Supplementary appendix to "Negotiation across multiple issues"

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Gabrielle Gayer
Departmentof Economics, Bar Ilan University

Dotan Persitz
Coller School of Management, Tel Aviv University

This appendix contains the nonemptiness characterizations of the sum of the cores of the individual issues ($\sum_{V_j \in \bar{V}} C(V_j)$) and of the core of the sum of individual issues ($C(\sum_{V_j \in \bar{V}} V_j)$). These characterizations use systems of multiweights, which makes them comparable to the nonemptiness characterization of the multicore (Theorem 2 in the paper). For this purpose, two additional sets of systems of multiweights are presented together with the systems of multiweights that appear in Definition 6 in the paper.

S1. Definitions

S1.1 *Multiweights*

A function $\tilde{\delta} : 2^N \times N \times \bar{V} \to \mathbb{R}_+$ that assigns a nonnegative real number to every triplet of coalition, agent, and issue is a system of multiweights.

We concentrate on systems of multiweights that satisfy "Zero to Nonmembers" and "Resource Exhaustion."

Definition S1. A system of multiweights, $\tilde{\delta}$, satisfies "Zero to Nonmembers" if $\forall V_j \in \bar{V}$, $\forall i \in N$, $\forall S \in 2^{N \setminus \{i\}}$: $\tilde{\delta}(S, i, V_j) = 0$.

"Zero to Nonmembers" entails a system of multiweights that assigns zero weight to all triplets where the agent is not a member of the coalition.

Definition S2. A system of multiweights, $\tilde{\delta}$, satisfies "Resource Exhaustion" if $\forall V_j \in \bar{V}$: $\sum_{i \in N} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) \chi^S = \chi^N$.

"Resource Exhaustion" implies that each agent is endowed with one unit of time per issue. When "Resource Exhaustion" and "Zero to Nonmembers" are imposed, we refer to a system of multiweights as an unrestricted system of balancing multiweights.[1]

Gabrielle Gayer: gabi.gayer@gmail.com
Dotan Persitz: persitzd@post.tau.ac.il

[1] To see that balancedness is imposed in each issue V_j, set $\delta(S) = \sum_{i \in N} \tilde{\delta}(S, i, V_j)$. Then "Resource Exhaustion" implies that in each issue V_j, $\sum_{S \in 2^N} \delta(S) \chi^S = \chi^N$. Observe that the identity of agent i is ignored

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The following two definitions impose across-issue restrictions on systems of multiweights. Definition S3 requires that the total weights (over coalitions) assigned to triplets that include agent i be constant across issues. Definition S4 compels the weights assigned to triplets that include agent i and coalition S to be the same across issues.

DEFINITION S3. A system of multiweights, $\tilde{\delta}$, satisfies "Constant Shares" if $\forall i \in N$, $\forall V_j, V_{j'} \in \bar{V}$: $\sum_{S \in 2^N} \tilde{\delta}(S, i, V_j)\chi^S = \sum_{S \in 2^N} \tilde{\delta}(S, i, V_{j'})\chi^S$.

DEFINITION S4. A system of multiweights, $\tilde{\delta}$, satisfies "Constant Allocations" if $\forall i \in N$, $\forall V_j, V_{j'} \in \bar{V}$, $\forall S \in 2^N$: $\tilde{\delta}(S, i, V_j) = \tilde{\delta}(S, i, V_{j'})$.

S1.2 *Systems*

We concentrate on the following three families of systems of balancing multiweights:

DEFINITION S5. A function $\tilde{\delta} : 2^N \times N \times \bar{V} \to \mathbb{R}_+$ that satisfies "Zero to Nonmembers" and "Resource Exhaustion" is one of the following families:

1. A system of unconstrained balancing multiweights (Δ_{UC} is the set of all systems of unconstrained balancing multiweights).
2. A system of balancing multiweights if it satisfies "Constant Shares" (Δ is the set of all systems of balancing multiweights).
3. A system of balancing multiweights with constant allocations if it satisfies "Constant Allocations" (Δ_{CA} is the set of all systems of balancing multiweights with constant allocations).

The "Constant Allocations" requirement implies the "Constant Shares" requirement, but not the opposite. Therefore, $\Delta_{\mathrm{UC}} \supseteq \Delta \supseteq \Delta_{\mathrm{CA}}$. The difference between the three definitions lies in the dependencies they impose on the weights across issues. The elements of Δ_{UC} are unrestricted across issues, so that $\tilde{\delta}(\cdot, \cdot, V_j)$ poses no restriction on the values of $\tilde{\delta}(\cdot, \cdot, V_{j'})$ for every $V_j, V_{j'} \in \bar{V}$. By contrast, for Δ_{CA}, $\tilde{\delta}(\cdot, \cdot, V_j)$ and $\tilde{\delta}(\cdot, \cdot, V_{j'})$ must be the same for every $V_j, V_{j'} \in \bar{V}$. The set Δ, which lies between these two sets, allows for some variation of $\tilde{\delta}(\cdot, \cdot, V_j)$ across issues, as long as they obey the "Constant Shares" requirement.[2]

The three sets, Δ_{UC}, Δ, and Δ_{CA}, coincide when the multi-issue problem consists of only one issue V. The correspondence above between standard weights and multiweights, establishes that any collection of coalitions that are assigned positive weights

in $\delta(S)$; therefore, when restricting attention to issue V_j, several systems of balancing multiweights are reduced to one system of balancing weights. Conversely, every system of balancing weights corresponds to at least one system of balancing multiweights (e.g., dividing $\delta(S)$ equally among the members of S).

[2]Put differently, consider the set of functions that assign weights to agent–coalition pairs restricted by two requirements: assigning zero to pairs where the agent is not an element of the coalition and allocating a total weight of 1 to each agent across coalitions,

$$F = \left\{ f : N \times 2^N \to \mathbb{R}_+ \;\middle|\; i \notin S \text{ implies } f(i, S) = 0, \forall i \in N : \sum_{S \in \{T \cup \{i\} | T \subseteq N \setminus \{i\}\}} \sum_{k \in S} f(k, S) = 1 \right\}.$$

in some system of balancing weights can also be assigned positive weights by any one of the three definitions above.

This observation is still true when concentrating on the weights of a specific issue in the multigame. However, once these weights are set, Definitions S5.2 and S5.3 confine the possible weights in the other issues.

S2. Example

Table 1 presents three examples of systems of balancing multiweights with $\tilde{\delta}_1$, $\tilde{\delta}_2$, and $\tilde{\delta}_3$, corresponding to the three definitions above in a two-issue–three-agent multigame. A row in this table corresponds to a triplet: coalition, agent, and issue.[3] The "Constant Allocation" condition is satisfied by $\tilde{\delta}_3$ since for every agent i and for every coalition S, $\tilde{\delta}_3(S,i,V_1) = \tilde{\delta}_3(S,i,V_2)$, whereas the two other functions do not satisfy it (e.g., agent 1 and coalition $\{1,2\}$). The "Constant Shares" condition is satisfied by both $\tilde{\delta}_2$ and $\tilde{\delta}_3$, but is violated by $\tilde{\delta}_1$ (agent 1).

S3. Results

Proposition S1. *The sum of the cores of the individual issues of $\bar{V}$, $\sum_{V_j\in\bar{V}} C(V_j)$, is nonempty if and only if every $\tilde{\delta}\in\Delta_{\text{UC}}$ satisfies*

$$\sum_{V_j\in\bar{V}} V_j(N) \geq \sum_{V_j\in\bar{V}} \sum_{i=1}^{n} \sum_{S\in 2^N} \tilde{\delta}(S,i,V_j)V_j(S).$$

Proposition S2. *The core of the sum of individual issues of $\bar{V}$, $C(\sum_{V_j\in\bar{V}} V_j)$, is nonempty if and only if every $\tilde{\delta}\in\Delta_{\text{CA}}$ satisfies*

$$\sum_{V_j\in\bar{V}} V_j(N) \geq \sum_{V_j\in\bar{V}} \sum_{i=1}^{n} \sum_{S\in 2^N} \tilde{\delta}(S,i,V_j)V_j(S).$$

For a given $V_j\in\bar{V}$, $\tilde{\delta}(S,i,V_j)$ satisfies "Zero to Nonmembers" and "Resource Exhaustion" if and only if it is an element of F.

Definition S5.1 states that Δ_{UC} is the set of systems of multiweights where, for each issue V_j, $\tilde{\delta}(S,i,V_j)$ is some element of F.

Definition S5.3 states that Δ_{CA} is the set of systems of multiweights where, for each issue V_j, $\tilde{\delta}(S,i,V_j)$ is the same element of F.

Let Π be a partition of F such that two functions f and f' belong to the same class if, for every pair of agents i and k,

$$\sum_{S\in\{T\cup\{i,k\}\mid T\subseteq N\setminus\{i,k\}\}} f(i,S) = \sum_{S\in\{T\cup\{i,k\}\mid T\subseteq N\setminus\{i,k\}\}} f'(i,S).$$

Definition S5.3 states that Δ is the set of systems of multiweights where, for each issue V_j, $\tilde{\delta}(S,i,V_j)$ belongs to the same class of Π.

[3]Every triplet that is not specified in the table is assigned zero weight. Notice that both "Zero to Nonmembers" and "Resource Exhaustion" are satisfied by all three systems.

Issue	Agent	Coalition	$\tilde{\delta}_1$	$\tilde{\delta}_2$	$\tilde{\delta}_3$
Issue V_1	Agent 1	$\{1\}$	0	0	0
		$\{1,2\}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
		$\{1,3\}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
		$\{1,2,3\}$	0	0	0
	Agent 2	$\{2\}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
		$\{1,2\}$	0	0	0
		$\{2,3\}$	0	0	0
		$\{1,2,3\}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$
	Agent 3	$\{3\}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$
		$\{1,3\}$	0	0	$\frac{1}{8}$
		$\{2,3\}$	0	0	$\frac{1}{8}$
		$\{1,2,3\}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$
Issue V_2	Agent 1	$\{1\}$	1	$\frac{1}{4}$	0
		$\{1,2\}$	0	0	$\frac{1}{4}$
		$\{1,3\}$	0	0	$\frac{1}{4}$
		$\{1,2,3\}$	0	$\frac{1}{4}$	0
	Agent 2	$\{2\}$	0	$\frac{1}{4}$	$\frac{1}{4}$
		$\{1,2\}$	0	0	0
		$\{2,3\}$	$\frac{1}{2}$	0	0
		$\{1,2,3\}$	0	$\frac{1}{4}$	$\frac{1}{4}$
	Agent 3	$\{3\}$	0	$\frac{1}{8}$	$\frac{1}{8}$
		$\{1,3\}$	0	$\frac{1}{8}$	$\frac{1}{8}$
		$\{2,3\}$	$\frac{1}{2}$	$\frac{1}{8}$	$\frac{1}{8}$
		$\{1,2,3\}$	0	$\frac{1}{8}$	$\frac{1}{8}$

TABLE 1. Three systems of balancing multiweights.

Both proofs rely directly on the Bondareva–Shapley theorem (Theorem 1 in the paper). Theorem 2 in the paper and Proposition S1 show that if there is no solution in the multicore, the sum of the cores of the individual issues is empty as well, since $\Delta \subseteq \Delta_{\text{UC}}$. Theorem 2 in the paper and Proposition S2 show that if the core of the sum of individual issues is empty, so is the multicore, since $\Delta_{\text{CA}} \subseteq \Delta$. These results are also established by Theorem 3 in the paper. The advantage of Propositions S1 and S2 is that they help identify the systems of balancing multiweights that violate the conditions above when either $\sum_{V_j \in \bar{V}} C(V_j) = \varnothing$ and $M(\bar{V}) \neq \varnothing$, or $M(\bar{V}) = \varnothing$ and $C(\sum_{V_j \in \bar{V}} V_j) \neq \varnothing$.

S4. Proof of Proposition S1

Proof. First, suppose $\sum_{V_j \in \bar{V}} C(V_j) \neq \varnothing$. For every system of unconstrained balancing multiweights, $\tilde{\delta} \in \Delta_{\mathrm{UC}}$, let us define $\delta_j(S) = \sum_{i=1}^{n} \tilde{\delta}(S, i, V_j)$. For every issue V_j, $\delta_j(S)$ is a system of balancing weights since by "Resource Exhaustion," $\sum_{S \in 2^N} \delta_j(S)\chi^S = \chi^N$.

Suppose there exists $\tilde{\delta}(S, i, V_j)$, such that

$$\sum_{V_j \in \bar{V}} V_j(N) < \sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) V_j(S).$$

Then there exists $V_j \in \bar{V}$ such that

$$V_j(N) < \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) V_j(S)$$

or

$$V_j(N) < \sum_{S \in 2^N} \delta_j(S) V_j(S).$$

By the Bondareva–Shapley theorem, $C(V_j) = \varnothing$ and, therefore, $\sum_{V_j \in \bar{V}} C(V_j) = \varnothing$. Thus, every $\tilde{\delta} \in \Delta_{\mathrm{UC}}$ satisfies

$$\sum_{V_j \in \bar{V}} V_j(N) \geq \sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) V_j(S).$$

For the other direction, suppose that every $\tilde{\delta} \in \Delta_{\mathrm{UC}}$ satisfies

$$\sum_{V_j \in \bar{V}} V_j(N) \geq \sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) V_j(S).$$

For every $V_j \in \bar{V}$ and for every system of balancing weights $\delta(S)$, define $\tilde{\delta}(S, i, V_l)$ as follows:

1. If $V_l \neq V_j$ and $S \neq N$, then for every $i \in N$, $\tilde{\delta}(S, i, V_l) = 0$.
2. If $V_l \neq V_j$ and $S = N$, then for every $i \in N$, $\tilde{\delta}(N, i, V_l) = 1/n$.
3. If $V_l = V_j$, then $\tilde{\delta}(S, i, V_j) = \delta(S)/|S|$ if $i \in S$ and 0 otherwise.

Note that $\tilde{\delta}$ satisfies the "Zero to Nonmembers" condition. Also, for $V_l \neq V_j$,

$$\sum_{i \in N} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_l)\chi^S = \sum_{i \in N} \tilde{\delta}(N, i, V_l)\chi^N = \sum_{i \in N} \frac{1}{n}\chi^N = \chi^N,$$

and for $V_l = V_j$,

$$\sum_{i \in N} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j)\chi^S = \sum_{S \in 2^N} \sum_{i \in S} \frac{\delta(S)}{|S|}\chi^S = \sum_{S \in 2^N} \delta(S)\chi^S = \chi^N.$$

Therefore, $\tilde{\delta}(S,i,V_l)$ also satisfies the "Resources Exhaustion" condition and, therefore, it is a system of unconstrained balancing multiweights.

Suppose, there exists an issue $V_j \in \bar{V}$ such that $C(V_j) = \varnothing$. Then, by the Bondareva–Shapley theorem, there exists a system of balancing weights, $\delta_j(S)$, such that $V_j(N) < \sum_{S\in 2^N} \delta_j(S)V_j(S)$. Consider the corresponding $\tilde{\delta}$:

$$
\begin{aligned}
&\sum_{V_l\in\bar{V}}\sum_{i=1}^{n}\sum_{S\in 2^N}\tilde{\delta}(S,i,V_l)V_l(S)\\
&\qquad = \sum_{V_l\in\bar{V}\setminus\{V_j\}}\sum_{i=1}^{n}\sum_{S\in 2^N}\tilde{\delta}(S,i,V_l)V_l(S) + \sum_{i=1}^{n}\sum_{S\in 2^N}\tilde{\delta}(S,i,V_j)V_j(S)\\
&\qquad = \sum_{V_l\in\bar{V}\setminus\{V_j\}}\sum_{i=1}^{n}\frac{1}{n}V_l(N) + \sum_{i=1}^{n}\sum_{S\in 2^N}\tilde{\delta}(S,i,V_j)V_j(S)\\
&\qquad = \sum_{V_l\in\bar{V}\setminus\{V_j\}} V_l(N) + \sum_{i=1}^{n}\sum_{S\in 2^N}\tilde{\delta}(S,i,V_j)V_j(S)\\
&\qquad = \sum_{V_l\in\bar{V}\setminus\{V_j\}} V_l(N) + \sum_{S\in 2^N}\sum_{i\in S}\frac{\delta(S)}{|S|}V_j(S)\\
&\qquad = \sum_{V_l\in\bar{V}\setminus\{V_j\}} V_l(N) + \sum_{S\in 2^N}\delta(S)V_j(S) > \sum_{V_l\in\bar{V}\setminus\{V_j\}} V_l(N) + V_j(N) = \sum_{V_l\in\bar{V}} V_l(N).
\end{aligned}
$$

Therefore, it must be that $\forall V_j \in \bar{V} : C(V_j) \neq \varnothing$ and,therefore, $\sum_{V_j\in\bar{V}} C(V_j) \neq \varnothing$. □

S5. Proof of Proposition S2

Proof. Suppose $C(\sum_{V_j\in\bar{V}} V_j) \neq \varnothing$. Assume by negation that there exists $\tilde{\delta} \in \Delta_{\mathrm{CA}}$ such that

$$\sum_{V_j\in\bar{V}} V_j(N) < \sum_{V_j\in V}\sum_{i=1}^{n}\sum_{S\in 2^N}\tilde{\delta}(S,i,V_j)V_j(S)$$

or

$$\sum_{V_j\in\bar{V}} V_j(N) < \sum_{S\in 2^N}\sum_{i=1}^{n}\sum_{V_j\in V}\tilde{\delta}(S,i,V_j)V_j(S).$$

Since $\tilde{\delta}$ is a system of balancing multiweights with constant allocation, for every agent i, coalition S, and two issues V_j and $V_{j'}$,

$$\tilde{\delta}(S,i,V_j) = \tilde{\delta}(S,i,V_j') \equiv \tilde{\delta}(S,i)$$

and, therefore,

$$\sum_{V_j \in \bar{V}} V_j(N) < \sum_{S \in 2^N} \sum_{i=1}^{n} \tilde{\delta}(S,i) \sum_{V_j \in \bar{V}} V_j(S).$$

Define $\delta(S) = \sum_{i=1}^{n} \tilde{\delta}(S,i)$. Due to the "Resource Exhaustion" property of $\tilde{\delta}$, $\delta(S)$ is a system of balancing weights

$$\sum_{S \in 2^N} \delta(S)\chi^S = \sum_{S \in 2^N} \left[\sum_{i=1}^{n} \tilde{\delta}(S,i)\right]\chi^S = \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S,i)\chi^S = \chi^N.$$

Therefore, the inequality above becomes,

$$\sum_{V_j \in \bar{V}} V_j(N) < \sum_{S \in 2^N} \delta(S) \sum_{V_j \in \bar{V}} V_j(S),$$

which by the Bondareva–Shapley theorem implies that $C(\sum_{V_j \in \bar{V}} V_j) = \varnothing$, which is a contradiction. Thus, if $C(\sum_{V_j \in \bar{V}} V_j) \neq \varnothing$, then every $\tilde{\delta} \in \Delta_{\mathrm{CA}}$ satisfies

$$\sum_{V_j \in \bar{V}} V_j(N) \geq \sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S,i,V_j)V_j(S).$$

For the other direction, suppose $C(\sum_{V_j \in \bar{V}} V_j) = \varnothing$. Then, by the Bondareva–Shapley theorem, there exists a system of balancing weights, $\delta(S)$, whereby $\sum_{S \in 2^N} \delta(S)\chi^S = \chi^N$ such that

$$\sum_{V_j \in \bar{V}} V_j(N) < \sum_{S \in 2^N} \delta(S) \sum_{V_j \in \bar{V}} V_j(S).$$

Define $\tilde{\delta}(S,i,V_j) = \delta(S)/|S|$ if $i \in S$ and $\tilde{\delta}(S,i,V_j) = 0$ otherwise. Obviously, $\tilde{\delta}$ satisfies the "Zero to Nonmembers" condition. Also, for every $V_j \in \bar{V}$,

$$\sum_{i \in N} \sum_{S \in 2^N} \tilde{\delta}(S,i,V_j)\chi^S = \sum_{S \in 2^N} \sum_{i \in S} \frac{\delta(S)}{|S|}\chi^S = \sum_{S \in 2^N} \delta(S)\chi^S = \chi^N.$$

Therefore, $\tilde{\delta}$ also satisfies the "Resources Exhaustion" condition. In addition, $\tilde{\delta}$ does not depend on any specific issue and, thus, it is a system of balancing multiweights with constant allocations:

$$\begin{aligned}\sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S,i,V_j)V_j(S) &= \sum_{S \in 2^N} \sum_{V_j \in \bar{V}} \sum_{i \in S} \frac{\delta(S)}{|S|} V_j(S) \\ &= \sum_{S \in 2^N} \delta(S) \sum_{V_j \in \bar{V}} V_j(S) > \sum_{V_j \in \bar{V}} V_j(N).\end{aligned}$$

Thus, if $C(\sum_{V_j \in \bar{V}} V_j) = \varnothing$ there exists $\tilde{\delta} \in \Delta_{\text{CA}}$ such that

$$\sum_{V_j \in \bar{V}} V_j(N) < \sum_{V_j \in \bar{V}} \sum_{i=1}^{n} \sum_{S \in 2^N} \tilde{\delta}(S, i, V_j) V_j(S). \qquad \square$$

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