Theoretical Economics 1 (2006), 143–166 1555-7561/20060143

Competitive markets with externalities

MITSUNORI NOGUCHI
Department of Economics, Meijo University

WILLIAM R. ZAME
Department of Economics, University of California, Los Angeles

This paper presents a general model of a competitive market with consumption externalities, and establishes the existence of equilibrium in the model, under assumptions comparable to those in classical models. The model allows production and indivisible goods. Examples illustrate the generality and applicability of the results.

KEYWORDS. Competitive equilibrium, externalities, distributional economies.

JEL CLASSIFICATION. D5.

1. INTRODUCTION

The classical model of competitive markets assumes that agents care only about their own consumption. However, the consumption of others may matter to me, because I am altruistic or spiteful, because I am a slave to fashion or a radical non-conformist—or because the consumption of others impinges on my own: my neighbors' use of coal to heat their houses pollutes the air I breathe; their use of household security services makes mine safer. For environments with a finite number of agents, it has been known for a long time that consumption externalities can be accommodated in a consistent market model. (See Shafer and Sonnenschein 1975 for the first proof and Laffont 1977 for discussion and applications.) However such models are not entirely satisfactory because they require individual preferences to be convex and exclude indivisible goods. (As Starrett 1972 argues, requiring individual preferences to be convex is especially problematical in the presence of consumption externalities.) Moreover, the price-taking assumptions inherent in the notion of competitive equilibrium are incompatible with the

Mitsunori Noguchi: noguchi@ccmfs.meijo-u.ac.jp
William R. Zame: zame@econ.ucla.edu

Earlier versions circulated under the title "Equilibrium Distributions with Externalities." We thank Robert Anderson for several enlightening conversations, and the Co-Editor Ed Green and several anonymous referees for very useful suggestions. Zame gratefully acknowledges financial support from the John Simon Guggenheim Foundation, the National Science Foundation (under grants SES-0079299 and SES-0317752), the UCLA Academic Senate Committee on Research, and the Caltech Social and Information Sciences Laboratory, and Caltech for sabbatical hospitality. Views expressed here are those of the authors and do not necessary reflect the views of any funding agency.

presence of agents who have market power—as all agents typically do when the total number of agents is finite.

Absent externalities, the work of Aumann (1964, 1966) and subsequent authors has shown that non-convex preferences, indivisible goods, and infinitesimal agents who have no market power are easily accommodated in a model with a continuum of agents. However, a satisfactory model that also incorporates consumption externalities has proved surprisingly elusive. This paper provides such a model.

The key to our approach is that we abandon Aumann's descriptions of the economy and of equilibrium in terms of maps from agent names to agent characteristics and consumptions, and instead use the descriptions (introduced by Hart et al. 1974) of the economy and of equilibrium in terms of probability distributions on agent characteristics and on agent characteristics and choices. In other words, our description is *distributional* rather than *individualistic*. This choice is born of necessity: as we show by examples, if we were to insist on an individualistic description of equilibrium then we would quickly be confronted with simple economies that admit no equilibrium at all. By contrast, our distributional approach, although perhaps less familiar than the individualistic approach, leads to a clean and elegant framework, easily accommodates both production and indivisible goods, and satisfies the basic consistency requirement that equilibrium exist.

A simple example, elaborated in Section 2, may help to orient the reader. Consider a large number of individuals who live on the bank of a long river. Each individual trades and consumes goods, but the waste products of that consumption create harmful pollution for everyone downstream. As we show, if we wish to guarantee that such an economy admits an equilibrium we must accept the possibility that two individuals who live in the same location, have the same preferences, are entitled to the same endowment, and experience the same externality, still choose *different* consumption bundles.

Previous papers on continuum economies with externalities all take Aumann's individualistic approach, and treat only exchange economies with divisible goods. Hammond et al. (1989) show that equilibrium allocations coincide with the *f-core* (a variation of the usual core in which improving coalitions are required to be finite). However, as our examples show, in their model, equilibrium may not exist and the f-core may be empty. Noguchi (2005) and Cornet and Topuzu (2005) prove that equilibrium exists if individual preferences are convex in own consumption and weakly continuous in others' consumptions. As we have noted, requiring that individual preferences be convex vitiates an important reason for treating continuum economies, and certainly excludes indivisibilities. As we discuss later, in the individualistic framework the requirement of weak continuity restricts the externalities agents experience to be linear in the consumption of others. Our examples in Section 5 show that, absent any of these assumptions, individualistic equilibrium need not exist. Balder (2003) demonstrates that equilibrium exists if the externality enters into the preferences of each individual in the same way. This requirement would seem to exclude all local externalities, all externalities that diminish with distance, and all externalities that have any directional aspect—indeed, most externalities that arise in practice. By contrast, our results allow for indivisible goods, for individual preferences that are not convex, and for general externalities that each agent experiences in a unique way.

Following this Introduction, we begin with an informal elaboration of the orienting example mentioned above. The detailed description of the model is in Section 3 and the existence theorem is in Section 4. Section 5 presents a number of illustrative examples. Proofs are collected in Appendix A. Appendix B sketches an alternative model, in which the consumption of others enters individual preferences as a parameter.

2. An orienting example

To orient the reader, we begin by describing, in the familiar individualistic framework, a simple economy that has no individualistic equilibrium. Our description is a bit informal.

Example 1. We consider a large number of agents living on the banks of a long river. We identify locations with the interval $T=[0,1]$, and assume the population is uniformly distributed along the river, so the population measure τ is Lebesgue measure.

There are two commodities, each perfectly divisible. Each agent is endowed with one unit of each good: $e(t)\equiv(1,1)$. Agents derive utility from their own consumption, but suffer a pollution externality from the consumption of others who live *upstream* from them. If we choose directions so that upstream from s means to the left of s, and write $f(t)=(f_1(t),f_2(t))$ for the consumption of an agent located at t, then the externality experienced by an agent located at s is

$$\eta(s,f)=\int_0^s f(t)\,dt.$$

Note that η is two-dimensional; write $\eta=(\eta_1,\eta_2)$ for the components of η. The utility of an agent located at s who consumes the bundle (x_1,x_2) when the consumption of all agents is described by f is

$$u_s(x_1,x_2,f)=[2-\eta_1(s,f)]x_1^2+[2-\eta_2(s,f)]x_2^2.$$

Note that $\eta_1(s,f)\leq 1$ for each i and that the exponents of x_1,x_2 are greater than 1, so utility is strictly increasing and *convex* in own consumption. In particular, individual preferences are *not* convex.

In the individualistic framework, an equilibrium consists of prices p_1,p_2 (without loss, normalize so that $p_1+p_2=1$) and a consumption allocation $f=(f_1,f_2):T\to\mathbb{R}^2_+$ so that almost every agent optimizes in his/her budget set and the market clears. We claim that no such equilibrium exists.

To see this, note first that each agent's wealth is $p_1\cdot 1+p_2\cdot 1=1$. Because the total supply of each good is 1, $\eta_i\leq 1$ for each i, so utility functions display increasing returns to scale in consumption of each good, and the optimal choice for an agent located at s is always either $(1/p_1,0)$ or $(0,1/p_2)$. (Of course the agent may be indifferent between the two choices.) An agent located at s strictly prefers $(1/p_1, 0)$ if and only if

$$\frac{2-\eta_1(s,f)}{p_1} > \frac{2-\eta_2(s,f)}{p_2}.$$

Similarly, an agent located at s strictly prefers $(0, 1/p_2)$ if and only if

$$\frac{2-\eta_1(s,f)}{p_1} < \frac{2-\eta_2(s,f)}{p_2}.$$

Set

$$\begin{aligned}
G_1 &= \left\{ s \in [0,1] : \frac{2-\eta_1(s,f)}{p_1} > \frac{2-\eta_2(s,f)}{p_2} \right\} \\
G_2 &= \left\{ s \in [0,1] : \frac{2-\eta_1(s,f)}{p_1} < \frac{2-\eta_2(s,f)}{p_2} \right\} \\
I &= \left\{ s \in [0,1] : \frac{2-\eta_1(s,f)}{p_1} = \frac{2-\eta_2(s,f)}{p_2} \right\}.
\end{aligned}$$

These sets are disjoint and their union is $[0,1]$. Because Lebesgue measure τ is non-atomic, both functions η_1, η_2 are continuous, so both sets G_1, G_2 are open (possibly empty).

If $G_1 \neq \emptyset$ then G_1 is the union of a countable collection of disjoint intervals, say $G_1 = \bigcup J_j$. Let $J = J_j$ be any one of these intervals, and let a be the left-hand endpoint of J. If $a \notin J$, then continuity guarantees that

$$\frac{2-\eta_1(s,f)}{p_1} = \frac{2-\eta_2(s,f)}{p_2}.$$

At each $s \in G_1$ the unique optimal choice of an agent located at s is to choose $(1/p_1, 0)$, so $2-\eta_1(s,f)$ is strictly decreasing on J and $2-\eta_2(s,f)$ is constant on J. Since these functions are equal at the left-hand endpoint of J we must have

$$\frac{2-\eta_1(s,f)}{p_1} < \frac{2-\eta_2(s,f)}{p_2}$$

at every point of J. Since this contradicts the definition of J, we conclude that $a \in J$. Since G_1 is open, this can only be the case if $a = 0$. Since J is arbitrary, we conclude that either $G_1 = \emptyset$ or G_1 itself is an interval and $0 \in G_1$.

We can apply the same reasoning to G_2 to conclude that either $G_2 = \emptyset$ or G_2 itself is an interval and $0 \in G_1$.

Because G_1, G_2 are disjoint, it follows that at least one of them must be empty; without loss of generality, say $G_2 = \emptyset$. If $G_1 \neq \emptyset$, let b be the right-hand endpoint of G_1. As we have noted above, at each $s \in G_1$ the unique optimal choice of an agent located at s is to choose $(1/p_1, 0)$. Hence if $b = 1$ then agents located at every point of $[0,1)$ choose only good 1, and the market for good 2 cannot clear. Hence $b < 1$. Because $G_1 \cup G_2 \cup I = [0,1]$, we conclude that $I = [b,1]$. Of course, if $G_1 = G_2 = \emptyset$ then $I = [0,1]$, so in either case we conclude that $I = [b,1]$ for some $b < 1$.

In particular, $1 \in I$ so

$$\frac{2-\eta_1(1,f)}{p_1} = \frac{2-\eta_2(1,f)}{p_2}.$$

Market clearing implies $\eta_1(1,f) = \eta_2(2,f) = 1$ so $p_1 = p_2 = \frac{1}{2}$. Since $\eta_1(0,f) = \eta_2(0,f) = 0$, it follows that

$$\frac{2-\eta_1(0,f)}{p_1} = \frac{2-\eta_2(0,f)}{p_2}.$$

Hence $0 \in I$ so $G_1 = \emptyset$ and $I = [0,1]$. Keeping in mind that $p_1 = p_2$ we conclude that $\eta_1(s,f) = \eta_2(s,f)$ for all $s \in [0,1]$ and hence that

$$0 = \eta_1(s,f) - \eta_2(s,f) = \int_0^s [f_1(t) - f_2(t)]\, dt$$

for every $s \in [0,1]$. Hence $f_1(t) = f_2(t)$ for almost all $t \in [0,1]$. However, as we have already noted, all agents choose either good 1 or good 2, so this is impossible. We have reached a contradiction, so we conclude that no individualistic equilibrium exists. However, we shall see in Section 5 that there *is* a distributional equilibrium. $\Diamond$

Non-convexity of preferences plays a crucial role in the above example. However, as we have argued in the Introduction, requiring preferences to be convex is problematical in a continuum economy, especially when there are externalities. Moreover, as Example 3 in Section 5 demonstrates, when the externality is non-linear in the consumption of others, even the requirement of convexity of preferences is not adequate to guarantee an individualistic equilibrium.

3. Economies with externalities

We consider economies with $L \geq 1$ divisible goods and $M \geq 0$ indivisible goods; indivisible goods are available only in integer quantities. (We allow $M = 0$—no indivisible goods—but we insist that $L \geq 1$, so there is at least one divisible good.) The commodity space and price space are both $\mathbb{R}^L \times \mathbb{R}^M = \mathbb{R}^{L+M}$. (Indivisibility enters into the description of consumption sets of individual agents but not into the description of the commodity or price space.) It is convenient to normalize prices to sum to 1; write

$$\overline{\Delta} = \left\{ p \in \mathbb{R}^{L+M} : \sum p_i = 1, p \geq 0 \right\}$$
$$\Delta = \left\{ p \in \mathbb{R}^{L+M} : \sum p_i = 1, p \gg 0 \right\}$$

for the simplex of normalized, positive prices and the simplex of normalized, strictly positive prices, respectively.

We follow McKenzie (1959) in describing the *production sector* as a closed convex cone $Y \subset \mathbb{R}^{L+M}$. As usual, we assume $Y \cap (-Y) = \{0\}$ (irreversibility), $-\mathbb{R}_+^{L+M} \subset Y$ (free disposal in production), and $Y \cap \mathbb{R}^{L+M} = \{0\}$ (no free production). Note that $Y = -\mathbb{R}_+^{L+M}$ for an exchange economy with no production.

For simplicity, we assume that individual consumption is constrained only to be non-negative and to respect the indivisibility requirement for the last M goods. If we write $\mathbb{Z}$ for the space of integers and $\mathbb{Z}_+$ for the space of non-negative integers, then individual consumption sets are $X = \mathbb{R}^L_+ \times \mathbb{Z}^M_+$.

We allow agent preferences to depend on the consumption of others. Because we describe the economy in distributional terms, the most obvious way to describe the consumption of others is as a distribution on the space of consumer characteristics and consumptions. However, if consumer characteristics include preferences, this approach is circular. To avoid this circularity, we follow the approach suggested by Mas-Colell (1984): we take as given an abstract space of observable characteristics and describe the consumption of others by a distribution on the product of the space of observable characteristics with the space of consumptions. Formally, we take as given a complete separable metric space T and a probability measure τ on T. We view T as the space of *observable characteristics* of agents and τ as the *distribution of observable characteristics* in the actual economy; we refer to (T,τ) as the *observable population characteristics*.

Because we assume consumptions are non-negative, we summarize the observable consumption of society as a distribution (probability measure) on $T \times X$; $\text{Prob}(T \times X)$ is the space of all such distributions. It is conceivable that agents care about *all* possible distributions of consumptions of others, but it is necessary for our purposes only that agents care about those distributions that involve a finite amount of total resources, shared among the actual population. To identify the relevant distributions, we say that $\sigma \in \text{Prob}(T \times X)$ is *integrable* if $\int |x|\, d\sigma < \infty$. Write $\mathscr{D}$ for the set of integrable distributions, $\mathscr{D}(\tau)$ for the subset of integrable distributions σ for which the marginal of σ on T is τ, and

$$\mathscr{D}^n(\tau) = \left\{ \sigma \in \mathscr{D}(\tau) : \int |x|\, d\sigma \le n \right\}.$$

Each $\mathscr{D}^n(\tau)$ is a weakly compact subset of $\text{Prob}(T \times X)$. Note that $\mathscr{D}(\tau) = \bigcup \mathscr{D}^n(\tau)$. For a given economy, we need to consider only distributions in some fixed $\mathscr{D}^n(\tau)$, so we give $\mathscr{D}(\tau) = \bigcup \mathscr{D}^n(\tau)$ the *direct limit topology*: $F \subset \mathscr{D}(\tau)$ is closed if and only if $F \cap \mathscr{D}^n(\tau)$ is closed for each n.

The description of preferences over pairs of own consumption and consumption of others is straightforward and familiar. A *preference relation with externalities*, or just a *preference relation*, is a subset $\rho \subset X \times \mathscr{D}(\tau) \times X \times \mathscr{D}(\tau)$. We usually write $(x,\sigma)\, \rho\, (x',\sigma')$ rather than $(x,\sigma,x',\sigma') \in \rho$. We restrict attention throughout to preference relations ρ that are irreflexive, transitive, negatively transitive, strictly monotone in own consumption (i.e., $(x,\sigma)\, \rho\, (x',\sigma)$ whenever $x > x'$) and continuous (i.e., open). Write $\mathscr{P}$ for the space of such preference relations. If ρ is a preference relation we write

$$\mathscr{B}(\rho) = \{(x,\sigma,x',\sigma') \in X \times \mathscr{D}(\tau) \times X \times \mathscr{D}(\tau) : (x,\sigma)\, \rho\, (x',\sigma')\}$$

for the *better-than set* and $\mathscr{B}(\rho)^c$ for its complement, the *not-better-than set*. We topologize $\mathscr{P}$ by closed convergence of not-better-than sets restricted to compact sets of consumption distributions. That is, $\rho_\alpha \to \rho$ exactly if for each compact $K \subset \mathscr{D}(\tau)$ we have

$$\mathscr{B}(\rho_\alpha)^c \cap K \to \mathscr{B}(\rho)^c \cap K$$

in the topology of closed convergence of subsets of $X \times K \times X \times K$. (Since the latter space is locally compact, closed convergence has its usual meaning.) The proof of the following useful proposition is in Appendix A.

PROPOSITION 1. *$\mathscr{P}$ is a complete, separable metric space.*

Agents are described by an observable characteristic, a preference relation, and an endowment, so the space of agent characteristics is

$$\mathscr{C} = T \times \mathscr{P} \times X.$$

In view of Proposition 1, $\mathscr{C}$ is a complete, separable metric space.

An *economy with preference externalities* or just an *economy* for short, is a tuple $\mathscr{E} = \langle T, \tau, Y, \lambda \rangle$ consisting of population characteristics (T, τ), a production sector Y, and a probability measure λ on $T \times \mathscr{P} \times X$ whose marginal on T is τ and that has the property that aggregate endowment is finite: $\int |e| d\lambda < \infty$. There is no loss in assuming that $e \neq 0$ for λ-almost all (t, ρ, e), because agents with 0 endowment necessarily consume 0 at equilibrium.

In the presence of indivisible goods, we need an assumption to guarantee that individual demand is well-behaved in prices. Say *endowments are desirable* if λ-almost all $(t, \rho, e) \in \mathscr{C}$ have the property that $(e, \sigma)\, \rho\, ((0, x_M), \sigma)$ for every $\sigma \in \mathscr{D}(\tau)$ and every $x_M \in \mathbb{Z}_+^M$. That is, for every fixed distribution of social consumption, the endowment is strictly preferred to any bundle that contains no divisible goods. Note that this property is automatically satisfied if all goods are divisible and almost all agents have non-zero endowment.

An *equilibrium* for the economy $\mathscr{E} = \langle T, \tau, Y, \lambda \rangle$ consists of a price $p \in \Delta$, an aggregate production vector $y \in Y$, and a probability measure μ on $T \times \mathscr{P} \times X \times X$ such that

(a) the marginal μ_{123} of μ on $T \times \mathscr{P} \times X$ equals λ

(b) almost all agents choose in their budget set

$$\mu\{(t, \rho, e, x) : p \cdot x > p \cdot e\} = 0$$

(c) production profit is maximized

$$p \cdot y = \sup\{p \cdot y' : y' \in Y\}$$

(d) markets clear

$$\int x \, d\mu = y + \int e \, d\lambda$$

(e) almost all agents optimize given prices p and the distribution of consumption μ_{14} (the marginal of μ on the first and fourth factors):

$$\mu\{(t, \rho, e, x) : \text{there exists } x' \in X, (x', \mu_{14})\, \rho\, (x, \mu_{14}),\ p \cdot x' \le p \cdot e\} = 0.$$

(Note that (a) and (d) together imply that the marginal $\mu_{14} \in \mathscr{D}(\tau)$.)

3.1 *Individualistic representations*

The descriptions above are entirely in distributional terms, but a description in individualistic terms is sometimes possible. We say that the economy $\mathscr{E} = \langle T, \tau, Y, \lambda \rangle$ *admits an individualistic representation* if there is a measurable function $\varphi : T \to \mathscr{P} \times X$ such that

$$\lambda = (\mathrm{id}_T, \varphi)_*(\tau)$$

where id_T is the identity map on T and $(\mathrm{id}_T, \varphi)_*(\tau)$ is the direct image measure. If $\mathscr{E}$ admits an individualistic representation then the equilibrium p, y, μ *admits an individualistic representation* if in addition there is a measurable function $f : T \to X$ such that

$$\mu = (\mathrm{id}_T, \varphi, f)_*(\tau).$$

Informally—but entirely correctly—$\mathscr{E}$ admits an individualistic representation exactly when almost all agents having the same observable characteristic have the same endowment and preferences, and an equilibrium p, y, μ admits an individualistic representation if and only if all agents having the same observable characteristic choose the same consumption bundle.

As we show in Section 5, there are economies that admit an individualistic representation for which no equilibrium admits an individualistic representation.

3.2 *Economies with finite support*

We should offer a *caveat* about interpretation. An economy $\mathscr{E} = \langle T, \tau, Y, \lambda \rangle$ for which τ and λ have finite support should *not* be interpreted as an economy with a finite number of agents, but rather as an economy with a finite number of *types* of agents. Theorems 1 and 2 of Section 4 are able to guarantee existence of equilibria for such economies only because our notion of equilibrium does not require agents having the same characteristics to choose the same consumption bundles.

4. EXISTENCE OF EQUILIBRIUM

For an economy $\mathscr{E} = \langle T, \tau, Y, \lambda \rangle$, we say that *all goods are available in the aggregate* if

$$\left[Y + \int e \, d\lambda \right] \cap \mathbb{R}^{L+M}_{++} \neq \emptyset.$$

That is, all goods are either represented in the endowment or can be produced.

Our main result guarantees existence of equilibrium.

THEOREM 1. *Every economy $\mathscr{E} = \langle T, \tau, Y, \lambda \rangle$ for which endowments are desirable and all goods are available in the aggregate admits an equilibrium.*

As Example 3 shows, convexity of preferences is not enough to guarantee the existence of an equilibrium with an individualistic representation, but strict convexity will do.

THEOREM 2. *If $\mathscr{E} = \langle T, \tau, Y, \lambda\rangle$ is an economy that admits an individualistic representation, $M = 0$ (so there are no indivisible goods), and λ-almost all agents have preferences that are strictly convex in own consumption, then every equilibrium for $\mathscr{E}$ admits an individualistic representation.*

5. EXAMPLES

In this section, we give a number of examples to illustrate the power and usefulness of our approach and to contrast our results with the results of Noguchi (2005), Cornet and Topuzu (2005), and Balder (2003). For ease of exposition, we do not stray far from the setting we have already discussed; the particular functional forms are chosen for ease of calculation, rather than for economic realism.

We begin by completing the discussion of Example 1.

EXAMPLE 1 CONTINUED. We begin by giving a distributional description of the economy. As before, all agents are endowed with one unit of each good: $e = (1, 1)$. If $\sigma \in \mathscr{D}(\tau)$ is the distribution of consumption, an agent located at $s \in [0, 1]$ who consumes (x_1, x_2) experiences the externality

$$\eta(s,\sigma) = \int_{T\times\mathbb{R}^2_+} x\, d\sigma$$

and enjoys utility

$$u_s(x_2, x_2, \sigma) = [2 - \eta_1(s,\sigma)]x_1^2 + [2 - \eta_2(s,\sigma)]x_2^2.$$

Hence λ is the image of τ under the map $s \mapsto (s, u_s, e)$.

The argument of Section 2 shows that, at equilibrium, we must have $p_1 = p_2 = \frac{1}{2}$ and $\eta_1(s,\sigma) = \eta_2(s,\sigma)$ for all $s \in [0, 1]$. This condition characterizes a unique equilibrium distribution; to describe it, define maps g^1, $g^2 : T \to T \times \mathscr{P} \times \mathbb{R}^2_+ \times \mathbb{R}^2_+$ by

$$g^1(s) = (s, u_s, (1, 1), (2, 0))$$
$$g^2(s) = (s, u_s, (1, 1), (0, 2)).$$

The unique equilibrium distribution is

$$\mu = \tfrac{1}{2} g^1_*(\tau) + \tfrac{1}{2} g^2_*(\tau).$$

The interpretation is simple: in every measurable set of locations, half of the agents choose $(2, 0)$ and half of the agents choose $(0, 2)$. Informally: half of the agents at each location choose $(2, 0)$ and half of the agents choose $(0, 2)$. Even more informally: "every other agent" chooses $(2, 0)$ and "every other agent" chooses $(0, 2)$. There is no measurable function with this property; there is, as Mas-Colell (1984) writes, "a measurability problem."[1] $\lozenge$

This example relies on non-convex preferences; as the following example shows, problems for individualistic representations can arise also from indivisible goods.

[1]The same point is made in Gretsky et al. (1992), and may also be familiar from the literature on the law of large numbers with a continuum of random variables (Feldman and Gilles 1985, Judd 1985).

EXAMPLE 2. Again, we consider a large number of agents living along a river. We continue to identify locations (the observable characteristic) with the interval $T=[0,1]$ and assume the population is uniformly distributed along the river, so τ is Lebesgue measure. There are two goods: good 1 is divisible and good 2 (speedboats) is indivisible, so individual consumption sets are $X=\mathbb{R}_+\times\mathbb{Z}_+$; write $x=(x_1,x_2)$ for a typical consumption bundle.

Speedboats are produced by a constant returns to scale technology: 1 unit of the consumption good produces 1 speedboat, so the production cone is

$$Y=\{(y_1,y_2): y_2\le -y_1\}.$$

Agents are each endowed with 2 units of the consumption good but no speedboats.

Agents care about the consumption good and about speedboats, but utility from speedboats is subject to a congestion externality. If $\sigma\in\mathscr{D}(\tau)$ is the distribution of social consumption, the congestion externality experienced by an agent located at $s\in(0,1]$ is

$$\eta(s,\sigma)=\frac{1}{s}\int_{T\times X}\mathbf{1}_{[0,s]}(t)\,x_2\,d\sigma(t,x_1,x_2).$$

(We leave $\eta(0,\sigma)$ undefined.) Note that, so long as all speedboat consumptions are bounded by 2 (which they are at any individually rational consumption distribution) the externality experienced by s is just the average of speedboat consumption upstream from s. The utility obtained by an agent located at s who consumes x_1 units of the divisible good and x_2 speedboats is

$$u_s(x_1,x_2;\sigma)=(x_1)^{1/2}\left(1+2[x_2-\eta(s,\sigma)]^+\right)^{1/2}.$$

Note that the externality matters only when speedboat consumption is strictly positive.[2]

To solve for equilibrium p,y,μ normalize so that $p_1=1$. Profit maximization entails that $p_2\le 1$ (else potential profit would be infinite). If no speedboats are produced then the equilibrium must be autarkic: every agent consumes his/her endowment $(2,0)$ and obtains utility $2^{1/2}$. However, if no speedboats are produced then no agent experiences a congestion externality, and each agent prefers to consume $(1,1)$ and obtain utility $3^{1/2}>2^{1/2}$. Hence speedboats must be produced at equilibrium, and now profit maximization entails that in fact $p_2=1$.

To solve for μ, write $\sigma=\mu_{14}$ for the marginal of μ on observable characteristics and speedboat consumption. Because τ is non-atomic, $\eta(t,\sigma)$ is a continuous function of t; we claim that $\eta(t,\sigma)\equiv\frac{1}{2}$ for every $t\in(0,1]$. To see this, consider the open set $\{s: E(s,\sigma)<\frac{1}{2}\}$. If this is not empty it is the countable union of disjoint intervals; let I be any one of these intervals, and let a be the left-hand endpoint of I. Check that if $\eta(s,\sigma)<\frac{1}{2}$ then the unique optimal choice for an agent located at s is to consume $(1,1)$. Thus, if $a=0$ then $\eta(s,\sigma)\equiv 1$ for every $s\in I$, which certainly contradicts the definition of I. Hence $a\neq 0$, and $a\notin I$ (because I is open). Continuity therefore guarantees that

[2] This utility function is not strictly increasing in x_2, but it is so in the relevant range.

$\eta(a,\sigma)=\frac{1}{2}$, so that $\eta(s,\sigma)$ is a strictly increasing function of I; in particular, $\eta(s,\sigma)>\frac{1}{2}$ for each $s\in I$, which again contradicts the definition of I. Since we have reached a contradiction whether or not $a=0$, we conclude that $\{s: E(s,\sigma)<\frac{1}{2}\}$ is empty. Similarly, we see that $\{s: E(s,\sigma)>\frac{1}{2}\}$ is empty, so $E(s,\sigma)\equiv\frac{1}{2}$ for every $s\in(0,1]$.

It follows immediately that equilibrium production is $y=(-\frac{1}{2},+\frac{1}{2})$.

If equilibrium had an individualistic representation then there would be a function $f=(f_1,f_2): T\to\mathbb{R}_+\times\mathbb{Z}_+$ for which

$$\frac{1}{s}\int_0^s f_2(t)\,d\tau(t)=\tfrac{1}{2}$$

for each s—but no such integer-valued function exists.

To define the unique equilibrium consumption distribution, define $g^1, g^2: T\to T\times\mathscr{P}\times X\times X$ by

$$\begin{aligned}g^1(s)&=(s,u_s,(2,0),(2,0))\\ g^2(s)&=(s,u_s,(2,0),(1,1)).\end{aligned}$$

The unique equilibrium distribution is

$$\mu=\tfrac{1}{2}g^1_*(\tau)+\tfrac{1}{2}g^2_*(\tau).$$

That is: at every location half of the agents consume their endowments and half of the agents use one unit of consumption to produce one speedboat. ◊

Examples 1 and 2 provide useful comparisons with Balder (2003), Noguchi (2005) and Cornet and Topuzu (2005). Balder assumes that the externality enters into each agent's utility function in the same way. This would only be the case if pollution (in Example 1) and congestion (in Example 2) had the same negative effects upstream as downstream. More generally, this assumption would obtain only for externalities that are purely global—but few, if any, externalities would seem to have this property. (Even a negative externality as wide-spread as global warming affects different regions of the earth in different ways.)

Noguchi (2005) and Cornet and Topuzu (2005) assume that individual preferences are convex. As we have discussed, this assumption is problematic in a continuum economy, and especially so in the presence of externalities. As Examples 1 and 2 demonstrate, if preferences are non-convex or some goods are indivisible, an individualistic equilibrium may not exist—but our distributional framework handles non-convex preferences and indivisible goods smoothly.

A more subtle point concerns the nature of the externality. Noguchi (2005) and Cornet and Topuzu (2005) assume that the externality experienced by each agent is continuous with respect to the topology of weak convergence of consumption allocations. If, as in Examples 1 and 2, the externality experienced by an agent is of the form

$$\eta(f)=\int_T E(f(t))\,dt,$$

weak continuity in f obtains only if E is linear (more precisely, affine) on the relevant range. This is evidently a strong assumption. In our framework, the corresponding externality is

$$\eta(\sigma) = \int_{T\times X} E(t,x)\, d\sigma(t,x)$$

and our requirement that preferences be continuous with respect to the topology of weak convergence of distributions is satisfied if E is measurable, bounded (or just uniformly integrable on the relevant range of distributions in $\mathscr{D}(\tau)$), and continuous in x. As the following simple example demonstrates, non-linear externalities are a problem for the individualistic approach but are handled smoothly in our distributional approach.

EXAMPLE 3. Again, $T = [0,1]$ and τ is Lebesgue measure. There are two goods, both divisible. When the distribution of consumption is $\sigma \in \mathrm{Prob}(T \times \mathbb{R}^2_+)$, an agent located at $s \in T$ who chooses the bundle (x_1, x_2) experiences the externality

$$\eta(s,\sigma) = \int_{T\times\mathbb{R}^2_+} \mathbf{1}_{[0,s]}(t)(x_1x_2 + x_1 - x_2)\, d\sigma(t, x_1, x_2)$$

and enjoys the utility

$$u_s(x_1, x_2; \sigma) = x_1 + (1 + \eta(s,\sigma))x_2.$$

All agents are endowed with one unit of each good: $e = (1,1)$. There is no production: $Y = -\mathbb{R}^2_+$.

To solve for equilibrium p, μ we follow a by-now-familiar strategy. Consider the disjoint open sets

$$W_1 = \left\{ t \in T : \frac{1}{p_1} > \frac{1+\eta(t,\sigma)}{p_2} \right\}$$
$$W_2 = \left\{ t \in T : \frac{1}{p_1} < \frac{1+\eta(t,\sigma)}{p_2} \right\}.$$

If $s \in W_1$ then every agent located at s strictly prefers to consume only good 1, so $\eta(\cdot,\sigma)$ is strictly increasing on every subinterval of W_1. If $s \in W_2$ then every agent located at s strictly prefers to consume only good 2, so $\eta(\cdot,\sigma)$ is strictly increasing on every subinterval of W_2. Arguing just as in the previous two examples, this quickly leads to a contradiction unless $W_1 = W_2 = \emptyset$.

We conclude that

$$\frac{1}{p_1} = \frac{1+\eta(s,\sigma)}{p_2}$$

for every $s \in T$. In particular, $\eta(\cdot,\sigma)$ is constant. Since $\eta(0,\sigma) = 0$ it follows that $\eta(\cdot,\sigma) \equiv$

0. Since total endowments are $(1,1)$ we have

$$\begin{aligned}0=\eta(1,\sigma)&=\int_{T\times\mathbb{R}^2_+} 1(x_1x_2+x_1-x_2)\,d\sigma(t,x_1,x_2)\\&=\int_{T\times\mathbb{R}^2_+}(x_1x_2)\,d\sigma(t,x_1,x_2)+\int_{T\times\mathbb{R}^2_+}(x_1-x_2)\,d\sigma(t,x_1,x_2)\\&=\int_{T\times\mathbb{R}^2_+}(x_1x_2)\,d\sigma(t,x_1,x_2).\end{aligned}$$

Because consumption is non-negative, this implies $x_1x_2=0$ almost everywhere with respect to σ: no agent consumes both goods. Given that endowments are $(1,1)$ it follows that the unique equilibrium price is $p_1=p_2=\frac{1}{2}$ and the unique equilibrium distribution is

$$\mu=\tfrac{1}{2}g^1_*(\tau)+\tfrac{1}{2}g^2_*(\tau)$$

where $g^1,g^2:T\to T\times\mathscr{P}\times X\times X$ are defined by

$$\begin{aligned}g^1(s)&=(s,u_s,(1,1),(2,0))\\g^2(s)&=(s,u_s,(1,1),(0,2)).\end{aligned}$$

Informally: at each location, half the agents choose $(2,0)$ and half the agents choose $(0,2)$. As before, this equilibrium does not admit an individualistic representation. ◊

We conclude with an example that illustrates that our distributional framework, although perhaps less familiar than the individualistic framework, is just as useful.

Example 4. It is natural to conjecture that negative externalities lead to sub-optimal equilibria; we verify that conjecture in a particular setting. (It is sometimes said that negative externalities *invariably* lead to sub-optimal equilibria; as we shall see, this is not so.)

We consider a more general environment than above. Let $T\subset\mathbb{R}^2$ be a compact set, and let τ be a probability measure on T with full support. We identify T as a set of locations (a city or neighborhood) and τ as the (relative) population density. (The assumption of full support means there are no unpopulated areas.)

For notational simplicity, assume that there are only two goods, both divisible, that all agents are endowed with one unit of the first good but that the second good must be produced.[3] One unit of the second good can be produced from one unit of the first good, so the production cone is:

$$Y=\{(y_1,y_2):y_1\le 0,y_2\le -y_1\}.$$

If the distribution of consumption is $\sigma\in\mathrm{Prob}(T\times\mathbb{R}^2_+)$, an agent located at $s\in T$ experiences the externality

$$\eta(s,\sigma)=\int_{T\times\mathbb{R}^2_+}E(s,t,x)\,d\sigma(t,x).$$

[3]More general specifications would add only notational complication.

If the agent consumes consumes the bundle (x_1, x_2) she enjoys utility

$$u_s(x_1, x_2, \sigma) = U(s, x_1, x_2, \eta(s, \sigma)).$$

For simplicity, assume E is real-valued. To guarantee that preferences satisfy the requirements of Theorem 1 and hence that an equilibrium exist, it suffices to assume that E is measurable, uniformly integrable on each $\mathscr{D}^n(\tau)$ (boundedness would be enough), and continuous in x, and that U is measurable, continuous in x_1, x_2, η, and strictly increasing in x_1, x_2.

Assume the derivatives of E and U obey

$$\frac{\partial E}{\partial x_1} = 0, \qquad \frac{\partial E}{\partial x_2} > 0, \qquad \frac{\partial U}{\partial x_i} > 0 \text{ for } i = 1, 2$$
$$\frac{\partial U}{\partial \eta}(s, x_1, x_2, \eta) < 0 \text{ if } x_2 > 0.$$

That is: only good 2 causes pollution, pollution at location s from consumption at location t is strictly increasing in consumption of good 2, utility is strictly increasing in consumption and strictly decreasing in experienced pollution.

These assumptions *do not* guarantee that equilibrium is inefficient. For example, suppose there are only two locations, so $T = \{s_1, s_2\}$, and that half the population lives at each location. Consumption of x_2 at any location creates an equal externality at every location

$$\eta(s, \sigma) = \int_{T \times \mathbb{R}^2_+} x_2 \, d\sigma.$$

Utility functions for agents who live at s_1 are

$$u_1(x_1, x_2, \eta) = x_1 + (3 - \eta) x_2$$

while utility functions for agents who live at s_2 are

$$u_2(x_1, x_2, \eta) = x_1 + (1 - \eta) x_2.$$

One equilibrium for this economy can be described in the following way: prices are $p_1 = p_2 = \frac{1}{2}$; agents who live at s_1 consume $(0, 1)$ and agents who live at s_2 consume $(1, 0)$. This equilibrium is efficient. The point is one made by Starrett (1972): agents at s_2 escape the negative effects of pollution by not consuming the second good at all.

However, these assumptions *do* guarantee that any equilibrium in which *some* agents consume *both* goods must be inefficient; in any such equilibrium, good 2 is overproduced.[4] To see this, let p, y, μ be an equilibrium in which some agents consume both goods, and let σ be the marginal of μ on $T \times \mathbb{R}^2_+$ (the distribution of locations and consumption). Because good 2 is produced, profit maximization entails that prices of goods 1, 2 must be equal: $p_1 = p_2$. Let

$$B_1 = \{(s, u_s, e, x) : x_1 > 0, x_2 > 0\}.$$

[4]We could guarantee that some agents consume both goods by requiring that marginal utilities of consumption at 0 be infinite.

By assumption, $\mu(B_1) > 0$. For $(s, u_s, e, x) \in B_1$ and $\varepsilon > 0$ small, set $x^\varepsilon = (x_1 + \varepsilon, x_2 - \varepsilon)$. Our differentiability assumptions imply that

$$\lim_{\varepsilon \to 0} \left| \frac{u_s(x^\varepsilon, \sigma) - u_s(x, \sigma)}{\varepsilon} \right| = 0 \tag{1}$$

$$\lim_{\varepsilon \to 0} \left[\frac{E(s, t, x^\varepsilon) - E(s, t, x)}{\varepsilon} \right] > 0. \tag{2}$$

A straightforward measure-theoretic argument shows that there is a subset $B_2 \subset B_1$ such that for (s, u_s, e, x), $(t, u_t, e, x) \in B_2$ convergence of the limits in (1) and (2) is uniform, the limit in (2) is bounded away from 0, and consumption of good 2 is bounded away from 0. Write B_2^c for the complement of B_2, define $h : B_2 \to B_2$ by $h(s, u_s, e, x) = (s, u_s, e, x^\varepsilon)$, and set

$$\hat{\mu} = \mu|_{B_2^c} + h_*(\mu|_{B_2})$$

Thus μ and $\hat{\mu}$ differ only in that agents who were initially in B_2 consume slightly more of good 1 and slightly less of good 2. If ε is small enough, these agents are better off because the loss in shifting own consumption x to x^ε (which is of order less than ε) is more than offset by the reduction in the externality (which is of order ε). In particular, μ is not Pareto optimal. ◊

Appendix

A. Proofs

We first isolate a useful lemma, then verify Proposition 1.

Lemma. *Every compact subset of $\mathscr{D}(\tau)$ is contained in some $\mathscr{D}^n(\tau)$.*

Proof. Suppose this is not true, so that there is a compact set $K \subset \mathscr{D}(\tau)$ and for each n there is a measure $\sigma_n \in K, \sigma_n \notin \mathscr{D}^n(\tau)$; there is no loss assuming that σ_n's are distinct. If E is any subset of $S = \{\sigma_n : n = 1, \ldots\}$ then for each m,

$$E \cap \mathscr{D}^m(\tau) \subset S \cap \mathscr{D}^m(\tau) \subset \{\sigma_n : n = 1, \ldots, m-1\}$$

so $E \cap \mathscr{D}^m(\tau)$ is finite, hence closed. By definition of the direct limit topology, therefore, E is a closed subset of $\mathscr{D}(\tau)$. Because E is arbitrary, this means that every subset of S is closed; i.e., S is a discrete set. On the other hand, compactness of K implies that $\{\sigma_n : n = 1, \ldots\}$ is compact. This is a contradiction, so we conclude that every compact subset of $\mathscr{D}(\tau)$ is contained in some $\mathscr{D}^n(\tau)$, as asserted. □

Proof of Proposition 1. Let $\mathscr{P}_0$ be the space of preference relations that are open, irreflexive, transitive and negatively transitive in own consumption (but not necessarily monotone). We first define a complete metric on $\mathscr{P}_0$. To this end, fix, for each n, a complete metric d^n on the space C^n of closed subsets of $X \times \mathscr{D}^n(\tau) \times X \times \mathscr{D}^n(\tau)$. For

$\rho \in \mathscr{P}$, recall that $B(\rho)$ is the better-than set and that the complement $B(\rho)^c$ is the non-better-than set. Set

$$W_\rho^n = B(\rho)^c \cap \left[X \times \mathscr{D}^n(\tau) \times X \times \mathscr{D}^n(\tau)\right]$$

and define

$$d_{\mathscr{P}}(\rho,\rho') = \sum_{n=1}^{\infty} 2^{-n} \frac{d^n(W_\rho^n, W_{\rho'}^n)}{1+d^n(W_\rho^n, W_{\rho'}^n)}.$$

It is straightforward to check that $d_{\mathscr{P}}$ is a complete metric on $\mathscr{P}_0$. That $\mathscr{P}_0$ is separable follows immediately from the fact that it is a subspace of the product $\prod_n C^n$. That this metric defines the given topology of $\mathscr{P} \subset \mathscr{P}_0$ follows immediately from the definitions.

Now choose countable dense sets $A \subset (X \setminus \{0\})$, $D \subset \mathscr{D}(\tau)$. Note that

$$\mathscr{P} = \bigcap_{a\in A, \sigma \in D} \{\rho \in \mathscr{P}_0 : (x+a,\sigma)\,\rho\,(x,\sigma)\}.$$

Each of the sets inside the intersection is open, so $\mathscr{P}$ is a countable intersection of open sets in a complete metric space. Hence the topology of $\mathscr{P}$ is defined by a complete metric. □

With these preliminaries in hand we turn to the proof of Theorem 1.[5]

Proof of Theorem 1. The proof proceeds along familiar lines. We construct auxiliary economies $\mathscr{E}_n$, use a fixed point argument to find equilibria $\langle p_n, y_n, \mu_n\rangle$ for these auxiliary economies, find a convergence subsequence $\langle p_n, y_n, \mu_n\rangle \to \langle p^*, y^*, \mu^*\rangle$, and show that the limit is an equilibrium for $\mathscr{E}$.

Step 1. Because $\mathscr{P}$, and hence $\mathscr{C}$, are complete separable metric spaces, every measure on $\mathscr{C}$ is tight. (See Billingsley 1968.) Use tightness of λ to construct an increasing sequence $H_1, H_2, \ldots$ of compact subsets of $T \times \mathscr{R} \times X$ such that, for each i, $\lambda(H_i) \geq 1-2^{-i}$. Because H_i is compact, $|e|$ is bounded on H_i.

Step 2. We construct the auxiliary economies $\mathscr{E}_n$. For each index n, let

$$Y_n = \left\{ y \in \mathbb{R}^{L+M} : \operatorname{dist}(y, Y) \leq \frac{|y|}{n} \right\}.$$

Note that Y_n is a closed convex cone and $-\mathbb{R}_+^{L+M} \subset Y \subset Y_n$. If n_0 is sufficiently large, then $Y_{n_0} \cap \mathbb{R}_+^{L+M} = \{0\}$ and so $Y_n \cap \mathbb{R}_+^{L+M} = \{0\}$ for $n \geq n_0$. In what follows, we restrict attention to such n. Write

$$Y_n^\circ = \{p \in \mathbb{R}^{L+M} : p \cdot y \leq 0 \text{ for all } y \in Y_n\}$$

for the *polar cone*, and set

$$\Delta_n = \Delta \cap Y_n^\circ = \overline{\Delta} \cap Y_n^\circ.$$

Note that if $p \in \Delta_n$ then $p_i \geq 1/n$ for each i. Set $\mathscr{E}_n = \langle T, \tau, Y_n, \lambda\rangle$.

[5] An alternative proof could be given along the lines of Balder (2005).

Step 3. Fix n. We construct compact convex spaces of prices and consumption distributions, and a correspondence on the space of prices and consumption distributions.

Set

$$K_n = \{(t,\rho,e,x) \in \mathscr{C} \times X : |x| \le Ln|e|\}$$
$$\mathscr{K}_n = \{\mu \in \mathrm{Prob}(\mathscr{C} \times X) : \mu_{\mathscr{C}} = \tau, \text{ and } \mu(K_n) = 1\}$$

where $\mu_{\mathscr{C}}$ is the marginal of μ on $\mathscr{C}$. It is evident that $\mathscr{K}_n$ is a closed subset of $\mathrm{Prob}(\mathscr{C} \times X)$, and tightness of λ entails that $\mathscr{K}_n$ is tight, so $\mathscr{K}_n$ is weakly compact. (See Billingsley 1968.) To see that $\mathscr{K}_n$ is non-empty, write

$$\mathrm{proj} : \mathscr{C} \times X \to \mathscr{C}$$

for the projection. For $B \subset \mathscr{C} \times X$ a Borel set, write

$$B_0 = B \cap (\mathscr{C} \times \{0\}).$$

Define the probability measure μ by $\mu(B) = \lambda\left(\mathrm{proj}(B_0)\right)$; note that $\mu \in \mathscr{K}_n$, so $\mathscr{K}_n$ is non-empty.

We define a correspondence Φ_n on $\Delta_n \times \mathscr{K}_n$ as the product of correspondences

$$\phi_n : \Delta_n \times \mathscr{K}_n \to \mathscr{K}_n$$
$$\psi_n : \Delta_n \times \mathscr{K}_n \to \Delta_n.$$

Given $(t,\rho,e) \in T \times \mathscr{P} \times X = \mathscr{C}$, $p \in \Delta_n$ and $\mu \in \mathrm{Prob}(\mathscr{C} \times X)$ having the property that $\mu_{14} \in \mathscr{D}(\tau)$ (recall that μ_{14} is the marginal of μ on observable characteristics and consumptions), define individual budget and demand sets by:

$$B(t,\rho,e;\mu,p) = \{x \in X : p \cdot x \le p \cdot e\}$$
$$d(t,\rho,e;\mu,p) = \{x \in B(t,\rho,e;\mu,p) : (x',\mu_{14})\,\rho\,(x,\mu_{14}) \Rightarrow p \cdot x' > p \cdot e\}.$$

Note that if $p \in \Delta_n$ then

$$d(t,\rho,e;\mu,p) \subset B(t,\rho,e;\mu,p) \subset K_n.$$

Finally, let $D(\mu,p)$ be the set of agents who choose in their demand set:

$$D(\mu,p) = \{(t,\rho,e,x) : x \in d(t,R,e;\mu,p)\}.$$

Now define the required correspondences by

$$\phi_n(p,\mu) = \{\nu \in \mathscr{K}_n : \nu(D(\mu,p)) = 1\}$$
$$\psi_n(p,\mu) = \operatorname{argmax}\left\{q \cdot \left(\int x\, d\mu - \int e\, d\mu\right) : q \in \Delta_n\right\}$$
$$\Phi_n(p,\mu) = \psi_n(p,\mu) \times \phi_n(p,\mu).$$

Step 4. We claim that ϕ_n, ψ_n, Φ_n are upper-hemi-continuous, and have compact, convex, non-empty values.

It is evident that ϕ_n has convex values. To show that it is upper-hemi-continuous, we first show that the individual demand is closed. To this end, let $(t,\rho,e) \in \mathscr{C}$, let $x^j \in d(t,\rho,e;\mu^j,p^j)$ and suppose $x^j \to x$; we must show $x \in d(t,\rho,e;\mu,p)$. Write $x = (x_L, x_M)$ where $x_L \in \mathbb{R}^L_+$ and $x_M \in \mathbb{Z}^M_+$. Because endowments are desirable, $x_L \neq 0$, else $(e,\mu_{14})\ \rho\ (x,\mu_{14})$ and x would not be in the demand set. Hence $p \cdot (x_L, 0) > 0$ and $p \cdot ((1-\epsilon)x_L, x_M) < p \cdot x$ for every $\epsilon > 0$. It then follows by a familiar continuity argument that $x \in d(t,\rho,e;\mu,p)$, as desired

Because $\Delta_n, \mathscr{K}_n$ are compact, to see that ϕ_n is upper-hemi-continuous and has compact values, it suffices to show that it has closed graph. To this end, let $\{(p^j,\mu^j)\}$ be a sequence in $\Delta_n \times \mathscr{K}_n$ converging to (p,μ); for each j, let $\nu^j \in \phi_n(p^j,\mu^j)$ and assume $\nu^j \to \nu$; we must show $\nu \in \phi_n(p,\mu)$. For each i, write $\mathscr{H}_i = H_i \times X$. By definition, $\nu^j[D(\mu^j,p^j)] = 1$, so

$$\nu^j[D(\mu^j,p^j) \cap \mathscr{H}_i] \geq 1 - 2^{-i}.$$

Convergence of ν^j to ν implies that

$$\nu\left(\limsup_{j\to\infty}\left[D(\mu^j,p^j) \cap \mathscr{H}_i\right]\right) \geq 1 - 2^{-i}.$$

On the other hand, it follows from closedness of the individual demand correspondence that

$$D(\mu,p) \cap \mathscr{H}_i \supset \limsup_{j\to\infty}\left[D(\mu^j,p^j) \cap \mathscr{H}_i\right]$$

so that $\nu[D(\mu,p) \cap \mathscr{H}_i] \geq 1 - 2^{-i}$. Since i is arbitrary, it follows that $\nu[D(\mu,p)] = 1$ so $\nu \in \phi_n(p,\mu)$, as desired. Hence ϕ_n is upper-hemi-continuous.

To see that ϕ_n has non-empty values, fix (p,μ). For each (t,ρ,e), let $f(t,\rho,e)$ be the unique lexicographically smallest element of $d(t,\rho,e,\mu,p)$. It is easily checked that f is a measurable function, and that the direct image measure $f_*\lambda$ belongs to $\phi_n(p,\mu)$.

That ψ_n is upper-hemi-continuous, and has compact, convex, non-empty values follows immediately from the usual argument for Berge's Maximum Theorem.

Finally, Φ_n is upper-hemi-continuous, and has compact, convex, non-empty values because ϕ_n, ψ_n enjoy these properties.

Step 5. Because Δ_n, $\mathscr{K}_n$ are compact and convex and Φ_n is upper-hemi-continuous and has compact, convex, non-empty values, Φ_n has a fixed point (p_n,μ_n). Set $y_n = \int x\,d\mu_n - \int e\,d\lambda$. We claim that $\langle p_n, y_n, \mu_n\rangle$ constitutes an equilibrium for the economy $\mathscr{E}_n$.

The construction guarantees that almost all agents optimize in their budget sets. By definition, p_n maximizes the value of y_n, which is aggregate excess demand at p_n,μ_n. Walras's Law guarantees that the value of excess demand at p_n,μ_n is 0: $p_n \cdot y_n = 0$. If $y_n \notin Y_n$ there would be a price $p \in Y_n^\circ$ such that $p \cdot y_n > 0$. However this would contradict the fact that p_n maximizes the value of excess demand. Hence $y_n \in Y_n$. By construction, $p_n \in Y_n^\circ$, so $p_n \cdot y \leq 0$ for every $y \in Y_n$; this is profit maximization. Hence $\langle p_n, y_n, \mu_n\rangle$ constitutes an equilibrium for the economy $\mathscr{E}_n$.

Step 6. We show that the equilibria $\langle p_n, y_n, \mu_n\rangle$ lie in compact sets of prices, production vectors, and distributions.

Note first that all prices p_n lie in $\overline{\Delta}$, which is compact. By construction, $y_n = \int x\, d\mu_n - \int e\, d\lambda$ so $y_n^- \le \int e\, d\lambda$. Because there is no free production ($Y_{n_0} \cap \mathbb{R}_+^{L+M} = \{0\}$), rates of transformation are bounded. Hence there is some constant A such that if $y \in Y_{n_0}$ then the positive and negative parts of y satisfy the inequality $|y^+| \le A|y^-|$. Thus $y_n^- \le \int e\, d\lambda$ and

$$|y_n^+| \le A|y_n^-| \le A\left|\int e\, d\lambda\right|. \tag{3}$$

Hence, the production vectors y_n lie in a bounded subset of $\mathbb{R}^{L+M}$.

To see that the distributions μ_n lie in a tight (hence relatively compact) set, recall the sets H_i constructed in Step 1. Fix i. For each k set

$$G_k = \{(t, R, e, x) \in H_i \times X : |x| \le k\}$$
$$J_k = \{(t, R, e, x) \in H_i \times X : |x| > k\}.$$

For each n

$$\int_{J_k} |x|\, d\mu_n \ge k\, \mu_n(J_k). \tag{4}$$

By construction,

$$\int x\, d\mu_n = y_n + \int e\, d\lambda \le y_n^+ + \int e\, d\lambda.$$

In view of (3), this implies

$$\int |x|\, d\mu_n = \left|\int x\, d\mu_n\right| \le (1+A)\left|\int e\, d\lambda\right|. \tag{5}$$

Combining (4) and (5) we conclude that

$$\mu_n(J_k) \le \frac{(1+A)}{k}\left|\int e\, d\lambda\right|.$$

Choose k large enough so that the right hand side is smaller than 2^{-i}. Since $G_k \cup J_k = H_i \times X$ and $\mu_n(H_i \times X) = \lambda(H_i) \ge 1 - 2^{-i}$, it follows that $\mu_n(G_k) > 1 - 2^{-i+1}$. Because G_k is compact, we conclude that $\{\mu_n\}$ is a uniformly tight family, hence relatively compact. We conclude that prices p_n, production vectors y_n, and consumption distributions μ_n lie in compact sets, as asserted.

Step 7. Some subsequence of $\langle p_n, y_n, \mu_n\rangle$ converges; say

$$\langle p_n, y_n, \mu_n\rangle \to \langle p^*, y^*, \mu^*\rangle.$$

By construction, $Y = \bigcap Y_n$. Since $p_n \in \overline{\Delta} \cap Y_n^\circ$ and $y_n \in Y_n$ for each n, it follows that $p^* \in \overline{\Delta} \cap Y^\circ$ and $y^* \in Y$. We claim that $p^* \in \Delta$ and that $\langle p^*, y^*, \mu^*\rangle$ is an equilibrium for $\mathscr{E}$.

To see that $p^* \in \Delta$, suppose not; say $p_j^* = 0$. We distinguish two cases, and obtain a contradiction in each case.

Case 1: $p^* \cdot \int e\,d\lambda = 0$. Because $p^* \neq 0$ and all goods are available in the aggregate, there is some $x \in \mathbb{R}_+^{L+M}$ such that $p^* \cdot x > 0$ and $\left(x - \int e\,d\lambda\right) \in Y$. Thus, $p^* \cdot \left(x - \int e\,d\lambda\right) > 0$. Since $p^* \in Y^\circ$, we have a contradiction.

Case 2: $p^* \cdot \int e\,d\lambda > 0$. Then there is some ℓ such that $p^*_\ell > 0$ and $\int e_\ell\,d\lambda > 0$. Let $E = \{(t,\rho,e) : e_\ell > 0\}$. Because $\int e_\ell\,d\lambda > 0$ it follows that $\lambda(E) > 0$. Because λ is regular and tight, it follows that there is a compact set $J \subset E$ such that $\lambda(J) > 0$. Define

$$Z = \left\{ \nu \in \operatorname{Prob}(\mathscr{C} \times X) : \nu_{\mathscr{C}} = \lambda, \int |x|\,d\nu \leq (1+A)\int |e|\,d\lambda \right\}.$$

Arguing as before shows that Z is weakly closed and uniformly tight, hence weakly compact.

We claim that as $n \to \infty$ aggregate demand is uniformly unbounded on Z: for every $\alpha > 0$ there is an integer n_0 such that if $n \geq n_0$, $(t,\rho,e) \in J$, $\nu \in Z$, and $z \in d(t,\rho,e;\zeta,p_n)$ then $|z| > \alpha$. To see this, suppose not. Then there is some $\alpha > 0$ such that for every n_0 there is some $n > n_0$, some $(t,\rho,e) \in J$, some $\zeta \in Z$, and some $z_n \in d(t,\rho,e;\zeta,p_n)$ such that $|z_n| \leq \alpha$. Letting n_0 tend to infinity, passing to limits of subsequences where necessary, and recalling the definition of the topology of $\mathscr{P}$, that J, Z are compact, and that preference relations are continuous in the distribution of consumption, and arguing as in Step 4, we find $(t^*,\rho^*,e^*) \in Z$, $\nu^* \in J$ and $z^* \in d(t^*,\rho^*,e^*;\nu^*,p^*)$ such that $|z^*| \leq \alpha$. Because $p^* \cdot e^* > 0$, $p^*_j = 0$ and ρ^* is strictly monotone, this is absurd. This contradiction establishes the claim.

Now apply the claim with $\alpha = 2\left|\int e\,d\lambda\right|/\lambda(J)$ to conclude that there is an n_0 such that for every $(t,\rho,e) \in J$, each $n \geq n_0$ and every $z \in d(t,\rho,e;\nu,p_n)$ we have

$$|z| > 2\left|\int e\,d\lambda\right|/\lambda(J).$$

If follows in particular that

$$\int_J \inf\{|z| : z \in d(t,\rho,e;\mu_n,p_n)\}\,d\lambda \geq 2\left|\int e\,d\lambda\right|$$

for each n. However, as we have shown above,

$$\int |x|\,d\mu_n \leq \left|\int e\,d\lambda\right|$$

so we have a contradiction.

Since we obtain contradictions in each case, we conclude that $p^* \in \Delta$, as asserted.

Arguing as in Step 4 shows that $\mu^*[D(\mu^*,p^*)] = 1$ and that $\int x\,d\mu^* = y^* + \int e\,d\lambda$, so $\langle p^*, y^*, \mu^* \rangle$ is an equilibrium for $\mathscr{E}$. The proof is complete. □

Proof of Theorem 2. Let p, y, μ be an equilibrium. Strict convexity of preferences guarantees that λ-almost all agents have unique demands at the given price p and distribution μ. Put differently, for almost all (t,ρ,e), the demand set $d(t,\rho,e,\mu,p)$ is a singleton. By assumption, $\mathscr{E}$ has an individualistic representation, so there is a measurable $\varphi : T \to \mathscr{P} \times X$ such that $\lambda = (\mathrm{id}_T, \varphi)_*(\tau)$. Define $f : T \to X$ by

$$f(t) = d(t, \varphi(t), \mu, p).$$

It is easily checked that f is measurable and that

$$\mu = (\mathrm{id}_T, \varphi, f)_*(\tau)$$

so p, y, μ admits an individualistic representation, as asserted. □

B. Externalities as a parameter

An alternative formulation of externalities in preferences has the distribution of consumption entering as a parameter. Here we give a brief sketch of this alternative. Commodities, consumption sets, production, and the distribution of consumption are all as in Section 3.

As in Hildenbrand (1974), write $\mathscr{P}^*_{mo}$ for the space of continuous, irreflexive, transitive, negatively transitive, strictly monotone preference relations on X; in the topology of closed convergence, $\mathscr{P}^*_{mo}$ is a complete separable metric space. We define a *parametrized preference* to be a map

$$R : \mathscr{D}(\tau) \to \mathscr{P}^*_{mo}.$$

We write $x\ R(\sigma)\ y$ to mean that the consumption bundle x is preferred to the consumption bundle y when σ is the distribution of consumption. The preference relation R is *continuous* if the set

$$\{(x, y, \sigma) \in X \times X \times \mathscr{D}(\tau) : x\ R(\sigma)\ y\}$$

is open. The following proposition records that continuity of the preference relation R is equivalent to continuity of R as a mapping.

Proposition 2. *For a parametrized preference relation R the following are equivalent.*

(i) R is continuous as a preference relation (in the sense above).

(ii) The mapping $R : \mathscr{D}(\tau) \to \mathscr{P}^*_{mo}$ is continuous.

(iii) For each n the restriction $R : \mathscr{D}^n(\tau) \to \mathscr{P}^*_{mo}$ is continuous.

Write $\mathscr{R}$ for the space of continuous parametrized preference relations, and give $\mathscr{R}$ the topology of uniform convergence on compact subsets of $\mathscr{D}(\tau)$. The following can be proved by combining the Lemma of Appendix A together with Proposition 2.

Proposition 3. *$\mathscr{R}$ is a complete, separable metric space.*

Agents are characterized by an observable characteristic, a preference relation, and an endowment, so for this description of preferences, the space of agent characteristics is

$$\mathscr{C}_P = T \times \mathscr{R} \times X.$$

In view of Proposition 3, $\mathscr{C}_P$ is a complete separable metric space.

An *economy with parametrized preference externalities* consists of an observable population (T,τ), a production sector Y, and a probability measure λ on $T\times\mathscr{R}\times X$ whose marginal on T is τ and which has the property that aggregate endowment is finite: $\int |e|\,d\lambda<\infty$ for each i. As before, we say *endowments are desirable* if λ-almost all $(t,R,e)\in\mathscr{C}_P$ have the property that $e\;R(\sigma)\,(0,x_M)$ for every $x_M\in\mathbb{Z}_+^M$.

For an economy $\mathscr{E}=\langle T,\tau,Y,\lambda\rangle$ with parametrized preferences, an *equilibrium* for $\mathscr{E}$ is a price $p\in\Delta$, a production vector $y\in Y$, and a probability measure μ on $T\times\mathscr{R}\times X\times X$ such that

(a) the marginal of μ on $T\times\mathscr{R}\times X$ is λ

(b) almost all agents choose in their budget set:

$$\mu\{(t,R,e,x):p\cdot x>p\cdot e\}=0$$

(c) production profit is maximized

$$p\cdot y=\sup\{p\cdot y':y'\in Y\}$$

(d) markets clear

$$\int x\,d\mu=y+\int e\,d\lambda$$

(e) almost all agents optimize given the price p and the distribution of consumption μ_{14}:

$$\mu\{(t,R,e,x):\text{ there exists } x'\in X,\, x'\;R(\mu_{14})\,x,\; p\cdot x'\leq p\cdot e\}=0.$$

(Again, (a) and (d) imply that the marginal μ_{14} of μ on $T\times X$ (the distribution of consumption) belongs to $\mathscr{D}(\tau)$.)

Existence of equilibrium follows almost exactly as in Theorem 1.

THEOREM 3. *Every economy $\mathscr{E}=\langle T,\tau,Y,\lambda\rangle$ with parametrized preferences for which endowments are desirable and all goods are available in the aggregate admits an equilibrium.*

As might be expected, there is a natural link between the two descriptions of economies. In particular, there is a natural continuous map $F:\mathscr{P}\to\mathscr{R}$ defined by

$$x\;F(\rho)(\sigma)\,x'\Leftrightarrow(x,\sigma)\,\rho\,(x',\sigma)$$

for all $x,x'\in X$, $\sigma\in\mathscr{D}(\tau)$. (This map "forgets" comparisons that involve different distributions of social consumption.) The map F induces a map

$$\lambda\mapsto\tilde{\lambda}:\mathrm{Prob}(T\times\mathscr{P}\times X)\to\mathrm{Prob}(T\times\mathscr{R}\times X)$$

by defining $\tilde{\lambda}=(\mathrm{id}_T,F,\mathrm{id}_X)_*(\lambda)$, where id_T is the identity on T and id_X is the identity on X. Given an economy with preference externalities $\mathscr{E}=\langle T,\tau,Y,\lambda\rangle$, define the economy with parametrized preferences $\tilde{\mathscr{E}}=\langle T,\tau,Y,\tilde{\lambda}\rangle$.

THEOREM 4. *If $\mathscr{E} = \langle T, \tau, Y, \lambda \rangle$ is an economy with preference externalities, then $\mathscr{E}$ and $\tilde{\mathscr{E}}$ have the same equilibrium prices and production plans.*[6]

PROOF. Given $\mu \in \mathrm{Prob}(T \times \mathscr{P} \times X \times X)$, define

$$\tilde{\mu} = (\mathrm{id}_T, F, \mathrm{id}_X, \mathrm{id}_X)_*(\mu) \in \mathrm{Prob}(T \times \mathscr{R} \times X \times X).$$

It is straightforward to check that if $\mathscr{E}$ is an economy with preference externalities and p, y, μ is an equilibrium for $\mathscr{E}$ then $p, y, \tilde{\mu}$ is an equilibrium for $\tilde{\mathscr{E}}$.

All that remains is to show that if p, y, ν is an equilibrium for $\tilde{\mathscr{E}}$, then we can find an equilibrium p, y, μ for $\mathscr{E}$ such that $\nu = \tilde{\mu}$. To this end, use disintegration of measures to find a measurable map

$$\xi : T \times \mathscr{R} \times X \to \mathrm{Prob}(X)$$

such that

$$\nu = \int_{T \times \mathscr{R} \times X} \xi(t, R, e)\, d\tilde{\lambda}.$$

Set

$$\mu = \int_{T \times \mathscr{P} \times X} \xi(t, F(\rho), e)\, d\lambda$$

and check that p, y, μ is an equilibrium for $\mathscr{E}$ and that $\tilde{\mu} = \nu$. □

REFERENCES

Aumann, Robert J. (1964), "Markets with a continuum of traders." *Econometrica*, 32, 39–50. [144]

Aumann, Robert J. (1966), "Existence of competitive equilibrium in markets with a continuum of traders." *Econometrica*, 34, 1–17. [144]

Balder, Erik (2003), "Existence of competitive equilibria in economies with a measure space of consumers and consumption externalities." Working Paper, University of Utrecht. [144, 151, 153]

Balder, Erik (2005), "More about equilibrium distributions for competitive markets with externalities." Working Paper, University of Utrecht. [158]

Billingsley, Patrick (1968), *Convergence of Probability Measures*. John Wiley and Sons, New York. [158, 159]

Cornet, Bernard and Mihaela Topuzu (2005), "Existence of equilibria for economies with externalities and a measure space of consumers." *Economic Theory*, 26, 397–421. [144, 151, 153]

[6]We do not claim that $\mathscr{E}$ and $\tilde{\mathscr{E}}$ have the same equilibrium consumption distributions. The problem is that F is not one-to-one, so many characteristics in $\mathscr{C}$ may correspond to the same characteristic in $\mathscr{C}_P$.

Feldman, Mark and Christian Gilles (1985), "An expository note on individual risk without aggregate uncertainty." *Journal of Economic Theory*, 35, 26–32. [151]

Gretsky, Neil E., Joseph M. Ostroy, and William R. Zame (1992), "The nonatomic assignment model." *Economic Theory*, 2, 103–127. [151]

Hammond, Peter J., Mamoru Kaneko, and Myrna Holtz Wooders (1989), "Continuum economies with finite coalitions: Core, equilibria, and widespread externalities." *Journal of Economic Theory*, 49, 113–134. [144]

Hart, Sergiu, Werner Hildenbrand, and Elon Kohlberg (1974), "On equilibrium allocations as distributions on the commodity space." *Journal of Mathematical Economics*, 1, 159–166. [144]

Hildenbrand, Werner (1974), *Core and Equilibria of a Large Economy*. Princeton University Press, Princeton. [163]

Judd, Kenneth L. (1985), "The law of large numbers with a continuum of IID random variables." *Journal of Economic Theory*, 35, 19–25. [151]

Laffont, Jean-Jacques (1977), *Effets Externes et Théorie Économique*. Éditions du Centre National de la Recherche Scientifique, Paris. [143]

Mas-Colell, Andreu (1984), "On a theorem of Schmeidler." *Journal of Mathematical Economics*, 13, 201–206. [148, 151]

McKenzie, Lionel W. (1959), "On the existence of general equilibrium for a competitive market." *Econometrica*, 27, 54–71. [147]

Noguchi, Mitsunori (2005), "Interdependent preferences with a continuum of agents." *Journal of Mathematical Economics*, 41, 665–686. [144, 151, 153]

Shafer, Wayne and Hugo Sonnenschein (1975), "Equilibrium in abstract economies without ordered preferences." *Journal of Mathematical Economics*, 2, 345–348. [143]

Starrett, David A. (1972), "Fundamental nonconvexities in the theory of externalities." *Journal of Economic Theory*, 4, 180–199. [143, 156]

Submitted 2004-12-27. Final version accepted 2005-10-31. Available online 2005-12-2.