Optimal auctions with ambiguity

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A crucial assumption in the optimal auction literature is that each bidder's valuation is known to be drawn from a unique distribution. In this paper, we study the optimal auction problem allowing for ambiguity about the distribution of valuations. Agents may be ambiguity averse (modeled using the maxmin expected utility model of Gilboa and Schmeidler 1989). When the bidders face more ambiguity than the seller we show that (i) given any auction, the seller can always (weakly) increase revenue by switching to an auction providing full insurance to all types of bidders, (ii) if the seller is ambiguity neutral and any prior that is close enough to the seller's prior is included in the bidders' set of priors then the optimal auction is a full insurance auction, and (iii) in general neither the first nor the second price auction is optimal (even with suitably chosen reserve prices). When the seller is ambiguity averse and the bidders are ambiguity neutral an auction that fully insures the seller is in the set of optimal mechanisms.

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JEL CLASSIFICATION. D44, D81.

1. INTRODUCTION

Optimal auctions for an indivisible object with risk neutral bidders and independently distributed valuations have been studied by Vickrey (1961), Myerson (1981), Harris and Raviv (1981), Riley and Samuelson (1981), and others. These papers show that the set of optimal mechanisms or auctions is quite large and contains both the first and second price auctions with reserve prices. One of the assumptions in this literature is that each

bidder's valuation is known to be drawn from a unique distribution. In this paper we relax this assumption and study how the design of the optimal auction is affected by the presence of ambiguity about the distribution from which the bidders' valuations are drawn.

The assumption of a unique prior is based on the subjective expected utility model, which has been criticized by Ellsberg (1961), among others. Ellsberg shows that lack of knowledge about the distribution over states, often referred to as ambiguity, can affect choices in a fundamental way that cannot be captured within the subjective expected utility framework.[1] Ellsberg and several subsequent researchers have demonstrated that in many situations decision makers exhibit ambiguity averse behavior.[2] Following Gilboa and Schmeidler (1989), we model ambiguity aversion using the maxmin expected utility (MMEU) model. The MMEU model is a generalization of the subjective expected utility model, and provides a natural and tractable framework to study ambiguity aversion. In MMEU agents have a set of priors (instead of a single prior) on the underlying state space, and their payoff is the minimum expected utility over the set of priors. Specifically, when an MMEU bidder is confronted with an auction, he evaluates each bid on the basis of the minimum expected utility over the set of priors, and then chooses the best bid. An MMEU seller evaluates each auction on the basis of its minimum expected revenue over the set of priors and chooses the best auction. In order to better contrast our results with the case of risk, we assume that the bidders and the seller are risk neutral (i.e. have linear utility functions).

Our main result, Proposition 1, is that when the bidders face more ambiguity than the seller, an auction that provides full insurance to the bidders[3] is always in the set of optimal mechanisms. Moreover, given any incentive compatible and individually rational mechanism, the seller can strictly increase his revenue by switching to a full insurance mechanism if the minimum expected utility of a bidder over the seller's set of priors is strictly larger than his expected utility over the bidders' set of priors for a positive measure of types.

To obtain some intuition for the main result, consider the special case where the seller is ambiguity neutral, i.e., his set of priors is a singleton. In this case the main result says that if an incentive compatible and individually rational mechanism is optimal for the seller then the minimizing set of distributions for all types of the bidders must include the seller's prior. Suppose this is not true for a positive measure of types and consider some such type θ. In this case, the seller and type θ of the bidder are willing to bet against each other. The seller recognizes that they have different beliefs about the

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[1]In one version of Ellsberg's experiment, a decision maker is offered two urns, one that has 50 black and 50 red balls, and one that has 100 black and red balls in unknown proportions. Faced with these two urns, the decision maker is offered a bet on black but can decide from which urn to draw the ball. Most decision makers prefer the first urn. The same is true when the decision maker is offered the same bet on red. This behavior is inconsistent with the expected utility model. Intuitively, decision makers do not like betting on the second urn because they do not have enough information or, put differently, there is too much ambiguity. Being averse to ambiguity, they prefer to bet on the first urn.

[2]For a survey see Camerer and Weber (1992).

[3]A full insurance auction keeps the bidders' payoffs constant for all reports of the other bidders and consequently keeps them indifferent between winning or losing the object.

underlying state space and will offer "side bets" using transfers. The crucial issue is that the modified mechanism has to maintain overall incentive compatibility. In our proof we address this issue by explicitly constructing the additional transfers that continue to satisfy incentive compatibility constraints while making the seller better off. Essentially, we show that these additional transfers (to the seller) can be chosen so that in the new mechanism, under truth telling type θ gets the minimum expected utility that he gets in the original mechanism in *every* state, and thus is fully insured against the ambiguity. Obviously then under truth telling, type θ is indifferent between the original mechanism and the new mechanism since he gets the same minimum expected utility under both. More interestingly, no other type wants to imitate type θ in the new mechanism. This is because the additional transfers in the new mechanism are constructed so as to have zero expected value under the minimizing set of distributions for type θ in the original mechanism, but to have strictly positive expected value under any other distribution. Therefore, if type θ' imitates type θ in the new mechanism, he gets at best what he would get by imitating type θ in the original mechanism. Hence, since the original mechanism is incentive compatible, the new mechanism is also incentive compatible. Moreover, since by assumption the seller's distribution is not in the minimizing set for type θ in the original mechanism, the additional transfers (to the seller) have strictly positive expected value under the seller's distribution. Since the original mechanism can be modified in this way for a positive measure of types, the seller strictly increases his revenue. In fact, for any incentive compatible and individually rational mechanism, by modifying the mechanism for all types as described above we can obtain a full insurance auction that is weakly preferred by the seller. Therefore, a full insurance auction is always in the set of optimal mechanisms.

There may be optimal selling mechanisms in addition to the full insurance mechanism, but in some cases the full insurance mechanism is the unique optimal mechanism. In Proposition 3 we show that if the seller is ambiguity neutral and any prior that is close enough to the seller's prior is included in the bidders' set of priors then the optimal auction must be a full insurance auction. In addition, we show in Proposition 5 that, in general, the first and the second price auctions are not optimal.

Some real life auctions resemble a full insurance auction.[4] In general, though, full insurance auctions are rarely observed in practice. Thus, the simplest incorporation of ambiguity aversion in auctions yields a surprising and somewhat negative result. The mismatch between theory and practice could be due to several reasons including the use of maxmin preferences in the theory or the relevance of ambiguity in real life market settings. As such the result presents a puzzle to the literature.

To highlight some of the economic implications of the above analysis, in Section 4 we explicitly derive the optimal mechanism when the seller is ambiguity neutral and the bidders' set of priors is the ε-contamination of the seller's prior. We show that the seller's revenue and efficiency both increase as ambiguity increases. We also describe an auction that implements the optimal mechanism.

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[4]For example, in the Amsterdam Auction studied by Goeree and Offerman (2004), the losing bidder is offered a premium.

When the seller is ambiguity averse and the bidders are ambiguity neutral we show that for every incentive compatible and individually rational selling mechanism there exists an incentive compatible and individually rational mechanism that provides deterministically the same payoff to the seller. From this it follows that when an optimal mechanism exists, an auction that fully insures the seller must be in the set of optimal mechanisms. A similar result was first shown by Eső and Futó (1999) for auctions (in independent private value environments) with a risk averse seller and risk (and ambiguity) neutral bidders. Hence, as long as bidders are risk and ambiguity neutral, ambiguity aversion on the part of the seller plays a similar role to that of risk aversion.

There is a small but growing literature on auction theory with non-expected utility starting with a series of papers by Karni and Safra (1986, 1989a,b) and Karni (1988). Salo and Weber (1995), Lo (1998), Volij (2002), and Ozdenoren (2001) study auctions with ambiguity averse bidders, and thus are closer to this paper. These papers look at specific auction mechanisms, such as the first and second price auctions, and not the optimal auction.

Matthews (1983) and Maskin and Riley (1984) study auctions with risk averse bidders. A more detailed comparison of our paper with Maskin and Riley (1984) is given in Section 7.

Finally, a strand of literature studies robust mechanism design. (See for example Bergemann and Morris 2005, Ely and Chung forthcoming, Jehiel et al. 2006, and Heifetz and Neeman 2006.) Even though there is some similarity between that literature and our work (as in that literature, we relax certain assumptions of the standard mechanism design framework), it is important to point out that our work differs significantly from this literature. Standard mechanism design—in particular Bayesian implementation—relies crucially on the underlying model being common knowledge. The focus of those papers is to study mechanism design while relaxing (some of) the common knowledge assumptions. In contrast, we maintain throughout the standard methodological assumption of considering the underlying *model*—which includes the modeling of the ambiguity—to be common knowledge and we relax assumptions on the preferences of the agents, in particular allowing the agents to exhibit ambiguity aversion.

2. THE OPTIMAL AUCTION PROBLEM

In this section we generalize the standard optimal auction problem by allowing the bidders and the seller to have MMEU preferences (Gilboa and Schmeidler 1989). There are two bidders (denoted as bidder 1 and bidder 2) and a seller. Bidders have one of a continuum of valuations $\theta \in \Theta = [0, 1]$. Let Σ be the Borel algebra on Θ. Each bidder knows his true valuation but not that of the other bidder. The set Δ_B^m is a non-empty set of probability measures on (Θ, Σ) with a corresponding set Δ_B of distribution functions. This set represents each bidder's belief about the other bidder's valuation. Bidders believe that valuations are generated independently, but they may not be confident about the probabilistic process that generates the valuations. This possible vagueness in the bidders' information is captured by allowing for a set of priors rather than a single prior.

We assume that both the bidders and the seller have linear utility functions.

The seller is also allowed to be ambiguity averse. The set Δ_S^m is a nonempty set of probability measures on (Θ, Σ) with a corresponding set Δ_S of distribution functions. This set represents the seller's belief about the bidders' valuations. That is, the seller believes that the bidders' valuations are generated independently from some distribution in Δ_S.[5] Each bidder's reservation utility is 0.

We consider a direct mechanism where bidders simultaneously report their types.[6] The mechanism stipulates a probability for assigning the item and a transfer rule as a function of reported types. Let $x_i(\tilde{\theta}, \theta')$ be the item assignment probability function and $t_i(\tilde{\theta}, \theta')$ the transfer rule for bidder $i \in \{1, 2\}$.[7] The first entry is i's report and the second entry is the other bidder's report. We assume that type θ of bidder i chooses a report $\tilde{\theta}$ to maximize $\inf_{G \in \Delta_B} \int (x_i(\tilde{\theta}, \theta')\theta - t_i(\tilde{\theta}, \theta')) dG(\theta')$.[8]

The seller's problem is to find a mechanism $\{(x_1, t_1), (x_2, t_2)\}$ that solves

$$\sup_{\{(x_1,t_1),(x_2,t_2)\}} \left[\inf_{F \in \Delta_S} \iint (t_1(\theta, \theta') + t_2(\theta', \theta)) \, dF(\theta) \, dF(\theta') \right] \tag{1}$$

subject to

$$\text{(IC)} \inf_{G \in \Delta_B} \int (x_i(\theta, \theta')\theta - t_i(\theta, \theta')) \, dG(\theta')$$

$$\geq \inf_{G \in \Delta_B} \int (x_i(\tilde{\theta}, \theta')\theta - t_i(\tilde{\theta}, \theta')) \, dG(\theta') \quad \text{for all } \theta, \tilde{\theta} \in \Theta \text{ and } i \in \{1, 2\} \tag{2}$$

and

$$\text{(IR)} \inf_{G \in \Delta_B} \int (x_i(\theta, \theta')\theta - t_i(\theta, \theta')) \, dG(\theta') \geq 0 \quad \text{for all } \theta \in \Theta \text{ and } i \in \{1, 2\} \tag{3}$$

where $x_1(\theta, \theta') + x_2(\theta', \theta) \leq 1$ for all $\theta, \theta' \in \Theta$. As is standard, we assume that all of the above is common knowledge. The first inequality gives the incentive compatibility (IC) constraints and the second inequality gives the individual rationality (IR) or participation constraint. These are the usual constraints except that the bidders compute their

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[5]Formally the seller's belief is the set of product measures $\mu \times \mu$ on the product space $(\Theta \times \Theta, \Sigma \times \Sigma)$ where $\mu \in \Delta_S^m$, even though for notational simplicity, throughout the paper we refer to the seller's belief simply as $\mu \in \Delta_S^m$. It is important, however, to keep this in mind, especially for a later result (Proposition 2) which talks about the seller's belief being in the interior of the set of the bidders' set of beliefs.

[6]The revelation principle holds in our setting.

[7]Note that we do not restrict attention to symmetric mechanisms. In the standard auction literature with ambiguity neutral (but not necessarily risk neutral) bidders, the seller can restrict attention to symmetric auctions without loss of generality. To see this suppose an asymmetric auction is optimal. Since bidders are ex ante symmetric, by exchanging the roles of the two bidders, the seller can obtain another optimal auction. But then by randomizing between these two asymmetric auctions with equal probability, the seller can obtain a symmetric and incentive compatible mechanism. This argument does not hold when the bidders are ambiguity averse. The reason is that ambiguity averse bidders may strictly prefer randomization. Therefore, even though they prefer truth-telling in the asymmetric mechanism, they may prefer to misrepresent their types in the randomized mechanism.

[8]Since bidders' preferences are linear in transfers, we consider only deterministic transfers.

utilities in the mechanism using the MMEU rule. For example, the IC constraint requires that the infimum expected utility a bidder of type θ gets reporting his type truthfully is at least as much as the infimum expected utility that he gets reporting any other type $\tilde{\theta}$.

One interpretation of the set of priors in the above formulation is "subjective": the players' preferences are common knowledge and the sets are subjective representations of the uncertainty (as well as the aversion to this uncertainty) the players face about the stochastic process that generates the valuations. Another interpretation is "objective": the players learn everything they can about the stochastic process that generates the types, but there are hard-to-describe factors that prevent them from learning the process completely. The objective interpretation is more restrictive than the subjective one for two reasons. First, when the set of priors is objectively fixed, bidders' attitudes to ambiguity are represented by the minimum functional *only*, which may be viewed as extreme. Second, the objective interpretation makes sense when both the seller and the buyers have the same set of priors (which is covered in our framework), since the set of priors is assumed to be common knowledge.

Note that our formulation differs slightly from that of Gilboa and Schmeidler since we use the infimum (supremum) instead of the minimum (maximum). However, we continue to refer to these preferences as maxmin since this terminology is standard. At the end of the next section we provide conditions on preferences and mechanisms that guarantee that the minimum over the sets of priors and an optimal auction exist.

3. FULL INSURANCE AUCTION

In this section we show that when $\Delta_S^m \subseteq \Delta_B^m$,[9] a full insurance auction is always in the set of optimal auctions and discuss when the seller can make strict gains by switching to a full insurance auction. In what follows, for a given mechanism $\{(x_1, t_1), (x_2, t_2)\}$, it is convenient to define

$$q_i(\theta, \theta') \equiv x_i(\theta, \theta')\theta - t_i(\theta, \theta')$$

for all $\theta, \theta' \in \Theta$. So $q_i(\theta, \theta')$ is the ex post payoff to type θ of bidder i from truth telling in the mechanism $\{(x_1, t_1), (x_2, t_2)\}$ when the other bidder reports θ'.[10]

We say that an event $\widetilde{\Theta} \subseteq \Theta$ has positive measure if $\inf_{\mu \in \Delta_S^m} \mu(\widetilde{\Theta}) > 0$ and zero measure otherwise. Next, we formally define a full insurance auction.

DEFINITION 1. A *full insurance mechanism* is one where the (ex post) payoff of a given type of a bidder does not vary with the report of the competing bidder. That is, $\{(x_1, t_1), (x_2, t_2)\}$ is a full insurance mechanism if, for almost all $\theta \in \Theta$, $q_i(\theta, \theta')$ is constant as a function of $\theta' \in \Theta$.

Next, we give the formal statement of the main proposition. All proofs are in the Appendix.

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[9] In particular, this covers two interesting cases. If Δ_S^m is a singleton set, then the seller is ambiguity neutral and the bidders are (weakly) ambiguity averse. On the other hand, if $\Delta_S^m = \Delta_B^m$, then both the seller and the bidders are (weakly) ambiguity averse with a common set of priors.

[10] Formally q_i should be indexed also by $\{(x_1, t_1), (x_2, t_2)\}$. Since the mechanism to which we are referring is always clear from the context, we drop this index to simplify the notation.

PROPOSITION 1. *Suppose that the seller's set of priors is Δ_S and the bidders' set of priors is Δ_B with $\Delta_S \subseteq \Delta_B$. Let $\{(x_1, t_1), (x_2, t_2)\}$ be an arbitrary incentive compatible and individually rational mechanism. There is always a full insurance mechanism, also satisfying incentive compatibility and individual rationality, that generates at least as much minimum expected revenue over the set of priors Δ_S for the seller. Moreover if there exists a bidder $i \in \{1, 2\}$ and some positive measure event $\widetilde{\Theta} \subseteq \Theta$ such that, for all $\theta \in \widetilde{\Theta}$,*

$$\inf_{G \in \Delta_S} \int_\Theta q_i(\theta, \theta') dG(\theta') > \inf_{H \in \Delta_B} \int_\Theta q_i(\theta, \theta') dH(\theta'),$$

then $\{(x_1, t_1), (x_2, t_2)\}$ is not optimal. In fact, the seller can strictly *increase his minimum expected revenue over the set of priors Δ_S by using a full insurance mechanism.*

To understand this result consider the case where the seller is ambiguity neutral, i.e., $\Delta_S = \{F\}$. Let

$$\Delta_i(\theta) = \arg \min_{H \in \Delta_B} \int_\Theta q_i(\theta, \theta') dH(\theta')$$

where for ease of exposition we assume that the minimum exists so that we write min instead of inf.[11] In this case Proposition 1 implies that if a mechanism $\{(x_1, t_1), (x_2, t_2)\}$ is optimal then F must be in $\Delta_i(\theta)$ for almost all $\theta \in \Theta$ and $i \in \{1, 2\}$. Suppose to the contrary that for some $i \in \{1, 2\}$ there exists a positive measure of types for which this is not true. Consider some such type $\tilde{\theta}$ for which $F \notin \Delta_i(\tilde{\theta})$. The seller can always adjust the transfers of type $\tilde{\theta}$ so that, under truth telling, type $\tilde{\theta}$ gets the *same* minimum expected utility as he gets in the original mechanism in every state, and thus is fully insured against ambiguity in the new mechanism. Furthermore, by construction, the difference between the transfers in the new mechanism and the original mechanism has weakly positive expected value[12] for any distribution in Δ_B. This is true because this difference has zero expected value under $\Delta_i(\tilde{\theta})$, the minimizing set of distributions in the original mechanism, and strictly positive expected value under any other distribution, i.e., for distributions in $\Delta_B - \Delta_i(\tilde{\theta})$. Obviously, under truth telling, type $\tilde{\theta}$ is indifferent between the original mechanism and the new mechanism since he gets the same minimum expected utility under both. More interestingly, no other type wants to imitate type $\tilde{\theta}$ in the new mechanism. This is true since the original mechanism is incentive compatible and imitation in the new mechanism is even worse given that the difference in transfers has weakly positive expected value under any distribution in Δ_B. Moreover, by assumption, the seller's distribution is not in the minimizing set for the original mechanism, which means the additional transfers (to the seller) must have strictly positive expected value under the seller's distribution. Thus the seller is strictly better off in the new mechanism, contradicting the optimality of the original mechanism.

When the infima and suprema in equations (1), (2), and (3) are replaced with minima and maxima, we can prove a stronger version of Proposition 1. The next proposition provides sufficient conditions for this.

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[11]See Proposition 2 below for conditions that guarantee that this assumption holds.

[12]Recall that these are transfers *to* the seller.

PROPOSITION 2. *Suppose the seller can use only mechanisms such that transfers are uniformly bounded and suppose that $\Delta_B^m \cup \Delta_S^m$ is weakly compact and convex and its elements are countably additive probability measures. Then the sets of minimizing priors in equations (1), (2), and (3) and the set of optimal mechanisms are nonempty.*

The following corollary strengthens Proposition 1 when the hypothesis of Proposition 2 holds.

COROLLARY 1. *Suppose that the hypothesis of Proposition 2 holds. For a given mechanism $\{(x_1, t_1), (x_2, t_2)\}$, let*

$$\Delta_S^{\min} = \arg\min_{G \in \Delta_S} \int_\Theta \int_\Theta \left[t_1(\theta, \theta') + t_2(\theta', \theta) \right] \, dG(\theta) \, dG(\theta').$$

If there exists a bidder $i \in \{1, 2\}$ and some positive measure event $\widetilde{\Theta} \subseteq \Theta$ such that for all $\theta \in \widetilde{\Theta}$ and for all $G \in \Delta_S^{\min}$,

$$\int_\Theta q_i(\theta, \theta') dG(\theta') > \min_{H \in \Delta_B} \int_\Theta q_i(\theta, \theta') \, dH(\theta'), \tag{4}$$

then the seller can strictly *increase his minimum expected revenue over the set of priors Δ_S using a full insurance mechanism. Moreover, there is a full insurance mechanism that is optimal for the seller.*

Corollary 1 is stronger than Proposition 1 in two ways. First, the inequality in (4) is checked only for the *minimizing* distributions for the seller (not all the distributions in Δ_S). Second, since an optimal mechanism exists, a full insurance mechanism is always optimal for the seller.[13]

In general there may be optimal selling mechanisms that differ from the full insurance mechanism. But if the seller's belief set is a singleton F with strictly positive density and if any prior that is close enough to F is in Δ_B, then the set of distributions that give the minimum expected utility does not include F unless the ex post payoffs are constant. In this case the unique optimal auction is a full insurance auction. The next proposition states this observation.

PROPOSITION 3. *Suppose that the seller is ambiguity neutral with $\Delta_S = \{F\}$ where F has strictly positive density. If there exists $\varepsilon > 0$ such that for any distribution H on Θ, $(1 - \varepsilon)F + \varepsilon H \in \Delta_B$, then the unique optimal auction is a full insurance auction.*

The usual (independent and private values) auction model, where the bidders and the seller are both risk and ambiguity neutral, corresponds in our model to the case where Δ_S and Δ_B are both singleton sets. In the usual model, every bidder takes the expectation of his payoffs (for different reports of his type) over his opponent's types and these interim expected payoffs are enough to characterize incentive compatibility

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[13] In contrast, Proposition 1 says that the seller can get arbitrarily close to the supremum in equation (1) using a full insurance mechanism.

and individual rationality and hence optimal auctions. Thus, in the usual model the dependence of bidders' (ex post) payoffs on the types of his opponents is not pinned down. In our model, when the buyers and the seller are possibly ambiguity averse, this dependence on the opponent's type is pinned down (uniquely under the conditions of Proposition 3). The requirements imposed by this dependence eliminate many auctions that are optimal in the usual model such as first and second price auctions (under the conditions of Proposition 5).

In the next two sections we provide some applications of the results in this section.

4. FULL INSURANCE UNDER ε-CONTAMINATION

In this section we explicitly derive the optimal mechanism in the case of ε-contamination when the seller is ambiguity neutral with $\Delta_S = \{F\}$. We assume that the seller's distribution F is a focal point, and bidders allow for an ε-order amount of noise around this focal distribution. We make the common assumptions that F has a strictly positive density f and

$$L(\theta) = \theta - \frac{1 - F(\theta)}{f(\theta)}$$

is strictly increasing in θ. We construct Δ_B as follows:

$$\Delta_B = \{G : G = (1 - \varepsilon)F + \varepsilon H \text{ for any distribution } H \text{ on } \Theta\}$$

where $\varepsilon \in (0, 1]$. By Proposition 3 we know that the unique optimal mechanism for the ε-contamination case is a full insurance mechanism. This implies that we can restrict ourselves to full insurance mechanisms in our search for the optimal mechanism.

Let $\{(x_1, t_1), (x_2, t_2)\}$ be a full insurance mechanism. I.e., for a given θ, $q_i(\theta, \theta')$ does not vary with θ'. Let $u_i(\theta) = q_i(\theta, \theta')$ for $i \in \{1, 2\}$. Next we define some useful notation. Let

$$X_i(\theta) = \int x_i(\theta, \theta') \, dF(\theta')$$

$$X_i^{\min}(\theta) = \inf_{G \in \Delta_B} \int x_i(\theta, \theta') \, dG(\theta')$$

$$X_i^{\max}(\theta) = \sup_{G \in \Delta_B} \int x_i(\theta, \theta') \, dG(\theta').$$

Using the IC constraint we obtain

$$
\begin{aligned}
u_i(\theta) &= \inf_{G \in \Delta_B} \int (x_i(\theta, \theta')\theta - t_i(\theta, \theta')) \, dG(\theta') \\
&\geq \inf_{G \in \Delta_B} \int (x_i(\tilde{\theta}, \theta')\theta - t_i(\tilde{\theta}, \theta')) \, dG(\theta') \\
&\geq u_i(\tilde{\theta}) + \inf_{G \in \Delta_B} \int (\theta - \tilde{\theta}) x_i(\tilde{\theta}, \theta') \, dG(\theta').
\end{aligned}
\tag{5}
$$

If $\theta > \tilde{\theta}$ then

$$u_i(\theta) \geq u_i(\tilde{\theta}) + (\theta - \tilde{\theta}) X_i^{\min}(\tilde{\theta}).\tag{6}$$

Exchanging the roles of θ and $\tilde{\theta}$ in (5) we obtain

$$u_i(\tilde{\theta}) \geq u_i(\theta) + \inf_{G \in \Delta_B} \int (\tilde{\theta} - \theta) x_i(\theta, \theta') dG(\theta').$$

Again if $\theta > \tilde{\theta}$ then

$$u_i(\tilde{\theta}) \geq u_i(\theta) + (\tilde{\theta} - \theta) X_i^{\max}(\theta).\tag{7}$$

Now observe that u is non-decreasing, since for $\theta > \tilde{\theta}$ by the IC constraint we have

$$u_i(\theta) \geq u_i(\tilde{\theta}) + (\theta - \tilde{\theta}) X_i^{\min}(\tilde{\theta}) \geq u_i(\tilde{\theta}).$$

The next lemma is useful in characterizing the optimal auction.

LEMMA 1. *The function u_i is Lipschitz.*

Since u_i is Lipschitz, it is absolutely continuous and therefore is differentiable almost everywhere. For $\theta > \tilde{\theta}$ we use (6) and (7) to obtain

$$X_i^{\max}(\theta) \geq \frac{u_i(\theta) - u_i(\tilde{\theta})}{\theta - \tilde{\theta}} \geq X_i^{\min}(\tilde{\theta}).$$

We take the limit as $\tilde{\theta}$ goes to θ to obtain for almost all θ that

$$X_i^{\max}(\theta) \geq \frac{\partial u_i}{\partial \theta} \geq X_i^{\min}(\theta).$$

Since an absolutely continuous function is the definite integral of its derivative,

$$\int_0^\theta X_i^{\max}(y) dy \geq u_i(\theta) - u_i(0) \geq \int_0^\theta X_i^{\min}(y) dy.\tag{8}$$

Inequality (8) suggests that the auctioneer may set

$$u_i(\theta) = \int_0^\theta X_i^{\min}(y) dy\tag{9}$$

and

$$t_i(\theta, \theta') = x_i(\theta, \theta')\theta - \int_0^\theta X_i^{\min}(y) dy,\tag{10}$$

since for a given allocation rule x_i, transfers as in (10) are the highest transfers the auctioneer can set without violating (8). Of course, (8) is only a necessary condition and for given allocation rules x_1 and x_2, the resulting mechanism $\{(x_1, t_1), (x_2, t_2)\}$ may not be incentive compatible. Fortunately, this difficulty does not arise if the allocation rules x_1 and x_2 are chosen optimally for transfers given as in (10). In other words, our strategy

is to find the optimal allocation rules x_1 and x_2 assuming that the transfers are given by (10), and then show that the resulting mechanism $\{(x_1, t_1), (x_2, t_2)\}$ is incentive compatible.

For a transfer function given by (10), we can rewrite the seller's revenue as

$$R = \sum_{i=1}^{2} \left[\int_0^1 \int_0^1 \left(\theta x_i(\theta, \theta') - \int_0^\theta X_i^{\min}(y) \, dy \right) d F(\theta') \, d F(\theta) \right].$$

Using integration by parts we obtain

$$R = \sum_{i=1}^{2} \left[\int_0^1 \theta X_i(\theta) f(\theta) \, d\theta - \int_0^1 (1 - F(\theta)) X_i^{\min}(\theta) \, d\theta \right]. \tag{11}$$

Define

$$L^\varepsilon(\theta) = \theta - (1 - \varepsilon) \frac{1 - F(\theta)}{f(\theta)}$$

and let $r \in (0, 1)$ be such that $L^\varepsilon(r) = 0$.

The following proposition characterizes the optimal allocation when the transfer function is given by (10).

PROPOSITION 4. *The unique optimal mechanism is symmetric and for any θ and θ' the allocation rule $x_1 = x_2 = x$ is given by*

$$x(\theta, \theta') = \begin{cases} 1 & \text{if } \theta > \theta' \text{ and } \theta \geq r \\ \frac{1}{2} & \text{if } \theta = \theta' \text{ and } \theta \geq r \\ 0 & \text{otherwise} \end{cases} \tag{12}$$

and the corresponding transfer function $t = t_1 = t_2$ is given by equation (10).

Note that the correspondence that maps ε to the set of optimal mechanisms is upper semicontinuous but not lower semicontinuous in ε at $\varepsilon = 0$. To see this note that when $\varepsilon = 0$, we are in the standard (independent and private value) setting with risk and ambiguity neutral bidders and seller. In this case, the set of optimal mechanisms consists of all mechanisms $\{(x_1, t_1), (x_2, t_2)\}$ where $x_1 = x_2 = x$ satisfies (12), r is given by $r - (1 - F(r))/f(r) = 0$, and the transfer functions satisfy

$$\int_0^1 t_i(\theta, \theta') \, d F(\theta') = X(\theta)\theta - \int_0^\theta X(y) \, dy.$$

Note that transfers are pinned down only at the interim level. Notice that setting $\varepsilon = 0$ and integrating over (10), we immediately see that the mechanism in Proposition 4 is optimal for the seller at $\varepsilon = 0$. Thus the correspondence that maps ε to the set of optimal mechanisms is upper semicontinuous at $\varepsilon = 0$. Yet, as is well known there are many other mechanisms (such as the second price auction with reserve price r) that are in the

set of optimal mechanisms at $\varepsilon = 0$. Obviously these mechanisms are not attained in the limit, so the correspondence is not lower semicontinuous.

It is interesting to note economic implications of the above analysis for revenue and efficiency. First, the seller's revenue increases as ambiguity increases. To see this note that under the above allocation rule $X^{\min}(\theta) = (1 - \varepsilon)X(\theta)$ for all $\theta < 1$. Plugging this into the revenue expression (11) we see that the revenue increases as ε increases. In fact, when ambiguity becomes extreme, i.e., when ε equals one, the seller extracts all the surplus.

Second, an increase in ambiguity helps efficiency. To see this note that L^{ε} shifts up as ε increases and since $L^{\varepsilon}(\theta)$ is an increasing function of θ, the cutoff type r decreases as ε increases. Again in the case of extreme ambiguity the seller does not exclude any types, and full efficiency is achieved.

Finally, a natural question to ask at this stage is how to implement the optimal mechanism described above. Several auctions implement the mechanism, and we describe one such auction here. Consider an auction where bidders submit bids for the object and the allocation rule is the usual one, namely, the highest bidder who bids above the reservation value r obtains the object. The payment scheme is as follows: the winning bidder pays to the auctioneer an amount equal to his bid, and all bidders (regardless of having won or lost) who have bid above the reservation price receives a gift from the seller. For a bidder who bids, say, b (where b is greater than r), the amount of the gift is $S(b) = (1 - \varepsilon)\int_r^b F(y)\,dy$. In this auction, the equilibrium strategy of a bidder with valuation θ is to bid his valuation. To see this note that the allocation rule is the same as the one in Proposition 4. Moreover, a bidder who bids θ pays $\theta - (1-\varepsilon)\int_r^{\theta} F(y)\,dy$ if he wins the auction and $-(1-\varepsilon)\int_r^{\theta} F(y)\,dy$ if he loses the auction, and these transfers are also the ones in Proposition 4. Since reporting one's true value is incentive compatible in the optimal mechanism, it is also optimal to bid one's true value in this auction as well.

5. The First and Second Price Auctions

Lo (1998) shows that the revenue equivalence result does not hold when bidders are ambiguity averse. In particular, the first price auction may generate more revenue than the second price auction. In this section we show that the first price auction is in general not optimal either.[14] In fact under rather general conditions, the first and second price auctions, as well as many other standard auctions are not optimal in this setting. The following proposition gives a weak condition on Δ_B that is sufficient for the non-optimality of a large class of auctions including the first and second price auctions.

PROPOSITION 5. *Suppose that Δ_S and Δ_B are weakly compact and convex with elements that are countably additive probability measures. Suppose that for any $G \in \Delta_S$ there exists some distribution $H \in \Delta_B$ such that H first-order stochastically dominates G. Now, if under some mechanism $\{(x_1, t_1), (x_2, t_2)\}$ with uniformly bounded transfers there exists a*

---

[14]When the type space is discrete, neither the first nor the second price auction is the optimal auction for reasons completely unrelated to the issues being studied in this paper.

bidder i and a positive measure subset $\widetilde{\Theta} \subseteq \Theta$ such that for all $\tilde{\theta} \in \widetilde{\Theta}$, $q_i(\tilde{\theta}, \theta)$ is weakly decreasing in θ and $q_i(\tilde{\theta}, \theta') < q_i(\tilde{\theta}, \theta'')$ for some $\theta', \theta'' \in \Theta$, then $\{(x_1, t_1), (x_2, t_2)\}$ is not optimal.

To apply the above proposition to the first and second price auctions, we need to show that in the direct mechanisms that correspond to these auction forms $q_i(\tilde{\theta}, \theta)$ is weakly decreasing in θ and $q_i(\tilde{\theta}, \theta') < q_i(\tilde{\theta}, \theta'')$ for some $\theta', \theta'' \in \Theta$ for a positive measure subset $\widetilde{\Theta} \subseteq \Theta$. First note that the payoff $q_i(\tilde{\theta}, \theta)$ of all types of a bidder is weakly decreasing in the report of the other bidder. Next consider a type $\tilde{\theta}$ that is greater than the reserve price and less than one, which is the highest possible valuation. The payoff of $\tilde{\theta}$ is strictly larger if the other bidder reports a type θ' that is less than $\tilde{\theta}$ as opposed to a type θ'' greater than $\tilde{\theta}$. This is because in both of these auctions if the other bidder reports more than $\tilde{\theta}$ the payoff of $\tilde{\theta}$ is zero, but if the other bidder reports less than $\tilde{\theta}$ the payoff of $\tilde{\theta}$ is strictly positive. This shows that $q_i(\tilde{\theta}, \theta') < q_i(\tilde{\theta}, \theta'')$. Therefore under the hypothesis of Proposition 5 the first and second price auctions are not optimal.

6. Ambiguity averse seller

Next, we consider the case where the seller is ambiguity averse and the bidders are ambiguity neutral. We restrict attention to symmetric mechanisms.[15] Our next result complements Proposition 1.

PROPOSITION 6. *Suppose that the seller is ambiguity averse, with a set of priors Δ_S, and the bidders are ambiguity neutral, with a prior $F \in \Delta_S$. For every incentive compatible and individually rational selling mechanism (x, t) there exists an incentive compatible and individually rational mechanism $(x, \tilde{t})$ that provides deterministically the same revenue to the seller, i.e. $\tilde{t}(\theta, \theta') + \tilde{t}(\theta', \theta)$ is constant for all $\theta, \theta' \in \Theta$. Moreover, if*

$$\inf_{G \in \Delta_S} \iint \left[t(\theta, \theta') + t(\theta', \theta) \right] dG(\theta) dG(\theta') < \iint \left[t(\theta, \theta') + t(\theta', \theta) \right] dF(\theta) dF(\theta')$$

then $(x, \tilde{t})$ strictly increases the minimum expected revenue of the seller over the set of priors Δ_S.

When an optimal mechanism exists, Proposition 6 implies that an auction that fully insures the seller must be in the set of optimal mechanisms. Eső and Futó (1999) prove a similar result for auctions with a risk averse seller in independent private values environments with risk (and ambiguity) neutral bidders.

---

[15]Since bidders are ambiguity neutral, we can now restrict attention to symmetric mechanisms without loss of generality. The argument is very similar to the standard one. If an asymmetric auction is optimal than by exchanging the roles of the bidders the seller obtains another optimal auction. Randomizing between these two asymmetric mechanisms generates a symmetric mechanism. Since bidders are ambiguity neutral, this symmetric mechanism is individually rational and incentive compatible. Moreover, it is easy to see that an ambiguity averse seller weakly prefers the symmetric mechanism.

The basic idea of the proof is simple. For any individually rational and incentive compatible mechanism (x, t), one can define a new mechanism $(\tilde{x}, \tilde{t})$ where the allocation rule $\tilde{x}$ is the same as x, but with the following transfers:

$$\tilde{t}(\theta, \theta') = T(\theta) - T(\theta') + \int T(i) \, dF(i)$$

where

$$T(\theta) = \int t(\theta, \theta') \, dF(\theta').$$

Note that in the new mechanism $\tilde{t}(\theta, \theta') + \tilde{t}(\theta', \theta)$ is always $2 \int T(i) \, dF(i)$, which is constant. It is straightforward to check that this mechanism is incentive compatible and individually rational as well. The reason this mechanism works in both risk and ambiguity settings is that since the bidders are risk and ambiguity neutral $(\tilde{x}, \tilde{t})$ is incentive compatible in either setting (risk or ambiguity) and provides full insurance to the seller against both risk and ambiguity.

7. COMPARISON OF OPTIMAL AUCTIONS WITH RISK AVERSE VS. AMBIGUITY AVERSE BIDDERS

Matthews (1983) and Maskin and Riley (1984), henceforth, MR, relax the assumption that bidders are risk neutral and replace it with risk aversion. Even though there is some similarity between risk aversion and ambiguity aversion, the two are distinct concepts. In particular, an environment with risk-averse bidders gives rise to optimal auctions that differ from the optimal auctions when bidders are ambiguity averse. In this section we contrast our results with those in MR, to highlight this distinction. To facilitate comparison, we assume, like MR, that the seller is risk and ambiguity neutral. Bidders, on the other hand, are risk averse and ambiguity neutral in MR and risk neutral and ambiguity averse in this paper.

MR define $u(-t, \theta)$ as the utility of a bidder of type θ when he wins and pays t, and $w(-t)$ as the utility when the bidder loses the auction (and pays t). Assuming u and w to be concave functions, they note that if the auction mechanism is such that the marginal utility u_1 is different from w_1, then keeping other things constant, a seller can gain by rearranging the payments in such a way that the bidder's expected utility remains the same while the expected value of the revenue increases. They note, however, that providing this insurance can change the incentives of the bidders; in particular when the marginal utility u_1 varies with θ, the seller can exploit this to earn higher revenue by exposing all but the highest type to some risk, thus, in effect, screening types better. MR define a mechanism called a *perfect insurance auction* where the marginal utility u_1 is equal to marginal utility w_1 for all types. Their results show that in general the optimal auction is not perfect insurance, the exception being the situation when bidders' preferences satisfy the condition $u_{12} = 0$, i.e. when the marginal utility u_1 does not vary with θ. (See their discussion following their Theorem 11.)

To contrast their result with ours, notice first that in our model (using their notation), $u(-t, \theta) = \theta - t$ and $w(-t) = -t$, so that $u_{12} = 0$, and more importantly, the

marginal utilities when a bidder wins and when he loses are equal to each other in all situations. (This is just restating the fact that we assume risk-neutral bidders in our model.) With ambiguity averse bidders, our results show that a full insurance auction is always within the set of optimal auctions and in some situations it is the uniquely optimal one. With risk-averse bidders, MR show that for the special case when $u(-t, \theta) = \theta - v(t)$ and $w(-t) = -v(t)$, so that $u_{12} = 0$, the optimal auction is a perfect insurance auction (given that v is a convex function).[16] Notice however, that our full insurance auction is *different* from their perfect insurance auction, since in a full insurance auction $x_i(\theta, \theta')\theta - t_i(\theta, \theta')$ is a function of θ only (i.e., does not vary with θ'), which means that the realized payoff when the bidder wins, $\theta - t_i(\theta, \theta')$, is the same as the realized payoff when he loses, $-t_i(\theta, \theta')$. Hence, the optimal auctions under the two situations are *different* mechanisms even when preferences in their model satisfy the restriction $u_{12} = 0$.[17]

Finally, note that in their framework, perfect insurance auctions do become full insurance auctions when preferences satisfy what they call Case 1. This is when $u(-t, \theta) = U(\theta - t)$ and $w(-t) = U(-t)$ (with U a concave function), so that equating marginal utilities implies equating utilities. However, in this situation, the perfect insurance auction (and hence the full insurance auction) is revenue equivalent to the second price auction (MR, Theorem 6). When U is strictly concave, both full insurance and second price auctions generate expected revenue that is strictly less than the expected revenue from the high bid auction (MR, Theorem 4 and Theorem 6; see in particular, the discussion at the bottom of page 1491). Hence the full insurance auction, which is the optimal (and in some cases, as mentioned above, the uniquely optimal) mechanism under ambiguity aversion is not the optimal mechanism under risk aversion.

8. CONCLUSION

We have analyzed the auctions that maximize a seller's profit when agents may not know the distribution from which the bidders' valuations are drawn. We have shown that when the bidders face more ambiguity than the seller, an auction that provides full insurance to the bidders is optimal and sometimes uniquely optimal. We have shown also that standard auctions such as the first and the second price auctions with reserve prices are not optimal in this setting. In addition, we have shown that when the bidders are ambiguity neutral but the seller is ambiguity averse, it is the seller who is perfectly insured.

The methods developed here maybe be used in other mechanism design problems with incomplete information in which agents are ambiguity averse. We believe that the results in this paper will naturally extend to these situations, especially in environments where the payoffs are quasilinear. For example in a bargaining problem (see Myerson 1979) we conjecture that the most efficient (from the mechanism designer's point of

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[16]Put differently, letting a and b be the payments when a bidder wins and loses the auction respectively, MR show that convexity of v implies that $a = b$ under the optimal mechanism.

[17]A further difference is that in our setting optimal mechanism may be asymmetric. See the discussion in footnote 6.

view) mechanism will require that some agent be fully insured against the ambiguity. In any case, and unlike the standard unique prior environment, the transfer and not just the allocation rule will play a crucial role in the design of the optimal mechanism in the presence of ambiguity. We hope to explore these extensions in future research.

<div align="center">APPENDIX</div>

<div align="center">A. PROOF OF PROPOSITION 1</div>

Fix a mechanism $\{(x_1, t_1), (x_2, t_2)\}$. Let

$$K_i(\theta) = \inf_{G \in \Delta_B} \int_\Theta q_i(\theta, \theta') \, dG(\theta'),$$

so that $K_i(\theta)$ is the maxmin expected payoff of type θ of bidder i. For any $\theta \in \Theta$, define the function $\delta_i(\theta, \cdot) : \Theta \to \mathbb{R}$ by

$$\delta_i(\theta, \theta') = q_i(\theta, \theta') - K_i(\theta) \text{ for all } \theta' \in \Theta.$$

Let $t_i'(\theta, \theta') = t_i(\theta, \theta') + \delta_i(\theta, \theta')$ and consider the mechanism $\{(x_1, t_1'), (x_2, t_2')\}$.

We prove the proposition in several steps. In the first step we show that $\{(x_1, t_1'), (x_2, t_2')\}$ is a full insurance mechanism. Furthermore, it leaves the bidders' payoffs unchanged under truth-telling and therefore is individually rational.

To see that $\{(x_1, t_1'), (x_2, t_2')\}$ is a full insurance mechanism consider an arbitrary type $\theta \in \Theta$ of bidder i and note that

$$
\begin{aligned}
x_i(\theta, \theta')\theta - t_i'(\theta, \theta') &= x_i(\theta, \theta')\theta - t_i(\theta, \theta') - \delta_i(\theta, \theta') \\
&= q_i(\theta, \theta') - q_i(\theta, \theta') + K_i(\theta) \\
&= K_i(\theta).
\end{aligned}
$$

Thus bidders' payoffs under truth telling are unchanged since

$$\inf_{G \in \Delta_B} \int_\Theta \left[x_i(\theta, \theta')\theta - t_i'(\theta, \theta') \right] dG(\theta') = K_i(\theta).$$

In the second step of the proof we show that $\{(x_1, t_1'), (x_2, t_2')\}$ is incentive compatible. The payoff for $\theta \in \Theta$ to deviate to an arbitrary $\tilde{\theta} \in \Theta$, $\theta \neq \tilde{\theta}$, is

$$
\begin{aligned}
\inf_{G \in \Delta_B} &\int_\Theta \left[x_i(\tilde{\theta}, \theta')\theta - t_i'(\tilde{\theta}, \theta') \right] dG(\theta') \\
&= \inf_{G \in \Delta_B} \int_\Theta \left[x_i(\tilde{\theta}, \theta')\theta - t_i(\tilde{\theta}, \theta') - \delta_i(\tilde{\theta}, \theta') \right] dG(\theta') \\
&\leq \inf_{G \in \Delta_B} \int_\Theta \left[x_i(\tilde{\theta}, \theta')\theta - t_i(\tilde{\theta}, \theta') \right] dG(\theta') - \inf_{G \in \Delta_B} \int_\Theta \delta_i(\tilde{\theta}, \theta') \, dG(\theta'). \qquad (13)
\end{aligned}
$$

The inequality above follows since the sum of the infimum of two functions is (weakly) less than the infimum of the sum of the functions. But note that

$$\inf_{G\in\Delta_B}\int_\Theta \delta_i(\tilde{\theta},\theta')\,dG(\theta')= \inf_{G\in\Delta_B}\int_\Theta \left[q_i(\tilde{\theta},\theta')-K_i(\tilde{\theta})\right]dG(\theta')=0.$$

Combining this with (13) implies that

$$\inf_{G\in\Delta_B}\int_\Theta \left[x_i(\tilde{\theta},\theta')\theta - t_i'(\tilde{\theta},\theta')\right]dG(\theta')\leq \inf_{G\in\Delta_B}\int_\Theta \left[x_i(\tilde{\theta},\theta')\theta - t_i(\tilde{\theta},\theta')\right]dG(\theta').$$

Now the payoff for type θ of bidder i from truth-telling in $\{(x_1, t_1'),(x_2, t_2')\}$ must be weakly larger than the last expression, because the mechanism $\{(x_1, t_1),(x_2, t_2)\}$ was assumed to be incentive compatible, and by the first step the truth telling payoffs are unchanged. Thus $\{(x_1, t_1'),(x_2, t_2')\}$ is incentive compatible.

In the third step we show that the seller is weakly better off using $\{(x_1, t_1'),(x_2, t_2')\}$. To see this first note

$$\inf_{G\in\Delta_S}\int_\Theta\int_\Theta \left[t_1'(\theta,\theta')+t_2'(\theta',\theta)\right]dG(\theta)\,dG(\theta')$$

$$= \inf_{G\in\Delta_S}\left[\int_\Theta\int_\Theta \left[t_1(\theta,\theta')+t_2(\theta,\theta')\right]dG(\theta)\,dG(\theta')\right.$$

$$\left.+ \int_\Theta\int_\Theta \delta_1(\theta,\theta')dG(\theta)\,dG(\theta')+ \int_\Theta\int_\Theta \delta_2(\theta,\theta')dG(\theta)\,dG(\theta')\right]$$

$$\geq \inf_{G\in\Delta_S}\int_\Theta\int_\Theta \left[t_1(\theta,\theta')+t_2(\theta,\theta')\right]dG(\theta)\,dG(\theta')$$

$$+ \inf_{G\in\Delta_S}\int_\Theta\int_\Theta \delta_1(\tilde{\theta},\theta')dG(\theta')\,dG(\tilde{\theta})+ \inf_{G\in\Delta_S}\int_\Theta\int_\Theta \delta_2(\tilde{\theta},\theta')dG(\theta')\,dG(\tilde{\theta}) \quad (14)$$

Moreover for any $G\in\Delta_S$,

$$\int_\Theta\int_\Theta \delta_i(\theta,\theta')dG(\theta')\,dG(\theta)$$

$$= \int_\Theta\int_\Theta \left[q_i(\theta,\theta')\,dG(\theta')-K_i(\theta))dG(\theta)\right.$$

$$= \int_\Theta\left[\int_\Theta q_i(\theta,\theta')dG(\theta')- \inf_{G'\in\Delta_B}\int_\Theta q_i(\theta,\theta')dG'(\theta')\right]dG(\theta)\geq 0 \quad (15)$$

where the inequality holds since by assumption $\Delta_S\subseteq\Delta_B$ and thus $G\in\Delta_B$. Combining equations (14) and (15) we see that the seller is weakly better off using $\{(x_1, t_1'),(x_2, t_2')\}$.

Finally we show that if there exists a bidder i and some positive measure $\widetilde{\Theta}\subseteq\Theta$ such that for any $\tilde{\theta}\in\widetilde{\Theta}$

$$\inf_{G\in\Delta_S}\int_\Theta q_i(\tilde{\theta},\theta')dG(\theta')> \inf_{H\in\Delta_B}\int_\Theta q_i(\tilde{\theta},\theta')dH(\theta')$$

then the seller strictly prefers $\{(x_1, t_1'), (x_2, t_2')\}$ to $\{(x_1, t_1), (x_2, t_2)\}$. To see this note that

$$
\inf_{G \in \Delta_S} \int_\Theta \int_\Theta \delta_i(\tilde{\theta}, \theta') \, dG(\theta') \, dG(\tilde{\theta})
$$

$$
= \inf_{G \in \Delta_S} \int_\Theta \left[\int_\Theta q_i(\tilde{\theta}, \theta') \, dG(\theta') - \inf_{H \in \Delta_B} \int q_i(\tilde{\theta}, \theta') \, dH(\theta') \right] dG(\tilde{\theta}) > 0. \quad (16)
$$

The strict inequality follows because for all $G \in \Delta_S$ the expression inside the integral is greater than zero for all $\tilde{\theta} \in \widetilde{\Theta}$ and, by assumption, the event $\widetilde{\Theta}$ gets strictly positive weight for all distributions in Δ_S. Combining equations (14) and (16) we conclude that the seller strictly prefers the mechanism $\{(x_1, t_1'), (x_2, t_2')\}$.

This completes the proof.

B. PROOF OF PROPOSITION 2

Note that since in this paper we deal in environments where the bidders' valuations are drawn independently, restricting attention to mechanisms where the transfers are uniformly bounded is without any loss of generality as far as a search for an optimal selling mechanism is concerned.

In our proof we use the following definitions and results. Suppose that p and q are conjugate indices, i.e. $1/p + 1/q = 1$. If $p = 1$ then the conjugate is $q = \infty$. Suppose that $f^n \in L_p(\Theta, \Sigma, \tilde{\mu})$ for $n \in \{1, 2, \ldots\}$. (From now on we will write L_p instead of $L_p(\Theta, \Sigma, \tilde{\mu})$ for notational simplicity.) We say that f^n converges weakly to $f \in L_p$ if $\int g f^n \, d\tilde{\mu}$ converges to $\int g f \, d\tilde{\mu}$ for all $g \in L_q$.

Denote by $\text{ca}(\Sigma)$ the set of countably additive probability measures on (Θ, Σ). Chateauneuf et al. (2005) prove that when $\Delta \subset \text{ca}(\Sigma)$ is weakly compact and convex then there is a measure $\tilde{\mu} \in \Delta$ such that all measures in Δ are absolutely continuous with respect to $\tilde{\mu}$. Using this result we fix $\tilde{\mu}$ to be a measure such that $\mu \ll \tilde{\mu}$ for all $\mu \in \Delta_B^m \cup \Delta_S^m$.

For each $\mu \in \Delta_B^m \cup \Delta_S^m$ there exists a Radon–Nikodym derivative $f \in L_1(\tilde{\mu})$. By the Radon–Nikodym Theorem, there is an isometric isomorphism between $\text{ca}(\tilde{\mu})$ and $L_1(\tilde{\mu})$ determined by the formula $\mu(A) = \int_A f \, d\tilde{\mu}$ (see Dunford and Schwartz 1958, p. 306 and Marinacci and Montrucchio 2004, Corollary 11). Hence, a subset is weakly compact in $\text{ca}(\tilde{\mu})$ if and only if it is in $L_1(\tilde{\mu})$ as well.

Let $\tilde{\Delta}_B$ and $\tilde{\Delta}_S$ be the set of Radon–Nikodym derivatives of measures in Δ_B^m and Δ_S^m with respect to $\tilde{\mu}$ respectively.

Finally, let

$$
\mathscr{B}_\infty^r = \{g \in L_\infty : \|g\|_\infty \leq r\}.
$$

By Theorem 19.4 in Billingsley (1995), $\mathscr{B}_\infty^r$ is weakly compact.

Now we turn to the proof.

PROOF OF PROPOSITION 2.   In this proof, to simplify notation, we drop the bidder subscript i. It is clear that all the arguments go through for asymmetric mechanisms as well. First we show that the minimizing set of priors in (1) is nonempty. Let

$$
g_{\tilde{\theta}\theta}(\theta') = x(\tilde{\theta}, \theta')\theta - t(\tilde{\theta}, \theta').
$$

Recall that we assume $|t(\theta,\theta')| \leq K$ for some $K > 0$. In other words transfers are uniformly bounded. Therefore by assumption $g_{\bar{\theta}\theta} \in L_\infty$.

Now suppose that $f^n \in \tilde{\Delta}_B$ is such that

$$\int g_{\bar{\theta}\theta} f^n \, d\tilde{\mu}$$

converges to

$$\inf_{f \in \tilde{\Delta}_B} \int g_{\bar{\theta}\theta} f \, d\tilde{\mu}.$$

Since $\tilde{\Delta}_B$ is weakly compact, by passing to a subsequence we can find $\bar{f} \in \tilde{\Delta}_B$ such that f^n weakly converges to $\bar{f}$. Thus

$$\bar{f} \in \arg\min_{f \in \tilde{\Delta}_B} \int g_{\bar{\theta}\theta} f \, d\tilde{\mu},$$

proving that the minimizing set of priors in the IC and IR constraints is nonempty.

Now, we show that the minimizing set of priors in the seller's objective function is nonempty. Suppose $f^n \in \tilde{\Delta}_S$ is such that

$$\iint t(\theta,\theta') f^n(\theta) f^n(\theta') \, d\tilde{\mu}(\theta) \, d\tilde{\mu}(\theta')$$

approaches

$$\inf_{f \in \tilde{\Delta}_S} \iint t(\theta,\theta') f(\theta) f(\theta') \, d\tilde{\mu}(\theta) \, d\tilde{\mu}(\theta').$$

Since $\tilde{\Delta}_S$ is weakly compact, by passing to a subsequence we can find $\bar{f} \in \tilde{\Delta}_S$ such that f^n weakly converges to $\bar{f}$. Thus $\int t(\theta,\theta') f^n(\theta) \, d\tilde{\mu}(\theta)$ converges to $\int t(\theta,\theta') \bar{f}(\theta) \, d\tilde{\mu}(\theta)$. Let

$$g^n(\theta') = \int t(\theta,\theta') f^n(\theta) \, d\tilde{\mu}(\theta) \qquad \text{and} \qquad g(\theta') = \int t(\theta,\theta') \bar{f}(\theta) \, d\tilde{\mu}(\theta).$$

Consider $\int g^n(\theta') f^n(\theta') \, d\tilde{\mu}(\theta')$. Note that

$$\left| \int g^n(\theta') f^n(\theta') \, d\tilde{\mu}(\theta') - \int g(\theta') \bar{f}(\theta') \, d\tilde{\mu}(\theta') \right|$$

$$\leq \left| \int g^n(\theta') f^n(\theta') \, d\tilde{\mu}(\theta') - \int g^n(\theta') \bar{f}(\theta') \, d\tilde{\mu}(\theta') \right|$$

$$+ \left| \int g^n(\theta') \bar{f}(\theta') \, d\tilde{\mu}(\theta') - \int g(\theta') \bar{f}(\theta') \, d\tilde{\mu}(\theta') \right|$$

$$\leq |K+1| \left| \int (f^n(\theta') - \bar{f}(\theta')) \, d\tilde{\mu}(\theta') \right| + \left| \int (g^n(\theta') - g(\theta')) \bar{f}(\theta') \, d\tilde{\mu}(\theta') \right|.$$

The first term goes to zero. To see that the second term also goes to zero note

$$\left| \int (g^n(\theta') - g(\theta')) \bar{f}(\theta') d\tilde{\mu}(\theta') \right|$$

$$= \left| \int \left(\int t(\theta, \theta') f^n(\theta) d\tilde{\mu}(\theta) - \int t(\theta, \theta') \bar{f}(\theta) d\tilde{\mu}(\theta) \right) \bar{f}(\theta') d\tilde{\mu}(\theta') \right|$$

$$= \left| \int \left(\int t(\theta, \theta') \bar{f}(\theta') d\tilde{\mu}(\theta') \right) f^n(\theta) d\tilde{\mu}(\theta) - \int \left(\int t(\theta, \theta') \bar{f}(\theta') d\tilde{\mu}(\theta') \right) \bar{f}(\theta) d\tilde{\mu}(\theta) \right|.$$

Thus

$$\bar{f} \in \arg\min_{f \in \tilde{\Delta}_S} \int\int t(\theta, \theta') f(\theta) f(\theta') d\tilde{\mu}(\theta) d\tilde{\mu}(\theta'),$$

proving that the minimizing set of priors in the seller's objective function is nonempty.

Next, we show that there exists a mechanism (x, t) that satisfies the IC and IR constraints and achieves the optimal revenue for the seller. Since transfers are bounded, the seller's revenue is bounded. Suppose that the value of the seller's problem (1) is R. This means that there exist a sequence of mechanisms $\{(x^n, t^n)\}$ such that (x^n, t^n) satisfies the IC and IR constraints for each n, and if we let

$$R^n = \min_{\mu \in \Delta_S^m} \int\int \left[t^n(\theta, \theta') + t^n(\theta', \theta) \right] d\mu(\theta) d\mu(\theta'),$$

then $R^n \to R$.

Note that $x^n \in \mathcal{B}_\infty^1$ and $t^n \in \mathcal{B}_\infty^K$. Therefore passing to subsequences, x^n converges weakly to x and t^n converges weakly to t. Clearly $x(\theta, \theta') + x(\theta', \theta) \leq 1$ for all $\theta, \theta' \in \Theta$.

Next, we show that (x, t) satisfies the IC and IR constraints. Note that it is sufficient to show that for any $\theta, \tilde{\theta} \in \Theta$,

$$\lim_{n \to \infty} \min_{\mu \in \Delta_B^m} \int \left[x^n(\tilde{\theta}, \theta') \theta - t^n(\tilde{\theta}, \theta') \right] d\mu(\theta') = \min_{\mu \in \Delta_B^m} \int \left[x(\tilde{\theta}, \theta') \theta - t(\tilde{\theta}, \theta') \right] d\mu(\theta').$$

To simplify notation let

$$g_{\tilde{\theta}\theta}^n(\theta') = x^n(\tilde{\theta}, \theta') \theta - t^n(\tilde{\theta}, \theta') \quad \text{and} \quad g_{\tilde{\theta}\theta}(\theta') = x(\tilde{\theta}, \theta') \theta - t(\tilde{\theta}, \theta')$$

for all $\theta, \tilde{\theta} \in \Theta$. Observe that $g_{\tilde{\theta}\theta}^n$ and $g_{\tilde{\theta}\theta}$ are both bounded by $K + 1$ and thus are both in L_∞. Moreover since x^n and t^n converge weakly to x and t, $g_{\tilde{\theta}\theta}^n$ converges weakly to $g_{\tilde{\theta}\theta}$.

Now note that for all $\hat{\mu} \in \Delta_B^m$,

$$\lim_{n \to \infty} \min_{\mu \in \Delta_B^m} \int g_{\tilde{\theta}\theta}^n(\theta') d\mu(\theta') \leq \lim_{n \to \infty} \int g_{\tilde{\theta}\theta}^n(\theta') d\hat{\mu}(\theta') = \int g_{\tilde{\theta}\theta}(\theta') d\hat{\mu}(\theta'), \qquad (17)$$

where the equality follows since $g_{\tilde{\theta}\theta}^n$ converges weakly to $g_{\tilde{\theta}\theta}(\theta')$. Thus

$$\lim_{n \to \infty} \min_{\mu \in \Delta_B^m} \int g_{\tilde{\theta}\theta}^n(\theta') d\mu(\theta') \leq \min_{\mu \in \Delta_B^m} \int g_{\tilde{\theta}\theta}(\theta') d\mu(\theta'). \qquad (18)$$

On the other hand, for each n let $f^n \in \tilde{\Delta}_B$ be such that

$$\int g_{\tilde{\theta}\theta}^n(\theta')f^n(\theta')\,d\tilde{\mu}(\theta') = \min_{f \in \tilde{\Delta}_B} \int g_{\tilde{\theta}\theta}^n(\theta')f(\theta')\,d\tilde{\mu}(\theta'). \tag{19}$$

(We know such an f^n exists since the minimizing set of priors is nonempty.) Since $\tilde{\Delta}_B$ is weakly compact again by passing to a subsequence, f^n converges weakly to $\bar{f} \in \tilde{\Delta}_B$. Note that

$$\left| \int g_{\tilde{\theta}\theta}^n(\theta')f^n(\theta')\,d\tilde{\mu}(\theta') - \int g_{\tilde{\theta}\theta}(\theta')\bar{f}(\theta')\,d\tilde{\mu}(\theta') \right|$$

$$\leq \left| \int g_{\tilde{\theta}\theta}^n(\theta')f^n(\theta')\,d\tilde{\mu}(\theta') - \int g_{\tilde{\theta}\theta}^n(\theta')\bar{f}(\theta')\,d\tilde{\mu}(\theta') \right|$$

$$+ \left| \int g_{\tilde{\theta}\theta}^n(\theta')\bar{f}(\theta')\,d\tilde{\mu}(\theta') - \int g_{\tilde{\theta}\theta}(\theta')\bar{f}(\theta')\,d\tilde{\mu}(\theta') \right|$$

$$\leq |K+1| \left| \int (f^n(\theta') - \bar{f}(\theta'))\,d\tilde{\mu}(\theta') \right| + \left| \int g_{\tilde{\theta}\theta}^n(\theta')\bar{f}(\theta')\,d\tilde{\mu}(\theta') - \int g_{\tilde{\theta}\theta}(\theta')\bar{f}(\theta')\,d\tilde{\mu}(\theta') \right|.$$

The last inequality follows from the fact that $|g_{\tilde{\theta}\theta}^n(\theta')| \leq K+1$. Since f^n weakly converges to $\bar{f} \in \tilde{\Delta}_B$ and $g_{\tilde{\theta}\theta}^n$ converges weakly to $g_{\tilde{\theta}\theta}$ both terms on the right hand side of the last inequality approach 0. This implies that by taking limits in equation (19),

$$\int g_{\tilde{\theta}\theta}(\theta')\bar{f}(\theta')\,d\tilde{\mu}(\theta') = \lim_{n \to \infty} \min_{f \in \tilde{\Delta}_B} \int \left[x^n(\tilde{\theta},\theta')\theta - t^n(\tilde{\theta},\theta') \right] f(\theta')\,d\tilde{\mu}(\theta'),$$

which in turn implies that

$$\min_{f \in \tilde{\Delta}_B} \int g_{\tilde{\theta}\theta}(\theta')f(\theta')\,d\tilde{\mu}(\theta') \leq \lim_{n \to \infty} \min_{f \in \tilde{\Delta}_B} \int \left[x^n(\tilde{\theta},\theta')\theta - t^n(\tilde{\theta},\theta') \right] f(\theta')\,d\tilde{\mu}(\theta').$$

The previous inequality together with (18) implies (17), which concludes the proof. □

C. Proof of Corollary 1

Suppose that for some mechanism $\{(x_1,t_1),(x_2,t_2)\}$, there exists a bidder i and some positive measure event $\widetilde{\Theta} \subseteq \Theta$ such that for all $\theta \in \widetilde{\Theta}$ and for all $G \in \Delta_S^{\min}$,

$$\int_\Theta q_i(\theta,\theta')\,dG(\theta') > \min_{H \in \Delta_B} \int_\Theta q_i(\theta,\theta')\,dH(\theta'). \tag{20}$$

We need to show that there exists a full insurance auction that is strictly preferred by the seller. Let $\{(x_1,t_1'),(x_2,t_2')\}$ be defined as in the proof of Proposition 1. We know that

$$\inf_{G \in \Delta_S} \int_\Theta \int_\Theta \left[t_1'(\theta,\theta') + t_2'(\theta',\theta) \right] dG(\theta)dG(\theta')$$

$$\geq \inf_{G \in \Delta_S} \int_\Theta \int_\Theta [t_1(\theta,\theta') + t_2(\theta,\theta')] \, dG(\theta)dG(\theta')$$

$$+ \inf_{G \in \Delta_S} \int_\Theta \int_\Theta \delta_1(\tilde{\theta},\theta')\,dG(\theta')dG(\tilde{\theta}) + \inf_{G \in \Delta_S} \int_\Theta \int_\Theta \delta_2(\tilde{\theta},\theta')\,dG(\theta')dG(\tilde{\theta}). \tag{21}$$

Let $\tilde{G} \in \arg\min_{G \in \Delta_S^m} \iint \left[t_1'(\theta, \theta') + t_2'(\theta', \theta) \right] dG(\theta) dG(\theta')$. We show the claim by considering two cases.

The first case is $\tilde{G} \in \Delta_S^{\min}$. In this case for all $\theta \in \widetilde{\Theta}$ equation (20) holds. Therefore

$$\int_\Theta \int_\Theta \delta_i(\theta, \theta') d\tilde{G}(\theta') d\tilde{G}(\theta)$$

$$= \int_\Theta \left[\int_\Theta q_i(\theta, \theta') d\tilde{G}(\theta') - \min_{H \in \Delta_B} \int_\Theta q_i(\theta, \theta') dH(\theta') \right] d\tilde{G}(\theta) > 0.$$

Using equations (14) and (15) we conclude that the seller strictly prefers the mechanism $\{(x_1, t_1'), (x_2, t_2')\}$.

The second case is $\tilde{G} \notin \Delta_S^{\min}$. In this case by definition of $\Delta_s^{\min}$,

$$\int_\Theta \int_\Theta \left[t_1(\theta, \theta') + t_2(\theta', \theta) \right] d\tilde{G}(\theta) d\tilde{G}(\theta')$$

$$> \min_{H \in \Delta_S} \int_\Theta \int_\Theta \left[t_1(\theta, \theta') + t_2(\theta', \theta) \right] dH(\theta) dH(\theta').$$

Again from equations (14) and (15) we observe that

$$\min_{G \in \Delta_S} \int_\Theta \int_\Theta \left[t_1'(\theta, \theta') + t_2'(\theta', \theta) \right] dG(\theta) dG(\theta')$$

$$= \int_\Theta \int_\Theta \left[t_1(\theta, \theta') + t_2(\theta', \theta) \right] d\tilde{G}(\theta) d\tilde{G}(\theta')$$

$$+ \int_\Theta \int_\Theta \delta_1(\tilde{\theta}, \theta') d\tilde{G}(\theta') d\tilde{G}(\tilde{\theta}) + \int_\Theta \int_\Theta \delta_2(\tilde{\theta}, \theta') d\tilde{G}(\theta') d\tilde{G}(\tilde{\theta})$$

$$> \min_{H \in \Delta_S} \int_\Theta \int_\Theta \left[t_1(\theta, \theta') + t_2(\theta', \theta) \right] dH(\theta) dH(\theta'),$$

and the seller strictly prefers $\{(x_1, t_1'), (x_2, t_2')\}$.

D. PROOF OF PROPOSITION 3

To obtain a contradiction, suppose that $\{(x_1, t_1), (x_2, t_2)\}$ is optimal but for bidder i and for a positive measure set of $\bar{\theta}$, $q_i(\bar{\theta}, \theta)$ is not constant. Since F has strictly positive density we have $\int q_i(\bar{\theta}, \theta') dF(\theta') > \inf_{\theta' \in \Theta} q_i(\bar{\theta}, \theta')$. So

$$\int q_i(\bar{\theta}, \theta') dF(\theta') > (1 - \epsilon) \int_\Theta q_i(\bar{\theta}, \theta') dF(\theta') + \epsilon \inf_{\theta' \in \Theta} q_i(\bar{\theta}, \theta')$$

$$= \inf_{H \in \Delta_B} \int_\Theta q_i(\bar{\theta}, \theta') dH(\theta').$$

By Proposition 1, $\{(x_1, t_1), (x_2, t_2)\}$ is not optimal.

E. Proof of Lemma 1

We need to show that there exists $M > 0$ such that

$$\left| u_i(\theta) - u_i(\tilde{\theta}) \right| \le M \left| \theta - \tilde{\theta} \right|.$$

We know that

$$(\theta - \tilde{\theta}) X_i^{\min}(\tilde{\theta}) \le u_i(\theta) - u_i(\tilde{\theta}) \le (\theta - \tilde{\theta}) X_i^{\max}(\tilde{\theta}).$$

So if $\theta > \tilde{\theta}$, using the fact that u_i is increasing we can conclude that

$$u_i(\theta) - u_i(\tilde{\theta}) \le (\theta - \tilde{\theta}) X_i^{\max}(\tilde{\theta}) \le \left| \theta - \tilde{\theta} \right|.$$

Similarly if $\theta < \tilde{\theta}$, then

$$-(u_i(\theta) - u_i(\tilde{\theta})) \le -(\theta - \tilde{\theta}) X_i^{\min}(\tilde{\theta}) \le \left| \theta - \tilde{\theta} \right|.$$

Together these imply that the Lipschitz condition holds with $M = 1$.

F. Proof of Proposition 4

First note that L^ε is increasing in θ if L is increasing in θ. To see this note that

$$\theta - \frac{1 - F(\theta)}{f(\theta)} > \theta' - \frac{1 - F(\theta')}{f(\theta')}$$

$$\Rightarrow \qquad \theta - \theta' > \frac{1 - F(\theta)}{f(\theta)} - \frac{1 - F(\theta')}{f(\theta')}$$

$$\Rightarrow \qquad \theta - \theta' > (1 - \varepsilon) \left(\frac{1 - F(\theta)}{f(\theta)} - \frac{1 - F(\theta')}{f(\theta')} \right)$$

$$\Rightarrow \qquad \theta - (1 - \varepsilon) \frac{1 - F(\theta)}{f(\theta)} > \theta' - (1 - \varepsilon) \frac{1 - F(\theta')}{f(\theta')}.$$

Note that $X_i^{\min}(\theta) \le X_i(\theta)$. Therefore if $X_i(\theta) = 0$, $X_i^{\min}(\theta) = 0$ as well. Letting $X_i^{\min}(\theta)/X_i(\theta) = 1$ whenever $X_i(\theta) = 0$, we define

$$M_i(\theta) = \theta - \frac{X_i^{\min}(\theta)}{X_i(\theta)} \cdot \frac{1 - F(\theta)}{f(\theta)}.$$

We can rewrite R as

$$R = \sum_{i=1}^{2} \left[\int_\Theta \int_\Theta M_i(\theta) x_i(\theta, \theta') f(\theta') f(\theta) \, d\theta' \, d\theta \right]. \tag{22}$$

Now we can show that the optimal allocation rule is given by $x_i(\theta,\theta')=1$ if $\theta > \theta'$ and $\theta \geq r$, $x_i(\theta,\theta')=\frac{1}{2}$ if $\theta = \theta'$ and $\theta \geq r$, and $x(\theta,\theta')=0$ otherwise. First note that, in the ε-contamination case, $X_i^{\min}(\theta) \geq (1-\varepsilon)X_i(\theta)$ for all θ such that $X_i(\theta) < 1$.[18]

Under the above allocation rule $X_i^{\min}(\theta) = (1-\varepsilon)X_i(\theta)$ for all θ such that $X_i(\theta) < 1$. Therefore this allocation rule maximizes $M_i(\theta)$. By construction $x_i(\theta,\theta')=1$ if and only if $M_i(\theta) > M_i(\theta')$ and $M_i(\theta) \geq 0$ therefore maximizing (22). Since the allocation rule is the same for both bidders, the optimal mechanism is symmetric and we drop the bidder subscripts in the rest of the proof.

Finally we show that (x,t) is incentive compatible. To this end first we show that if $X^{\min}$ is non-decreasing, selecting u as in (9) satisfies IC. We check two cases.

If $\theta > \tilde{\theta}$,

$$u(\theta) - u(\tilde{\theta}) = \int_{\tilde{\theta}}^{\theta} X^{\min}(y)\, dy \geq X^{\min}(\tilde{\theta})(\theta - \tilde{\theta})$$

and if $\theta < \tilde{\theta}$,

$$u(\tilde{\theta}) - u(\theta) = \int_{\theta}^{\tilde{\theta}} X^{\min}(y)\, dy \leq X^{\min}(\tilde{\theta})(\tilde{\theta} - \theta).$$

So in either case,

$$u(\theta) \geq u(\tilde{\theta}) + \inf_{G \in \Delta_B} \int_{\Theta} (\theta - \tilde{\theta})x(\tilde{\theta},\theta')\, dG(\theta')$$

$$= \inf_{G \in \Delta_B} \int_{\Theta} (x(\tilde{\theta},\theta')\theta - t(\tilde{\theta},\theta'))\, dG(\theta').$$

which is the IC constraint.

Now, note that for the allocation rule in the statement of Proposition 4, $X^{\min}$ is non-decreasing, and thus the mechanism (x,t) is incentive compatible.

G. PROOF OF PROPOSITION 5

Suppose that given a mechanism $\{(x_1,t_1),(x_2,t_2)\}$, there exists a bidder i and a positive measure subset $\widetilde{\Theta} \subseteq \Theta$ such that for all $\tilde{\theta} \in \widetilde{\Theta}$, $q_i(\tilde{\theta},\theta)$ is weakly decreasing in θ and $q_i(\tilde{\theta},\theta') < q_i(\tilde{\theta},\theta'')$ for some $\theta',\theta'' \in \Theta$. First note that if $H \in \Delta_B$ first-order stochastically dominates $G \in \Delta_S$ then

$$\int_{\Theta} q_i(\tilde{\theta},\theta')\, dH(\theta') < \int_{\Theta} q_i(\tilde{\theta},\theta')\, dG(\theta').$$

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[18]This is true since

$$X_i^{\min}(\theta) = \inf_{G \in \Delta_b} \int_{\Theta} x_i(\theta,\theta')\, dG(\theta')$$

$$= (1-\varepsilon)\int_{\Theta} x_i(\theta,\theta')\, d\tilde{\mu}(\theta') + \varepsilon \inf_{\hat{\mu} \ll \mu} \int_{\Theta} x_i(\theta,\theta')\, d\hat{\mu}(\theta')$$

$$\geq (1-\varepsilon)\int_{\Theta} x_i(\theta,\theta')\, dF(\theta').$$

So there exists $H \in \Delta_B$ such that

$$\int_\Theta q_i(\theta,\theta')\,dH(\theta') < \min_{G \in \Delta_S} \int_\Theta q_i(\theta,\theta')\,dG(\theta')$$

which in turn implies that

$$\min_{H \in \Delta_B} \int_\Theta q_i(\theta,\theta')\,dH(\theta') < \min_{G \in \Delta_S} \int_\Theta q_i(\theta,\theta')\,dG(\theta').$$

Note that since Δ_S and Δ_B are weakly compact and convex with elements that are countably additive probability measures and the transfers are uniformly bounded, the minimums above exist by Proposition 2. Finally, by Proposition 1, $\{(x_1,t_1),(x_2,t_2)\}$ is not optimal.

H. PROOF OF PROPOSITION 6

Let (x,t) be an arbitrary symmetric, incentive compatible, and individually rational mechanism. Define $T(\theta)$ as bidder θ's expected transfer under F, that is,

$$T(\theta) = \int_\Theta t(\theta,\theta')\,dF(\theta').$$

Now let

$$\tilde{t}(\theta,\theta') = T(\theta) - T(\theta') + \int_\Theta T(i)\,dF(i).$$

First, we show that the mechanism $(x,\tilde{t})$ makes the seller (weakly) better off, leaves the bidders' payoffs unchanged under truthtelling, and is incentive compatible.

To see that the seller is (weakly) better off under $(x,\tilde{t})$, note that the seller's payoff in the mechanism $(x,\tilde{t})$ is

$$\inf_{G \in \Delta_S} \int_\Theta \int_\Theta \left[\tilde{t}(\theta,\theta') + \tilde{t}(\theta',\theta)\right]\,dG(\theta)dG(\theta')$$

$$= \inf_{G \in \Delta_S} \int_\Theta \int_\Theta \left[T(\theta) - T(\theta') + \int T(i)\,dF(i)\right.$$

$$\left. + T(\theta') - T(\theta) + \int_\Theta T(j)\,dF(j)\right]\,dG(\theta)dG(\theta')$$

$$= \inf_{G \in \Delta_S} \int_\Theta \int_\Theta \left[2\int T(i)\,dF(i)\right]\,dG(\theta)dG(\theta')$$

$$= 2\int_\Theta T(i)\,dF(i)$$

$$= \int_\Theta \int_\Theta \left[t(\theta,\theta') + t(\theta',\theta)\right]\,dF(\theta)dF(\theta')$$

$$\geq \inf_{G \in \Delta_S} \int_\Theta \int_\Theta \left[t(\theta,\theta') + t(\theta',\theta)\right]\,dG(\theta)dG(\theta') \tag{23}$$

where the last inequality follows since $F \in \Delta_S$. Hence the seller weakly prefers $(x, \tilde{t})$.

Next we show that $(x, \tilde{t})$ leaves the bidders' payoffs unchanged under truth-telling. By construction,

$$\int_\Theta \tilde{t}(\theta, \theta') dF(\theta') = \int_\Theta \left[T(\theta) - T(\theta') + \int_\Theta T(i) dF(i) \right] dF(\theta')$$

$$= T(\theta) - \int_\Theta T(\theta') dF(\theta') + \int_\Theta T(i) dF(i)$$

$$= T(\theta)$$

$$= \int_\Theta t(\theta, \theta') dF(\theta').$$

Finally we show that $(x, \tilde{t})$ is incentive compatible. Note that,

$$\int \left[\theta x(\tilde{\theta}, \theta') - t(\tilde{\theta}, \theta') \right] dF(\theta') = \int \theta x(\tilde{\theta}, \theta') dF(\theta') - \int t(\tilde{\theta}, \theta') dF(\theta')$$

$$= \int \theta x(\tilde{\theta}, \theta') dF(\theta') - T(\tilde{\theta})$$

$$= \int \theta x(\tilde{\theta}, \theta') dF(\theta') - \int \tilde{t}(\tilde{\theta}, \theta') dF(\theta').$$

So the payoff for type θ to pretend to be $\tilde{\theta}$ is the same in both the mechanisms (x, t) and $(x, \tilde{t})$ and since (x, t) is incentive compatible, $(x, \tilde{t})$ is as well. Since, by construction, $\tilde{t}(\theta, \theta') + \tilde{t}(\theta', \theta)$ is constant for all $\theta, \theta' \in \Theta$, the first part of the proof is complete. Next suppose

$$\inf_{G \in \Delta_S} \int_\Theta \int_\Theta \left[t(\theta, \theta') + t(\theta', \theta) \right] dG(\theta) dG(\theta') < \int_\Theta \int_\Theta \left[t(\theta, \theta') + t(\theta', \theta) \right] dF(\theta) dF(\theta').$$

Then the weak inequality in (23) is strict and the seller is strictly better off.

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