of the proposed strategy. This involves identifying the appropriate values for $\hat{s}_1$ and $\hat{s}_2$. In defining these two values below, we also determine η^*, the maximal bid.

To identify the preceding values, we first derive several preliminary results. In particular, we define four functions: $b_1(s)$, $b_2(s)$, $b_\tau(s)$, and $b_\psi(s)$. For $k \in \{1, 2\}$, $b_k(s)$ has domain $[r_k, \bar{s}]$ and is defined as

$$b_k(s) := s - \int_{r_k}^{s} \left[\frac{F_k(z)}{F_k(s)}\right]^{N_k - 1} dz. \tag{A.1}$$

The function $b_k(s)$ is the equilibrium bidding strategy in a symmetric first-price auction and it constitutes the lower portion of the strategy defined in Theorem 2a.

The functions $b_\tau(s)$ and $b_\psi(s)$ are defined in Remark A.2 below. Intuitively, $b_\tau(s)$ corresponds to the utility-maximizing bid above r_2 of a type-s group-1 agent conjecturing

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that all group-2 bidders bid according to $b_2(s)$. Its definition relies on the function $\tau(s)$, which is defined below (Lemma A.1). Likewise, $b_\psi(s)$ corresponds to the bid above r_2 that satisfies an indifference condition for bidders in group 1 (Lemma A.8). This is the indifference condition that supports the discontinuous increase in bid among group-1 members. The definition of $b_\psi(s)$ relies on the function $\psi(s)$, which is defined below (Lemma A.4).[1]

Lemmas A.5 and A.6 prove some additional properties of $\psi(s)$ that are useful elsewhere in the analysis, particularly in understanding the relative positions and geometry of $b_1(s)$, $b_2(s)$, $b_\tau(s)$, and $b_\psi(s)$. A sketch of the resulting configuration is provided in Figure A.2.

Identification of $\hat{s}_1$, $\hat{s}_2$, and η^* Having defined $b_\psi(s)$ and $b_2(s)$, we adapt the argument of Lebrun (1999) to simultaneously identify $\hat{s}_1$, $\hat{s}_2$, and η^* (Lemma A.11 and Remarks A.4 and A.5). Intuitively, this argument proceeds as follows: For each candidate value of η^* we look at solutions to the system of differential equations defined in the theorem's statement. This system characterizes bidding near the top of the equilibrium bid distribution. By varying η^* we look for a solution to this system that "hits" $b_\psi(s)$ and $b_2(s)$ at a common level, b^*. Figure A.4 conveys this intuition and illustrates the final outcome. The values $\hat{s}_1$ and $\hat{s}_2$ are then defined as $\hat{s}_1 := b_\psi^{-1}(b^*)$ and $\hat{s}_2 := b_2^{-1}(b^*)$. We note that the argument in Lemmas A.10 and A.11 and Remarks A.4 and A.5 is more complex notationally as the system of differential equations characterizing equilibrium bidding is defined with reference to *inverse* bid functions.

This definition of $\hat{s}_1$ and $\hat{s}_2$ ensures that a group-2 bidder's equilibrium strategy is continuous at $\hat{s}_2$. Additionally, a group-1 bidder of type $\hat{s}_1$ is indifferent between the bids $b_1(\hat{s}_1)$ and $b_\psi(\hat{s}_1)$. Furthermore, since $b_\psi(\hat{s}_1) \le b_\tau(\hat{s}_1)$, a group-1 bidder has no incentive to bid $b \in [r_2, b_\psi(\hat{s}_1))$ given the strategy adopted by bidders in group 2.

Verification of equilibrium The proof's final step involves confirming that the defined strategy profile is an equilibrium (Lemma A.15). The proof of this lemma mirrors the case-by-case analysis from the proof of Theorem 1a. The proof relies on several preliminary results (Lemmas A.12–A.14).

Preliminaries As noted in the outline above, a key preliminary step is the definition of two functions, $\tau(s)$ and $\psi(s)$. To introduce these functions define

$$p(s) := \int_{r_1}^{s} F_2(r_2)^{N_2} F_1(z)^{N_1-1}\, dz,$$

$$q_s(t) := F_1(s)^{N_1-1} F_2(t)^{N_2}(s-t) + F_1(s)^{N_1-1} F_2(t) \int_{r_2}^{t} F_2(z)^{N_2-1}\, dz.$$

The function $p(s)$ coincides with the expected payoff of a type-s group-1 bidder when he and others in group 1 bid according to $b_1(\cdot)$. The function $q_s(t)$ is the expected payoff of a type-s group-1 agent when he places the same bid as a type-t group-2 agent, $b_2(t)$,

[1] In this supplement, the function ψ is distinct from the equilibrium allocation rule, defined in Section 3 in the main text. No confusion should result.

and all group-1 agents of type $s' < s$ ($s' > s$) bid less (more) than $b_2(t)$. To simplify notation, we write $q'_s(t) := \frac{\partial}{\partial t} q_s(t)$. At boundaries, $q'_s(t)$ is understood to be the right or left derivative as appropriate.

LEMMA A.1. *Let* $\tau(s) := \arg\max_{t\in[r_2,\bar{s}]} q_s(t)$. *Then for all* $s \in [r_2, \bar{s}]$,

(i) $\tau(s) \neq \varnothing$ *and is single valued*

(ii) $\tau(r_2) = r_2$, $\tau(\bar{s}) = \bar{s}$, *and for all* $s \in (r_2, \bar{s})$, $s < \tau(s) \leq \bar{s}$

(iii) $t < \tau(s) \implies q'_s(t) > 0$ *and* $\bar{s} \geq t > \tau(s) \implies q'_s(t) < 0$

(iv) $\tau(s)$ *is nondecreasing.*

PROOF. The function $q_s(t)$ is continuous; hence, for all s, $\tau(s) \neq \varnothing$. To confirm the other points of the lemma, we begin by computing the derivative of $q_s(t)$ with respect to t:

$$q'_s(t) = F_1(s)^{N_1-1} f_2(t) \underbrace{\left(N_2(s-t)F_2(t)^{N_2-1} + \int_{r_2}^{t} F_2(z)^{N_2-1}\, dz \right)}_{\hat{q}_s(t)}.$$

From the preceding expression, $\operatorname{sgn}(q'_s) = \operatorname{sgn}(\hat{q}_s)$. Fix $s \in [r_2, \bar{s}]$ and choose $\tau \in \arg\max_{t\in[r_2,\bar{s}]} q_s(t)$. There are three cases.

Case 1. Suppose $\tau = r_2$. Then $q'_s(r_2) \leq 0 \implies \hat{q}_s(r_2) \leq 0 \implies N_2(s - r_2)F_2(r_2)^{N_2-1} \leq 0 \implies s \leq r_2 \implies s = r_2$. Hence, r_2 can maximize $q_s(t)$ only when $s = r_2$. Then for all $t > r_2$,

$$\begin{aligned}\hat{q}_{r_2}(t) &= N_2(r_2 - t)F_2(t)^{N_2-1} + \int_{r_2}^{t} F_2(z)^{N_2-1}\, dz \\ &< (r_2 - t)F_2(t)^{N_2-1} + \int_{r_2}^{t} F_2(z)^{N_2-1}\, dz < 0.\end{aligned}$$

Since for all $t > r_2$, $q'_{r_2}(t) < 0$, we conclude that indeed $\tau(r_2) = r_2$.

Case 2. Suppose $\tau \in (r_2, \bar{s})$. At an interior maximum, τ must satisfy the first-order condition $q'_s(\tau) = 0$. If $\hat{q}_s(t) = 0$ and $t > r_2$, then

$$0 = N_2(s - r_2)F_2(t)^{N_2-1} + \underbrace{\int_{r_2}^{t} F_2(z)^{N_2-1} - N_2 F_2(t)^{N_2-1}\, dz}_{<0}.$$

Thus, $s > r_2$. Moreover, by inspection, if $t \leq s$, then $\hat{q}_s(t) > 0$. Thus, $\tau > s > r_2$.

First, we argue that if for some $t^* \in (s, \bar{s})$, $q'_s(t^*) = \hat{q}_s(t^*) = 0$, then t^* must be a local maximum of $q_s(t)$. Taking the derivative of $\hat{q}_s(t)$ with respect to t gives

$$\hat{q}'_s(t) = -(N_2 - 1)F_2(t)^{N_2-2}\big(F_2(t) + N_2(t - s)f_2(t)\big).$$

Thus, $t > s \implies \hat{q}_s'(t) < 0$. Let $\epsilon > 0$ be small. Then $\hat{q}_s(t^* - \epsilon) > \hat{q}_s(t^*) = 0 > \hat{q}_s(t^* + \epsilon)$. Hence, $q_s'(t^* - \epsilon) > 0 = q_s'(t^*) > q_s'(t^* + \epsilon)$. Therefore, if $q_s'(t^*) = 0$, then t^* must be a local maximum of $q_s(t)$.

Since $q_s'(\tau) = 0$, τ must be the only local maximum of $q_s(t)$ for all $t \in [s, \bar{s}]$. If there were other local maxima on the interval, then there would also exist a local minimum at some $t^* \in (s, \bar{s})$, satisfying $q_s'(t^*) = 0$, which would be a contradiction since such points must be local maxima. Therefore, $q_s'(t) > 0$ for all $t < \tau$, $q_s'(t) < 0$ for all $t > \tau$, and $\tau(s)$ is single valued.

Case 3. Suppose $\tau = \bar{s}$. Then $q_s'(\bar{s}) \geq 0$. Recalling the arguments in the preceding case, if there exists $t^* \in (s, \bar{s})$ that is also a maximizer of $q_s(t)$, then there would also exist a local minimum on this interval—but that would be a contradiction. Thus, for all $t < \bar{s}$, $q_s'(t) > 0$.

In each of the preceding cases, there is a unique maximizer of $q_s(t)$. It was also shown that $\tau(r_2) = r_2$ and when $s \in (r_2, \bar{s})$, $\tau(s) > s$ since $q_s'(s) > 0$. The desiderata concerning the derivative of $q_s(t)$ were also shown to apply in each case. To show that $\tau(s)$ is nondecreasing, note that when $\tau(s) \in (r_2, \bar{s})$, it must satisfy $\hat{q}_s(\tau(s)) = 0$. However, if $s' > s$, then $0 = \hat{q}_s(\tau(s)) < \hat{q}_{s'}(\tau(s))$. Thus, $q_{s'}'(\tau(s)) > 0$ and, therefore, $\tau(s') > \tau(s)$. □

Lemma A.2 is a preliminary result leading to Lemma A.3, which defines $\check{s}$ and $\tilde{s}$. These values are used in the definition of $\psi(s)$ in Lemma A.4 below.

LEMMA A.2. *For all $s \in [r_2, \bar{s}]$, (i) $s \mapsto q_s(s) - p(s)$ is continuous and strictly increasing, and (ii) $s \mapsto q_s(\tau(s)) - p(s)$ is continuous and strictly increasing.*

PROOF. (i) Let $q(s) := q_s(s)$. To show that $q_s(s) - p(s)$ is strictly increasing, it is sufficient to show that $q'(s) > p'(s)$ for all $s > r_2$:

$$\begin{aligned} q'(s) &= \left(\frac{d}{ds} F_1(s)^{N_1 - 1}\right)\left(F_2(s) \int_{r_2}^{s} F_2(z)^{N_2 - 1}\, dz\right) \\ &\quad + F_1(s)^{N_1 - 1} f_2(s) \int_{r_2}^{s} F_2(z)\, dz + F_1(s)^{N_1 - 1} F_2(s)^{N_2} \\ &> F_1(s)^{N_1 - 1} F_2(s)^{N_2} \\ &> F_1(s)^{N_1 - 1} F_2(r_2)^{N_2} = p'(s). \end{aligned}$$

(ii) To show that $q_s(\tau(s)) - p(s)$ is strictly increasing, fix $s^* \in (r_2, \bar{s})$ and let $\tau^* = \tau(s^*)$. From the previous lemma, $\tau^* \geq s^*$. Consider the derivative

$$\begin{aligned} &\frac{d}{ds}\big(q_s(\tau^*) - p(s)\big) \\ &= F_1(s)^{N_1 - 2} F_1(s)\big(F_2(\tau^*)^{N_2} - F_2(r_2)^{N_2}\big) \\ &\quad + (N_1 - 1) f_1(s) F_2(\tau^*) F_1(s)^{N_1 - 2}\left(\big(s - \tau^*\big) F_2(\tau^*)^{N_2 - 1} + \int_{r_2}^{\tau^*} F_2(z)^{N_2 - 1}\, dz\right). \end{aligned}$$

We note that $\tau^* > r_2$ and when $s = s^*$, $(s^* - \tau^*)F_2(\tau^*)^{N_2-1} + \int_{r_2}^{\tau^*} F_2(z)^{N_2-1}\,dz \geq N_2(s^* - \tau^*)F_2(\tau^*)^{N_2-1} + \int_{r_2}^{\tau^*} F_2(z)^{N_2-1}\,dz = \hat{q}_{s^*}(\tau^*) \geq 0$. Hence $\frac{d}{ds}(q_s(\tau^*) - p(s))|_{s=s^*} > 0$ and for $s > s^*$, $\frac{d}{ds}(q_s(\tau^*) - p(s)) > 0$. Now take $s < s'$. Then $q_s(\tau(s)) - p(s) < q_{s'}(\tau(s)) - p(s') \leq q_{s'}(\tau(s')) - p(s')$. □

LEMMA A.3. *(i) There exists a unique $\tilde{s} \in (r_2, \bar{s})$ such that $p(\tilde{s}) = q_{\tilde{s}}(\tilde{s})$. (ii) There exists a unique $\check{s} \in (r_2, \tilde{s}]$ such that $q_{\check{s}}(\tau(\check{s})) - p(\check{s}) = 0$.*

PROOF. (i) By inspection, $p(r_2) > 0 = q_{r_2}(r_2)$ and

$$p(\bar{s}) = \int_{r_1}^{\bar{s}} F_2(r_2)^{N_2} F_1(z)^{N_1-1}\,dz < \int_{r_2}^{\bar{s}} F_2(z)^{N_2-1}\,dz = q_{\bar{s}}(\bar{s}).$$

Since $q_s(s) - p(s)$ is continuous and strictly increasing, by the intermediate value theorem there exists $\tilde{s} \in (r_2, \bar{s})$ such that $p(\tilde{s}) = q(\tilde{s})$.

(ii) Similarly, note that $q_{r_2}(\tau(r_2)) - p(r_2) < 0$ and $q_{\tilde{s}}(\tau(\tilde{s})) - p(\tilde{s}) \geq q_{\tilde{s}}(\tilde{s}) - p(\tilde{s}) = 0$. Since $q_s(\tau(s)) - p(s)$ is continuous and strictly increasing, the conclusion follows. □

LEMMA A.4. *Let $\Psi := \{(s,t) \in [r_2, \bar{s}]^2 : 0 \leq q_s(t) - p(s)\}$ and define the function $\psi : [\check{s}, \bar{s}] \to [r_2, \bar{s}]$ as*

$$\psi(s) := \inf_{t \geq r_2} \big\{(s,t) \in \Psi\big\},$$

where $\psi(s)$ is continuous, $\psi(\tilde{s}) = \tilde{s}$, and for all $t \in (\psi(s), \tau(s)]$, $(s,t) \in \Psi$. Moreover, $p(s) = q_s(\psi(s))$.

The proof is an immediate consequence of continuity and the preceding two lemmas.

The following two lemmas identify some useful properties of the $\psi(\cdot)$ function defined in Lemma A.4.

LEMMA A.5. *The function $\psi(\cdot)$ is decreasing.*

PROOF. Let $s \geq \check{s}$ be fixed and, to simplify notation, let $\bar{\psi} = \psi(s)$. Note that $\bar{\psi} \leq \tau(s)$. To show that $\psi(s)$ is decreasing, it is sufficient to show that $s \mapsto q_s(\bar{\psi}) - p(s)$ is nondecreasing. Since $\bar{\psi} \geq r_2$, it is sufficient to show that $(s - \bar{\psi})F_2(\bar{\psi})^{N_2-1} + \int_{r_2}^{\bar{\psi}} F_2(z)^{N_2-1}\,dz \geq 0$. This is obviously true if $s \geq \bar{\psi}$. Suppose instead that $s < \bar{\psi}$. Then $(s - \bar{\psi})F_2(\bar{\psi})^{N_2-1} + \int_{r_2}^{\bar{\psi}} F_2(z)^{N_2-1}\,dz \geq N_2(s - \bar{\psi})F_2(\bar{\psi})^{N_2-1} + \int_{r_2}^{\bar{\psi}} F_2(z)^{N_2-1}\,dz = \hat{q}_s(\bar{\psi}) \geq 0$. The final inequality follows from the fact that $\bar{\psi} \leq \tau(s)$. □

LEMMA A.6. *Let $s > \check{s}$. $\psi(s) = r_2 \iff s - \int_{r_1}^{s} [\frac{F_1(z)}{F_1(s)}]^{N_1-1}\,dz \geq r_2$.*

PROOF. ($\Rightarrow$) Suppose $\psi(s) = r_2$. Then $(s, r_2) \in \Psi$, so $q_s(r_2) \geq p(s)$. Thus,

$$\begin{aligned} q_s(r_2) \geq p(s) \quad &\iff \quad F_2(r_2)^{N_2} F_1(s)^{N_1-1}(s - r_2) \geq \int_{r_1}^{s} F_2(r_2)^{N_2} F_1(z)^{N_1-1}\,dz \\ &\iff \quad s - \int_{r_1}^{s} \left[\frac{F_1(z)}{F_1(s)}\right]^{N_1-1} dz \geq r_2. \end{aligned}$$

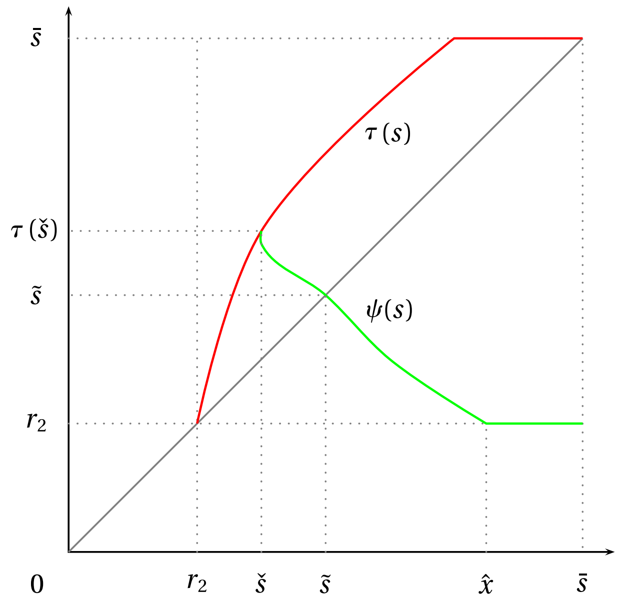

FIGURE A.1. An illustration of $\tau(s)$ and $\psi(s)$.

($\Leftarrow$) Suppose $s - \int_{r_1}^{s} [\frac{F_1(z)}{F_1(s)}]^{N_1-1} dz \geq r_2$. Then $q_s(r_2) \geq p(s)$. Thus, $(s, r_2) \in \Psi$. Hence, $\psi(s) \leq r_2$. By definition, $\psi(s) \geq r_2$. Therefore, $\psi(s) = r_2$. □

REMARK A.1. Throughout the remainder of our analysis we draw all diagrams with the assumption that there exists an $\hat{x}$ such that for all $s \geq \hat{x}$, $\psi(s) = r_2$, and for all $s < \hat{x}$, $\psi(s) > r_2$. This assumption is made for illustration and does not affect the conclusions of our analysis. Figure A.1 sketches the relationship between $\tau(s)$ and $\psi(s)$; $\psi(s)$ is illustrated as equal to r_2 for $s \geq \hat{x}$.

REMARK A.2. We defined $b_1(s)$ and $b_2(s)$ in expression (A.1). Given $b_2(s)$, let $b_\tau \colon [r_2, \bar{s}] \to [r_2, \bar{s}]$ be defined as $b_\tau(s) := b_2(\tau(s))$. Likewise, let $b_\psi \colon [\check{s}, \bar{s} \to [r_2, \bar{s}]$ be $b_\psi(s) := b_2(\psi(s))$. It is straightforward to verify that b_1, b_2, and b_τ are nondecreasing and continuous; b_ψ is nonincreasing and continuous. Since $\tau(s) > s$ for $s \in (r_2, \bar{s})$, $b_\tau(s) > b_2(s)$.

To simplify notation, let $\tilde{b} = b_2(\tilde{s}) = b_\psi(\tilde{s})$ and $\check{b} = b_\tau(\check{s}) = b_\psi(\check{s}) = b_2(\check{s}')$, where $\check{s}$ and $\tilde{s}$ are defined above. It is easy to confirm that $\check{s}' > \check{s}$.

LEMMA A.7. *Define $\tilde{s}$ as in Lemma A.3. Then $b_1(\tilde{s}) < r_2$.*

PROOF. Suppose $b_1(\tilde{s}) \geq r_1$. This implies $\int_{r_1}^{\tilde{s}} F_1(z)^{N_1-1} dz < F_1(\tilde{s})^{N_1-1}(\tilde{s} - r_2)$. But then $p(\tilde{s}) = q_{\tilde{s}}(\tilde{s}) > q_{\tilde{s}}(r_2)$, which implies $\int_{r_1}^{\tilde{s}} F_2(r_2)^{N_2} F_1(z)^{N_1-1} dz > F_1(\tilde{s})^{N_1-1} F_2(r_2)^{N_2}(\tilde{s} - r_2)$, which is a contradiction. □

Figure A.2 sketches the functions $b_1(s)$, $b_2(s)$, $b_\tau(s)$, and $b_\psi(s)$.

The following lemma verifies an indifference condition for bidders in group 1 when $s > \check{s}$.

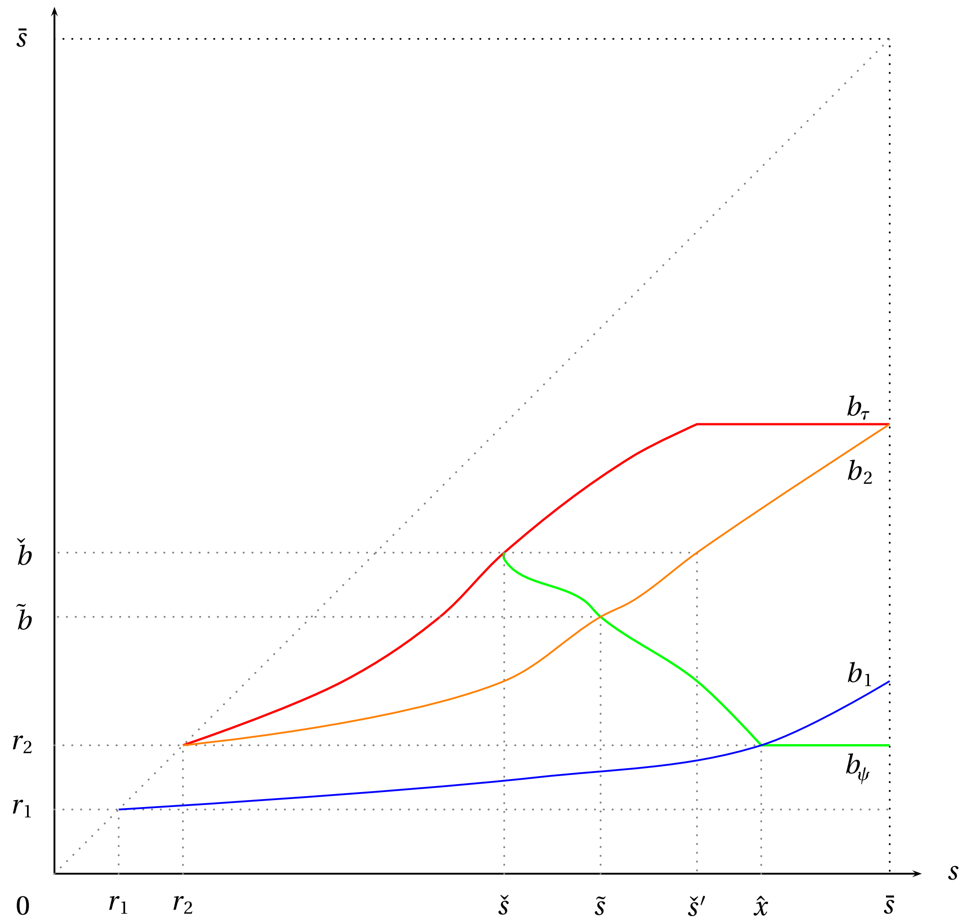

FIGURE A.2. The functions $b_1(s)$, $b_2(s)$, $b_\tau(s)$, and $b_\psi(s)$. The illustration assumes that Remark A.1 applies.

LEMMA A.8. *Let $s > \check{s}$ and suppose $b_1(s) < r_2$. Then*

$$F_1(s)^{N_1-1}F_2(r_2)^{N_2}\big(s - b_1(s)\big) = F_1(s)^{N_1-1}F_2\big(b_2^{-1}\big(b_\psi(s)\big)\big)^{N_2}\big(s - b_\psi(s)\big).$$

PROOF. The left-hand side is

$$F_1(s)^{N_1-1}F_2(r_2)^{N_2}\big(s - b_1(s)\big) = \int_{r_1}^{s} F_2(r_2)^{N_2}F_1(z)^{N_1-1}\,dz = p(s).$$

After some algebra, one can show that the right-hand side is

$$\begin{aligned} &F_1(s)^{N_1-1}F_2\big(b_2^{-1}\big(b_\psi(s)\big)\big)^{N_2}\big(s - b_\psi(s)\big) \\ &\quad = F_1(s)^{N_1-1}F_2\big(\psi(s)\big)^{N_2}\big(s - \psi(s)\big) + F_1(s)^{N_1-1}F_2\big(\psi(s)\big)\int_{r_2}^{\psi(s)} F_2(z)^{N_2-1}\,dz = q_s\big(\psi(s)\big). \end{aligned}$$

Thus, the result follows from the definition of $\psi(s)$ since $p(s) = q_s(\psi(s))$. $\square$

Identification of $\hat{s}_1$, $\hat{s}_2$, and η^* In this section we formally identify $\hat{s}_1$, $\hat{s}_2$, and η^* (Remarks A.4 and A.5). Prior to those remarks, we note some properties of the system of differential equations that characterize equilibrium bidding. The following result follows from the analysis of Lebrun (1999).

LEMMA A.9 (Lebrun (1999)). *Consider the system of differential equations*

$$\phi_k'(b) = \frac{1}{N_k + N_j - 1} \frac{F_k\big(\phi_k(b)\big)}{f_k\big(\phi_k(b)\big)} \times \left[\frac{1}{\phi_k(b) - b} + \frac{N_j\big(\phi_k(b) - \phi_j(b)\big)}{\big(\phi_k(b) - b\big)\big(\phi_j(b) - b\big)}\right], \tag{A.2}$$

$k \in \{1, 2\}$, $k \neq j$, defined in the domain $(b, \phi_1, \phi_2) \in [0, \bar{s}]^3$.

(i) For each $\eta \in (0, \bar{s})$, there exists a unique, strictly increasing solution to (A.2) satisfying the boundary condition $\phi_1(\eta) = \phi_2(\eta) = \bar{s}$. Denote this solution as $\phi_k^\eta(b)\colon [\underline{b}_\eta, \eta] \to [0, \bar{s}]$, where $[\underline{b}_\eta, \eta]$ is the solution's maximal domain.

(ii) There exists $\bar{\eta}$ such that for all $\eta < \bar{\eta}$, $\underline{b}_\eta = 0$ and for all $\eta > \bar{\eta}$, $\phi_k^\eta(\underline{b}_\eta) = \underline{b}_\eta$.

(iii) If $\eta' > \eta$, then for all b on which both $\phi_k^{\eta'}(b)$ and $\phi_k^\eta(b)$ are defined, $\phi_k^{\eta'}(b) > \phi_k^\eta(b)$.

REMARK A.3. Since the solutions of (A.2) depend continuously on η and they are monotone in η, for any (b, s_2) such that $s_2 \leq b$ there exists an η such that $\phi_2^\eta(b) = s_2$.

LEMMA A.10. *Let $(\phi_1^{\check{\eta}}, \phi_2^{\check{\eta}})$ be the solution of (A.2) satisfying the condition $\phi_2^{\check{\eta}}(\check{b}) = \check{s}'$. Then $\phi_1^{\check{\eta}}(\check{b}) \geq \check{s}$.*

PROOF. To prove this lemma, it suffices to show that $\tau(\phi_1^{\check{\eta}}(\check{b})) \geq \check{s}'$. First if $\phi_1^{\check{\eta}}(\check{b}) \geq \phi_2^{\check{\eta}}(\check{b}) = \check{s}'$, then the conclusion is satisfied since $\check{s}' \geq \check{s}$. Suppose instead that $\phi_1^{\check{\eta}}(\check{b}) < \phi_2^{\check{\eta}}(\check{b})$. Since $\phi_1^{\check{\eta}}$ is nondecreasing, we can refer to (A.2) to see that

$$\begin{aligned}
&\left[\frac{1}{\phi_1^{\check{\eta}}(\check{b}) - \check{b}} + \frac{N_2\big(\phi_1^{\check{\eta}}(\check{b}) - \phi_2^{\check{\eta}}(\check{b})\big)}{\big(\phi_1^{\check{\eta}}(\check{b}) - \check{b}\big)\big(\phi_2^{\check{\eta}}(\check{b}) - \check{b}\big)}\right] \geq 0 \\
&\implies \quad N_2\big(\phi_2^{\check{\eta}}(\check{b}) - \phi_1^{\check{\eta}}(\check{b})\big) \leq \phi_2^{\check{\eta}}(\check{b}) - \check{b} \\
&\implies \quad N_2\big(\check{s}' - \phi_1^{\check{\eta}}(\check{b})\big) \leq \check{s}' - \left(\check{s}' - \int_{r_2}^{\check{s}'} \left[\frac{F_2(z)}{F_2(\check{s}')}\right]^{N_2 - 1} dz\right) \\
&\implies \quad 0 \leq N_2 F_2(\check{s}')^{N_2 - 1}\big(\phi_1^{\check{\eta}}(\check{b}) - \check{s}'\big) + \int_{r_2}^{\check{s}'} F_2(z)^{N_2 - 1}\, dz \\
&\implies \quad 0 \leq \hat{q}_{\phi_1^{\check{\eta}}(\check{b})}(\check{s}') \\
&\implies \quad \check{s}' \leq \tau\big(\phi_1^{\check{\eta}}(\check{b})\big).
\end{aligned}$$

□

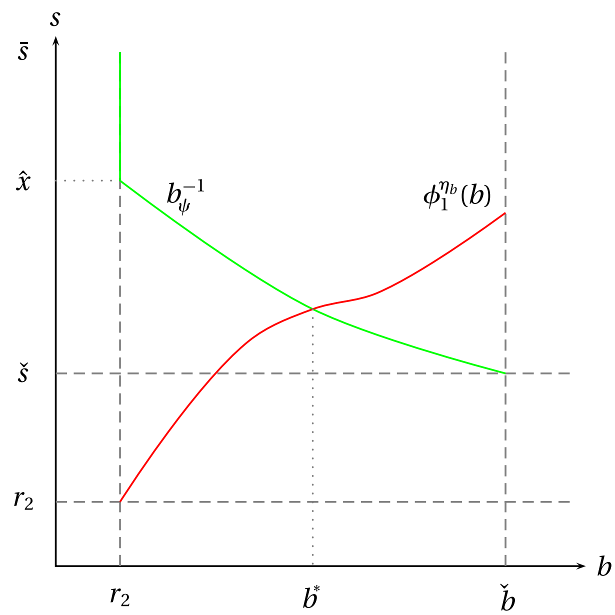

FIGURE A.3. An illustration of the argument in Lemma A.11. b_ψ^{-1} is illustrated assuming the conditions of Lemma A.6 are satisfied.

LEMMA A.11. *For each $b \in [r_2, \check{b}]$, let $(\phi_1^{\eta_b}, \phi_2^{\eta_b})$ be the solution of (A.2) such that $\phi_2^{\eta_b}(b) = b_2^{-1}(b)$. Then there exists $b^* \in (r_2, \check{b})$ such that $b_\psi^{-1}(b^*) = \phi_1^{\eta_{b^*}}(b^*)$.*

PROOF. Since η_b varies continuously in b and the solutions of the system $(\phi_1^{\eta_b}, \phi_2^{\eta_b})$ also vary continuously in η_b, it follows that $\phi_2^{\eta(\cdot)}(\cdot)\colon [r_2, \check{b}] \to [r_2, \bar{s}]$ is also continuous. From Lebrun (1999) it is known that $\phi_2^{\eta_{r_2}}(r_2) = r_2$ while from Lemma A.10 $\phi_2^{\check{\eta}}(\check{b}) \geq \check{s}$. Since $b_\psi^{-1}(b)$ is downward sloping, continuous, $b_\psi^{-1}(\check{b}) = \check{s}$, and, for every $s \geq \check{s}$, there exists b such that $b_\psi^{-1}(b) = s$, there exists b^* such that $b_\psi^{-1}(b^*) = \phi_1^{\eta_{b^*}}(b^*)$. See Figure A.3. □

REMARK A.4. Let $b^* \in [r_2, \check{b}]$ be any value satisfying the conditions in Lemma A.11. Define the constants $\hat{s}_1 = b_\psi^{-1}(b^*)$ and $\hat{s}_2 = b_2^{-1}(b^*)$. Note that $b_\psi(\hat{s}_1) = b_2(\psi(\hat{s}_1)) = b^* = b_2(\hat{s}_2)$ and thus $\psi(\hat{s}_1) = \hat{s}_2$.

REMARK A.5. Let b^* be defined as in Remark A.4 and let η^* be the associated value of η from Lemma A.11. Given the equations $(\phi_1^{\eta^*}, \phi_2^{\eta^*})$, which are the solution of (A.2) satisfying the conditions $\phi_1^{\eta^*}(b^*) = \hat{s}_1$ and $\phi_2^{\eta^*}(b^*) = \hat{s}_2$, and the terminal boundary condition $\phi_1^{\eta^*}(\eta^*) = \phi_2^{\eta^*}(\eta^*) = \bar{s}$, let $\hat{b}_k^{\eta^*}\colon [\hat{s}_k, \bar{s}] \to [b^*, \eta^*]$ be defined as the inverse of $\phi_k^{\eta^*}$. Figure A.4 places the functions $(\hat{b}_1^{\eta^*}, \hat{b}_2^{\eta^*})$ in the context of our preceding definitions.

REMARK A.6. Since $\hat{b}_1^{\eta^*}(\hat{s}_1) = b_\psi(\hat{s}_1)$ and $\hat{s}_1 > \check{s}$, by Lemma A.8 we see that

$$\begin{aligned} U_1\big(b_1(\hat{s}_1)|\hat{s}_1\big) &= F_1(\hat{s}_1)^{N_1-1} F_2(r_2)^{N_2}\big(\hat{s}_1 - b_1(\hat{s}_1)\big) \\ &= F_1(\hat{s}_1)^{N_1-1} F_2\big(b_2^{-1}\big(b_\psi(\hat{s}_1)\big)\big)^{N_2}\big(\hat{s}_1 - b_\psi(\hat{s}_1)\big) \end{aligned}$$

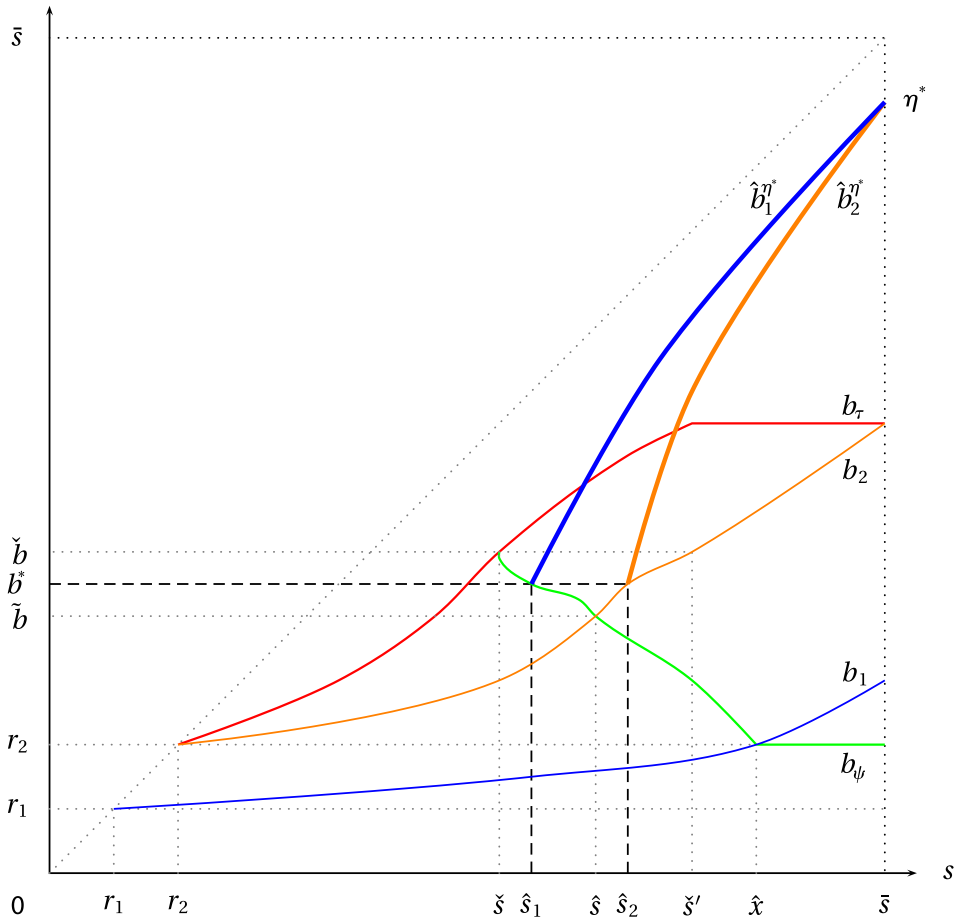

Figure A.4. The functions $\{b_1, b_2, b_\tau, b_\psi\}$ and $(\hat{b}_1^{\eta^*}, \hat{b}_2^{\eta^*})$. For illustration, $b^* \in (\tilde{b}, \check{b})$.

$$
\begin{aligned}
&= F_1(\hat{s}_1)^{N_1-1} F_2(\hat{s}_2)^{N_2}\big(\hat{s}_1 - \hat{b}_1^{\eta^*}(\hat{s}_1)\big) \\
&= U_1\big(\hat{b}_1^{\eta^*}(\hat{s}_1)|\hat{s}_1\big).
\end{aligned}
$$

Verification of equilibrium In this subsection, we verify that the constructed strategy profile is an equilibrium of the auction (Lemma A.15). Lemmas A.12–A.14 record some preliminary results, which are used in the proof of Lemma A.15.

Lemma A.12. *Let $\hat{s}_1$ be as defined in Remark A.4. Let $q_s^*(t)\colon [r_2, \bar{s}] \to \mathbb{R}$ be defined as*

$$
\begin{aligned}
q_s^*(t) = {} & F_1(\hat{s}_1)^{N_1-1} F_2(t)^{N_2}(s-t) \\
& + F_1(\hat{s}_1)^{N_1-1} F_2(t) \int_{r_2}^{t} F_2(z)^{N_2-1}\, dz.
\end{aligned}
$$

Define $\tau(s)$ as in Lemma A.1. Then for all $s \in [r_2, \bar{s}]$,

(i) $\tau(s) = \arg\max_{t\in[r_2,\bar{s}]} q_s^*(t)$ and $q_s^*(\tau(s))$ is continuous

(ii) for all $t < \tau(s)$, $\frac{d}{dt}q_s^*(t) > 0$; for all $t > \tau(s)$, $\frac{d}{dt}q_s^*(t) < 0$

(iii) $s \mapsto q_s^*(\tau(s)) - p(s)$ is continuous and strictly increasing for all $s \le \hat{s}_1$

(iv) there exists a unique $\check{s}^*$ such that $q_{\check{s}^*}^*(\tau(\check{s}^*)) = p(\check{s}^*)$ and for all $s \in [\check{s}^*, \hat{s}_1]$, $q_s^*(\tau(s)) \ge p(s)$.

Proof. Much of this proof follows from the proof of Lemma A.1 since $q_s^*(t) = [\frac{F(\hat{s}_1)}{F(s)}]^{N_1-1} q_s(t)$. As with $q_s(t)$, $q_s^*(t)$ is continuous in (s,t) and is differentiable.

(i) Suppose $\tau^* \in \arg\max_{t\in[r_2,\bar{s}]} q_s^*(t) \iff \tau^* \in \arg\max_{t\in[r_2,\bar{s}]} [\frac{F(\hat{s}_1)}{F(s)}]^{N_1-1} q_s(t) \iff \tau^* \in \arg\max_{t\in[r_2,\bar{s}]} q_s(t)$. Since $\tau(s) = \arg\max_{t\in[r_2,\bar{s}]} q_s(t)$, the result follows. Continuity of $q_s^*(\tau(s))$ is trivial.

(ii) Suppose $t < \tau(s)$, then $\frac{d}{dt}q_s^*(t) = \frac{d}{dt}([\frac{F(\hat{s}_1)}{F(s)}]^{N_1-1} q_s(t)) = [\frac{F(\hat{s}_1)}{F(s)}]^{N_1-1} \frac{d}{dt} q_s(t) > 0$. The case of $t > \tau(s)$ is analogous.

(iii) When $s < \hat{s}_1$, $\frac{d}{ds}(q_s^*(t) - p(s)) = F_1(\hat{s}_1)^{N_1-1} F_2(t)^{N_2} - F_1(s)^{N_1-1} F_2(r_2)^{N_2} > 0$. If $s < s' \le \hat{s}_1$, then $q_s^*(\tau(s)) - p(s) < q_{s'}^*(\tau(s)) - p(s') \le q_{s'}^*(\tau(s')) - p(s')$.

(iv) We have $q_{r_2}^*(\tau(r_2)) - p(r_2) < 0$ and $q_{\hat{s}_1}^*(\tau(\hat{s}_1)) - p(\hat{s}_1) = q_{\hat{s}_1}(\tau(\hat{s}_1)) - p(\hat{s}_1) > q_{\hat{s}_1}(\hat{s}_1) - p(\hat{s}_1) = 0$. Thus, there exists $\check{s}^*$ such that $q_{\check{s}^*}^*(\tau(\check{s}^*)) - p(\check{s}^*) = 0$. Uniqueness of $\check{s}^*$ follows from $q_s^*(\tau(s)) - p(s)$ being strictly increasing; therefore, for all $s \in [\check{s}^*, \hat{s}_1]$, $q_s^*(\tau(s)) \ge p(s)$. □

Lemma A.13. *Define $\psi^*\colon [\check{s}^*, \hat{s}_1] \to [r_2, \bar{s}]$ as $\psi^*(s) = \inf_{t \ge r_2}\{(s,t) \in p(s) \le q_s^*(t)\}$. Then $\psi^*(s)$ is decreasing and for all $s \in [\check{s}^*, \hat{s}_1]$, $\psi^*(s) \ge \hat{s}_2$.*

Proof. From Lemma A.12, $s \mapsto q_s^*(t) - p(s)$ is increasing when $s \in [\check{s}^*, \hat{s}_1]$. Thus, $\psi^*(s)$ is decreasing. Finally, $q_{\hat{s}_1}(\psi^*(\hat{s}_1)) = p(\hat{s}_1) \implies \psi^*(\hat{s}_1) = \psi(\hat{s}_1) \implies \psi^*(\hat{s}_1) = \hat{s}_2$. Thus, for all $s \in [\check{s}^*, \hat{s}_1]$, $\psi^*(s) \ge \psi^*(\hat{s}_1) \ge \hat{s}_2$. □

Lemma A.14. *For all $s \le \hat{s}_1$ and for all $t \in [r_2, \hat{s}_2]$, $q_s^*(t) \le p(s)$.*

The proof is a direct consequence of Lemmas A.12 and A.13.

Lemma A.15. *For each k, let*

$$\beta_k(s) = \begin{cases} \ell & \text{if } s < r_k, \\ b_k(s) & \text{if } s \in [r_k, \hat{s}_k], \\ \hat{b}_k^{\eta^*}(s) & \text{if } s \in (\hat{s}_k, \bar{s}]. \end{cases}$$

The strategy profile $\beta = (\beta_1, \beta_2)$ is a Bayesian–Nash equilibrium.

Proof. Figure A.5 presents a sketch of $\beta_1(s)$ and $\beta_2(s)$, building on the preceding discussion. For illustration, $\hat{s}_1 < \hat{s}_2$. The converse is also possible and the argument is

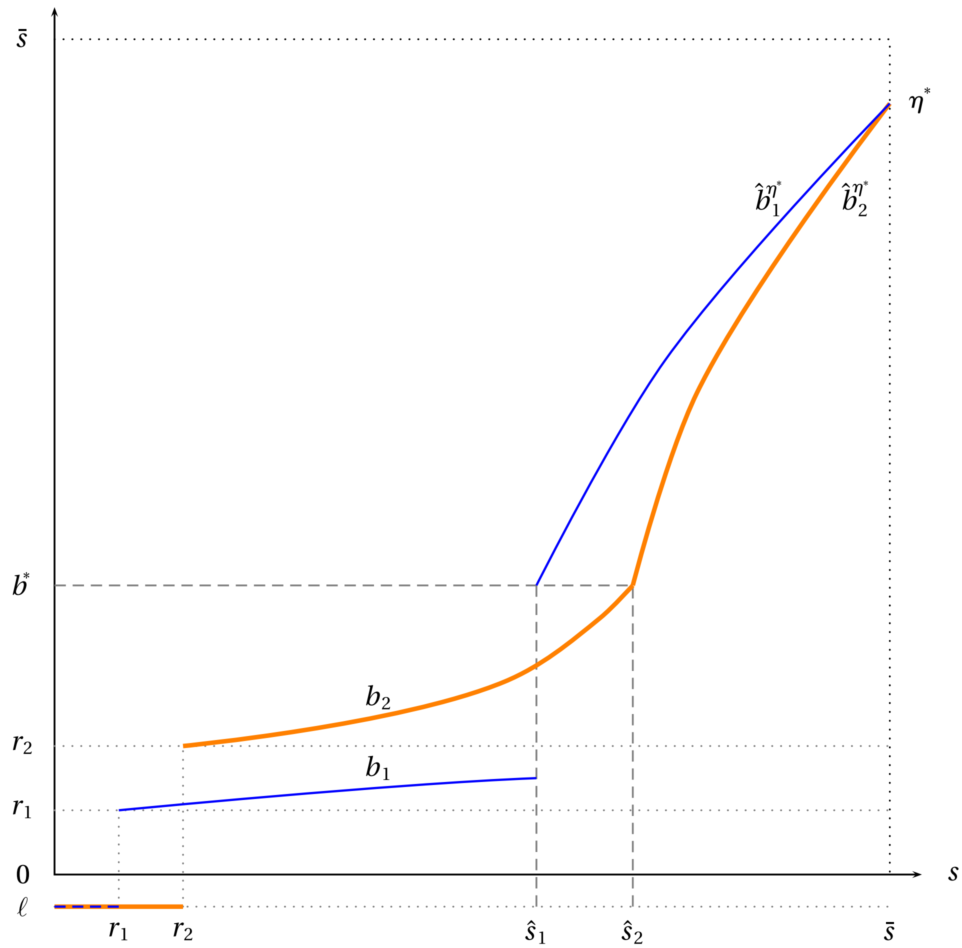

FIGURE A.5. The equilibrium strategy profile in Lemma A.15. The case where $\hat{s}_1 < \hat{s}_2$.

unchanged. As in the symmetric case, the argument proceeds in cases due to the discontinuity in the proposed equilibrium strategy. For notation, let $\gamma_k : [\hat{s}_k, \bar{s}] \to [\hat{s}_j, \bar{s}]$ be defined as $\gamma_k(z) = (\phi_j^{\eta^*} \circ \hat{b}_k^{\eta^*})(z)$, $j \neq k$; $\gamma_k(z)$ is nondecreasing.

Suppose all bidders follow the prescribed strategy. By a standard argument, appealing to the envelope theorem, we can write the expected utility of a bidder in group 1 when he bids $\beta_1(s)$ as

$$U_1\big(\beta_1(s)|s\big) = \begin{cases} 0 & \text{if } s < r_1, \\ \displaystyle\int_{r_1}^{s} F_1(z)^{N_1-1} F_2(r_2)^{N_2-1}\, dz & \text{if } s \in [r_1, \hat{s}_1], \\ \displaystyle\int_{r_1}^{\hat{s}_1} F_1(z)^{N_1-1} F_2(r_2)^{N_2-1}\, dz & \\ \qquad + \displaystyle\int_{\hat{s}_1}^{s} F_1(z)^{N_1-1} F_2\big(\gamma_1(z)\big)^{N_2}\, dz & \text{if } \hat{s}_1 < s. \end{cases}$$

The expected utility of a bidder in group 2 when bidding $\beta_2(s)$ is

$$U_2\big(\beta_2(s)|s\big)=\begin{cases}0 & \text{if } s<r_2,\\ \displaystyle\int_{r_1}^{s}F_2(z)^{N_2-1}F_1(\hat{s}_1)^{N_1}\,dz & \text{if } s\in[r_2,\hat{s}_2],\\ \displaystyle\int_{r_1}^{\hat{s}_2}F_2(z)^{N_2-1}F_1(\hat{s}_1)^{N_1}\,dz & \\ \qquad+\displaystyle\int_{\hat{s}_2}^{s}F_2(z)^{N_2-1}F_1\big(\gamma_2(z)\big)^{N_1}\,dz & \text{if } \hat{s}_2<s,\end{cases}$$

It is clear that all group-k bidders of type $s<r_k$ are best off bidding ℓ and no bidder of type $s\geq r_k$ can benefit from bidding ℓ. It is sufficient to rule out profitable deviations for this latter set of bidders to alternative bids in the range of β. We consider four cases.

Case 1: A group-1 bidder of type $s\in[r_1,\hat{s}_1]$.

(a) A bidder in this class has no profitable deviation to a bid $b_1(t)\in[r_1,\hat{s}_1]$. The argument is identical to the symmetric case.

(b) Suppose this bidder bids $b_2(t)$, $t\in[r_2,\hat{s}_2]$. The expected payoff from placing this bid is

$$\begin{aligned}U_1\big(b_2(t)|s\big)&=F_1(\hat{s}_1)^{N_1-1}F_2(t)^{N_2}\big(s-b_2(t)\big)\\ &=F_1(\hat{s}_1)^{N_1-1}F_2(t)^{N_2}(s-t)+F_1(\hat{s}_1)^{N_1-1}F_2(t)\int_{r_2}^{t}F_2(z)^{N_2-1}\,dz\\ &=q_s^*(t)\\ &\leq p(s)\\ &=U_1\big(b_1(s)|s\big).\end{aligned}$$

The inequality follows from Lemma A.14. Thus, $U_1(b_2(t)|s)\leq U_1(b_1(s)|s)$.

(c) Suppose this bidder bids $\hat{b}_1^{\eta^*}(t)$, $t\in(\hat{s}_2,\bar{s}]$. Then

$$\begin{aligned}&U_1\big(\hat{b}_1^{\eta^*}(t)|s\big)\\ &\quad=F_1(t)^{N_1-1}F_2\big(\gamma_1(t)\big)^{N_2}\big(s-\hat{b}_1^{\eta^*}(t)\big)\\ &\quad=F_1(t)^{N_1-1}F_2\big(\gamma_1(t)\big)^{N_2}(s-t)+U_1\big(b_1(\hat{s}_1)|\hat{s}_1\big)\\ &\qquad+\int_{\hat{s}_1}^{t}F_1(z)^{N_1-1}F_2\big(\gamma_1(z)\big)^{N_2}\,dz\\ &\quad=U_1\big(b_1(\hat{s}_1)|\hat{s}_1\big)+F_1(t)^{N_1-1}F_2\big(\gamma_1(t)\big)^{N_2}(s-\hat{s}_1)\\ &\qquad+F_1(t)^{N_1-1}F_2\big(\gamma_1(t)\big)^{N_2}(\hat{s}_1-t)+\int_{\hat{s}_1}^{t}F_1(z)^{N_1-1}F_2\big(\gamma_1(z)\big)^{N_2}\,dz\end{aligned}$$

$$
\begin{aligned}
&\leq U_1\big(b_1(\hat{s}_1)|\hat{s}_1\big) + F_1(t)^{N_1-1}F_2\big(\gamma_1(t)\big)^{N_2}(s-\hat{s}_1)\\
&= U_1\big(b_1(s)|s\big) + \int_s^{\hat{s}_1} F_1(z)^{N_1-1}F_2(r_2)^{N_2}\,dz + F_1(t)^{N_1-1}F_2\big(\gamma_1(t)\big)^{N_2}(s-\hat{s}_1)\\
&\leq U_1\big(b_1(s)|s\big).
\end{aligned}
$$

Thus, $U_1(\hat{b}_1^{\eta^*}(t)|s) \leq U_1(b_1(s)|s)$ and this deviation is not profitable.

Therefore, a bidder in group 1 of type $s \in [r_1, \hat{s}_1]$ does not have a profitable deviation from $\beta_1(s)$.

Case 2: A group-1 bidder of type $s \in (\hat{s}_1, \bar{s}]$.

(a) Suppose this bidder bids $b_1(t)$, $t \in [r_1, \hat{s}_1]$. Then

$$
\begin{aligned}
&U_1\big(b_1(t)|s\big) - U_1\big(\hat{b}_1^{\eta^*}(s)|s\big)\\
&\quad= F_1(t)^{N_1-1}F_2(r_2)^{N_2}\big(s-b_1(t)\big) - U_1\big(b_1(\hat{s}_1)|\hat{s}_1\big)\\
&\qquad - \int_{\hat{s}_1}^{s} F_1(z)^{N_1-1}F_2\big(\gamma_1(z)\big)^{N_2}\,dz\\
&\quad= F_1(t)^{N_1-1}F_2(r_2)^{N_2}(s-\hat{s}_1) + F_1(t)^{N_1-1}F_2(r_2)^{N_2}(\hat{s}_1-t)\\
&\qquad + \int_{r_1}^{t} F_1(z)^{N_1-1}F_2(r_2)^{N_2}\,dz - \int_{r_1}^{t} F_1(z)^{N_1-1}F_2(r_2)^{N_2}\,dz\\
&\qquad - \int_{t}^{\hat{s}_1} F_1(z)^{N_1-1}F_2(r_2)^{N_2}\,dz - \int_{\hat{s}_1}^{s} F_1(z)^{N_1-1}F_2\big(\gamma_1(z)\big)^{N_2}\,dz\\
&\quad\leq F_1(t)^{N_1-1}F_2(r_2)^{N_2}(s-\hat{s}_1) - \int_{\hat{s}_1}^{s} F_1(z)^{N_1-1}F_2\big(\gamma_1(z)\big)^{N_2}\,dz \leq 0.
\end{aligned}
$$

The inequality follows from $t \leq \hat{s}_1 \leq z$ and $r_2 \leq \gamma_1(z)$. Thus, $U_1(b_1(t)|s) \leq U_1(\hat{b}_1^{\eta^*}(s)|s)$ and this deviation is not profitable.

(b) Suppose this bidder bids $b_2(t)$, $t \in [r_2, \hat{s}_2]$. Then

$$
\begin{aligned}
&U_1\big(b_2(t)|s\big) - U_1\big(\hat{b}_1^{\eta^*}(s)|s\big)\\
&\quad= F_1(\hat{s}_1)^{N_1-1}F_2(t)^{N_2}\big(s-b_2(t)\big) - U_1\big(\hat{b}_1^{\eta^*}(s)|s\big)\\
&\quad= F_1(\hat{s}_1)^{N_1-1}F_2(t)^{N_2}(s-\hat{s}_1) - U_1\big(\hat{b}_1^{\eta^*}(s)|s\big)\\
&\qquad + \left[F_1(\hat{s}_1)^{N_1-1}F_2(t)^{N_2}(\hat{s}_1-t) + F_1(\hat{s}_1)^{N_1-1}F_2(t)\int_{r_2}^{t} F_2(z)^{N_2-1}\,dz\right]\\
&\quad\leq F_1(\hat{s}_1)^{N_1-1}F_2(t)^{N_2}(s-\hat{s}_1) - U_1\big(\hat{b}_1^{\eta^*}(s)|s\big)\\
&\qquad + \left[\int_{r_1}^{\hat{s}_1} F_1(z)^{N_1-1}F_2(r_2)^{N_2}\,dz\right]
\end{aligned}
$$

$$= F_1(\hat{s}_1)^{N_1-1}F_2(t)^{N_2}(s-\hat{s}_1) - \int_{\hat{s}_1}^{s} F_1(z)^{N_1-1}F_2\big(\gamma_1(z)\big)^{N_2}\,dz$$
$$= \int_{\hat{s}_1}^{s}\big(F_1(\hat{s}_1)^{N_1-1}F_2(t)^{N_2} - F_1(z)^{N_1-1}F_2\big(\gamma_1(z)\big)^{N_2}\big)\,dz \le 0.$$

The first inequality is from Lemma A.14. The second inequality is because $z \ge \hat{s}_1$ and $\gamma_1(z) \ge \hat{s}_2 \ge t$. Thus, $U_1(b_2(t)|s) \le U_1(\hat{b}_1^{\eta^*}(s)|s)$. Hence, this deviation is not profitable.

(c) This bidder has no profitable deviation to a bid $b_1(t) \in (\hat{s}_1, \bar{s}]$. The argument is identical to the case of a standard asymmetric first-price auction.

Therefore, a bidder in group 1 of type $s \in (\hat{s}_1, \bar{s}]$ does not have a profitable deviation from $\beta_1(s) = \hat{b}_1^{\eta^*}(s)$.

Case 3: A group-2 bidder of type $s \in [r_2, \hat{s}_2]$.

(a) This bidder has no profitable deviation to a bid $b_2(t) \in [r_2, \hat{s}_2]$. The argument is identical to the case of a symmetric auction.

(b) Suppose this bidder bids $b_2(t)$, where $t \in [\hat{s}_2, \bar{s}]$. By an argument fully parallel to Case 1(c) above, we can show that $U_2(\hat{b}_2^{\eta^*}(t)|s) \le U_2(b_2(s)|s)$.

Therefore, a bidder in group 2 of type $s \in [r_2, \hat{s}_2]$ does not have a profitable deviation from $\beta_2(s) = b_2(s)$.

Case 4: A group-2 bidder of type $s \in (\hat{s}_2, \bar{s}]$.

(a) Suppose this bidder bids $b_2(t)$, where $t \in [r_2, \hat{s}]$. Then

$$U_2\big(b_2(t)|s\big) - U_2\big(\hat{b}_2^{\eta^*}(s)|s\big)$$
$$= F_1(\hat{s}_1)^{N_1}F_2(t)^{N_2-1}\big(s - b_2(t)\big) - U_2\big(\hat{b}_2^{\eta^*}(s)|s\big)$$
$$= F_1(\hat{s}_1)^{N_1}F_2(t)^{N_2-1}(s-t) + \int_{r_2}^{t} F_1(\hat{s}_1)^{N_1}F_2(z)^{N_2-1}\,dz - U_2\big(\hat{b}_2^{\eta^*}(s)|s\big)$$
$$= F_1(\hat{s}_1)^{N_1}F_2(t)^{N_2-1}(s-\hat{s}_2) - \int_{\hat{s}_2}^{s} F_1\big(\gamma_2(z)\big)^{N_1}F_2(z)^{N_2-1}\,dz$$
$$+ F_1(\hat{s}_1)^{N_1}F_2(t)^{N_2-1}(\hat{s}_2 - t) + \int_{\hat{s}_2}^{t} F_1(\hat{s}_1)^{N_1}F_2(z)^{N_2-1}\,dz$$
$$= \int_{\hat{s}_2}^{s}\big(F_1(\hat{s}_1)^{N_1}F_2(t)^{N_2-1} - F_1\big(\gamma_2(z)\big)^{N_1}F_2(z)^{N_2-1}\big)\,dz$$
$$+ \int_{t}^{\hat{s}_2}\big(F_1(\hat{s}_1)^{N_1}F_2(t)^{N_2-1} - F_1(\hat{s}_1)^{N_1}F_2(z)^{N_2-1}\big)\,dz \le 0.$$

Thus, this is not a profitable deviation.

(b) Suppose this bidder bids $b_2^{\eta^*}(t)$, $t \in (\hat{s}_2, \bar{s}]$. It can be shown that this is not a profitable deviation. The argument is identical to the case of an asymmetric auction (Lebrun 1999).

Therefore a bidder in group 2 of type $s \in (\hat{s}_2, \bar{s}]$ does not have a profitable deviation from $\beta_2(s) = \hat{b}_2^{\eta^*}(s)$.

Since bids outside of the range of β_1 and β_2 are dominated, the preceding argument exhausts all relevant cases. Thus, β is an equilibrium strategy profile. $\square$

A.2 *Proof of Theorem 2b (the pooling equilibrium)*

It is simple to verify that a group-1 bidder will not wish to deviate to a bid in the range of β_1 and that a group-2 bidder will not wish to deviate to a bid in the range of β_2. (The argument is identical to a symmetric first-price, sealed-bid auction.) Therefore, we check that a bidder in group 1 will not wish to bid $\beta_2(t)$, where $t \in [r_2, \bar{s}]$.

Let $U_k(b|s)$ be the expected utility of a bidder in group k of type s if he bids b and all other bidders are following the theorem's prescribed strategy. Then

$$\begin{aligned} U_1\big(\beta_2(t)|s\big) &= F_1(\bar{s})^{N_1-1}F_2(t)^{N_2}\big(s-\beta_2(t)\big) \\ &= F_1(\bar{s})^{N_1-1}F_2(t)^{N_2}(s-t) + F_1(\bar{s})^{N_1-1}F_2(t)\int_{r_2}^{t} F_2(z)^{N_2-1}\,dz \\ &\le \int_{r_2}^{s} F_2(r_2)^{N_2}F_1(z)^{N_1-1}\,dz \\ &= U_1\big(\beta_1(t)|s\big). \end{aligned}$$

The inequality follows from an adaptation of Lemmas A.12 and A.14 (replace $\hat{s}_1$ with $\bar{s}$ as needed) from the semi-separating case. (See also the proof of Lemma A.15, Case 1(b).) Therefore, β defines a group-symmetric equilibrium.

A.3 *Proof of Theorem 3 (equilibrium uniqueness)*

To verify the uniqueness of the auction's equilibrium, we adapt prior results on equilibrium uniqueness in asymmetric first-price auctions. The following analysis draws heavily on Lebrun (1997, 1999) and Maskin and Riley (2003). For brevity, we reference these authors' relevant results without repeating their arguments in detail. Throughout, we let $\beta_k^i(s)$ be an equilibrium strategy of bidder i in group k for some fixed equilibrium β.

Asymmetric first-price auctions can exhibit equilibria where an agent bids more than his valuation (Kaplan and Zamir 2015). This issue does not arise in our particular model.

Lemma A.16. *There does not exist an equilibrium where a bidder places a bid that strictly exceeds his valuation with strictly positive probability.*

Proof. Suppose to the contrary and assume that bidder i in group k of type $s > 0$ bids $\beta_k^i(s) > s$, $\beta_k^i(s) \geq r_k$, in equilibrium. As this bid leads to a negative payoff conditional on winning, it must win with probability 0. Given the full support of the valuation distribution, the event that all bidders other than i have a valuation less than s occurs with positive probability. Thus, there must exist some bidder $j \neq i$ who bids more than $\beta_k^i(s)$ with positive probability conditional on his valuation being less than s. Moreover, without loss of generality, bidder j must win the auction with this bid when others' valuations are less than s. But this implies bidder j receives a negative payoff conditional on winning the auction. Thus, he has a profitable deviation to a bid less than his valuation—a contradiction.

If a bidder of type $s = 0$ bids $\beta_k^i(0) > 0$, $\beta_k^i(0) \geq r_k$, then he wins the auction with strictly positive probability. (With positive probability all other bidders have valuations less than $\beta_k^i(0)$ and, by the previous part, bid less than $\beta_k^i(0)$ with probability 1.) Thus, $\beta_k^i(0)$ cannot be an equilibrium bid since a bid of 0 yields a greater payoff. □

From Lemma A.16, we can infer that $\beta_k^i(s) = \ell$ for all $s < r_k$ and $r_k \leq \beta_k^i(s) \leq s$ for all $s \geq r_k$. Standard arguments show that each $\beta_k^i(s)$ is nondecreasing and differentiable almost everywhere. To simplify notation, let $\beta_k^i(s^-) := \lim_{x\to s^-} \beta_k^i(x)$ and $\beta_k^i(s^+) := \lim_{x\to s^+} \beta_k^i(x)$ denote the left- and right-hand limits of $\beta_k^i(\cdot)$ at s.

The following lemma confirms that the distribution of equilibrium bids cannot have any mass points at values different from the reserve prices.[2] Its proof is standard.

Lemma A.17. *Fix an equilibrium and let $\beta_k^i(s)$ be the strategy of bidder i in group k. If $\beta_k^i(s') > r_k$, then $s'' > s' \implies \beta_k^i(s'') > \beta_k^i(s')$.*

For the proof, see the proof of Proposition 3 in Maskin and Riley (2000).

Suppose henceforth that β is the strategy profile from an equilibrium where all group-1 bidders place a bid above r_2 with strictly positive probability. We show that this equilibrium is characterized by the strategy reported in Theorem 2a.

When all bidders bid above r_2 with positive probability, Lebrun (1999) and Maskin and Riley (2003) show that there is a common maximal bid submitted by all bidders in the auction, say η^*. This is a consequence of the common support of the valuation distributions. Near this common maximal bid, the agents' bidding strategies are characterized by a system of differential equations. As shown by Lebrun (1997, Section 5) and Lebrun (1999), under the assumptions of Theorem 2a, this system has a unique solution (due to the symmetry among bidders in each group) and this system of differential equations simplifies to the expression stated in Theorem 2a. Sufficiently close to the maximal bid, all agents in group k follow the same strategy, i.e., $\beta_k^i(s) = \beta_k^j(s)$ for all i and j when s is close to $\bar{s}$.

The strategy of each bidder i in group 1, $\beta_1^i(s)$, must be strictly increasing, except possibly when $\beta_1^i(s) = r_1$. Thus, there exists a unique $\hat{s}_1^i$ for each bidder i in group 1 such that $s < \hat{s}_1^i \implies \beta_i^1(s) < r_2$ and $s > \hat{s}_1^i \implies \beta_i^1(s) > r_2$.

Lemma A.18. *(a) For each j, the function $\beta_2^j(s)$ is continuous at each $s > r_2$.*

[2] When there is one agent in group k, he may bid r_k for a range of valuations.

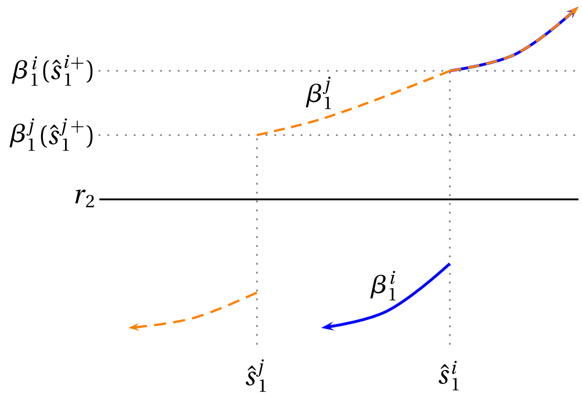

FIGURE A.6. A situation where $\hat{s}_1^i > \hat{s}_1^j$.

(b) *For each i, the function $\beta_1^i(s)$ is continuous at each $s > r_1$, except possibly at $\hat{s}_1^i$.*

(c) *For all i and j, $\hat{s}_1 = \hat{s}_1^i = \hat{s}_1^j$.*

PROOF. The continuity of each $\beta_1^i(s)$ and $\beta_2^j(s)$ at all points specified in the lemma follows from Lebrun (1999, pp. 136–137).[3] To prove part (c), suppose $\hat{s}_1^i > \hat{s}_1^j$ are the two greatest distinct elements in $\{\hat{s}_1^{j'}\}_{j'=1,\ldots,N_1}$. Let $\beta_1^i(s)$ and $\beta_1^j(s)$ be the corresponding bidding strategies. For all $s > \hat{s}_1^i$, $\beta_1^i(s) = \beta_1^j(s)$.[4] Since $\beta_1^j(s)$ is strictly increasing and continuous, $\beta_1^j(\hat{s}_1^{j+}) < \beta_1^j(\hat{s}_1^i) = \beta_1^i(\hat{s}_1^{i+})$ and bidder j in group 1 with a valuation $s \in (\hat{s}_1^j, \hat{s}_1^i)$ must place a bid $b \in (\beta_1^j(\hat{s}_1^{j+}), \beta_1^i(\hat{s}_1^{i+}))$. Furthermore, because $\beta_1^j(\hat{s}_1^{j+}) \geq r_2$, it follows that β_1^i has a jump discontinuity at $\hat{s}_1^i$, i.e., $\beta_1^i(\hat{s}_1^+) > r_2 \geq \beta_1^i(\hat{s}_1^-)$. An instance of such a situation is illustrated in Figure A.6. This situation satisfies the conditions shown to be incompatible with equilibrium bidding by Lebrun (1999, pp. 136–137). □

REMARK A.7. Since the strategies of all group-1 bidders coincide when s is sufficiently large, $\beta_1^i(s) = \beta_1^j(s)$ for all i and j and $s > \hat{s}_1$. Let $b^* = \beta_1^i(\hat{s}_1^+)$ for some i. For each i, let $\hat{s}_2^i$ be the largest value where $\beta_2^i(\hat{s}_2^{i+}) = b^*$. Since the strategies of all group-2 bidders coincide when s is sufficiently large, there exists an $\hat{s}_2$ such that $\hat{s}_2 = \hat{s}_2^i = \hat{s}_2^j$ and $\beta_2^i(s) = \beta_2^j(s)$ for all $s \geq \hat{s}_2$ for all i and j. By continuity, $b^* = \beta_2^i(\hat{s}_2)$.

LEMMA A.19. *(a) If $N_2 = 1$, then $b^* = r_2$.*

[3]Lebrun's (1999) argument does not necessarily imply continuity of β_1^i at $\hat{s}_1^i$. The augment involves bounding the change in the probability of winning when some bidders place a hypothetically lower bid. In our setting, group 2 bidders cannot bid less than r_2. This constraint precludes applying Lebrun's reasoning under all circumstances. Specifically, discontinuities where β_1^i "jumps over" r_2 cannot be ruled out when the strategy of *all* group-1 bidders has a jump discontinuity at a common value, i.e., $\hat{s}_1^i = \hat{s}_1^j$ for all i and j. Otherwise, even when $\hat{s}_1^i \neq \hat{s}_1^j$, Lebrun's argument applies.

[4]This follows from the common maximal bid for each agent. The associated system of differential equations that characterizes equilibrium bidding in the neighborhood of this maximal bid has a unique solution with the property that all group-1 agents adopt the same strategy.

(b) *If $N_2 \geq 2$, then either (i) $b^* = r_2$ and $\hat{s}_2 = r_2$ or (ii) $b^* > r_2$ and $\hat{s}_2 > r_2$.*

Proof. (a) Let $N_2 = 1$ and suppose $b^* > r_2$. Thus, no group-1 bidder places a bid in the range (r_2, b^*). Therefore, the single group-2 bidder has a profitable deviation from each bid $b \in (r_2, b^*]$ to an infinitesimally lower one. Thus, his equilibrium strategy must satisfy $\beta_2^i(s) = r_2$ for all $s \leq \hat{s}_2$. Thus, $b^* = \beta_2^i(\hat{s}_2) = r_2$.

(b) Suppose $b^* = r_2$ but $\hat{s}_2 > r_2$. Thus, two group-2 bidders bid r_2 whenever their valuations are $s \in (r_2, \hat{s}_2)$. Thus, the equilibrium bid distribution has an atom at r_2 and at least one of the group-2 bidders can increase his payoff by increasing his bid slightly—a contradiction. Hence, $\hat{s}_2 = r_2$. Alternatively, suppose $b^* > r_2$. Then $b^* = \beta_2^i(\hat{s}_2) > r_2$. Since $\beta_2^i(s) \leq s$, it necessarily follows that $\hat{s}_2 > r_2$. □

Lemma A.20. *For every bidder i in group 2, $\beta_2^i(s) = s - \int_{r_2}^{s} [\frac{F_2(z)}{F_2(s)}]^{N_2 - 1}\, dz$ for all $s \in [r_2, \hat{s}_2]$.*

Proof. When $N_2 = 1$, Lemma A.19 implies that $b^* = r_2$. Hence, $\beta_2^i(s) = r_2$ for all $s \in [r_2, \hat{s}_2]$, as required. Suppose $N_2 \geq 2$. From Lemma A.19, there are two cases. If $b^* = r_2$, then $\hat{s}_2 = r_2$ and the strategy defined above reduces to $\beta_2^i(r_2) = r_2$. If $b^* > r_2$ instead, then $\hat{s}_2 > r_2$. Let $\phi_2^i(\cdot)$ denote the inverse of the equilibrium bidding strategy of bidder i in group 2. When this bidder places a bid $b \in (r_2, b^*)$, his expected payoff is $F_1(\hat{s}_1)^{N_1} \prod_{j \neq i} F_2(\phi_2^j(b))(s - b)$. This bidder faces direct competition from other group-2 bidders. He is certain to defeat all group-1 bidders with a valuation less than $\hat{s}_1$ due to the discontinuity in their bidding strategies (recall $b^* = \beta_1^i(\hat{s}_1^+) > r_2 \geq \beta_1^i(\hat{s}_1^-)$). Group-1 bidders with a valuation greater than $\hat{s}_1$ and group-2 bidders with a valuation greater than $\hat{s}_2$ bid above b^*; hence, they do not affect the local incentives faced by i in group 2 when placing a bid below b^*. As shown by Lebrun (1999), the inverse bid functions must solve the system of differential equations

$$\frac{d}{db} \sum_{j \neq i} \log\big(F_2\big(\phi_2^j(b)\big)\big) + \log\big(\phi_2^i(b) - b\big) = 0, \quad i = 1, \dots, N_2,$$

subject to the boundary condition $\phi_2^i(b^*) = \hat{s}_2$ for all $i = 1, \dots, N_2$. This system satisfies the standard assumptions of the fundamental theorem of differential equations. Thus, it admits a unique solution. Due to the situation's symmetry, this solution satisfies $\phi_2^i(b) = \phi_2^j(b)$ for all i and j. That is, all group-2 bidders follow the same strategy.

Since $\phi_2^i(b)$ defines an equilibrium strategy that is symmetric for all bidders in group 2, it must also satisfy the boundary condition $\phi_2^i(r_2^+) = r_2$. Computing the inverse of ϕ_2^i, as in a standard first-price auction with risk-neutral symmetric bidders, we conclude that the equilibrium strategy for every bidder i in group 2 must be $\beta_2^i(s) = s - \int_{r_2}^{s} [\frac{F_2(z)}{F_2(s)}]^{N_2 - 1}\, dz$ when $s \in [r_2, \hat{s}_2]$. □

Lemma A.21. *For every bidder i in group 1, $\beta_1^i(s) = s - \int_{r_1}^{s} [\frac{F_1(z)}{F_1(s)}]^{N_1 - 1}\, dz$ for all $s \in [r_1, \hat{s}_1]$.*

Proof. First, observe that $\beta_1^i(\hat{s}_1^-) = \beta_1^j(\hat{s}_1^-)$ for each bidder i and j in group 1. This conclusion follows as a corollary to Maskin and Riley (2003, Lemma 10) when applied to a first-price auction with valuations distributed on $[0, \hat{s}_1]$ with the c.d.f. $F_1(s)/F_1(\hat{s}_1)$.

Given that each bidder's equilibrium bidding strategy must coincide at $\hat{s}_1^-$, the same reasoning as in the proof of Lemma A.20 lets us conclude that the equilibrium bid for bidder i in group 1 must $\beta_1^i(s) = s - \int_{r_1}^s [\frac{F_1(z)}{F_1(s)}]^{N_1-1}\,dz$ for all $s \in [r_1, \hat{s}_1]$. (Of course, when there is only one group-1 bidder, this function reduces to $\beta_1^i(s) = r_1$.) □

The preceding lemmas together confirm that if there is an equilibrium where all group-1 bidders bid above r_2 with strictly positive probability, then the equilibrium strategy is characterized by the strategy outlined in Theorem 2a.

Lemma A.22. *There exists at most one equilibrium where all group-*1 *bidders bid above* r_2 *with strictly positive probability.*

Proof. Let β and $\tilde{\beta}$ be two distinct equilibria where all group-1 bidders bid above r_2 with strictly positive probability. As established above, β and $\tilde{\beta}$ are both characterized by a strategy conforming to the description in Theorem 2a. We use the following notation:

(i) Let $\hat{s}_1$ be the point such that $\beta_1(\hat{s}_1^-) \le r_2 \le \beta_1(\hat{s}_1^+)$. For all $s \in [r_1, \hat{s}_1)$, $\beta_1(s) = s - \int_{r_1}^s [\frac{F_1(z)}{F_1(s)}]^{N_1-1}\,dz$. Similarly, define $\tilde{s}_1$ as the point such that $\tilde{\beta}_1(\tilde{s}_1^-) \le r_2 \le \tilde{\beta}_1(\tilde{s}_1^+)$. As above, $\tilde{\beta}_1(s) = s - \int_{r_1}^s [\frac{F_1(z)}{F_1(s)}]^{N_1-1}\,dz$ for all $s \in [r_1, \tilde{s}_1)$.

(ii) Let $\hat{s}_2$ be the transition points in the bidding strategy of a typical group-2 bidder. Then $\beta_2(s) = s - \int_{r_2}^s [\frac{F_2(z)}{F_2(s)}]^{N_2-1}\,dz$ for all $s \in [r_2, \hat{s}_2]$ and $\beta_2(\hat{s}_2) = \beta_1(\hat{s}_1^+)$. The value $\tilde{s}_2$ is defined analogously, i.e., $\tilde{\beta}_2(s) = s - \int_{r_2}^s [\frac{F_2(z)}{F_2(s)}]^{N_2-1}\,dz$ for all $s \in [r_2, \tilde{s}_2]$ and $\tilde{\beta}_2(\tilde{s}_2) = \tilde{\beta}_1(\tilde{s}_1^+)$.

(iii) Let $\eta^* = \beta_1(\bar{s}) = \beta_2(\bar{s})$ be the common maximal bid submitted in the β equilibrium. Analogously, $\tilde{\eta}^* = \tilde{\beta}_1(\bar{s}) = \tilde{\beta}_2(\bar{s})$.

(iv) Let $U_k(b|s)$ denote the expected utility of a type-s group-k bidder when he bids b and all others bid according to β. We define $\tilde{U}_k(b|s)$ similarly but assume that all others bid according to $\tilde{\beta}$ instead.

In this proof, we make use of the following property of the equilibrium strategies due to Lebrun (1997, Lemma A2-8). He shows that the solutions to the system of differential equations that characterize equilibrium bidding in the range where agents from both groups bid (above r_2) are monotone in the maximal bid submitted in the auction.[5] Thus, $\tilde{\beta}_1$ and β_1 cannot cross in the range above r_2 and given $s' \ge \hat{s}_1$, $\tilde{\beta}_1(s') > \beta_1(s') > r_2$ implies that $\tilde{\beta}_1(s) > \beta_1(s) > r_2$ for all $s \ge s'$ (in particular, $\tilde{\eta}^* > \eta^*$). An analogous relationship applies to $\tilde{\beta}_2$ and β_2 in the relevant range.

We consider several cases depending on the relative values of $\hat{s}_1$, $\tilde{s}_1$, $\hat{s}_2$, and $\tilde{s}_2$. First, suppose $\hat{s}_2 = \tilde{s}_2$. Thus, $\beta_2(\hat{s}_2) = \tilde{\beta}_2(\tilde{s}_2)$. It follows that $\beta_1(\hat{s}_1^+) = \tilde{\beta}_1(\tilde{s}_1^+)$. Without loss

[5] Consider two different solutions to the system of differential equations, and let β_k and $\tilde{\beta}_k$ be the associated bidding strategies for bidders in group k. If η^* ($\tilde{\eta}^*$) is the maximal bid submitted under β_k ($\tilde{\beta}_k$), then $\tilde{\eta}^* > \eta^* \implies \tilde{\beta}_k(s) > \beta_k(s)$ for all values of s such that $\beta_k(s) \ge r_2$. Thus, increasing the maximal bid increases the associated bidding strategy at each valuation (in the relevant domain). The strategies presented in Figures A.7 and A.8 illustrate this monotonicity property.

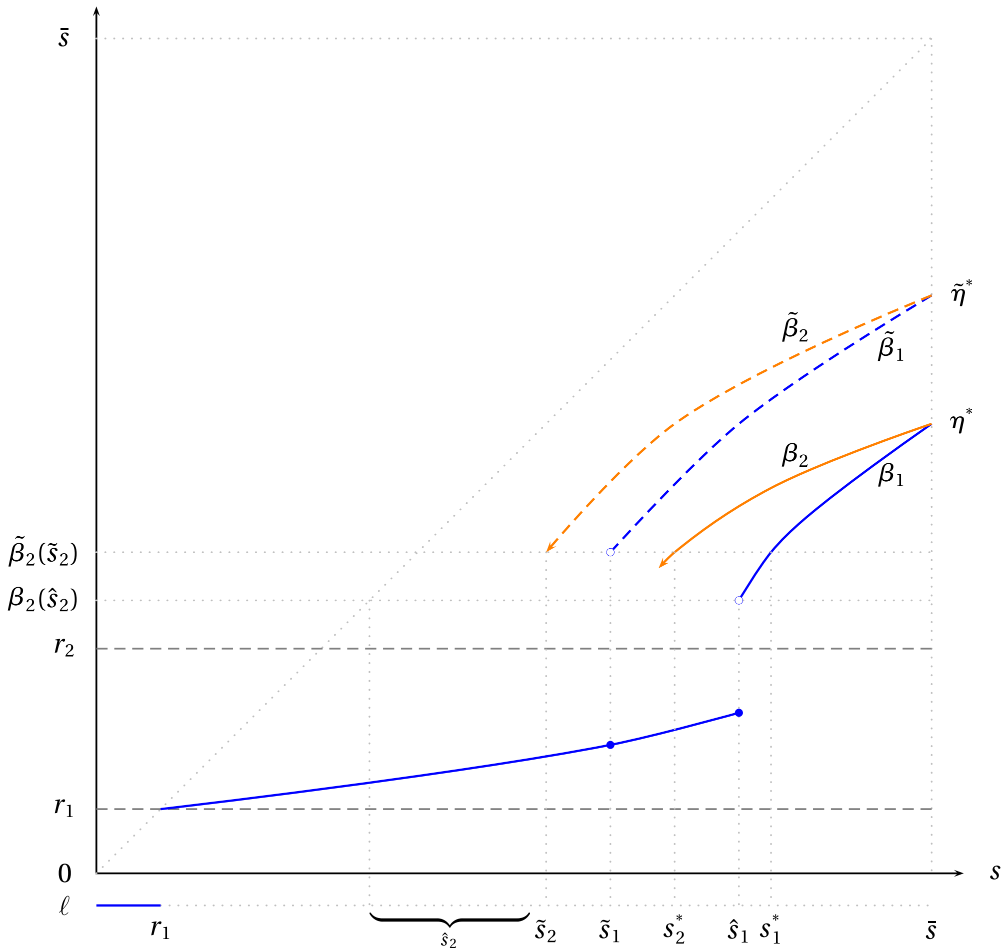

FIGURE A.7. Equilibrium strategies when $\tilde{s}_1 \leq \hat{s}_1$ and $\hat{s}_2 < \tilde{s}_2$; case (A). For clarity, we illustrate the strategy of group-2 bidders only for bids above $\tilde{\beta}_2(\tilde{s}_2)$. The value $\hat{s}_2$ is located on the identified interval. By assumption it is less than $\tilde{s}_2$.

of generality, suppose $\hat{s}_1 < \tilde{s}_1$. By strict monotonicity, $\beta_1(\tilde{s}_1) > \tilde{\beta}_1(\tilde{s}_1^+)$. Consequently, $\eta^* > \tilde{\eta}^*$. However, by the monotonicity of the equilibrium strategies in the maximal bid, $\beta_2(\hat{s}_2) > \tilde{\beta}_2(\tilde{s}_2)$, which is a contradiction.

Henceforth, assume that $\hat{s}_2 < \tilde{s}_2$. This assumption is without loss of generality. By point (ii) above, $\beta_2(\hat{s}_2) = \tilde{\beta}_2(\hat{s}_2) < \tilde{\beta}_2(\tilde{s}_2)$. There are two further cases depending on $\hat{s}_1$ and $\tilde{s}_1$.

Case 1. Suppose $\hat{s}_1 < \tilde{s}_1$. Point (i) above implies that $\beta_1(s) = \tilde{\beta}_1(s)$ for all $s < \hat{s}_1$ and $\beta_1(\hat{s}_1^-) = \tilde{\beta}_1(\hat{s}_1) < \tilde{\beta}_1(\tilde{s}_1^-) < r_2$.

Consider a type-$\hat{s}_1$ bidder in the $\tilde{\beta}$ equilibrium. Suppose this bidder places the bid $\beta_1(\hat{s}_1^+) > r_2$ instead of $\tilde{\beta}_1(\hat{s}_1)$. With the higher bid, the agent defeats all bidders in group 2 with a valuation less than $\hat{s}_2$. He also defeats all bidders in group 1 who are bidding less than r_2. These bidders have a valuations less

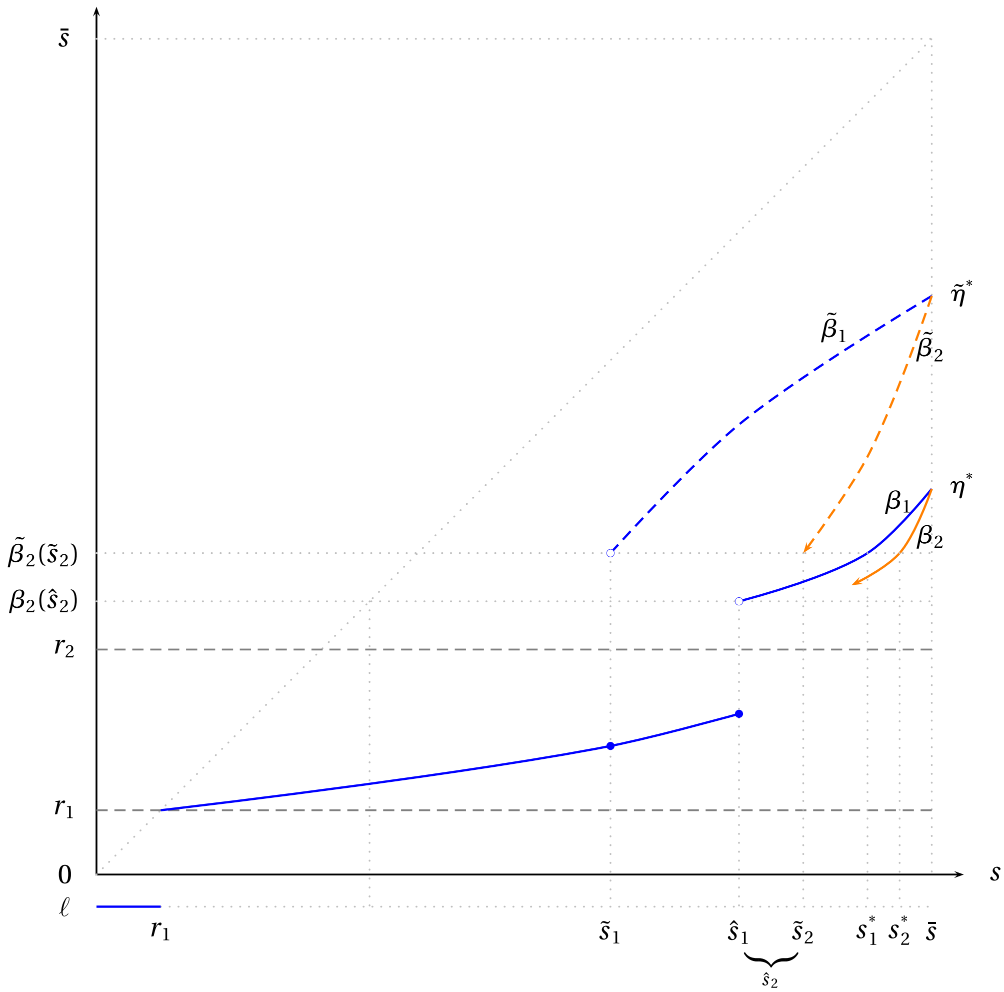

FIGURE A.8. Equilibrium strategies when $\tilde{s}_1 \leq \hat{s}_1$ and $\hat{s}_2 < \tilde{s}_2$; case (B). For clarity, we illustrate the strategy of group-2 bidders only for bids above $\tilde{\beta}_2(\tilde{s}_2)$. The value $\hat{s}_2$ is located on the identified interval. By assumption it is less than $\tilde{s}_2$. It is greater than $\hat{s}_1$ because $\beta_2(s)$ is bounded above by $\beta_1(s)$.

than $\tilde{s}_1$. He does not defeat any bidders from group 1 who are bidding above r_2 since $\tilde{\beta}_1(\tilde{s}_1^+) = \tilde{\beta}_2(\tilde{s}_2) > \beta_2(\hat{s}_2) = \beta_1(\hat{s}_1^+)$. We observe that

$$\begin{aligned}
\tilde{U}_1\big(\beta_1(\hat{s}_1^+)|\hat{s}_1\big) &= F_2(\hat{s}_2)^{N_2} F_1(\tilde{s}_1)^{N_1-1}\big(\hat{s}_1 - \beta_1(\hat{s}_1^+)\big) \\
&> F_2(\hat{s}_2)^{N_2} F_1(\hat{s}_1)^{N_1-1}\big(\hat{s}_1 - \beta_1(\hat{s}_1^+)\big) \\
&= F_2(r_2)^{N_2} F_1(\hat{s}_1)^{N_1-1}\big(\hat{s}_1 - \beta_1(\hat{s}_1^-)\big) \\
&= F_2(r_2)^{N_2} F_1(\hat{s}_1)^{N_1-1}\big(\hat{s}_1 - \tilde{\beta}_1(\hat{s}_1)\big) \\
&= \tilde{U}_1\big(\tilde{\beta}_1(\hat{s}_1)|\hat{s}_1\big).
\end{aligned}$$

The inequality follows from the assumption that $\tilde{s}_1 > \hat{s}_1$. The subsequent equality is because a type-$\hat{s}_1$ bidder is indifferent between the bids $\beta_1(\hat{s}_1^-)$ and $\tilde{\beta}_1(\hat{s}_1^+)$ in the β equilibrium. The next equality follows from $\beta_1(\hat{s}_1^-) = \tilde{\beta}_1(\hat{s}_1)$. Thus, a group-1 bidder of type $\hat{s}_1$ has a profitable deviation in the $\tilde{\beta}$ equilibrium, which is a contradiction.

Case 2. Suppose $\tilde{s}_1 \leq \hat{s}_1$. Thus, $\beta_1(s) = \tilde{\beta}_1(s)$ for all $s < \tilde{s}_1$ and $\tilde{\beta}_1(\tilde{s}_1^-) = \beta_1(\tilde{s}_1) \leq \beta_1(\hat{s}_1^-) < r_2$.

Recall that $\tilde{\beta}_2(\tilde{s}_2) = \tilde{\beta}_1(\tilde{s}_1^+)$ and $\beta_2(\hat{s}_2) = \beta_1(\hat{s}_1^+)$. Because $\tilde{\beta}_2(\tilde{s}_2) > \beta_2(\hat{s}_2)$, we conclude that $\tilde{\beta}_1(\tilde{s}_1^+) > \beta_1(\hat{s}_1^+)$. Since $\hat{s}_1 \geq \tilde{s}_1$, by monotonicity $\tilde{\beta}_1(\hat{s}_1^+) > \tilde{\beta}_1(\tilde{s}_1^+) > \beta_1(\hat{s}_1^+)$. Since equilibrium strategies are monotone in the maximal bid (Lebrun 1997, Lemma A2-8), $\tilde{\beta}_1(\hat{s}_1^+) > \beta_1(\hat{s}_1^+) \implies \tilde{\beta}_1(s) > \beta_1(s)$ for all $s \geq \hat{s}_1$ and, in particular, $\tilde{\eta}^* = \tilde{\beta}_1(\bar{s}) > \beta_1(\bar{s}) = \eta^*$. Furthermore, $\tilde{\eta}^* = \tilde{\beta}_2(\bar{s})$ and $\eta^* = \beta_2(\bar{s})$.

To simplify notation, for each group k, let

$$s_k^* := \begin{cases} \beta_k^{-1}\big(\tilde{\beta}_2(\tilde{s}_1)\big) & \text{if } \eta^* > \tilde{\beta}_2(\tilde{s}_2), \\ \bar{s} & \text{if } \eta^* \leq \tilde{\beta}_2(\tilde{s}_2). \end{cases}$$

When $\eta^* > \tilde{\beta}_2(\tilde{s}_1)$, then $s_k^* < \bar{s}$. By Lebrun (1999, Corollary 3), there are two possible cases:

(A) $\beta_2(s) \geq \beta_1(s)$ for all $s \geq s_1^*$ and $\tilde{\beta}_2(s) \geq \tilde{\beta}_1(s)$ for all $s \geq \tilde{s}_1$.

(B) $\beta_2(s) \leq \beta_1(s)$ for all $s \geq s_2^*$ and $\tilde{\beta}_2(s) \leq \tilde{\beta}_1(s)$ for all $s \geq \tilde{s}_2$.[6]

In Figures A.7 and A.8 we illustrate cases (A) and (B), respectively. In each figure the solid lines depict the β equilibrium while the dashed curves depict the $\tilde{\beta}$ equilibrium, at bids above r_2. For clarity, we illustrate the strategy of bidders from group 2 only for bids above $\tilde{\beta}_2(\tilde{s}_2)$.

In each case, $\beta_1(\hat{s}_1^+) < \tilde{\beta}_2(\tilde{s}_2)$ and by monotonicity of β_1, $\hat{s}_1 < s_1^*$. Similarly, $\hat{s}_2 < s_2^*$. (These inequalities also hold when $\eta^* \leq \tilde{\beta}_2(\tilde{s}_1)$ and $s_k^* = \bar{s}$.)

Now consider a type-$\tilde{s}_1$ bidder in the β equilibrium. Suppose this bidder bids $\tilde{\beta}_1(\tilde{s}_1^+) = \tilde{\beta}_2(\tilde{s}_2)$ instead of $\beta_1(\tilde{s}_1)$. With this higher bid he defeats all group-1 bidders with a valuation $s \leq s_1^*$ and all group-2 bidders with a valuation $s \leq s_2^*$. Thus,

$$\begin{aligned} U_1\big(\tilde{\beta}_1(\tilde{s}_1^+)|\tilde{s}_1\big) &= F_2\big(s_2^*\big)^{N_2} F_1\big(s_1^*\big)^{N_1-1}\big(\tilde{s}_1 - \tilde{\beta}_1(\tilde{s}_1^+)\big) \\ &> F_2(\tilde{s}_2)^{N_2} F_1(\tilde{s}_1)^{N_1-1}\big(\tilde{s}_1 - \tilde{\beta}_1(\tilde{s}_1^+)\big) \\ &= F_2(r_2)^{N_2} F_1(\tilde{s}_1)^{N_1-1}\big(\tilde{s}_1 - \tilde{\beta}_1(\tilde{s}_1^-)\big) \end{aligned}$$

[6]We can restrict attention to these two cases since the maintained regularity condition, i.e., $\frac{d}{ds}\big(\frac{F_k(s)}{F_{k'}(s)}\big) < 0$ for all $s > r_2$, ensures that $\beta_1(\cdot)$ and $\beta_2(\cdot)$ (and $\tilde{\beta}_1(\cdot)$ and $\tilde{\beta}_2(\cdot)$) are strictly ordered (Lebrun 1999, Corollary 3). This conclusion follows from the properties of the solutions to the differential equations characterizing equilibrium bidding.

$$
\begin{aligned}
&= F_2(r_2)^{N_2} F_1(\tilde{s}_1)^{N_1-1}\big(\tilde{s}_1 - \beta_1(\tilde{s}_1)\big) \\
&= U_1\big(\beta_1(\tilde{s}_1)|\tilde{s}_1\big).
\end{aligned}
$$

Therefore, the bidder has a profitable deviation, which is a contradiction.

The preceding cases exhaust all possibilities. Therefore, there exists at most one equilibrium where all group-1 bidders bid above r_2. □

The following lemma shows that if all group-1 bidders bid exclusively below r_2, the auction's equilibrium is characterized by the bidding strategy reported in Theorem 2b.

Lemma A.23. *Consider an equilibrium where $\beta_1^i(\bar{s}) \le r_2$ for all bidders i in group 1. Then*

$$
\beta_k^i(s) = \begin{cases} \ell & \text{if } s < r_k, \\ s - \displaystyle\int_{r_k}^{s} \left[\frac{F_k(z)}{F_k(s)}\right]^{N_k-1} dz & \text{if } s \ge r_k, \end{cases} \tag{A.3}
$$

for all bidders i in group $k \in \{1, 2\}$.

Proof. If all bidders in group 1 bid below r_2 in equilibrium, their presence has no effect on the incentives faced by bidders in group 2. From their point of view, the auction is equivalent to a symmetric auction with N_2 bidders and reserve price r_2. As shown by Lebrun (1999, Corollary 3) and Maskin and Riley (2003, Proposition 2), the equilibrium in such an auction is unique and is given by (A.3) with $k = 2$.

Now consider bidders in group 1. Since β is an equilibrium, bidder i must not wish to deviate to any other bid in the range of the other bidders' strategies or to any bid above r_2. Since all of the bids of agents in group 1 are bounded above by r_2 it follows that these strategies of group-1 bidders also define an equilibrium in a symmetric first-price sealed-bid auction with N_1 bidders and a reserve price of r_1. But such an auction has a unique equilibrium where all bidders bid according to (A.3) with $k = 1$. □

Lemma A.24. *There does not exist any equilibrium where some group-1 bidders bid above r_2 with positive probability and others always bid less than r_2.*

This situation is equivalent to that addressed by Lemma A.18 with $\hat{s}_1^i = \bar{s}$. Therefore, it is ruled out by the argument provided by Lebrun (1999, pp. 136–137).

Lemma A.25. *Suppose there exists an equilibrium where all group-1 bidders bid above r_2 with strictly positive probability. Then there does not exist an equilibrium where all bidders in group 1 bid exclusively less than r_2.*

Proof. Suppose there exists an equilibrium where all group-1 bidders bid above r_2 with positive probability. Given the preceding results, we may assume that this equilibrium is in group-symmetric strategies. Let β_1 and β_2 denote the equilibrium strategies for groups 1 and 2, respectively. Thus, $\beta_1(\hat{s}_1^-) \le r_2 \le \beta_1(\hat{s}_1^+)$ for some $\hat{s}_1 < \bar{s}$. From above, we know that $\beta_k(s) = s - \int_{r_k}^{s} [\frac{F_k(z)}{F_k(s)}]^{N_k-1}\, dz$ for all $s \in [r_k, \hat{s}_k)$ and $\beta_1(\hat{s}_1^+) = \beta_2(\hat{s}_2)$.

Let $\tilde{\beta}_1$ and $\tilde{\beta}_2$ denote the strategies of bidders in groups 1 and 2, respectively, in an alternative equilibrium where $\tilde{\beta}_1(s) \leq r_2$ for all s. From above, we known that $\tilde{\beta}_k(s) = s - \int_{r_k}^{s} [\frac{F_k(z)}{F_k(s)}]^{N_k-1}\, dz$ for all $s \in [r_k, \bar{s}]$.

The expected utility of a type-s group-1 bidder in the "$\tilde{\beta}$" equilibrium is

$$\tilde{U}_1\big(\tilde{\beta}_1(s)|s\big) = F_2(r_2)^{N_2} \int_{r_1}^{s} F_1(z)^{N_1-1}\, dz.$$

This agent must not have a profitable deviation to any bid in the range of $\tilde{\beta}_2$. If he bids $\tilde{\beta}_2(t)$, $t \geq r_2$, his expected payoff is

$$\tilde{U}_1\big(\tilde{\beta}_2(t)|s\big) = F_2(t)^{N_2}\left(s - t + \int_{r_2}^{t}\left[\frac{F_2(z)}{F_2(t)}\right]^{N_2-1} dz\right).$$

In particular,

$$\tilde{U}_1\big(\tilde{\beta}_1(s)|s\big) \geq \tilde{U}_1\big(\tilde{\beta}_2(\hat{s}_2)|s\big) \tag{A.4}$$

for all $s \geq r_2$.

Given the indifference condition supporting the β equilibrium and the definitions of β_k and $\tilde{\beta}_k$, we observe that at $s = \hat{s}_1$,

$$\tilde{U}_1\big(\tilde{\beta}_1(\hat{s}_1)|\hat{s}_1\big) = U_1\big(\beta_1(\hat{s}_1^-)|\hat{s}_1\big) = U_1\big(\beta_1(\hat{s}_1^+)|\hat{s}_1\big) = U_1\big(\beta_2(\hat{s}_2)|\hat{s}_1\big) \leq \tilde{U}_1\big(\tilde{\beta}_2(\hat{s}_2)|\hat{s}_1\big).$$

The final inequality is because a group-1 bidder defeats all other group-1 bidders with the bid $\beta_2(\hat{s}_2) = \tilde{\beta}_2(\hat{s}_2)$ in the $\tilde{\beta}$ equilibrium. Combined with (A.4), we conclude that

$$\tilde{U}_1\big(\tilde{\beta}_1(\hat{s}_1)|\hat{s}_1\big) = \tilde{U}_1\big(\tilde{\beta}_2(\hat{s}_2)|\hat{s}_1\big).$$

However,

$$\frac{d}{ds}\tilde{U}_1\big(\tilde{\beta}_1(s)|s\big) = F_2(r_2)^{N_2} F_1(s)^{N_1-1} < F_2(\hat{s}_2)^{N_2} = \frac{d}{ds}\tilde{U}_1\big(\tilde{\beta}_1(\hat{s}_2)|s\big).$$

Thus, for $s' > \hat{s}_1$, $\tilde{U}_1(\tilde{\beta}_1(s')|s') < \tilde{U}_1(\tilde{\beta}_2(\hat{s}_2)|s')$, which contradicts (A.4). $\square$

Thus, we conclude that the equilibrium identified by Theorems 2a and 2b is this auction's unique equilibrium in each case.

Corollary 1. *If $F_1 = F_2$, the auction has a unique equilibrium.*

Proof. Uniqueness of the equilibrium in the symmetric case follows from the preceding analysis. The sole necessary qualification concerns the regularity condition $\frac{d}{ds}(\frac{F_k(s)}{F_{k'}(s)}) < 0$ that was imposed in our analysis of the asymmetric case.

The preceding argument continues to apply once we observe the following. First, due to the common support of valuations, all bidders in a semi-separating equilibrium submit a common maximal bid. Near this common maximal bid, it is well known that the agents' bidding strategies are characterized by a system of differential equations. As shown by Lebrun (1997, Section 5) and Lebrun (1999), this system has a unique solution that is symmetric across bidders. Thus, sufficiently close to $\bar{s}$, $\beta^i_k(s) = \beta^j_{k'}(s)$ for all bidders i, j, and $k, k' \in \{1,2\}$. This solution varies monotonically with the maximal bid submitted in the auction (Lebrun 1999). The remainder of the above argument is unchanged with $\hat{s}_1 = \hat{s}_2 = \hat{s}$ defining the critical point in the strategy of a group-1 bidder. □

B. Proofs from Section 4

Proof of Theorem 6. The proof proceeds similarly to the case of reserve prices. Let $U_k(b|s)$ be the expected utility of a group-k bidder of type s when he bids b given that all others follow the strategy prescribed in the theorem.

(i) Consider a group-1 bidder of type $s < \check{s}_1$. Suppose this bidder enters the auction and places the bid $\beta_1(t)$, $t \in [\check{s}_1, \check{s}_2)$. His expected payoff is

$$
\begin{aligned}
&U_1\big(\beta_1(t)|s\big)\\
&\quad= F(t)^{N_1-1}F(\check{s}_2)^{N_2}\left(s - t + \int_{\check{s}_1}^{t}\left[\frac{F(z)}{F(t)}\right]^{N_1-1} dz + \frac{c_1}{F(t)^{N_1-1}F(\check{s}_2)^{N_2}}\right) - c_1\\
&\quad= F(\check{s}_2)^{N_2}\left(F(t)^{N_1-1}(s-\check{s}_1) - F(t)^{N_1-1}(t-\check{s}_1) + \int_{\check{s}_1}^{t} F(z)^{N_1-1}\,dz\right)\\
&\quad= F(\check{s}_2)^{N_2}\left(F(t)^{N_1-1}(s-\check{s}_1) + \int_{\check{s}_1}^{t}\big[F(z)^{N_1-1} - F(t)\big]\,dz\right) \le 0.
\end{aligned}
$$

If instead this bidder places the bid $\beta_1(t)$, $t \ge \check{s}_2$, his expected payoff is

$$
\begin{aligned}
U_1\big(\beta_1(t)|s\big) &= F(t)^{N_1+N_2-1}\left(s - t + \int_{\check{s}_2}^{t}\left[\frac{F(z)}{F(t)}\right]^{N_1+N_2-1} dz + \frac{c_2}{F(t)^{N_1+N_2-1}}\right) - c_1\\
&= F(t)^{N_1+N_2-1}(s-\check{s}_2) - F(t)^{N_1+N_2-1}(t-\check{s}_2)\\
&\quad + \int_{\check{s}_2}^{t} F(z)^{N_1+N_2-1}\,dz + F(\check{s}_2)^{N_2}\int_{\check{s}_1}^{\check{s}_2} F(z)^{N_1-1}\,dz\\
&= F(t)^{N_1+N_2-1}(s-\check{s}_1) + F(t)^{N_1+N_2-1}(\check{s}_1-\check{s}_2)\\
&\quad + F(\check{s}_2)^{N_2}\int_{\check{s}_1}^{\check{s}_2} F(z)^{N_1-1}\,dz\\
&\quad + \int_{\check{s}_2}^{t} F(z)^{N_1+N_2-1}\,dz - F(t)^{N_1+N_2-1}(t-\check{s}_2)\\
&= F(t)^{N_1+N_2-1}(s-\check{s}_1) + \int_{\check{s}_1}^{\check{s}_2}\big[F(\check{s}_2)^{N_2}F(z)^{N_1-1} - F(t)^{N_1+N_2-1}\big]\,dz\\
&\quad + \int_{\check{s}_2}^{t}\big[F(z)^{N_1+N_2-1} - F(t)^{N_1+N_2-1}\big]\,dz \le 0.
\end{aligned}
$$

Therefore, it is optimal for a type $s < \check{s}_1$ bidder in group 1 to not bid in the auction given the strategies adopted by the other bidders.

(ii) Consider a group-1 bidder of type $s \in [\check{s}_1, \check{s}_2)$. When this bidder and others follow the prescribed strategy, his expected payoff is $U_1(\beta_1(s)|s) = \int_{\check{s}_1}^{s} F(\check{s}_2)^{N_2} \times F(z)^{N_1-1}\, dz$.

Suppose this agent bids $\beta_1(t)$, $t \in [\check{s}_1, \check{s}_2)$. Then

$$U_1\big(\beta_1(t)|s\big) = F(\check{s}_2)^{N_2}\bigg(F(t)^{N_1-1}(s-t) + \int_{\check{s}_1}^{t} F(z)^{N_1-1}\, dz\bigg).$$

Let $\Delta(t,s) := U_1(\beta_1(t)|s) - U_1(\beta_1(s)|s)$. For all $t \in [\check{s}_1, \check{s}_2)$,

$$\begin{aligned}\Delta(t,s) &= F(\check{s}_2)^{N_2}\bigg(F(t)^{N_1-1}(s-t) + \int_{s}^{t} F(z)^{N_1-1}\, dz\bigg) \\ &= F(\check{s}_2)^{N_2}\bigg(\int_{s}^{t}\big[F(z)^{N_1-1} - F(t)^{N_1-1}\big]\, dz\bigg) \le 0.\end{aligned}$$

If instead this bidder places the bid $\beta_1(t)$, $t \ge \check{s}_2$, his expected payoff is

$$U_1\big(\beta_1(t)|s\big) = F(t)^{N_1+N_2-1}(s-t) + \int_{\check{s}_2}^{t} F(z)^{N_1+N_2-1}\, dz + c_2 - c_1.$$

Let $\Delta(t,s) := U_1(\beta_1(t)|s) - U_1(\beta_1(s)|s)$. Hence,

$$\begin{aligned}\Delta(t,s) &= F(t)^{N_1+N_2-1}(s-t) + \int_{\check{s}_2}^{t} F(z)^{N_1+N_2-1}\, dz \\ &\quad + F(\check{s}_2)^{N_2}\int_{\check{s}_1}^{\check{s}_2} F(z)^{N_1-1}\, dz - F(\check{s}_2)^{N_2}\int_{\check{s}_1}^{s} F(z)^{N_1-1}\, dz \\ &= F(t)^{N_1+N_2-1}(s-t) + \int_{\check{s}_2}^{t} F(z)^{N_1+N_2-1}\, dz + F(\check{s}_2)^{N_2}\int_{s}^{\check{s}_2} F(z)^{N_1-1}\, dz \\ &= F(t)^{N_1+N_2-1}(s-\check{s}_2) + F(\check{s}_2)^{N_2}\int_{s}^{\check{s}_2} F(z)^{N_1-1}\, dz \\ &\quad + F(t)^{N_1+N_2-1}(\check{s}_2 - t) + \int_{\check{s}_2}^{t} F(z)^{N_1+N_2-1}\, dz \\ &\le F(\check{s}_2)^{N_2}\int_{s}^{\check{s}_2}\big[F(z)^{N_1-1} - F(t)^{N_1-1}\big]\, dz \\ &\quad + \int_{\check{s}_2}^{t}\big[F(z)^{N_1+N_2-1} - F(t)^{N_1+N_2-1}\big]\, dz \le 0.\end{aligned}$$

Therefore, $\beta_1(s)$ is the utility-maximizing bid for a type $s \in [\check{s}_1, \check{s}_2)$ bidder in group 1 given the other bidders' strategies.

(iii) Consider a group-1 bidder of type $s \geq \check{s}_2$. When this bidder and others follow the prescribed strategy, his expected payoff is

$$U_1\big(\beta_1(s)|s\big) = F(\check{s}_2)^{N_2}\int_{\check{s}_1}^{\check{s}_2} F(z)^{N_1-1}\,dz + \int_{\check{s}_2}^{s} F(z)^{N_1+N_2-1}\,dz.$$

If this agent bids $\beta_1(t)$, $t \geq \check{s}_2$, then

$$\begin{aligned} &U_1\big(\beta_1(t)|s\big) \\ &\quad = F(t)^{N_1+N_2-1}(s-t) + \int_{\check{s}_2}^{t} F(z)^{N_1+N_2-1}\,dz + F(\check{s}_2)^{N_2}\int_{\check{s}_1}^{\check{s}_2} F(z)^{N_1-1}\,dz. \end{aligned}$$

Let $\Delta(t,s) := U_1(\beta_1(t)|s) - U_1(\beta_1(s)|s)$. Hence,

$$\begin{aligned} \Delta(t,s) &= F(t)^{N_1+N_2-1}(s-t) + \int_{s}^{t} F(z)^{N_1+N_2-1}\,dz \\ &= \int_{s}^{t}\big[F(z)^{N_1+N_2-1} - F(t)^{N_1+N_2-1}\big]\,dz \leq 0. \end{aligned}$$

Suppose this agent bids $\beta_1(t)$, $t \in [\check{s}_1, \check{s}_2)$. Then

$$U_1\big(\beta_1(t)|s\big) = F(\check{s}_2)^{N_2}\bigg(F(t)^{N_1-1}(s-t) + \int_{\check{s}_1}^{t} F(z)^{N_1-1}\,dz\bigg).$$

As $t < s$, the same reasoning as in case (ii) confirms that $U_1(\beta_1(t)|s) - U_1(\beta_1(s)|s) \leq 0$.

(iv) Consider a group-2 bidder of type $s < \check{s}_2$. For this bidder, $U_2(\beta_2(s)|s) = 0$. Suppose that he bids $\beta_1(t)$, $t \in [\check{s}_1, \check{s}_2)$. Then

$$\begin{aligned} &U_2\big(\beta_1(t)|s\big) \\ &\quad = F(t)^{N_1}F(\check{s}_2)^{N_2-1}\bigg(s - t + \int_{\check{s}_1}^{t}\bigg[\frac{F(z)}{F(t)}\bigg]^{N_1-1} dz + \frac{c_1}{F(t)^{N_1-1}F(\check{s}_2)^{N_2}}\bigg) - c_2 \\ &\quad \leq F(t)^{N_1-1}F(\check{s}_2)^{N_2}\bigg(s - t + \int_{\check{s}_1}^{t}\bigg[\frac{F(z)}{F(t)}\bigg]^{N_1-1} dz + \frac{c_1}{F(t)^{N_1-1}F(\check{s}_2)^{N_2}}\bigg) - c_2 \\ &\quad = F(t)^{N_1-1}F(\check{s}_2)^{N_2}(s-t) + F(\check{s}_2)^{N_2}\int_{\check{s}_1}^{t} F(z)^{N_1-1}\,dz + c_1 - c_2. \end{aligned}$$

Differentiating the final line with respect to t gives $(N_1-1)F(t)^{N_1-2}f(t)F(\check{s}_2)^{N_2} \times (s-t) - F(t)^{N_1-1}F(\check{s}_2) + F(\check{s}_2)F(t)^{N_1-1}$, which is positive when $t < s$ and negative when $t > s$. Hence, the final expression above achieves a maximum at $t = s$. Thus,

$$\begin{aligned} U_2\big(\beta_1(t)|s\big) &\leq F(\check{s}_2)^{N_2}\int_{\check{s}_1}^{s} F(z)^{N_1-1}\,dz + c_1 - c_2 \\ &= F(\check{s}_2)^{N_2}\int_{\check{s}_1}^{s} F(z)^{N_1-1}\,dz - F(\check{s}_2)^{N_2}\int_{\check{s}_1}^{\check{s}_2} F(z)^{N_1-1}\,dz \leq 0. \end{aligned}$$

If instead this agent bids $\beta_2(t)$, $t \geq \check{s}_2$, his expected payoff is

$$
\begin{aligned}
&U_2\big(\beta_2(t)|s\big) \\
&\quad = F(t)^{N_1+N_2-1}\left(s - t + \int_{\check{s}_2}^{t}\left[\frac{F(z)}{F(t)}\right]^{N_1+N_2-1} dz + \frac{c_2}{F(t)^{N_1+N_2-1}}\right) - c_2 \\
&\quad = F(t)^{N_1+N_2-1}(s-t) + \int_{\check{s}_2}^{t} F(z)^{N_1+N_2-1}\, dz \\
&\quad = F(t)^{N_1+N_2-1}(s-\check{s}_2) + \int_{\check{s}_2}^{t}\big[F(z)^{N_1+N_2-1} - F(t)^{N_1+N_2-1}\big]\, dz \leq 0.
\end{aligned}
$$

Thus, given the strategy adopted by the other bidders, it is optimal for this bidder not to enter the auction.

(v) Consider a group-2 bidder of type $s \geq \check{s}_2$. When this bidder and others follow the prescribed strategy, his expected payoff is $U_2(\beta_2(s)|s) = \int_{\check{s}_2}^{s} F(z)^{N_1+N_2-1}\, dz$. Reasoning parallel to that from case (iii) shows that this bidder cannot gain from a deviation to any bid $\beta_2(t)$, $t \geq \check{s}_2$.

If he bids $\beta_1(t)$, $t \in [\check{s}_1, \check{s}_2)$, his expected payoff is

$$
\begin{aligned}
&U_2\big(\beta_1(t)|s\big) \\
&\quad = F(t)^{N_1} F(\check{s}_2)^{N_2-1}\left(s - t + \int_{\check{s}_1}^{t}\left[\frac{F(z)}{F(t)}\right]^{N_1-1} dz + \frac{c_1}{F(t)^{N_1-1}F(\check{s}_2)^{N_2}}\right) - c_2.
\end{aligned}
$$

Let $\Delta(t,s) := U_2(\beta_1(t)|s) - U_2(\beta_2(s)|s)$. Then

$$
\begin{aligned}
\Delta(t,s) &\leq F(t)^{N_1-1}F(\check{s}_2)^{N_2}(s-t) + F(\check{s}_2)^{N_2}\int_{\check{s}_2}^{t} F(z)^{N_1-1}\, dz \\
&\qquad - F(\check{s}_2)^{N_2}\int_{\check{s}_1}^{\check{s}_2} F(z)^{N_1-1}\, dz - \int_{\check{s}_2}^{s} F(z)^{N_1+N_2-1}\, dz \\
&\leq F(\check{s}_2)^{N_2}F(t)^{N_1-1}(s-\check{s}_2) - \int_{\check{s}_2}^{s} F(z)^{N_1+N_2-1}\, dz \\
&\qquad + F(\check{s}_2)^{N_2}F(t)^{N_1-1}(\check{s}_2 - t) + F(\check{s}_2)^{N_2}\int_{\check{s}_2}^{t} F(z)^{N_1-1}\, dz \\
&= F(\check{s}_2)\int_{\check{s}_2}^{s}\big[F(t)^{N_1-1} - F(z)^{N_1-1}\big]\, dz \\
&\qquad + F(\check{s}_2)^{N_2}\int_{\check{s}_2}^{t}\big[F(z)^{N_1-1} - F(t)^{N_1-1}\big]\, dz \leq 0.
\end{aligned}
$$

The final inequality follows since $t \leq \check{s}_2 \leq s$. Therefore, a group-2 bidder of type $s \geq \check{s}_2$ does not have a profitable deviation from the prescribed strategy.

As the preceding cases exhaust all possibilities, the proposed strategy is an equilibrium. $\square$

Proof of Corollary 1. Let $\check{s}_1$ and $\check{s}_2$ be the cutoff type submitting a competitive bid in the auction with entry fees $c_1 \le c_2$. Let $\check{s}_1'$ and $\check{s}_2'$ be the cutoff types submitting competitive bids given entry fees $c_1' < c_1$ and $c_2' = c_2$. To prove the claim, it is sufficient to show that $\check{s}_1' < \check{s}_1 \le \check{s}_2 < \check{s}_2'$.

From Theorem 6, we can characterize the cutoff types participating in an auction with entry fees c_1 and c_2 as the values of s_1 and s_2 where the following curves intersect in (s_1, s_2) space:

$$c_1 = F(s_1)^{N_1-1}F(s_2)^{N_2}s_1 \tag{B.1}$$

$$c_2 = F(s_2)^{N_2}\int_{s_1}^{s_2} F(z)^{N_1-1}\,dz + F(s_1)^{N_1-1}F(s_2)^{N_2}s_1. \tag{B.2}$$

Both curves are downward sloping in (s_1, s_2) space, with (B.2) being "steeper" than (B.1). Reducing c_1 shifts (B.1) up for each s_1, while (B.2) remains constant. Thus, the curves intersect at a point $(\check{s}_1', \check{s}_2')$ such that $\check{s}_1' < \check{s}_1 \le \check{s}_2 < \check{s}_2'$. □

Proof of Corollary 2. The equilibrium allocation rule with entry fees—say, ψ^E—differs from the allocation rule with reserve prices—say, ψ^R—only in the following way. In reference to Figure 3 (see the main text), when $(\tilde{s}_1, \tilde{s}_2) \in B$, the item is allocated to the agent with the highest valuation among all bidders when entry fees apply, i.e., $\psi_1^E(\tilde{s}_1, \tilde{s}_2) = 1$ and $\psi_2^E(\tilde{s}_1, \tilde{s}_2) = 0$. With reserve prices, the item is allocated to the agent in group 2 with the highest valuation, i.e., $\psi_1^R(\tilde{s}_1, \tilde{s}_2) = 0$ and $\psi_2^R(\tilde{s}_1, \tilde{s}_2) = 1$. As $J(s)$ is nondecreasing, $\sum_k \psi_k^E(\tilde{s}_1, \tilde{s}_2)J(\tilde{s}_k) \ge \sum_k \psi_k^R(\tilde{s}_1, \tilde{s}_2)J(\tilde{s}_k)$ for all $(\tilde{s}_1, \tilde{s}_2) \in B$. Thus, the auction with entry fees generates greater revenues. □

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