Theoretical Economics 1 (2006), 369–393 1555-7561/20060369

# Ex post implementation in environments with private goods

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We prove by construction that ex post incentive compatible mechanisms exist in a private goods setting with multi-dimensional signals and interdependent values. The mechanism shares features with the generalized Vickrey auction of one-dimensional signal models. The construction implies that for environments with private goods, informational externalities (i.e., interdependent values) are compatible with ex post equilibrium in the presence of multi-dimensional signals.

KEYWORDS. Ex post incentive compatibility, multi-dimensional information, interdependent values.
JEL CLASSIFICATION. D44.

## 1. INTRODUCTION

In models of mechanism design with interdependent values, each player's information is usually one dimensional. While this is convenient, it might not capture a significant element of the setting. For instance, suppose that agent $A$'s reservation value for an object is the sum of a private value, which is idiosyncratic to this agent, and a common value, which is the same for all agents in the model. Agent $A$'s private information consists of an estimate of the common value and a separate estimate of his private value. As other agents care only about $A$'s estimate of the common value, a one-dimensional statistic does not capture all of $A$'s private information that is relevant to every agent (including $A$).[1]

Therefore, it is important to test whether insights from the literature are robust to relaxing the assumption that an agent's private information is one dimensional. Building on earlier work by Maskin (1992), Jehiel and Moldovanu (2001) show that if agents have multi-dimensional information, interdependent values, and independent signals then,

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An earlier version of this paper was circulated under the title "The Limits of Ex Post Implementation Revisited." I am grateful to Jeffrey Ely, Benny Moldovanu, Joe Ostroy, Bill Zame, and two anonymous referees for comments that led to substantial improvements in this paper. This research was supported by the National Science Foundation under Grant No. SES-0422317.

[1] A $d$-dimensional, $d \geq 2$, private signal $s_A$ can be mapped, without any loss of information, into a single dimension using a one-to-one function $f : \mathbb{R}^d \to \mathbb{R}$. However, agents' values will not be non-decreasing or continuous in the signal $f(s_A)$. Hence, the assumption of one-dimensional signals is a limitation only in conjunction with assumptions commonly made in the literature that a buyer's (one-dimensional) signal is ordered so that a higher realization is more favorable, or that the valuation is a continuous function of the signal.

unlike in models with one-dimensional information, every Bayesian Nash equilibrium is (generically) inefficient.[2] Jehiel et al. (2006) call into question the existence of ex post equilibrium when agents have multi-dimensional information and interdependent values. They show that ex post incentive compatible mechanisms do not generically exist (except, of course, trivial mechanisms that disregard the reports of players).[3]

We prove an existence result for non-trivial ex post incentive compatible mechanisms in a private goods setting when buyers have interdependent values and multi-dimensional signals. To reconcile this with Jehiel et al.'s result, we note that their non-existence result depends on the assumption that for any pair of outcomes there exist at least two agents who are not indifferent between that pair of outcomes. In a private goods environment, as agents care only about their own allocation, this assumption is not satisfied. To see this, consider the sale of one indivisible object to two buyers, 1 and 2. There are three possible outcomes: $a_i$, the good is assigned to buyer $i$, $i = 1, 2$, and $a_0$, neither gets the good. Buyer 1 is indifferent between $a_2$ and $a_0$ and buyer 2 is indifferent between $a_1$ and $a_0$. There exist pairs of outcomes (namely $(a_0, a_1)$ and also $(a_0, a_2)$) between which all agents except one is indifferent. Consequently, preferences over private goods are non-generic in the space of social choice settings considered by Jehiel et al.; their definition of genericity requires the presence of externalities. Therefore, even if buyers have multi-dimensional signals, the possibility of existence of non-trivial ex post incentive compatible selling mechanisms in generic private goods models is not precluded.

We prove an existence result for ex post incentive compatible mechanisms for the sale of an indivisible (private) good to $n$ buyers with multi-dimensional signals and interdependent values. The assumptions of the model are not non-generic in private goods environments. In the constructed mechanism, the rule for deciding whether buyer 1, say, should be assigned the object is as follows. (The mechanism is illustrated in Figure 1 for the case of $n = 2$ buyers.) Fix the other buyers' signals at some realization. Partition buyer 1's set of possible signal realizations into equivalence classes or "indifference curves" such that buyer 1's reservation value is constant on an indifference curve. These indifference curves are completely ordered by buyer 1's value. If a generalization of the single-crossing property is satisfied then there exists a pivotal indifference curve for buyer 1 with the property that it is ex post incentive compatible to award the object to buyer 1 if and only if buyer 1's signal realization is on an indifference curve that is greater than the pivotal one. On the pivotal indifference curve, as illustrated in Figure 1, the maximum of the other buyers' values is equal to buyer 1's value. If buyer 1 wins, the price paid by him is equal to his value on the pivotal indifference curve; if he loses he pays nothing. This mechanism is non-trivial and can be extended to multiple objects when buyer preferences over objects are subadditive.

[2]See also Harstad et al. (1996), who obtain sufficient conditions under which an efficient allocation is attained by common auction forms. McLean and Postlewaite (2004) show that efficient Bayesian implementation is possible when signals are correlated.

[3]In a recent working paper, Mezzetti and Parreiras (2005) obtain sufficient conditions for existence of differentiable ex post mechanisms.

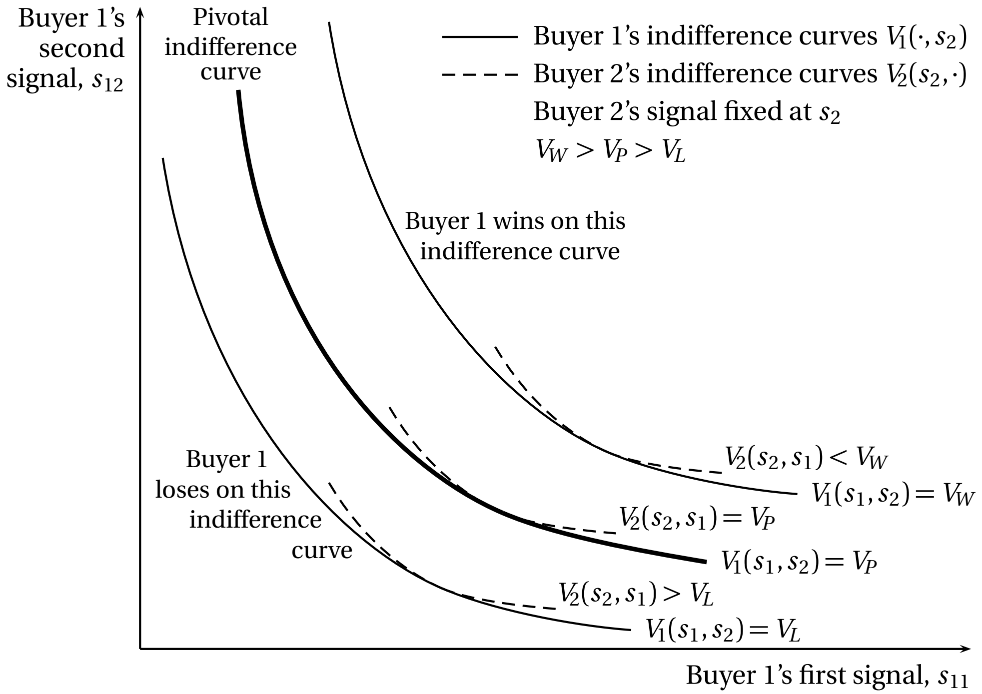


FIGURE 1. Indifference curves in buyer 1 signal space, $s_1 = (s_{11}, s_{12})$

The mechanism shares the feature with the generalized Vickrey auction of one-dimensional information models that the price paid by the winning buyer is equal to this buyer's value at the lowest possible signal (i.e., on the pivotal indifference curve) at which this buyer would just win. Thus, ex post equilibria in auction models with one-dimensional signals are robust in that non-trivial ex post equilibria exist even when buyers have multi-dimensional signals.

In a multi-dimensional signal setting the pivotal indifference curve for a buyer consists of a continuum of this buyer's signal realizations whereas in a one-dimensional setting there is exactly one pivotal signal realization for this buyer. Consequently, when the highest two buyer values are close to each other no buyer's signal is above his pivotal indifference curve. This ensures that the subset of buyers' signals in which one buyer gets the object does not share a common boundary with the (disjoint) subset of buyers' signals in which another buyer gets the object.[4] The social cost of incentive compatibility in our model is that the object is retained by the auctioneer and gains from trade are not realized when the highest two buyer valuations are close to each other.

There are only private goods in our model. Thus, in environments with private goods, informational externalities (i.e., interdependent values) alone do not preclude the existence of ex post equilibrium in the presence of multi-dimensional signals. One needs consumption externalities or public goods, in addition to information externali-

[4]It is precisely the existence of such a common boundary that is used by Jehiel et al. to show the non-existence of ex post incentive compatible mechanisms in a setting in which auctions are non-generic.

ties, for generic non-existence. Ex post equilibrium has been employed mostly in auction models with private goods;[5] hence, these models are robust to relaxing the assumption of one-dimensional signals.

The mechanism described above is conditionally efficient in that whenever it allocates the object, it is to the buyer with the highest valuation. Furthermore, it is more efficient than any other conditionally efficient mechanism. However, this mechanism need not be efficient, subject to incentive constraints, and a constrained efficient mechanism need not be conditionally efficient. Contrast this to one-dimensional signal models, where constrained, conditional, and first-best efficiency are attained in the same mechanism because incentive constraints do not bind.

The paper is organized as follows. A model with two buyers is presented in Section 2, together with preliminary results. An existence result for ex post incentive compatible mechanisms in a model with two buyers and one private good is proved in Section 3. This result is generalized to $n$ buyers, and possible extensions to models with many buyers and many objects are explored in Section A of the appendix. A sufficient condition and a necessary condition for constrained efficiency is provided in Section 4. Section 5 concludes.

## 2. The model

The main idea can be seen in a model with two buyers, $i = 1, 2$, and one indivisible object. The information state is denoted $s$. Each buyer $i$ observes a $d^i \geq 2$ dimensional private signal $s_i = (s_{i1}, s_{i2}, \ldots, s_{id^i})$ about the information state. The domain of $s_i$ is $S_i = [0,1]^{d^i}$. Without loss of generality, the buyers' information jointly determines the information state with $s = (s_1, s_2)$.[6] Buyer $i$'s reservation value for the object in information state $s = (s_i, s_j)$ is $V_i(s_i, s_j)$. Buyers have quasilinear utility. If buyer $i$ gets the object in state $s$ and pays $t$, then his utility is $V_i(s) - t$; if he does not get the object and pays $t$, his utility is $-t$.

The outcome in which buyer $i$, $i = 1, 2$, is allocated the object is denoted $a_i$, and the outcome in which no buyer gets the object is $a_0$. A (deterministic) mechanism consists of an allocation rule $h$ and payment functions $\hat{t}_i$, $i = 1, 2$. The allocation rule $h : S \to \{a_0, a_1, a_2\}$ is a function from the buyers' reported signals to an outcome; the payment function $\hat{t}_i : S \to \mathbb{R} \cup \{\infty\}$ is a function from the buyers' reported signals to a monetary payment by buyer $i$.

A mechanism $(h, \hat{t})$ is *ex post incentive compatible* if for $i, j = 1, 2$, $i \neq j$,

$$V_i(s_i, s_j) 1_{\{h(s_i, s_j) = a_i\}} - \hat{t}_i(s_i, s_j) \geq V_i(s_i, s_j) 1_{\{h(s'_i, s_j) = a_i\}} - \hat{t}_i(s'_i, s_j) \qquad \forall s_i, \forall s'_i, \forall s_j, \tag{1}$$

where $1_A$ is the indicator function of the event $A$. In other words, at each information state if buyer $j$ truthfully reports his signal then buyer $i$ can do no better than truth-

[5] See, for example, Crémer and McLean (1985), Ausubel (1999), Dasgupta and Maskin (2000), Perry and Reny (2002), and Bergemann and Välimäki (2002).

[6] An information state will also be denoted as $s = (s_i, s_j)$, where $i, j \in \{1, 2\}$, $i \neq j$.

fully report his signal.[7] If a mechanism $(h,\hat{t})$ satisfies (1) then the allocation rule $h$ is *implementable* and $\hat{t}$ is said to *implement* $h$.

Clearly, ex post incentive compatibility implies that $\hat{t}_i(s_i,s_j)=\hat{t}_i(s_i',s_j)$ if $h(s_i,s_j)=h(s_i',s_j)$; otherwise, if say $\hat{t}_i(s_i,s_j)<\hat{t}_i(s_i',s_j)$ then buyer $i$ has an incentive to misreport $s_i$ at information state $(s_i',s_j)$. In effect, the payment function $\hat{t}_i$ maps $\{a_0,a_1,a_2\}\times S_j$ to $\mathbb{R}\cup\{\infty\}$. The following characterization of ex post implementability is due to Chung and Ely (2003).

Lemma 1 (Chung and Ely 2003). *An allocation rule $h$ is implementable if and only if for each $i$, $a_k$, and $s_j$, $j\neq i$, there exist transfers $\hat{t}_i(a_k,s_j)\in\mathbb{R}\cup\{\infty\}$ such that*

$$h(s)\in\arg\max_{a_k}\{V_i(s_i,s_j)1_{\{h(s_i,s_j)=a_i\}}-\hat{t}_i(a_k,s_j)\}.$$

Without loss of generality we assume that $\hat{t}_i(a_0,s_j)=\hat{t}_i(a_j,s_j)$ for all $s_j$.[8] Thus, buyer $i$'s monetary payment depends only on whether or not buyer $i$ is assigned the object and on buyer $j$'s reported signal, $j\neq i$. We restrict attention to mechanisms in which a buyer pays nothing if he does not get the object; that is, $\hat{t}_i(a_k,s_j)=0$ if $a_k\neq a_i$. From Lemma 1 it is clear that this does not decrease the set of implementable allocation rules. With this restriction on monetary payments we may write

$$\hat{t}_i(a_k,s_j)\equiv\begin{cases}t_i(s_j) & \text{if } a_k=a_i\\ 0 & \text{otherwise.}\end{cases}$$

If $h(s_i,s_j)\neq a_i$ for all $s_i$ then let $t_i(s_j)=\infty$.

For mechanisms in which losing buyers pay nothing, the requirement of ex post incentive compatibility, i.e., condition (1), is rewritten as follows. For $i=1,2$, $i\neq j$,

$$\left[V_i(s_i,s_j)-t_i(s_j)\right]1_{\{h(s_i,s_j)=a_i\}}\geq\left[V_i(s_i,s_j)-t_i(s_j)\right]1_{\{h(s_i',s_j)=a_i\}}\quad\forall s_i,\forall s_i',\forall s_j.\tag{2}$$

The function $t_i(s_j)$ is buyer $i$'s payment conditional on getting the object. One may think of $t_i(s_j)$ as a *personalized price* at which the object is available to buyer $i$ at information states $(\cdot,s_j)$. Let $t=(t_1,t_2)$.

Thus, any implementable allocation rule may be implemented with personalized prices. One may also ask what type of personalized prices implement some allocation rule. To this end, define a pair of personalized price functions $t_i(s_j)$, $t_j(s_i)$ to be *admissible* if in each information state at most one buyer's value exceeds his personalized price:

$$V_i(s_i,s_j)>t_i(s_j)\Longrightarrow V_j(s_j,s_i)\leq t_j(s_i)\qquad\forall s_i,s_j.$$

[7] Ex post incentive compatibility is the same as uniform equilibrium of d'Aspremont and Gérard-Varet (1979) and uniform incentive compatibility of Holmström and Myerson (1983).

[8] If, say, $\hat{t}_i(a_0,s_j)<\hat{t}_i(a_j,s_j)$ for some $s_j$, then from Lemma 1 we see that $h(s_i,s_j)\neq a_j$ for any $s_i$. Therefore, letting $\hat{t}_i'(a_j,s_j)\equiv\hat{t}_i(a_0,s_j)$, the transfers $\hat{t}_i(a_0,s_j)$, $\hat{t}_i'(a_j,s_j)$, $\hat{t}_i(a_i,s_j)$ satisfy the argmax condition of Lemma 1 at $s_j$.

Define an allocation rule supported by admissible personalized prices $(t_1, t_2)$:

$$h(s_1,s_2) \equiv \begin{cases} a_1 & \text{if } V_1(s_1,s_2) > t_1(s_2) \\ a_2 & \text{if } V_2(s_2,s_1) > t_2(s_1) \\ a_0 & \text{otherwise.} \end{cases} \tag{3}$$

As $t$ is admissible, $h$ is a well-defined allocation rule. Using (2) it is easily verified that $(h,t)$ is ex post incentive compatible.

A mechanism $(h,t)$ is *non-trivial* if there exist at least two distinct outcomes, each of which is in the range of $h$ at a positive (Lebesgue) measure of information states. Any pair of personalized prices implements an ex post implementable allocation rule. However, the mechanism may be trivial. Existence of a non-trivial ex post incentive compatible mechanism is proved in the next section.

## 3. Existence

We prove that under reasonable assumptions, a non-trivial ex post incentive compatible mechanism exists in the model described in the previous section. An extension of this result to $n$ buyers is straightforward and sketched out in Section A of the appendix. The possibility of non-trivial mechanisms when many objects are to be allocated is explored in Section B of the appendix.

We assume that higher signals correspond to better news. That is, players' reservation values do not decrease as buyer signals increase.[9] In order to simplify the proofs, we assume also that buyers' reservation values are continuous.

Assumption 1. For $i = 1, 2$, $V_i$ is (a) non-decreasing and (b) continuous.

The assumption that $V_i$ is non-decreasing in $s_j$ can be dropped provided one assumes that $V_i$ is increasing in $s_i$. The next assumption is a generalization of the single-crossing property.[10]

Assumption 2. For $i, j = 1, 2$, for any $s_j$ we have

$$V_i(s_i', s_j) - V_i(s_i, s_j) \geq V_j(s_j, s_i') - V_j(s_j, s_i) \qquad \forall s_i' > s_i.$$

As buyer $i$'s signal increases from $s_i$ to $s_i'$, the increase in $i$'s value is not less than the increase in buyer $j$'s value. That is, buyer $i$'s value is at least as responsive as buyer $j$'s value to changes in buyer $i$'s signal. In models with one-dimensional signals, Assumption 2 is a version of the single-crossing property that is a sufficient condition for existence of an efficient mechanism in such models (see Maskin 1992).

The next assumption rules out the uninteresting case where the efficient rule is trivial. As any trivial rule is ex post incentive compatible, if Assumption 3 is violated then the efficient rule is ex post incentive compatible.

[9] The following terminology regarding monotonicity of a function $f : \mathbb{R}^n \to \mathbb{R}$ is adopted. For $x, x' \in \mathbb{R}^n$, $x' > x$ denotes that $x'$ is at least as large as $x$ in every co-ordinate and $x' \neq x$. If $f(x') \geq f(x)$ whenever $x' > x$ then $f$ is *non-decreasing*. If $f(x') > f(x)$ whenever $x' > x$ then $f$ is *increasing*.

[10] An equivalent assumption is that for each $s_j$, $V_i(s_i, s_j) - V_j(s_j, s_i)$ is a non-decreasing function of $s_i$.

ASSUMPTION 3. For each buyer, there exists a positive measure of information states at which this buyer's valuation is strictly greater than the other buyer's valuation.

We construct an admissible pair of personalized prices under Assumptions 1 and 2. If, in addition, Assumption 3 holds then the allocation rule implemented by this pair of admissible prices is non-trivial.

Fix buyer $j$'s signal at some level $s_j$. The domain of $s_i$, $i \neq j$, is the unit cube in $\mathbb{R}_+^{d_i}$ and each buyer's valuation is non-decreasing in $s_i$. Therefore, with buyer $j$'s signal fixed at $s_j$, the maximum of either buyer's reservation value as a function of buyer $i$'s signal is attained when $s_i = \mathbf{1}$, where $\mathbf{1}$ denotes the point $(1,1,\dots,1)$ in $\mathbb{R}_+^{d_i}$. Similarly, the minimum of either buyer's value as a function of $s_i$ is attained at $s_i = \mathbf{0} \equiv (0,0,\dots,0)$. Define $S_i(\lambda, s_j)$, the set of signals of buyer $i$ that lead to the same reservation value for buyer $i$ as the signal $\hat{s}_i = \lambda\mathbf{1}$, where $\lambda \in [0,1]$. That is,

$$S_i(\lambda, s_j) \equiv \{s_i \in S_i \mid V_i(s_i, s_j) = V_i(\lambda\mathbf{1}, s_j)\} \qquad 0 \leq \lambda \leq 1.$$

Thus, for a fixed $s_j$, buyer $i$'s signal space is partitioned into equivalence classes or "indifference" curves, $S_i(\lambda, s_j)$, one for each $\lambda \in [0,1]$, with $V_i(\lambda'\mathbf{1}, s_j) \geq V_i(\lambda\mathbf{1}, s_j)$ whenever $\lambda' > \lambda$.

While buyer $i$'s value (as a function of $s_i$) is constant on his indifference curve $S_i(\lambda, s_j)$, buyer $j$'s value is, in general, not constant on this set. The maximum of buyer $j$'s value on buyer $i$'s indifference curve $S_i(\lambda, s_j)$ is

$$V_{ij}^m(\lambda, s_j) \equiv \max_{s_i \in S_i(\lambda, s_j)} V_j(s_j, s_i), \tag{4}$$

and the maximum is achieved at

$$s_{ij}^m(\lambda, s_j) \in \arg \max_{s_i \in S_i(\lambda, s_j)} V_j(s_j, s_i).$$

Thus, $V_{ij}^m(\lambda, s_j) = V_j(s_j, s_{ij}^m(\lambda, s_j))$ and $V_i(s_{ij}^m(\lambda, s_j), s_j) = V_i(\lambda\mathbf{1}, s_j)$. As $S_i$ is compact and $V_i(\cdot, s_j)$ is continuous, $S_i(\lambda, s_j)$ is compact. This, together with the continuity of $V_j(s_j, \cdot)$ implies that $V_{ij}^m(\lambda, s_j)$ exists. The continuity of the valuations (Assumption 1b) implies that $V_{ij}^m(\lambda, s_j)$ is continuous in $\lambda$ and $s_j$. As $V_j$ is non-decreasing in $s_i$ (Assumption 1a), $V_{ij}^m(\lambda, s_j)$ is non-decreasing in $\lambda$.

Figure 1 depicts indifference curves of buyers 1 and 2 in buyer 1's (two-dimensional) signal space, keeping buyer 2's signal fixed at some value $s_2$. By Assumption 1a, indifference curves are negatively sloped. However, (i) the indifference curves need not be convex; (ii) indifference curves may touch the axes; and (iii) the maximum value for buyer 2 in buyer 1's indifference curve may be attained at more than one point. "Thick" indifference curves are not ruled out, unless we strengthen Assumption 1a to require that buyers' valuations be (strictly) increasing in signals. At any other value of buyer 2's signal, $s_2' \neq s_2$, buyer 1's indifference curves in $s_1$ space are different from, and may intersect with, the ones depicted in Figure 1.[11]

[11]The two sets of buyer 1 indifference curves for two different values of buyer 2 signals do not intersect if buyer 1's valuation is separable in $s_1$ and $s_2$, i.e., $V_1(s_1, s_2) = u(s_1) + v(s_2)$.

Ex post incentive compatibility imposes the following necessary condition. If buyer 1, say, is allocated the object at information state $(s_1, s_2)$, then he should also be allocated the object at any information state $(s_1', s_2)$ such that $V_1(s_1', s_2) > V_1(s_1, s_2)$. Otherwise, buyer 1 would have an incentive to report $s_1$ instead of $s_1'$ at the information state $(s_1', s_2)$. That is, an implementable allocation rule must be weakly monotone.[12]

We construct an ex post incentive compatible mechanism in which, for each value of $s_j$, there exists a $\lambda^*_{ij}(s_j) \in [0,1]$ such that buyer $i$ wins if and only if his signal is in an indifference curve (with an index) greater than $\lambda^*_{ij}(s_j)$. Clearly, this allocation rule satisfies weak monotonicity. Call $S_i(\lambda^*_{ij}(s_j), s_j)$ the *pivotal indifference curve* for buyer $i$ at $s_j$. Any $s_i$ in the pivotal indifference curve is a *pivotal signal* for buyer $i$. Buyer $i$'s personalized price is defined to be $V^m_{ij}(\lambda^*_{ij}(s_j), s_j)$, the maximum of buyer $j$'s value in buyer $i$'s pivotal indifference curve (and this is usually equal to buyer $i$'s value on the pivotal indifference curve).[13] These personalized prices are admissible if $V_i(\lambda\mathbf{1}, s_j) - V^m_{ij}(\lambda, s_j)$ is non-decreasing in $\lambda$. This is shown in the next lemma.

LEMMA 2. *If Assumptions 1 and 2 are satisfied then for any $s_j$ and $1 \geq \lambda' > \lambda > 0$,*

$$V_i(\lambda'\mathbf{1}, s_j) - V^m_{ij}(\lambda', s_j) \geq V_i(\lambda\mathbf{1}, s_j) - V^m_{ij}(\lambda, s_j).$$

PROOF. To simplify notation, we write $s^m_{ij}(\lambda')$, $s^m_{ij}(\lambda)$ for $s^m_{ij}(\lambda', s_j)$, $s^m_{ij}(\lambda, s_j)$.

Let $\lambda^m \in [0,1]$ be such that $\lambda^m s^m_{ij}(\lambda') \in S_i(\lambda, s_j)$. To see that $\lambda^m$ exists, define $f(x) \equiv V_i(x s^m_{ij}(\lambda'), s_j)$, where $x \in [0,1]$, and note that $f(1) = V_i(s^m_{ij}(\lambda'), s_j) = V_i(\lambda'\mathbf{1}, s_j) \geq V_i(\lambda\mathbf{1}, s_j) \geq V(0, s_j) = f(0)$. By Assumption 1b, $f(x)$ is a continuous function of $x$, and therefore there exists $\lambda^m$ such that $f(\lambda^m) = V_i(\lambda^m s^m_{ij}(\lambda'), s_j) = V_i(\lambda\mathbf{1}, s_j)$. (As shown in Figure 2, $\lambda^m s^m_{ij}(\lambda')$ is on the line joining $s^m_{ij}(\lambda')$ to the origin.) Hence,

$$\begin{aligned} V_i(\lambda\mathbf{1}, s_j) - V^m_{ij}(\lambda, s_j) &= V_i(\lambda^m s^m_{ij}(\lambda'), s_j) - V^m_{ij}(\lambda, s_j) \\ &\leq V_i(\lambda^m s^m_{ij}(\lambda'), s_j) - V_j(s_j, \lambda^m s^m_{ij}(\lambda')) \\ &\leq V_i(s^m_{ij}(\lambda'), s_j) - V_j(s_j, s^m_{ij}(\lambda')) \\ &= V_i(\lambda'\mathbf{1}, s_j) - V^m_{ij}(\lambda', s_j) \end{aligned}$$

where the first inequality follows from the fact that $\lambda^m s^m_{ij}(\lambda') \in S_i(\lambda, s_j)$ and (4), and the second inequality from Assumption 2. □

For $\lambda \in [0,1]$, define

$$g_{ij}(\lambda; s_j) \equiv V_i(\lambda\mathbf{1}, s_j) - V^m_{ij}(\lambda, s_j).$$

[12] See Bikhchandani et al. (2006) for conditions under which weak monotonicity is also sufficient for incentive compatibility.

[13] In Figure 1, $V_1(s_1, s_2) = V_P$ is the pivotal indifference curve for buyer 1 when buyer 2's signal is at $s_2$. On this indifference curve, the highest valuation of buyer 2 is also $V_P$, which is buyer 1's personalized price at $s_2$.

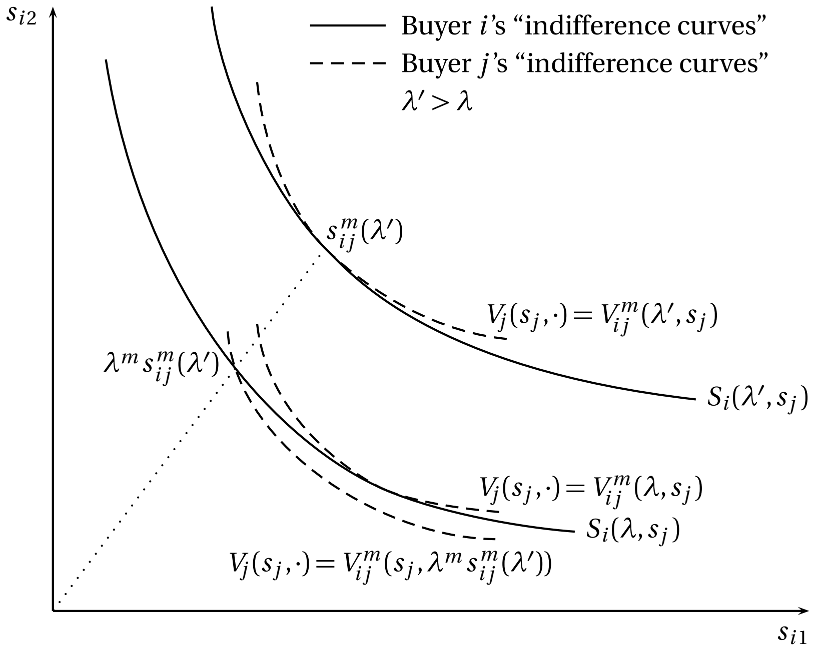


FIGURE 2. Proof of Lemma 2.

Lemma 2 implies that $g_{ij}(\lambda; s_j)$ is a non-decreasing function of $\lambda$. The continuity of $V_i$ and of $V_{ij}^m$ implies that $g_{ij}(\lambda; s_j)$ is continuous (in $\lambda$). Thus, the following is well defined:

$$\lambda_{ij}^*(s_j) \equiv \begin{cases} 1 & \text{if } g_{ij}(1; s_j) < 0 \\ \max\{\lambda \in [0,1] \mid g_{ij}(\lambda; s_j) = 0\} & \text{if } g_{ij}(1; s_j) \geq 0 \geq g_{ij}(0; s_j) \\ 0^- & \text{if } g_{ij}(0; s_j) > 0, \end{cases}$$

where $0^-$ is a negative number arbitrarily close to 0. Hence, $V_i(\lambda \mathbf{1}, s_j) > V_{ij}^m(\lambda, s_j)$ if and only if $\lambda > \lambda_{ij}^*$.[14] Define $V_{ij}^m(0^-, s_j) = V_{ij}^m(0, s_j)$. Then, as $V_{ij}^m(\lambda, s_j)$ is non-decreasing in $\lambda$, we have

$$V_i(\lambda \mathbf{1}, s_j) > V_{ij}^m(\lambda_{ij}^*, s_j) \text{ if and only if } V_i(\lambda \mathbf{1}, s_j) > V_{ij}^m(\lambda, s_j) \text{ if and only if } \lambda > \lambda_{ij}^*. \quad (5)$$

Let

$$t_i^*(s_j) \equiv V_{ij}^m(\lambda_{ij}^*, s_j) \quad (6)$$

be buyer $i$'s personalized price as a function of $s_j$.[15] Theorem 1 shows that the following allocation rule is non-trivial and implementable: buyer $i$ wins if and only if his valuation exceeds the personalized price defined in (6).

[14] Hereafter, the dependence of $\lambda_{ij}^*$ on $s_j$ is usually suppressed to simplify the notation.

[15] Note that if $\lambda_{ij}^* \in [0,1)$ then $V_i(\lambda_{ij}^* \mathbf{1}, s_j) = V_{ij}^m(\lambda_{ij}^*, s_j)$ and therefore $t_i^*(s_j) = V_i(\lambda_{ij}^* \mathbf{1}, s_j)$. If $\lambda_{ij}^* = 0^-$ then $V_i(\mathbf{0}, s_j) > V_{ij}^m(0, s_j) = t_i^*(s_j)$ and if $\lambda_{ij}^* = 1$ then $V_i(\mathbf{1}, s_j) \leq V_{ij}^m(1, s_j) = t_i^*(s_j)$.

THEOREM 1. *The personalized prices $t^* = (t_1^*, t_2^*)$ defined in (6) are admissible. The mechanism $(h^*, t^*)$, where $h^*$ is supported by $t^*$, is non-trivial and ex post incentive compatible.*

PROOF. Suppose that the information state is $(s_i, s_j)$. Let $\lambda_i$ be defined by $V_i(\lambda_i \mathbf{1}, s_j) = V_i(s_i, s_j)$. Note that (4) implies

$$V_2(s_2, s_1) \leq V_{12}^m(\lambda_1, s_2) \quad \text{and} \quad V_1(s_1, s_2) \leq V_{21}^m(\lambda_2, s_1). \tag{7}$$

Suppose that $V_1(s_1, s_2) > t_1^*(s_2) = V_{12}^m(\lambda_{12}^*, s_2)$. By (5), $\lambda_1 > \lambda_{12}^*$ and $V_1(s_1, s_2) = V_1(\lambda_1 \mathbf{1}, s_2) > V_{12}^m(\lambda_1, s_2)$. Hence, (7) implies that $V_{21}^m(\lambda_2, s_1) > V_2(s_2, s_1) = V_2(\lambda_2 \mathbf{1}, s_1)$. From (5) we have $\lambda_{21}^* \geq \lambda_2$ and, therefore, $V_2(s_2, s_1) = V_2(\lambda_2 \mathbf{1}, s_1) \leq V_{21}^m(\lambda_{21}^*, s_1) = t_2^*(s_1)$. An identical argument implies that if, instead, $V_2(s_2, s_1) > t_2^*(s_1)$, then $V_1(s_1, s_2) \leq t_1^*(s_2)$. Thus, $t^*$ is admissible. Define an allocation rule

$$h^*(s_1, s_2) \equiv \begin{cases} a_1 & \text{if } V_1(s_1, s_2) > t_1^*(s_2) \\ a_2 & \text{if } V_2(s_2, s_1) > t_2^*(s_1) \\ a_0 & \text{otherwise.} \end{cases}$$

Clearly, the mechanism $(h^*, t^*)$ is feasible and ex post incentive compatible.

To complete the proof, we show that $(h^*, t^*)$ is non-trivial. Let information state $s^1 = (s_1^1, s_2^1)$ be such that $V_1(s_1^1, s_2^1) > V_2(s_1^1, s_2^1)$. Assumption 3 guarantees that there exists a positive measure of such information states. By Assumption 2, $V_1(\mathbf{1}, s_2^1) > V_2(s_2^1, \mathbf{1})$ and by Assumption 1a, $V_2(s_2^1, \mathbf{1}) = V_{12}^m(1, s_2^1)$. Thus, $V_1(\mathbf{1}, s_2^1) > V_{12}^m(1, s_2^1)$, which implies $\lambda_{12}^*(s_2^1) < 1$. Hence buyer 1 gets the object at $(s_1, s_2^1)$ for all $s_1 \in S_1(\lambda, s_2^1)$, $\lambda \in (\lambda_{12}^*(s_2^1), 1]$. As there is a positive measure of information states at which buyer 1's value is strictly greater than buyer 2's value, there is a positive measure of information states at which buyer 1 is allocated the object. A similar argument establishes that buyer 2 is allocated the object at a positive measure of information states. Hence, the mechanism is non-trivial. □

Neither buyer is allocated the object when their valuations are close to each other. This occurs at information states $s = (s_i, s_j)$ such that $\lambda_i(s) < \lambda_{ij}^*(s_j)$ and $\lambda_j(s) < \lambda_{ji}^*(s_i)$. Such information states have positive measure unless either (i) buyers' indifference curves in $S_i$ space for each fixed $s_j$, $i = 1, 2$, $i \neq j$ are identical or (ii) Assumption 3 is not satisfied. From Jehiel and Moldovanu (2001) we know that any incentive compatible mechanism, including this one, is (generically) inefficient.[16] The source of the inefficiency in the current mechanism is the cost of enforcing incentives when buyer valuations are close.

A mechanism is *conditionally efficient* if, when the object is allocated, it is given to the buyer with the highest valuation. Conditional efficiency is a desirable property for a seller who wishes to prevent resale. The mechanism constructed in Theorem 1 is conditionally efficient. To see this, suppose that buyer $i$ is allocated the object at information state $s = (s_i, s_j)$. Let $\lambda_i(s_i, s_j)$ and $\lambda_{ij}^*(s_j)$ be defined at this state in the usual manner. Then, from the proof of Theorem 1 it is clear that $\lambda_i(s_i, s_j) > \lambda_{ij}^*(s_j)$ and therefore

[16] Eső and Maskin (2000) describe a multi-dimensional signal environment in which an efficient mechanism exists.

$V_i(s_i,s_j) > V_{ij}^m(\lambda_i(s_i,s_j),s_j) \geq V_j(s_j,s_i)$. In the next section we show that this mechanism allocates the object at more information states, and is therefore more efficient, than any other conditionally efficient mechanism.

Recall that any signal $s_i$ in the pivotal indifference curve $S_i(\lambda_{ij}^*,s_j)$ is a pivotal signal for buyer $i$ at $s_j$. With buyer $j$'s signal fixed at $s_j$, buyer $i$ wins (loses) at signals greater (less) than a pivotal signal. Thus a pivotal signal is an infimum of winning signals. The price paid by a winning buyer $i$ equals the valuation of this buyer at a pivotal signal (provided that $\lambda_{ij}^* \in [0,1)$). This is similar to the generalizations of Vickrey auctions in Ausubel (1999) and in Dasgupta and Maskin (2000), where buyers have one-dimensional signals.[17] However, unlike in these models, in the mechanism of Theorem 1 the valuations of buyers $i$ and $j$ need not be equal at a pivotal signal of buyer $i$; at a pivotal signal of buyer $i$, buyer $i$'s valuation equals the most that buyer $j$'s valuation can be in the pivotal indifference curve of buyer $i$. The difference arises because in one-dimensional models, indifference curves of buyer $i$ signals are singletons and hence for a given realization of $s_j$ there can be only one pivotal signal for buyer $i$. A second difference is the role of the single-crossing property or Assumption 2. With one-dimensional signals, the single-crossing property is sufficient for the existence of an efficient ex post incentive compatible mechanism whereas with multi-dimensional signals Assumption 2 is sufficient for the existence of an ex post incentive compatible mechanism (which is inefficient).

Next, we illustrate the construction in Theorem 1 with an example from Jehiel et al. (2006).

Example 1. Two buyers compete for an indivisible object. Each gets a pair of signals $(p_i,c_i)$, $i=1,2$. Buyer $i$'s valuation for the object is $V_i(p_i,c_i,p_j,c_j) = p_i + c_i c_j$, $j \neq i$. Further, each buyer's signal lies in the unit square: $(p_i,c_i) \in [0,1]^2$, $i=1,2$.

Fix buyer $j$'s signals at $(p_j,c_j)$. Buyer $i$'s indifference curves in $c_i,p_i$ space are straight lines with slope $-c_j$. (Buyer $j$'s indifference curves are vertical lines, as $V_j$ does not depend upon $p_i$.) It may be verified that the pivotal indifference curve goes through the point $p_i = c_i = (p_j+c_j)/(1+c_j)$ and $V_i = V_{ij}^m = p_j + c_j$ on this indifference curve.

Therefore, using (6), define personalized prices $t_i^*(p_j,c_j) = p_j + c_j$, $t_j^*(p_i,c_i) = p_i + c_i$. Theorem 1 implies (and it may be directly verified) that these prices are admissible. Let $h^*$ be the allocation rule supported by the prices $t_1^*$, $t_2^*$. In the ex post incentive compatible mechanism $(h^*,t^*)$, the buyers report their private signals. The mechanism designer allocates the object to buyer $i$ for a payment equal to his personalized price $t_i^*(p_j,c_j) = p_j + c_j$ if and only if $V_i(p_i,c_i,p_j,c_j) = p_i + c_i c_j$ exceeds $t_i^*(p_j,c_j)$. As noted in the discussion after Theorem 1, this mechanism is efficient conditional on the object being allocated to a buyer. This is checked directly as $V_i = p_i + c_i c_j > p_j + c_j \geq p_j + c_i c_j = V_j$.

[17]There is one difference in Dasgupta and Maskin (2000). The mechanism designer (auctioneer) does not know the mapping from buyer signals to valuations. Hence, buyers submit contingent bids rather than report their private signals.

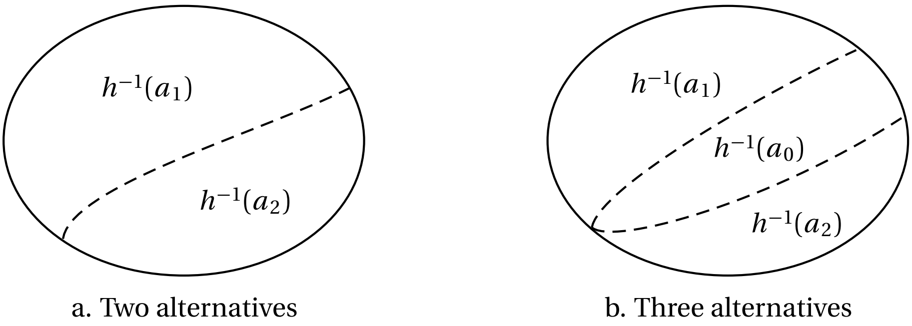


FIGURE 3. Domain of buyers' signals, $S$

Let $(h^*)^{-1}(a_k)$ be the set of information states that are mapped onto $a_k$ by this allocation mechanism. We have

$$(h^*)^{-1}(a_i) = \{(p_i, c_i, p_j, c_j) \in [0,1]^4 \mid p_i - p_j > c_j - c_i c_j\}, \quad i = 1,2$$
$$(h^*)^{-1}(a_0) = \{(p_i, c_i, p_j, c_j) \in [0,1]^4 \mid c_i c_j - c_i \leq p_i - p_j \leq c_j - c_i c_j\}.$$

The boundary between the sets $(h^*)^{-1}(a_1)$ and $(h^*)^{-1}(a_2)$ is

$$\overline{(h^*)^{-1}(a_1)} \cap \overline{(h^*)^{-1}(a_2)} = \{(p_1, c_1, p_2, c_2) \in [0,1]^4 \mid p_1 = p_2,\ c_1 = c_2 = 0\},$$

where $\overline{A}$ is the closure of set $A$. The boundary between two subsets of the unit cube in $\mathbb{R}^4$ can be three-dimensional; here, the boundary between between $(h^*)^{-1}(a_1)$ and $(h^*)^{-1}(a_2)$ is one-dimensional. We refer to this as the boundary being of less than full dimension, and return to this point below. ◊

*Relationship with Jehiel et al. (2006)*

Consider a setting where the object is always allocated, i.e., the outcome $a_0$ is ruled out. Define $\mu^1(s) \equiv V_1(s)$, $\mu^2(s) \equiv -V_2(s)$, the difference between the two buyers' valuations for outcomes $a_1$ and $a_2$. The domain of signals, $S$, is shown schematically in Figure 3a. Any non-trivial allocation rule $h$ partitions $S$ into two subsets, depending on whether $h(s) = a_1$ or $h(s) = a_2$. The boundary between these two sets is the broken line in Figure 3a. Jehiel et al. show that for any non-trivial ex post incentive compatible allocation rule this boundary is full dimensional and ex post incentive compatibility implies that the gradients of $\mu^1(s)$ and $\mu^2(s)$ on the boundary must be, roughly speaking, co-directional. They show that this condition is impossible to satisfy for generic reservation values and hence the generic non-existence of ex post incentive compatible mechanisms.

The Jehiel et al. proof depends on the assumption that each buyer is not indifferent between the two outcomes $a_1$ and $a_2$. Suppose we add back the third outcome, $a_0$. For their argument to extend it must be case that each buyer is not indifferent between any two of the three outcomes. This assumption does not hold in the private goods model of this section as buyer $i$ is indifferent between $a_j$, $j \neq i$, and $a_0$ at every information state. Now consider a non-trivial ex post incentive compatible allocation rule $h$ that yields each of the three outcomes $a_0$, $a_1$, and $a_2$. Jehiel et al.'s theorem implies that the boundary between $h^{-1}(a_1)$ and $h^{-1}(a_2)$ has less than full dimension. Their theorem does not impose any restriction on the dimensionality of the boundary between $a_0$ and $a_i$, $i = 1, 2$. In particular, the possibility that $h$ partitions $S$ as shown in Figure 3b is not ruled out. In fact, this figure is a schematic representation of Example 1 where we demonstrated existence of an ex post incentive compatible mechanism in which the boundary between $h^{-1}(a_1)$ and $h^{-1}(a_2)$ is of less than full dimension.

More generally, suppose there are $i = 1,2,\dots,n$ buyers and $L$ outcomes labeled $a_\ell$, $\ell = 1,2,\dots,L$. Suppose that there exist two outcomes $a_\ell$ and $a_k$ such that all buyers except one are indifferent between these two outcomes at every information state:

$$a_\ell(s) \sim_i a_k(s) \quad \forall s, \forall i \neq j. \tag{8}$$

The Jehiel et al. theorem does not rule out generic existence of a non-trivial ex post incentive compatible mechanism with outcomes $a_\ell$ and $a_k$.

Consider the allocation of a bundle of private goods to $n$ buyers. Each outcome is an assignment of objects among the $n$ buyers, where we allow the possibility that not all objects are allocated to the buyers. Let $a_\ell$ be any assignment and let $a_k$ be another assignment that differs from $a_\ell$ only in the allocation that buyer $j$ receives.[18] Condition (8) is satisfied for each assignment $a_\ell$. A full range ex post incentive compatible mechanism is a possibility.[19]

Thus, in order to obtain their non-existence result, the notion of genericity that Jehiel et al. consider necessarily includes perturbations that add externalities to the model. Theorem 1 is true for a class of models, namely those satisfying Assumptions 1 to 3, that is not non-generic in the space of quasilinear models without allocative externalities.

Theorem 1 is easily extended to allow for $n$ buyers (see Section A of the appendix). Non-trivial ex post incentive compatible mechanisms exist also when there are several objects for sale and buyers have subadditive preferences (see Section B).

## 4. Constrained efficiency

We obtain a necessary condition and a sufficient condition for second-best efficient mechanisms within the class of ex post incentive compatible mechanisms. Throughout this section, strengthened versions of Assumptions 1 and 2 (of Section 3) are assumed, with strict inequality replacing weak inequality. Thus, $V_i$ is continuous, increasing, and strict single-crossing holds.

A mechanism *dominates* another mechanism if at each information state the sum of reservation values attained under the first mechanism is weakly greater than the sum of reservation values attained under the second mechanism, with strict improvement at

[18] Thus, not all the objects are allocated in at least one of the two assignments $a_\ell$, $a_k$.

[19] A mechanism has full range if each outcome is implemented at a positive measure of information states.

some state. An ex post incentive compatible mechanism is (ex post) *constrained efficient* if it is undominated by another ex post incentive compatible mechanism. The definition of constrained efficiency is due to Holmström and Myerson (1983).

In Section 2 we saw that any implementable allocation rule can be implemented through admissible prices.[20] The allocation rules supported by a pair of admissible prices differ along pivotal indifference curves. We modify (3), the definition of an allocation rule supported by admissible prices, so as to allocate the object to buyers on pivotal indifference curves whenever possible. Define

$$h(s_1,s_2)\equiv\begin{cases} a_1 & \text{if } V_1(s_1,s_2)>t_1(s_2)\\ a_1 & \text{if } V_1(s_1,s_2)=t_1(s_2) \text{ and } V_2(s_2,s_1)\le t_2(s_1)\\ a_2 & \text{if } V_2(s_2,s_1)>t_2(s_1)\\ a_2 & \text{if } V_2(s_2,s_1)=t_2(s_1) \text{ and } V_1(s_1,s_2)<t_1(s_2)\\ a_0 & \text{otherwise.}\end{cases} \tag{9}$$

This allocation rule $h$ is supported by admissible prices $t$, i.e., $(h,t)$ is ex post incentive compatible; moreover, it dominates the allocation rule defined in (3) and is undominated by any other allocation rule supported by $t$.

Let $(t_i,t_j)$ be admissible prices. Fix buyer $j$'s signal at $s_j$. At any buyer $i$ pivotal signal $s_i$, we have $V_i(s_i,s_j)=t_i(s_j)$. Admissibility, together with continuity of valuations in signals, implies that for any $s_i$ that is pivotal at $s_j$, we have $V_j(s_j,s_i)\le t_j(s_i)$, i.e., $s_j$ is (weakly) less than pivotal. With this background, consider the following definitions.

A pair of signals $(s_i,s_j)$ is *mutually pivotal* under admissible prices $(t_i,t_j)$ if $s_i$ is pivotal for buyer $i$ at $s_j$ and $s_j$ is pivotal for buyer $j$ at $s_i$. If at every $s_j$, $j=1,2$, for which there is a buyer $i$ pivotal signal, there exist mutually pivotal signals $(s_i,s_j)$, then admissible prices $(t_i,t_j)$ are *mutually pivotal.* This is a necessary condition for constrained efficiency.

THEOREM 2. *Let $(h,t)$ be an ex post incentive compatible mechanism in which the admissible prices are continuous functions. If $(h,t)$ is constrained efficient then the admissible prices must be mutually pivotal.*

PROOF. Suppose that continuous admissible prices $(t_i,t_j)$ are not mutually pivotal. That is, there exists $s_j$ such that

$$V_i(\mathbf{0},s_j)\le t_i(s_j)\le V_i(\mathbf{1},s_j) \tag{10}$$

$$\text{for any } s_i,\ V_i(s_i,s_j)=t_i(s_j)\Longrightarrow V_j(s_j,s_i)<t_j(s_i). \tag{11}$$

Inequalities (10) imply that there exists at least one buyer $i$ pivotal signal at $s_j$, and (11) states that $s_j$ is not buyer $j$ pivotal at any of these buyer $i$ pivotal signals. Define $t'_j\equiv t_j$, and $t'_i$ to be identical to $t_i$ except at $s_j$ where $t'_i(s_j)=t_i(s_j)-\epsilon$. The continuity of $V_i$, $V_j$,

[20]For this reason, we sometimes write that admissible prices $t$ dominate admissible prices $t'$ or that admissible prices $t$ are constrained efficient.

and $t_j$ implies that if $\epsilon > 0$ is small then $t'$ is admissible. (Of course $t'$ is not continuous.) When buyer $j$'s signal is not $s_j$, (the allocation rules implemented by) $t'$ and $t$ lead to identical outcomes. However, at $s_j$ the subset of buyer $i$ signals at which buyer $i$ gets the object is strictly larger under $t'$ than under $t$ whereas the subset of buyer $i$ signals at which buyer $j$ gets the object is identical under $t$ and $t'$. Thus, $t$ is dominated by $t'$. □

Consider the following definition. Admissible personalized prices $(t_i, t_j)$ are *strongly mutually pivotal* if for each $s_j$

(i) if $V_i(\mathbf{0}, s_j) \leq t_i(s_j) \leq V_i(\mathbf{1}, s_j)$ then there exists $s_i$ such that

$$t_i(s_j) = V_i(s_i, s_j) = V_j(s_j, s_i) = t_j(s_i)$$

(ii) if, instead, $V_i(\mathbf{0}, s_j) > t_i(s_j)$ then $V_i(\mathbf{0}, s_j) \geq V_j(s_j, \mathbf{0})$

(iii) if, instead, $V_i(\mathbf{1}, s_j) < t_i(s_j)$ then $V_i(\mathbf{1}, s_j) \leq V_j(s_j, \mathbf{1})$, and $\forall s_i$, $V_j(s_j, s_i) > t_j(s_i)$.

Condition (i) strengthens the requirement of mutually pivotal prices to require that there exist mutually pivotal signals at which the two buyers' valuations are the same. Conditions (ii) and (iii) impose restrictions when there do not exist buyer $i$ pivotal signals and either buyer $i$ always wins or buyer $i$ always loses at $s_j$. As valuations are continuous, strongly mutually pivotal prices are continuous for the range of signals in which (i) holds.

For one-dimensional signals, a generalized Vickrey auction is first-best efficient (provided single crossing holds); it is also strongly mutually pivotal. This condition is sufficient for constrained efficiency when signals are multi-dimensional.

THEOREM 3. *An ex post incentive compatible mechanism $(h, t)$ is constrained efficient if $(t_1, t_2)$ are strongly mutually pivotal.*

PROOF. Suppose that $(t_i, t_j)$ are strongly mutually pivotal. Fix buyer $j$'s signal at $s_j$ and suppose that a pivotal signal exists for buyer $i$ (case (i) of the definition of strongly mutually pivotal). Then there exists $s_i$ such that $t_i(s_j) = V_i(s_i, s_j) = V_j(s_j, s_i) = t_j(s_i)$. Suppose that $s_i \neq \mathbf{0}$ or $\mathbf{1}$. For small enough $\delta, \hat{\delta} \in \mathbb{R}_+^{d_i}$, $\delta$, $\hat{\delta} \neq \mathbf{0}$, we have $s_i - \delta$, $s_i + \hat{\delta} \in [0,1]^{d_i}$. The strict single-crossing property implies that

$$V_i(s_i - \delta, s_j) < V_j(s_j, s_i - \delta)$$
$$V_i(s_i + \hat{\delta}, s_j) > V_j(s_j, s_i + \hat{\delta}).$$

Let $(t_i', t_j')$ be another pair of admissible personalized prices. Suppose that $t_i'(s_j) < t_i(s_j)$ at $s_j$. For small enough $\delta$ the object is allocated to buyer $j$ under $t$ and to buyer $i$ under $t'$ at $(s_i - \delta, s_j)$. The first inequality above implies that $t'$ does not dominate $t$. Suppose instead that $t_i'(s_j) > t_i(s_j)$. For small enough $\hat{\delta}$ the object is allocated to buyer $i$ under $t$ and to buyer $j$ under $t'$ at $(s_i + \hat{\delta}, s_j)$. The second inequality above implies that $t'$ does not dominate $t$. The proof for $s_i = \mathbf{0}$ or $\mathbf{1}$ is similar, with only one of the above inequalities being used in each case.

Next, suppose that there is no buyer $i$ pivotal signal at $s_j$. First, consider case (ii). By single-crossing $V_i(s_i, s_j) \geq V_j(s_j, s_i)$ for all $s_i$. Under $t$ buyer $i$ gets the object at $(s_i, s_j)$ for all $s_i$ and hence the allocation cannot be improved. Suppose, instead, we are in case (iii). By single-crossing, for all $s_i$ the best use of the object at $(s_i, s_j)$ is to allocate it to buyer $j$, which is the allocation under $t$. □

The notion of constrained efficiency is different from that of conditional efficiency. There usually does not exist a mechanism that satisfies both. As already noted, the mechanism constructed in Theorem 1 is conditionally efficient. However, it need not satisfy the mutually pivotal prices condition of Theorem 2 and therefore is, in general, not constrained efficient. (Recall that the admissible prices in this mechanism are continuous.) This is illustrated next, in a continuation of Example 1.

EXAMPLE 2. Consider the same setting and valuation functions as in Example 1 of the previous section. The allocation rule supported by the personalized prices $t_i^*(p_i, c_i) = p_i + c_i$ is the mechanism of Theorem 1. We directly verified that this mechanism is conditionally efficient. Theorem 4 below implies that it dominates any other conditionally efficient mechanism. However, as shown below, this mechanism is not constrained efficient.

Fix buyer 2's signal at $(p_2, c_2)$, $c_2 > 0$. Buyer 1's pivotal indifference curve is $p_1 = p_2 + c_2 - c_1 c_2$. For any $(p_1, c_1)$ on this indifference curve,

$$t_2^*(p_1, c_1) = p_1 + c_1 = p_2 + c_2 - c_1 c_2 + c_1 > p_2 + c_1 c_2 = V_2(p_2, c_2, p_1, c_1).$$

Thus, (11) implies that $t_1^*$ and $t_2^*$ are not mutually pivotal. As $t_1^*$ and $t_2^*$ are continuous, Theorem 2 implies that this mechanism is not constrained efficient.

*A constrained efficient mechanism* Define personalized prices $t_i(p_j, c_j) \equiv p_j + c_j^2$ and $t_j(p_i, c_i) \equiv p_i + c_i^2$. Suppose that

$$V_i(p_i, c_i, p_j, c_j) = p_i + c_i c_j > p_j + c_j^2 = t_i(p_j, c_j).$$

Then $p_i - p_j > c_j^2 - c_i c_j \geq c_i c_j - c_i^2$ implies that

$$V_j(p_j, c_j, p_i, c_i) = p_j + c_i c_j < p_i + c_i^2 = t_j(p_i, c_i).$$

Consequently, personalized prices $p_2 + c_2^2$ for buyer 1 and $p_1 + c_1^2$ for buyer 2 are admissible. Let $h$ be the allocation rule supported by the prices $t_1, t_2$ (as defined in (9)). The mechanism $(h, t)$ is ex post incentive compatible. From the fact $t_i(p_j, c_j) \leq t_i^*(p_j, c_j)$, with strict inequality whenever $c_j < 1$, we conclude that (i) $(h, t)$ dominates $(h^*, t^*)$, and (ii) $(h, t)$ is non-trivial.[21]

We show that $(t_1, t_2)$ are strongly mutually pivotal. Fix buyer 2's signal at $(p_2, c_2) = (p, c) \in [0, 1]^2$. As $V_1((0,0), (p, c)) \leq t_1(p_2, c_2) = p + c^2 \leq V_1((1,1), (p, c))$, condition (i) of the

[21] In fact, if $(p_j, c_j) \neq (0,0)$ or (1,1) then each of the outcomes $a_0$, $a_1$, and $a_2$ is implemented for a positive Lebesgue measure of buyer $i$ signals.

definition applies. Observe that $(p_1,c_1)=(p,c)$ and $(p_2,c_2)=(p,c)$ are mutually pivotal signals that satisfy the restriction in condition (i):

$$V_1((p,c),(p,c))=p+c^2=V_2((p,c),(p,c)).$$

Theorem 3 implies that $(h,t)$ is constrained efficient.

Next observe that at $(p_1,c_1)=(0.5,0.2)$, $(p_2,c_2)=(0.4,0.8)$, $V_1=0.66>0.56=V_2$. However, as $t_1=1.04$ and $t_2=0.54$, buyer 2 is allocated the object at the information state $(p_1,c_1,p_2,c_2)=(0.5,0.2,0.4,0.8)$. Thus, the mechanism is not conditionally efficient. ◊

The next result shows that any mechanism that dominates the mechanism of Theorem 1 must be conditionally inefficient in that a buyer with a lower valuation is sometimes allocated the object. Thus, constrained efficiency and conditional efficiency are conflicting objectives.

THEOREM 4. *The allocation rule supported by the admissible prices defined in (6) dominates any other conditionally efficient implementable allocation rule.*

PROOF. Let $t^*$ be defined in (6), and let $h^*$ be a mechanism supported by $t^*$. Let $(h',t')$ be any other ex post incentive compatible mechanism. Suppose that $t_i^*(s_j)>t_i'(s_j)$ for some $s_j$. Let $\lambda^*$ and $\lambda'$ be the indexes of buyer $i$ indifference curves that are pivotal at $s_j$ under $t_i^*$ and $t_i'$, respectively. Clearly, $\lambda^*>\lambda'$. Let $s_i^m\in S(\lambda^*,s_j)$ be the point on the $\lambda^*$-indifference curve at which $V_j$ is maximized. That is, $V_j(s_j,s_i^m)=V_{ij}^m(\lambda^*,s_j)$. Further, as $S(\lambda^*,s_j)$ is pivotal at $s_j$ under $t^*$, the construction in Theorem 1 implies that $V_i(s_i^m,s_j)=V_j(s_j,s_i^m)$. Take any $\lambda\in(\lambda',\lambda^*)$ and $s_i\in S(\lambda,s_j)$ such that $s_i<s_i^m$. As $V_i(s_i^m,s_j)=V_j(s_j,s_i^m)$, single crossing implies that $V_i(s_i,s_j)<V_j(s_j,s_i)$. However, at $(s_i,s_j)$ buyer $i$ gets the object under $(h',t')$. Thus, $(h',t')$ is not conditionally efficient.

Hence, for any other conditionally efficient mechanism $(h,t)$, we have $t_i^*(\cdot)\le t_i(\cdot)$, $i=1,2$. Therefore, whenever the object is allocated (to a buyer) under $(h,t)$, it is also allocated under $(h^*,t^*)$, with both mechanisms allocating to the buyer with the highest valuation. As $(h^*,t^*)\neq(h,t)$, at some information state $(h^*,t^*)$ allocates the object while $(h,t)$ does not. □

Consequently, among all conditionally efficient mechanisms, the set of information states at which the object is allocated is the largest in the mechanism of Theorem 1. However, this mechanism is usually not constrained efficient as it is unlikely to satisfy the necessary condition of Theorem 2. To see this, we write the mutually pivotal condition for this mechanism. Take any $s_j$ for which (10) holds. Let $\lambda_i^*$ be the index of the buyer $i$ pivotal indifference curve at $s_j$. That is, $\lambda_i^*$ is such that there exists $s_i^m\in S_i(\lambda_i^*,s_j)$ satisfying

$$V_i(s_i^m,s_j)=V_j(s_j,s_i^m)\ge V_j(s_j,s_i')\quad\forall s_i'\in S_i(\lambda_i^*,s_j).$$

Further, there exists $s_i\in S_i(\lambda_i^*,s_j)$ at which $s_j$ is buyer $j$ pivotal. That is, letting $\lambda_j^*$ be the index of the buyer $j$ pivotal indifference curve at $s_i$, we have $s_j\in S_j(\lambda_j^*,s_i)$ where

$$\exists s_j^m\in S_j(\lambda_j^*,s_i)\text{ s.t. }V_j(s_j,s_i)=V_j(s_j^m,s_i)=V_i(s_i,s_j^m)\ge V_i(s_i,s_j')\quad\forall s_j'\in S_j(\lambda_j^*,s_i).$$

This condition appears to be difficult to satisfy, as we saw in Example 2. Whenever that is the case, Theorem 2 implies that the mechanism of Theorem 1 is not constrained efficient.

## 5. Concluding remarks

Jehiel et al. (2006) question the existence of ex post equilibrium in models with multi-dimensional signals and interdependent values. Our paper shows that ex post equilibrium exists in such models with private goods. Existence is proved under the assumption that buyers' information satisfies a generalization of the single-crossing property. The mechanism shares the feature with the generalized Vickrey auction of one-dimensional information models that the price paid by the winning buyer is equal to this buyer's value at the lowest possible signal (indifference curve) at which this buyer wins. Thus, ex post equilibria in auction models with one-dimensional models are robust in that non-trivial ex post equilibria exist even when buyers have multi-dimensional signals.

To reconcile our positive result with the negative result of Jehiel et al., observe that no consumption externality is a natural assumption in many economic models. But in the space of all preferences (with and without externalities), the absence of externalities is non-generic. At a tiny perturbation away from the no externalities assumption, (8) is not satisfied by any pair of outcomes. Jehiel et al.'s theorem would then imply generic non-existence of ex post incentive compatible mechanisms. However, non-trivial mechanisms that are "approximately" ex post incentive compatible still exist at these tiny perturbations away from selfish preferences. Thus, ex post incentive equilibrium is a robust equilibrium concept for models with private goods. Under a small departure from the usual assumption of selfish preferences in private goods models, many results in economics would be only approximately true.

Three possible notions of efficiency are (i) first-best efficiency without regard to incentive constraints, (ii) efficiency subject to incentive constraints, and (iii) conditional efficiency subject to incentive constraints. In one-dimensional signal models, the incentive constraints do not bind and all three types of efficiency are attainable in one mechanism. With multi-dimensional signals, incentive constraints preclude first-best efficiency (Jehiel and Moldovanu 2001) and we illustrate that, in the class of implementable allocation rules, there is a tension between constrained efficiency and conditional efficiency. The mechanism of Theorem 1 dominates any other conditionally efficient mechanism but is not usually constrained efficient. Conceivably, as the differences in how the buyers aggregate the multiple dimensions of the signals decreases, the mechanism of Theorem 1 approaches both constrained efficiency and first-best efficiency.

## Appendix

### A. Extension to many buyers

We outline the minor changes in notation, assumptions, and analysis required to extend Theorem 1 to many buyers.

Each buyer's valuation depends on the (possibly multi-dimensional) signals of all $n$ buyers. The information states are denoted $s=(s_1,s_2,\ldots,s_n)=(s_i,s_{-i})$. Change $s_j$ to $s_{-i}$ in Assumption 2, and require the assumption to hold for every $V_i$ and $V_j$. Assumption 3 is required to hold for two distinct buyers, i.e., there exist two sets of information states, $A_i$ and $A_j$, each set with positive measure, such that buyer $i$'s [$j$'s] value is strictly greater than all other buyers' values on the set $A_i$ [$A_j$].

We write $V_i(s_i,s_{-i})$, $V_{ij}^m(\lambda,s_{-i})$, $g_{ij}(\lambda;s_{-i})$ instead of $V_i(s_i,s_j)$, $V_{ij}^m(\lambda,s_j)$, $g_{ij}(\lambda;s_j)$, etc. The definition of $\lambda_{ij}^*(s_{-i})$ is

$$\lambda_{ij}^*(s_{-i})\equiv\begin{cases}1 & \text{if } g_{ij}(1;s_{-i})<0\\ \max\{\lambda\in[0,1]\mid g_{ij}(\lambda;s_{-i})=0\} & \text{if } g_{ij}(1;s_{-i})\geq 0\geq g_{ij}(0;s_{-i})\\ 0 & \text{if } g_{ij}(0;s_{-i})>0,\end{cases}$$

where $g_{ij}(\lambda;s_{-i})\equiv V_i(\lambda\mathbf{1},s_{-i})-V_{ij}^m(\lambda,s_{-i})$. Buyer $i$'s personalized price is

$$t_i^*(s_{-i})\equiv\max_{j\neq i}V_{ij}^m(\lambda_{ij}^*(s_{-i}),s_{-i}).$$

Once again, buyer $i$'s personalized price equals the maximum valuation of all other buyers on the pivotal indifference curve, which equals $i$'s valuation at a pivotal signal whenever $\max_{j\neq i}\lambda_{ij}^*(s_{-i})\in[0,1)$.

### B. Many buyers and many objects

There are $n$ buyers indexed by $i$ or $j$, and $K$ objects indexed by $k$ or $\ell$. Each buyer $i$ receives a $d^i$-dimensional signal $s_i\in[0,1]^{d^i}$. Buyer $i$'s valuation for object $k$ alone is $V_i^k(s_i,s_{-i})$; his valuation for a subset $L\subseteq\{1,2,\ldots,K\}$ is denoted $V_i^L(s_i,s_{-i})$. Each buyer's preferences over subsets of objects are subadditive (defined in (14) below).

We write $S_i^k(\lambda,s_{-i})$, $V_{ij}^{m,k}(\lambda,s_{-i})$, $g_{ij}^k(\lambda;s_{-i})$, $\lambda_{ij}^k(s_{-i})$ instead of $S_i(\lambda,s_j)$, $V_{ij}^m(\lambda,s_j)$, $g_{ij}(\lambda;s_j)$, $\lambda_{ij}^*(s_{-i})$, etc. Note that only $V_i^k(s_i,s_{-i})=V_i^k(\lambda\mathbf{1},s_{-i})$ for $s_i\in S_i^k(\lambda,s_{-i})$; in general, $V_i^\ell(s_i,s_{-i})\neq V_i^\ell(\lambda\mathbf{1},s_{-i})$ when $s_i\in S_i^k(\lambda,s_{-i})$, $\ell\neq k$.

The following generalizations of Assumptions 1 and 2 of Section 3 are sufficient for existence of admissible prices.

Assumption 1*. For all $i$ and $k$, $V_i^k$ is (a) non-decreasing and (b) continuous.

Assumption 2*. For all $i$, $j$, and $k$, for any $s_{-i}$ we have

$$V_i^k(s_i',s_{-i})-V_i^k(s_i,s_{-i})\geq V_j^k(s_i',s_{-i})-V_j^k(s_i,s_{-i})\qquad\forall s_i'>s_i.$$

Personalized prices (which we show to be admissible for subadditive preferences over subsets of objects) for each object are defined by

$$t_i^k(s_{-i})\equiv\max_{j\neq i}V_{ij}^{m,k}(\lambda_{ij}^k(s_{-i}),s_{-i})\qquad\forall s_{-i},\forall k,i.\tag{12}$$

Note that $t_i^k(s_{-i}) \geq 0$. With Assumptions 1* and 2*, the results of Section 3 generalize so that for any $s$,

$$\text{if } V_i^k(s) > t_i^k(s_{-i}) \text{ then } V_j^k(s) \leq t_j^k(s_{-j}) \text{ for all } j \neq i. \tag{13}$$

Consider the following mechanism, which gives each buyer a surplus maximizing bundle at personalized prices $t_i^k$ that satisfy (13) (for instance, the prices defined in (12)). Buyers report their signals. Personalized prices $t_i^k$ are computed for each buyer and each object. Every buyer gets a minimal element in his demand set at the reported signals at these personalized prices. Thus, if buyers report $s = (s_1, s_2, \ldots, s_n)$ then buyer $i$ gets $L_i \subseteq \{1, 2, \ldots, K\}$ such that

$$V_i^{L_i}(s) - \sum_{k \in L_i} t_i^k(s_{-i}) \geq V_i^L(s) - \sum_{k \in L} t_i^k(s_{-i}) \quad \forall L \subseteq \{1, 2, \ldots, K\},$$

and if $V_i^{L_i}(s) - \sum_{k \in L_i} t_i^k(s_{-i}) = V_i^L(s) - \sum_{k \in L} t_i^k(s_{-i})$ for some other $L$ then $L \not\subset L_i$. Call this mechanism a *demand mechanism*, because each buyer gets a (minimal) element of his demand set.

No matter what preferences buyers have over subsets of objects, because at each information state each buyer gets an element of his demand set, the demand mechanism "satisfies" the ex post incentive compatibility constraints. However, in general this mechanism may not be feasible as demand for an object may exceed its supply (of one unit). We show that for subadditive preferences (defined below), the demand mechanism is feasible: each object is allocated to at most one buyer. Moreover, we exhibit examples of subadditive preferences in which this mechanism is non-trivial.

*Subadditive preferences* The value of the union of two disjoint subsets is no greater than the sum of the values of the two subsets. That is, for all $s$

$$V_i^{L \cup L'}(s) \leq V_i^L(s) + V_i^{L'}(s) \qquad \forall L, L' \subset \{1, 2, \ldots, K\}, \ L \cap L' = \emptyset. \tag{14}$$

We mention two special cases of subadditive preferences. Clearly, *additive preferences*, where the valuation of a subset is the sum of the valuations of objects in the subset, satisfy (14) with equality. A second example is that of *unit demand preferences*, i.e., the preferences of the assignment model. Each buyer has utility for at most one object. If a buyer is given a subset of objects $L$, he selects an object with the highest valuation and throws away the rest. Thus, his reservation value for any subset $L \subseteq \{1, 2, \ldots, K\}$ is

$$V_i^L(s) \equiv \max_{k \in L}\{V_i^k(s)\}.$$

Note that the object that attains the maximum may vary with $s$. It is easily verified that unit demand preferences also satisfy (14).

LEMMA 3. *If Assumptions 1* and 2* are satisfied and buyers' preferences are subadditive then the demand mechanism is feasible and ex post incentive compatible.*

PROOF. Let $L_i$ and $L_j$ be the subsets allocated by the demand mechanism to buyers $i$ and $j$, $j \neq i$, at some information state $s$. We show that $L_i \cap L_j = \emptyset$. If $L_i = \emptyset$ there is nothing to prove. Therefore, suppose that $L_i \neq \emptyset$. If $L_i = \{k\}$ for some $k$, then minimality of $L_i$ implies that $V_i^k(s) - t_i^k(s_{-i}) > 0$. Next, suppose that $|L_i| \geq 2$. Then for any $k \in L_i$,

$$V_i^{L_i}(s) - \sum_{\ell \in L_i} t_i^\ell(s_{-i}) > V_i^{L_i \setminus k}(s) - \sum_{\ell \in L_i \setminus k} t_i^\ell(s_{-i})$$
$$\implies \quad V_i^{L_i}(s) - V_i^{L_i \setminus k}(s) > t_i^k(s_{-i})$$

where the first inequality follows from the fact that $L_i$ is minimal. By subadditivity,

$$V_i^k(s) \geq V_i^{L_i}(s) - V_i^{L_i \setminus k}(s) > t_i^k(s_{-i}).$$

Therefore, if $|L_i| \geq 1$ then for any $k \in L_i$ we have $V_i^k(s) > t_i^k(s_{-i})$; (13) implies that $V_j^k(s) \leq t_j^k(s_{-j})$. This, together with subadditivity, implies that for any subset of objects $L'$ such that there exists $k \in L' \cap L_i$,

$$\begin{aligned} V_j^{L'}(s) - \sum_{\ell \in L'} t_j^\ell(s_{-j}) &\leq V_j^{L' \setminus k}(s) + V_j^k(s) - \sum_{\ell \in L' \setminus k} t_j^\ell(s_{-j}) - t_j^k(s_{-j}) \\ &\leq V_j^{L' \setminus k}(s) - \sum_{\ell \in L' \setminus k} t_j^\ell(s_{-j}). \end{aligned}$$

Hence, $L'$ cannot be a minimal element of $j$'s demand set. Therefore, $L_j \cap L_i = \emptyset$. The demand mechanism is feasible for subadditive preferences.

Under truth-telling, each buyer gets an element of his demand set at the realized information state. Therefore, the mechanism is ex post incentive compatible. □

It may be verified that the demand mechanism is non-trivial for additive preferences, provided that the analog of Assumption 3 of Section 3 holds. The following example, which builds on Example 2, exhibits two examples of strictly subadditive preferences for which the demand mechanism is non-trivial.

EXAMPLE 3. There are three buyers, 1, 2, and 3, and two objects, $a$ and $b$. Buyer $i$ gets signal $(p_{ai}, p_{bi}, c_i)$, $i = 1, 2, 3$. Buyer $i$'s valuation for the object $k$ as a function of buyer signals is[22]

$$V_i^k \equiv p_{ki} + w_k c_i \max\{c_j, c_{j'}\}, \qquad k = a, b,$$

where $w_a$, $w_b \geq 0$ are constants. We specify buyer $i$'s reservation value for the bundle $\overline{ab}$ later.

Define

$$t_i^k \equiv \max\{p_{kj} + w_k c_j^2, p_{kj'} + w_k c_{j'}^2\}, \qquad k = a, b. \tag{15}$$

Let $t_i = (t_i^a, t_i^b)$ be the personalized prices at which the two objects are available to buyer $i$, $i = 1, 2$. Suppose that at some information state, buyer $i$'s value for object $k$ exceeds his personalized price. This implies that $V_i^k = p_{ki} + w_k c_i c_j > t_i^k \geq p_{kj} + w_k c_j^2$,

[22]We assume that $i, j, j' = 1, 2, 3$ are three distinct buyers, that is $j \neq j' \neq i \neq j$.

| $(p_{a1}, p_{b1}, c_1)$ | (0.9,0.1,0.5) | (0.1,0.9,0.5) | (0.9,0.9,0.5) |
|---|---|---|---|
| $V_1^a$ | 1.1 | 0.3 | 1.1 |
| $V_1^b$ | 0.4 | 1.2 | 1.2 |
| $V_1^{ab}$ | 1.4 | 1.4 | 2.2 |
| $t_1^a$ | 0.56 | 0.56 | 0.56 |
| $t_1^b$ | 0.64 | 0.64 | 0.64 |
| Buyer 1's allocation | $\overline{a}$ | $\overline{b}$ | $\overline{ab}$ |
| $V_2^a$ | 0.6 | 0.6 | 0.6 |
| $V_2^b$ | 0.4 | 0.4 | 0.4 |
| $V_2^{ab}$ | 0.9 | 0.9 | 0.9 |
| $t_2^a$ | 1.15 | 0.35 | 1.15 |
| $t_2^b$ | 0.64 | 1.275 | 1.275 |
| Buyer 2's allocation | $\emptyset$ | $\overline{a}$ | $\emptyset$ |
| $V_3^a$ | 0.3 | 0.3 | 0.3 |
| $V_3^b$ | 0.7 | 0.7 | 0.7 |
| $V_3^{ab}$ | 0.9 | 0.9 | 0.9 |
| $t_3^a$ | 1.15 | 0.56 | 1.15 |
| $t_3^b$ | 0.475 | 1.275 | 1.275 |
| Buyer 3's allocation | $\overline{b}$ | $\emptyset$ | $\emptyset$ |

TABLE 1. Subadditive preferences.

$j \neq i$. Then, mimicking the steps in Example 2, it is easily shown that $V_j^k < p_{ki} + w_k c_i^2 \leq t_j^k$. Thus, the personalized prices defined in (15) satisfy (13). By Lemma 3, the demand mechanism at these personalized prices is ex post incentive compatible for any specification of $V_i^{ab}$ that satisfies the subadditive inequality (14).[23]

In the rest of this example we assume that $w_a = 1$ and $w_b = 1.5$. Thus, buyer $i$'s reservation values for exactly one object are $V_i^a = p_{ai} + c_i \max\{c_j, c_{j'}\}$ and $V_i^b = p_{bi} + 1.5 c_i \max\{c_j, c_{j'}\}$. We give two sets of preferences over the bundle $\overline{ab}$ for which this mechanism is non-trivial.

*Subadditive preferences* Buyer $i$'s valuation for the bundle $\overline{ab}$ is

$$V_i^{ab} \equiv p_{ai} + p_{bi} + 2c_i \max\{c_j, c_{j'}\}, \qquad i = 1, 2, 3.$$

These preferences are subadditive. In the demand mechanism, buyer $i$ is allocated $L_i = \emptyset$ if none of the subsets $\overline{a}$, $\overline{b}$ or $\overline{ab}$ yields a strictly positive surplus at the personalized prices at the realized (i.e., reported) information state. Otherwise he is allocated a smallest $L_i \in \{\overline{a}, \overline{b}, \overline{ab}\}$ that maximizes his surplus.

To see that the mechanism is non-trivial, fix buyer 2's signals at $(p_{a2}, p_{b2}, c_2) = (0.4, 0.1, 0.4)$ and buyer 3's signals at $(p_{a3}, p_{b3}, c_3) = (0.1, 0.4, 0.4)$. Thus, $t_1^a = \max\{0.4 +$

[23]These prices of (15) have not been defined as in (12). The proof of Lemma 3 requires only that the personalized prices used in the demand mechanism satisfy (13).

| $(p_{a1}, p_{b1}, c_1)$ | (0.9,0.1,0.5) | (0.1,0.9,0.5) |
|---|---|---|
| $V_1^a$ | 1.1 | 0.3 |
| $V_1^b$ | 0.4 | 1.2 |
| $t_1^a$ | 0.56 | 0.56 |
| $t_1^b$ | 0.64 | 0.64 |
| Buyer 1's allocation | $\overline{a}$ | $\overline{b}$ |
| $V_2^a$ | 0.6 | 0.6 |
| $V_2^b$ | 0.4 | 0.4 |
| $t_2^a$ | 1.15 | 0.35 |
| $t_2^b$ | 0.64 | 1.275 |
| Buyer 2's allocation | $\emptyset$ | $\overline{a}$ |
| $V_3^a$ | 0.3 | 0.3 |
| $V_3^b$ | 0.7 | 0.7 |
| $t_3^a$ | 1.15 | 0.56 |
| $t_3^b$ | 0.475 | 1.275 |
| Buyer 3's allocation | $\overline{b}$ | $\emptyset$ |

TABLE 2. Unit demand preferences.

$(0.4)^2, 0.1+(0.4)^2\} = 0.56$, and $t_1^b = \max\{0.1+1.5\times(0.4)^2, 0.4+1.5\times(0.4)^2\} = 0.64$. Further, fix $c_1 = 0.5$. Hence, $V_2^a = 0.6$, $V_2^b = 0.4$, and $V_2^{ab} = 0.9$, and $V_3^a = 0.3$, $V_3^b = 0.7$, and $V_2^{ab} = 0.9$.

Allocations under the demand mechanism at three different values of buyer 1's private signals are shown in Table 1 (on the previous page). With $(p_{a2}, p_{b2}, c_2)$ fixed at (0.4,0.1,0.4), and $(p_{a3}, p_{b3}, c_3)$ at (0.1,0.4,0.4), we see that (i) buyer 1 gets object $a$, 2 gets nothing, and 3 gets $b$ when $(p_{a1}, p_{b1}, c_1) = (0.9, 0.1, 0.5)$, (ii) buyer 1 gets object $b$, 2 gets $a$, and 3 gets nothing when $(p_{a1}, p_{b1}, c_1) = (0.1, 0.9, 0.5)$, and (iii) buyer 1 gets both the objects when $(p_{a1}, p_{b1}, c_1) = (0.9, 0.9, 0.5)$. Moreover, at each of these three information states, buyers' demand sets are singletons. Hence, the allocation attained by the demand mechanism is constant in a positive measure neighborhood around each information state. Thus, the demand mechanism is non-trivial. By symmetry, the mechanism has full range.

*Unit demand preferences* Buyer $i$'s valuation for the bundle $\overline{ab}$ is

$$V_i^{ab} \equiv \max\{V_i^a, V_i^b\}, \qquad i = 1, 2, 3.$$

The demand mechanism allocates to buyer $i$ the object that maximizes his surplus, $V_i^k - t_i^k$, $k = a, b$, provided this surplus is positive; if $V_i^k - t_i^k \leq 0$ for $k = a, b$ then buyer $i$ gets nothing. To check that the mechanism is non-trivial, fix buyer 2's signals at $(p_{a2}, p_{b2}, c_2) = (0.4, 0.1, 0.4)$, and buyer 3's signals at $(p_{a3}, p_{b3}, c_3) = (0.1, 0.4, 0.4)$. Table 2 shows that when $(p_{a1}, p_{b1}, c_1) = (0.9, 0.1, 0.5)$ buyer 1 gets object $a$ and 3 gets $b$, and when $(p_{a1}, p_{b1}, c_1) = (0.1, 0.9, 0.5)$, buyer 1 gets $b$ and buyer 2 gets $a$. Moreover, as the demand set of each buyer is a singleton at each of the two information states, the two allocations are implemented at a positive measure of information states. Hence, the mechanism is non-trivial, and, by symmetry, has full range. ◊

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Submitted 2005-12-14. Final version accepted 2006-6-30. Available online 2006-6-30.