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Group formation and voter participation

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We present a model of participation in large elections in which the formation of voter groups is endogenous. Partisan citizens decide whether to become leaders (activists) and try to persuade impressionable citizens to vote for the leaders' preferred party. In the (unique) pure strategy equilibrium, the number of leaders favoring each party depends on the cost of activism and the importance of the election. In turn, the expected turnout and the winning margin in an election depend on the number of leaders and the strength of social interactions. The model predicts a nonmonotonic relationship between the expected turnout and the winning margin in large elections.

KEYWORDS. Voter mobilization, endogenous leaders, turnout, winning margin.

JEL CLASSIFICATION. D72.

"Most people are other people. Their thoughts are someone else's opinions, their lives a mimicry, their passions a quotation." (Oscar Wilde, De Profundis, 1905)

"Si nos habitudes naissent de nos propres sentimens dans la retraite, elles naissent de l'opinion d'autrui dans la Société. Quand on ne vit pas en soi, mais dans les autres, ce sont leurs jugemens qui règlent tout ..." (Jean-Jacques Rousseau, Lettre à M. d'Alembert, 1758)

1. INTRODUCTION

It has been observed that an individual's decision to cast a vote in a large election is strongly correlated with indicators of the degree to which the individual is integrated in society. Empirical research from the US and other countries has found that citizens who are better educated, older, more religious, married, and less mobile are more likely

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to vote.[1] The decision to cast a vote, it has also been observed, is weakly related to the individual benefit of voting; the significant of variations in the cost of voting due to, say, weather conditions, has been shown to be marginal.[2]

Overall, the evidence suggests that in order to understand voter participation in large elections we must look beyond individual voters to the groups they belong to. This is hardly surprising. If voters are motivated only by the effect their actions may have on the result of an election and there is only a small cost involved in the act of voting then game-theoretic models predict a dismally low turnout—a prediction clearly at odds with mass participation in elections.[3] Thus, for empirical and theoretical reasons, attention has turned to modeling participation in elections as a group activity.

The earliest group-based models of voter turnout (e.g. Uhlaner 1989 and Morton 1991) effectively substitute a game between relatively few players (groups or rather the leaders of groups) for a game between many players (voters at large). Early models emphasize either side-payments, social pressure, or "group identity" as explanations of why individual voters would follow group leaders. More recently, Feddersen and Sandroni (2006), following Harsanyi (1980), propose an ethical theory of group behavior in elections. Coate and Conlin (2004) propose an ethical model similar to that of Feddersen and Sandroni (2006). Incidentally, Coate and Conlin assume that the distribution of supporters of each party is the beta distribution, while we obtain endogenously a beta distribution of supporters by aggregating the influence of leaders.

The group-based models of voter turnout proposed so far have in common the idea that the electorate is divided into mutually exclusive prearranged groups. Each group has a leader that spends effort to persuade group members (the leader's potential followers) to vote. In this paper we take leaders and their groups as endogenous: leaders emerge out of the population and compete for followers. We look at the implications of having endogenous leaders/groups for electoral participation in large elections. We propose a model in which there is a continuum of citizens distributed uniformly over a circle that represents the social network. Each of a countable infinity of citizens has an unbending preference for a party and is a potential opinion leader, while the rest, the overwhelming majority, are followers whose preferences can be influenced by opinion leaders. Citizens with strong preferences must decide whether to become leaders and mobilize groups of voters in support of their preferred candidate in an election. If a citizen becomes a leader, the citizen is assigned an interval of influence on the circle. The length of the interval of influence of each leader is random, as leaders cannot know exactly what their leadership effort will accomplish, e.g. how far the leader's influence will spread in the social network. Nonetheless, leaders compete for followers so the length of a leader's interval of influence depends negatively on the number of other leaders and

[1] The landmark study of Wolfinger and Rosenstone (1980) for the US shows the importance of education, age, marriage, and mobility. The recent study by Blais (2000) in nine democratic countries, including the US, maintains that education and age are important factors as are religiosity, income, and marriage.

[2] Knack (1994) shows that the impact of rain on voter turnout in the US is minimal. See the excellent discussion in Blais (2000).

[3] See e.g. Palfrey and Rosenthal (1983, 1985), who build on earlier work by Ledyard (1984). An elegant treatment of the subject is provided by Myerson (1998).

possibly also on exogenous breaks in the circle, which represent weak spots in social interactions. Leaders are able to bring influenceable citizens in their leadership group to the voting booth. While becoming a leader is costly, it has the potential benefit of swinging the election in the direction favored by the leader.

The game between potential leaders has some analogies with the Palfrey and Rosenthal (1983, 1985) voter participation game. Leaders in our model compare the probability of being decisive with the cost/benefit ratio of political activism, just as citizens in the Palfrey–Rosenthal model compare the probability of being decisive with the cost/benefit ratio of voting. Like citizens in the Palfrey–Rosenthal model, potential leaders in our model become activists in low (that is, finite) numbers because the probability of being decisive declines with participation. However, since each leader commands broad support in the society, voter turnout in our model is relatively high.

What distinguishes our game between potential leaders from a voter participation game is that group leaders are uncertain about how much support they will muster, and the introduction of a new leader is likely to reduce the expected support of existing leaders. These features of the model are important in terms of comparative statics. When there are few leaders in relation to the number of exogenous breaks, the main effect of adding new leaders (say, in response to a decrease in the cost/benefit ration of activism) is to bring voters from abstention rather than to reduce the support of existing leaders. Thus, if there are initially few leaders, increasing the number of leaders increases the expected margin of victory. Intuitively, the variance in the difference between the number of votes for one party and the number of votes for the other increases if we add new leaders whose support is approximately independent of the support of other leaders. If there are many leaders in relation to the number of exogenous breaks, we get the opposite result. With many leaders, the main effect of adding new leaders is to take away supporters from existing leaders, which reduces the expected margin of victory. Intuitively, the variance in the difference between the number of votes for one party and the number of votes for the other is reduced because of the increased (negative) covariance of the supports of the two parties.

In our model, increasing the number of leaders for each party increases the expected winning margin if the expected turnout is below 50% and decreases the expected winning margin if the expected turnout is above 50%. Since increasing the number of leaders always increases turnout, we get a nonmonotonic relationship between the winning margin and the expected turnout. For commonly observed turnout levels, though, the model predicts a positive association between turnout and the closeness of the election result, in accordance with the evidence. Our model illustrates the ability of group-based theories of voting to generate nontrivial, testable implications about the relationship between turnout and winning margin, and it also suggests the need to gather evidence about the winning margin in elections with relatively low turnout rates.

Another comparative static result we obtain is that stronger social interactions among citizens imply that each leader has a greater influence on followers, which in turn increases the voter turnout and the winning margin. The empirical implication of this result is that more integrated societies should have higher voter turnout. We can interpret the strength of social interactions in our model as reflecting the size of the polity, which introduces the possibility of segmented support for potential leaders, or the quality of group leaders in terms of their ability to address larger audiences. The strength of local interactions can also be correlated with the extent of overlapping memberships in social institutions such as unions and churches. In our model, fractionalization of unions and churches, even keeping total union or church membership constant, should depress turnout and lead to tighter elections.

Our model has some resemblance to the citizen-candidate models pioneered by Osborne and Slivinski (1996) and Besley and Coate (1997). Our model has in common with theirs the idea of endogenizing political activism. However, the type of political activism we try to capture in our model is radically different. Citizen-candidate models focus on the formation of party platforms and consider political activists who are candidates themselves, but they do not consider the issue of voter turnout. Our model, by comparison, takes party platforms as given and focuses on the issue of voter turnout. Further, political activists in our model are not candidates themselves, but rather citizens interested in influencing the election outcome. Finally, we assume that most voters, unlike political activists, have very weak policy preferences and may vote for one candidate or the other. In the conclusions, we discuss some implications of this assumption versus the alternative assumption that voters are arranged in a one-dimensional policy space.

We can also compare our model with the work by Shachar and Nalebuff (1999) and Morton (1991) on voter mobilization. In our model, leaders are not exogenously given and groups are not fixed or mutually exclusive. The margin of analysis on which we focus is that of the decision to try to influence all (or most) citizens, rather than the effort invested in mobilizing pre-assigned followers.

Our work is also related to the literature on social interactions pioneered by Glaeser et al. (1996), among others.[4] We borrow from them the arrangement of citizens on a circle and the idea that most citizens imitate the behavior of the group to which they belong, while some act independently. We deviate in that the number of citizens who act as leaders is derived endogenously in the model. When both the costs and benefits of an activity such as casting a vote or issuing an opinion are very low, *most* people are impressionable[5] and may be content to follow the lead of a *few*, as expressed, perhaps not without certain pessimism, by Rousseau and Wilde in our introductory quotations.

2. The model

We consider a winner-take-all election with two alternatives or parties, A and B. There is a continuum of citizens of measure one. The members of a countably infinite subset of the citizens are committed party A partisans, the members of another countably infinite subset of citizens are committed party B partisans, and the remainder, the vast majority, are uncommitted influenceable citizens whom we refer to as followers. Partisans enjoy a net gain of $G > 0$ if their preferred party wins the election. The party favored by the

[4] See Becker and Murphy (2000) and Durlauf and Young (2001) for more recent contributions.

[5] The assumption that a large (possibly decisive) set of voters is impressionable has been used before by Grossman and Helpman (2001), Baron (1994), and others.

larger number of votes wins the election. In case both parties obtain the same number of votes, each wins with probability $\frac{1}{2}$.

The sequence of events is as follows. First, every partisan citizen chooses whether to become an active supporter of a party or not. We refer to an active supporter as an opinion *leader*. Becoming a leader in favor of a particular party involves a utility cost $c > 0$. Second, followers are assigned to (or influenced by) a leader or no leader at all according to a random process described in detail below. In the benchmark model, all the followers influenced by a leader are mobilized to vote for the party that the leader supports. Finally, citizens who are not assigned to any leader abstain while all partisans vote for the party they support. As will become clear later, the overwhelming majority of partisan citizens do not become leaders but remain a negligible part of the voting population.

We want to capture two important aspects of any leader's influence. First, a leader is uncertain about the number of followers the leader can influence to vote for the party the leader favors. Second, the impact of a leader is diminished by the presence other leaders: the leader's personal influence on followers fades the more leaders there are. We capture these two aspects of a leader's influence by assuming that leaders are dropped uniformly on a circle of measure one that represents the population of citizens. A fixed number O of exogenous interruptions is also dropped uniformly on the circle. Each leader is assigned the arc of the circle of citizens adjacent to the leader in a clockwise direction until the arc is interrupted by an exogenous interruption or by another leader, as per Glaeser et al. (1996). The inverse of the number of exogenous interruptions represents the strength of social interactions or the reach of a leader's influence absent other leaders. When social interactions are strong, the influence of a leader is likely to extend over the circle representing society until it is contested by some other leader. When social interactions are weak, the influence of a leader is likely to die out independently of the influence of other leaders.[6]

Figure 1 represents one possible realization of the influence of two party A leaders and three party B leaders with two exogenous interruptions. In the benchmark model, the sum of the blue arcs represents the voter turnout for party A, the sum of the red arcs represents the voter turnout for party B, and the sum of the white arcs represents the percentage of abstainers.

It is easy to check that only A-partisans want to become leaders in favor of party A, and similarly for B. Let L_A and L_B represent the (finite) numbers of A and B-partisans who become leaders. Using the model, we can calculate the probability that A wins the election, say $P(L_A, L_B)$. We define

$$\begin{aligned} P_A(L_A, L_B) &= P(L_A, L_B) - P(L_A - 1, L_B) \\ P_B(L_A, L_B) &= P(L_A, L_B - 1) - P(L_A, L_B) \end{aligned}$$

as the probabilities that a leader for party A and for party B, respectively, are decisive in

[6]The assumption that the number of exogenous interruptions is fixed is made only for simplicity. As we explain later (in Section 5), our results hold if the number of exogenous interruptions is random.

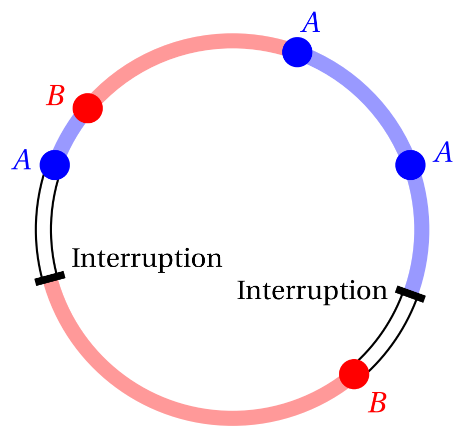

FIGURE 1. Influence of two party A leaders and three party B leaders.

favor of their party. We refer to these probabilities as *decisiveness in favor of party A* and *decisiveness in favor of party B*, respectively.[7]

An *equilibrium* is a pair of nonnegative integers L_A^*, L_B^* such that

$$\begin{aligned} P_A(L_A^*+1, L_B^*) &\leq c/G \\ P_A(L_A^*, L_B^*) &> c/G \quad \text{if } L_A^* \geq 1 \\ P_B(L_A^*, L_B^*+1) &\leq c/G \\ P_B(L_A^*, L_B^*) &> c/G \quad \text{if } L_B^* \geq 1. \end{aligned}$$

Note that the definition of equilibrium makes no reference to the identities of the partisans who become leaders. That is, we make no distinction between two situations in which different citizens become leaders, as long as the number of leaders for each party is the same. Note also that an equilibrium of the model is a pure strategy Nash equilibrium with the additional requirement that partisans become leaders only if they are strictly better off doing so. Without this restriction, the game has multiple equilibria for a countably infinite number of values of c/G. With the restriction, it has a unique equilibrium for all values of c/G. We describe the equilibrium of the model in the next section.

3. Decisiveness and Equilibrium

THEOREM 1. *There is a unique equilibrium. If $c/G < \frac{1}{2}$, in equilibrium $L_A^* = L_B^* = L^*$, where $L^* \geq 1$ is the largest integer satisfying the inequality*

$$\frac{1}{2^{2L-1}} \frac{(2L-2)!}{(L-1)!(L-1)!} > c/G.$$

[7] For L_A or L_B countably infinite, we simply assume that the probability of being decisive is zero. This guarantees that there is no equilibrium in which more than finitely many citizens become leaders.

If $c/G \geq \frac{1}{2}$, $L_A^* = L_B^* = 0$. *If* $c/G < \frac{1}{2}$, *the expected voter turnout in equilibrium is*

$$\frac{2L^*}{2L^* + O}$$

and the expected winning margin (the difference between the numbers of votes for the two parties) in equilibrium is

$$\frac{2L^*}{2L^* + O}\left(\frac{1}{2^{2L^*}}\frac{(2L^*)!}{L^*!L^*!}\right).$$

As will become clear from the proofs, the expression

$$\frac{1}{2^{2L-1}}\frac{(2L-2)!}{(L-1)!(L-1)!}$$

represents decisiveness when there are L leaders for each party. Note that this expression is strictly decreasing in L and takes the value $\frac{1}{2}$ when $L = 1$. If $c/G \leq \frac{1}{2}$ and there is some L' such that

$$c/G = \frac{1}{2^{2L'-1}}\frac{(2L'-2)!}{(L'-1)!(L'-1)!},$$

then according to the theorem the equilibrium is $L_A^* = L_B^* = L' - 1$. In this case the equilibrium is not strict; partisans who do not become leaders are indifferent between becoming leaders or not.[8] In every other case the equilibrium is strict.

The absence of asymmetric equilibria is due to the symmetry of decisiveness as described below. Intuitively, if one party has more leaders than another party and their partisans would be strictly worse off by decreasing the number of their leaders, then necessarily a partisan of the party with less support would be strictly better off by becoming an additional leader.

Before the proof of the theorem, we state a series of lemmas, the proofs of which are in the Appendix.

First, we consider the question of how a pair L_A, L_B of numbers of leaders for the parties maps into the distribution of votes.

LEMMA 1. *If* L_A, L_B, *and* O *are positive, the joint probability density function of the fraction* a *of votes for party A and the fraction* b *for party* B *is*

$$h_{L_A,L_B}(a,b) = \frac{(L_A + L_B + O - 1)!}{(L_A - 1)!(L_B - 1)!(O - 1)!}\, a^{L_A - 1} b^{L_B - 1}(1 - a - b)^{O-1}$$

for $0 \leq a + b \leq 1$.

If L_A *and* L_B *are positive and* $O = 0$, *the probability density function of the fraction* a *of votes for party A is*

$$h^0_{L_A,L_B}(a) = \frac{(L_A + L_B - 1)!}{(L_A - 1)!(L_B - 1)!}\, a^{L_A - 1}(1 - a)^{L_B - 1}$$

for $0 \leq a \leq 1$, *and for any value of* a, *the fraction of votes for party* B *is* $1 - a$.

[8] If the equilibrium definition did not preclude partisans from becoming leaders whenever indifferent, any pair $(L_A^*, L_B^*) \in \{L' - 1, L'\}^2$ would be an equilibrium.

Lemma 1 establishes that the joint distribution of the fractions of votes going to parties A and B is equal to the joint distribution of the L_A-th order statistic and the difference between the $(L_A + L_B)$-th and the L_A-th order statistic of a sample of size $L_A + L_B + O - 1$ drawn from a uniform distribution over the unit interval.[9] The idea of the proof is the following. We can pick the location in the circle of any leader for party A and consider that location point 0 (from the left) and point 1 (from the right). Thus, the remainder of leaders and exogenous breaks is a uniform sample of size $L_A + L_B + O - 1$. Due to a symmetry property of uniform order statistics, we can calculate the fractions of votes for parties A and B as if all the remaining leaders for party A came first in the unit interval and all the leaders for party B came second.

The distribution of party supporters we just derived is the bivariate beta distribution (also know as the Dirichlet distribution). In the special case $O = 0$ it reduces to the univariate beta, which is incidentally the distribution of party supporters taken (with exogenous parameters) as a starting point in Coate and Conlin (2004). In our model the parameters of the beta distribution represent the leaders exactly: any potential leader must assess exactly how much the change of parameters that his leadership induces in the beta distribution increases the probability that the leader's preferred party will win the election. Next, we calculate the probability that party A wins the election.

LEMMA 2. *If L_A and L_B are positive, the probability of party A winning the election is*

$$P(L_A, L_B) = 1 - \sum_{k=1}^{L_B} \left(\frac{1}{2}\right)^{L_A+L_B-1} \frac{(L_A + L_B - 1)!}{(L_A + k - 1)!(L_B - k)!}.$$

Note that the probability of party A winning the election is independent of the number of exogenous interruptions. Following the line of the previous intuitive argument, the probability that the L_A-th order statistic is larger than the difference between the $(L_A + L_B)$-th and L_B-th order statistics is independent of the size of the sample.

Next, we use Lemma 2 to calculate the decisiveness in favor of party A.

LEMMA 3. *If L_A and L_B are positive, the decisiveness in favor of party A is*

$$P_A(L_A, L_B) = \frac{1}{2^{L_A+L_B-1}} \frac{(L_A + L_B - 2)!}{(L_A - 1)!(L_B - 1)!}.$$

This expression for decisiveness is exceedingly simple and plays a key role in the proof of the theorem.

Finally, we derive some useful properties of decisiveness from this expression.

LEMMA 4. *If L_A and L_B are positive, decisiveness satisfies the following properties.*

(i) (*Near single-peakedness*)

$$P_A(L_A, L_B) \gtreqless P_A(L_A + 1, L_B) \iff L_A + 1 \gtreqless L_B.$$

[9] We follow the convention of naming the kth smallest element in the sample the kth-order statistic, for any k.

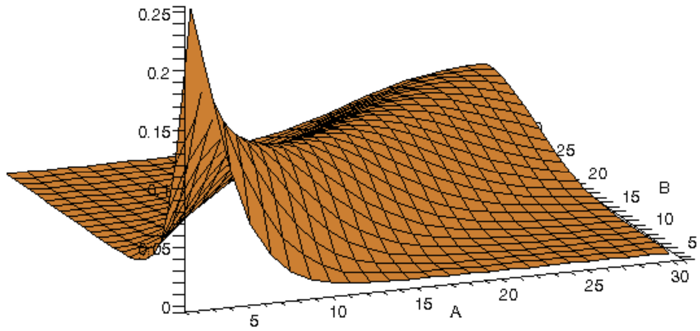

FIGURE 2. Decisiveness of a leader.

(ii) (*Symmetry*)

$$P_A(L_A, L_B) = P_A(L_B, L_A) = P_B(L_B, L_A).$$

Lemma 4 establishes, in particular, that decisiveness in favor of a party achieves its maximum value when the party has either the same number of leaders as the other party or one leader less.

Figure 2 illustrates the decisiveness of a leader of party A or party B as a function of the number of leaders of either party. To construct the plot we use the gamma function, which is conveniently defined over the reals, rather than the factorial function.

The decisiveness peaks on the diagonal axis ($L_A = L_B$), which is when the election is most likely to end up in a tie, illustrating the fact that there is little use in becoming an additional leader if one's preferred party is already very likely *to win or lose* the election. Also, the decisiveness decreases along the diagonal because of increased competition between leaders, which implies a smaller average influence of leaders.

We are now ready to prove the theorem.

PROOF OF THEOREM 1. Using the equilibrium conditions, any equilibrium L_A^*, L_B^* with positive numbers of leaders for parties A and B must be such that

$$\begin{aligned} P_A(L_A^*, L_B^*) &> c/G \geq P_A(L_A^* + 1, L_B^*) \\ P_B(L_A^*, L_B^*) &> c/G \geq P_B(L_A^*, L_B^* + 1). \end{aligned}$$

Using symmetry (Lemma 4*ii*), these conditions are equivalent to

$$\begin{aligned} P_A(L_A^*, L_B^*) &> c/G \geq P_A(L_A^* + 1, L_B^*) \\ P_A(L_B^*, L_A^*) &> c/G \geq P_A(L_B^* + 1, L_A^*). \end{aligned}$$

Using near single-peakedness (Lemma 4*i*), we get

$$L_A^* + 1 > L_B^* \text{ and } L_B^* + 1 > L_A^*,$$

which together imply

$$L_A^* = L_B^*.$$

Thus, any equilibrium with positive numbers of leaders for parties A and B must be such that $L_A^* = L_B^* = L^*$, where L^* satisfies

$$P_A(L^*, L^*) > c/G \geq P_A(L^* + 1, L^*).$$

Using symmetry, this condition is equivalent to

$$P_A(L^*, L^*) > c/G \geq P_A(L^*, L^* + 1).$$

Using near single-peakedness (Lemma 4i), the equilibrium condition is

$$P_A(L^*, L^*) > c/G \geq P_A(L^* + 1, L^* + 1).$$

Now, from Lemma 3 we get

$$P_A(L, L) = \frac{1}{2^{2L-1}} \frac{(2L-2)!}{(L-1)!(L-1)!}.$$

Note that this expression is strictly decreasing in L and takes the value $\frac{1}{2}$ when $L = 1$. Thus, there are equilibria with positive numbers of leaders for parties A and B if and only if $c/G < \frac{1}{2}$, and they are as described by the statement of the theorem.

Consider now an equilibrium in which there is a positive number L_A^* of leaders for party A and no leaders for party B. The equilibrium conditions are

$$\begin{gathered} P_A(L_A^*, 0) > c/G \geq P_A(L_A^* + 1, 0) \\ c/G \geq P_B(L_A^*, 1). \end{gathered}$$

Recall that if both parties receive the same fraction of votes, the election is resolved by a fair coin toss. Since neither party receives votes when there are no leaders, we have $P(0,0) = \frac{1}{2}$. Since only party A receives a positive fraction of votes when L_A^* is positive and L_B^* is zero, we have $P(L_A^*, 0) = 1$ for $L_A^* \geq 1$. Thus, $P_A(1,0) = \frac{1}{2}$ and $P_A(L_A^*, 0) = 0$ for $L_A^* \geq 2$. Using the first equilibrium condition above, we get $L_A^* = 1$. From symmetry and Lemma 3, we get $P_B(1,1) = P_A(1,1) = \frac{1}{2}$. Thus, for $L_A^* = 1$ the first equilibrium condition above implies $\frac{1}{2} > c/G$ and the second equilibrium condition implies $c/G \geq \frac{1}{2}$, a contradiction. It follows that there are no equilibria in which only party A has a positive number of leaders. A similar argument shows that there are no equilibria in which only party B has a positive number of leaders.

Finally, consider an equilibrium in which neither party has a positive number of leaders. The equilibrium conditions are

$$\begin{gathered} c/G \geq P_A(1,0) \\ c/G \geq P_B(0,1). \end{gathered}$$

We established in the previous paragraph that $P_A(1,0) = \frac{1}{2}$. A similar argument shows that $P_B(0,1) = \frac{1}{2}$. Thus, there is an equilibrium in which neither party has a positive number of leaders if and only if $c/G \geq \frac{1}{2}$.

For the remainder of the proof we assume that there is a positive number of leaders L for each party. With respect to voter participation, for $O=0$ it is easy to see that voter turnout is 1, as results from the formula provided in the statement of the theorem. For $O\geq 1$, we have that the expected turnout in equilibrium is

$$\mathrm{E}(a+b)=\int_0^1\int_0^{1-a}(a+b)h_{L,L}(a,b)\,db\,da.$$

Using Lemma 1,

$$\mathrm{E}(a+b)=\frac{(2L+O-1)!}{(L-1)!(L-1)!(O-1)!}\int_0^1\int_0^{1-a}(a+b)a^{L-1}b^{L-1}(1-a-b)^{O-1}\,db\,da.$$

Or equivalently,

$$\mathrm{E}(a+b)=\frac{(2L+O-1)!}{(L-1)!(L-1)!(O-1)!}\times \int_0^1\int_0^{1-a}\left(a^Lb^{L-1}(1-a-b)^{O-1}+a^{L-1}b^L(1-a-b)^{O-1}\right)db\,da.$$

Using Lemma 1 again,

$$\mathrm{E}(a+b)=\frac{(2L+O-1)!}{(L-1)!(L-1)!(O-1)!}\times\frac{(L-1)!L!(O-1)!}{(2L+O)!}\times \left(\int_0^1\int_0^{1-a}h_{L+1,L}(a,b)\,db\,da+\int_0^1\int_0^{1-a}h_{L,L+1}(a,b)\,db\,da\right).$$

Since $h_{L+1,L}(a,b)$ and $h_{L,L+1}(a,b)$ are bivariate probability density functions with support $0\leq a+b\leq 1$, we get

$$\mathrm{E}(a+b)=\frac{(2L+O-1)!}{(L-1)!(L-1)!(O-1)!}\times\frac{(L-1)!L!(O-1)!}{(2L+O)!}\times 2.$$

The result on voter turnout in the statement of the theorem follows.

With respect to the closeness of the election, suppose that $O\geq 1$. (The proof for $O=0$ is similar.) Using Lemma 1, the expected winning margin is

$$\begin{aligned}\mathrm{E}(|a-b|)&=2\mathrm{E}(a-b\mid a>b)\\&=2\int_0^1\int_0^{\min\{a,1-a\}}(a-b)h_{L,L}(a,b)\,db\,da\\&=\frac{2(2L+O-1)!}{(L-1)!(L-1)!(O-1)!}\int_0^1\int_0^{\min\{a,1-a\}}(a-b)a^{L-1}b^{L-1}(1-a-b)^{O-1}\,db\,da\\&=\frac{2(2L+O-1)!}{(L-1)!(L-1)!(O-1)!}\times\\&\qquad\int_0^1\int_0^{\min\{a,1-a\}}\left(a^Lb^{L-1}(1-a-b)^{O-1}-a^{L-1}b^L(1-a-b)^{O-1}\right)db\,da.\end{aligned}$$

Using Lemma 1 again,

$$\begin{aligned}\mathrm{E}(|a-b|) = {} & \frac{2(2L+O-1)!}{(L-1)!(L-1)!(O-1)!} \times \frac{(L-1)!L!(O-1)!}{(2L+O)!} \times \\ & \left(\int_0^1 \int_0^{\min\{a,1-a\}} h_{L+1,L}(a,b)\,db\,da - \int_0^1 \int_0^{\min\{a,1-a\}} h_{L,L+1}(a,b)\,db\,da\right).\end{aligned}$$

Note that

$$P(L_A, L_B) = \int_0^1 \int_0^{\min\{a,1-a\}} h_{L_A,L_B}(a,b)\,db\,da.$$

Thus, we get

$$\mathrm{E}(|a-b|) = \frac{2L}{2L+O}(\mathrm{P}(L+1,L) - \mathrm{P}(L,L+1)).$$

Using the definitions of decisiveness in favor of party A and party B and symmetry (Lemma 4*ii*),

$$\begin{aligned}\mathrm{E}(|a-b|) &= \frac{2L}{2L+O}(P_A(L+1,L+1) + P_B(L+1,L+1)) \\ &= \frac{4L}{2L+O} P_A(L+1,L+1).\end{aligned}$$

Using Lemma 3,

$$\mathrm{E}(|a-b|) = \frac{2L}{2L+O}\left(\frac{1}{2^{2L}}\frac{(2L)!}{L!L!}\right),$$

as stated in the theorem. □

4. Turnout and winning margin

In this section we analyze the relationship between the parameters of the model, namely the cost/benefit ratio of political activism (c/G) and the strength of social interactions ($1/O$), and the endogenous, observable variables of the model, namely the expected voter turnout and the expected winning margin.

Theorem 1 calculates the equilibrium number of leaders for each party as a (decreasing) function of the cost/benefit ratio of activism. The effects of changes in the cost/benefit ratio of activism and the strength of social interactions on the equilibrium expected voter turnout follow immediately from Theorem 1.

Corollary 1. *An increase in the strength of social interactions increases the expected voter turnout and the expected winning margin. A reduction in the cost/benefit ratio of activism (weakly) increases the expected voter turnout.*

Stronger social interactions imply more effective leadership while the equilibrium number of leaders remains unchanged. A stronger average influence of each leader implies a higher voter turnout for each party and a higher variance in the number of votes for each party which in turn implies a higher winning margin. While the effect of the cost/benefit ratio of activism on voter turnout is unambiguous, its effect on the closeness of the election result is more complex.

COROLLARY 2. *If the cost/benefit ratio of activism decreases, then the expected winning margin (weakly) decreases if initially the expected turnout is above* $\frac{1}{2}$ *and it (weakly) increases if initially the expected turnout is below* $\frac{1}{2}$.

PROOF. From Theorem 1, we have that the expected winning margin with $L+1$ leaders is

$$\frac{L+1}{L+1+O/2}\left(\frac{1}{2^{2L+2}}\frac{(2L+2)!}{(L+1)!(L+1)!}\right),$$

and with L leaders is

$$\frac{L}{L+O/2}\left(\frac{1}{2^{2L}}\frac{(2L)!}{L!L!}\right).$$

The ratio of these two expressions is

$$\frac{1+1/(2L)}{1+1/(L+O/2)},$$

which is smaller than one if and only if $O<2L$ (or equivalently, if and only if expected turnout is smaller than $\frac{1}{2}$) and is larger than one if and only if $O>2L$. □

Intuitively, if the number of voters is expected to be smaller than the number of abstainers, then extra leaders are more likely to persuade abstainers to vote rather than to steal voters from other leaders, thereby increasing the variance of the difference between the votes received by the parties. If, instead, the fraction of voters is expected to be larger than the fraction of abstainers, then new leaders tend to steal voters from each other rather than persuading abstainers to vote, thereby reducing the variance of the difference between the votes received by the parties. In sum, elections with higher voter turnout are closer elections only when the turnout is expected to be high (the majority votes), while in elections for which the turnout is expected to be low (the majority abstains), a higher actual turnout suggests a higher expected margin of victory.

In most democracies, the average voter turnout in elections at the national level is above 50% (see e.g. Blais 2000). In these circumstances, our model predicts that movements in the cost/benefit ratio of activism lead to movements in the winning margin in the same direction. If, as seems likely, the costs and benefits of activism experience more short-term variations than the strength of social interactions, we should expect to see a negative short-term correlation between the winning margin and voter turnout. Long term trends in voter turnout, however, are likely to be affected by trends in both the cost/benefit ratio of activism and the strength of social interactions.

The relation between the winning margin (or rather, the election closeness) and voter turnout has been the subject of empirical literature reviewed recently by Blais (2000). In this author's words,

> the verdict is crystal clear with respect to closeness: closeness has been found to increase turnout in 27 out of the 32 studies that have tested the relationship, in many different settings and with diverse methodologies. (Blais 2000, p. 60)

Blais goes on to state that the importance of election closeness is not captured by direct measures of the very small probability of a single vote being decisive. The positive relationship between election closeness and voter turnout, it seems, should be addressed at the group/leader level.

Let T be the expected turnout and W be the expected winning margin. Using Theorem 1, we can obtain an approximation for the parameters of the model when there is a lot of activism (that is, when the cost/benefit ratio of activism is small):

COROLLARY 3. *For fixed* $1/O$,

$$\lim_{c/G\to 0}\left(\frac{c/G}{W/(2T)}\right)=1 \quad \textit{and} \quad \lim_{c/G\to 0}\left(\frac{1/O}{\pi W^2/(2T(1-T))}\right)=1.$$

(See the proof in the Appendix.) Corollary 3 suggests that, for any given democracy, the ratio of winning margin to turnout should be positively correlated with other measures of the cost of political activism. Similarly, the ratio of the square of the winning margin to the product of participation and abstention rates should be positively correlated with other measures of the strength of local interactions, such as the extent of overlapping memberships in social institutions.

5. Extensions

Randomness in the number of exogenous interruptions in the circle is easily incorporated into the model. Though the model assumes leaders who are identical except for their party preferences, it can be extended to allow for heterogeneous cost/benefit ratios of activism under some conditions.

5.1 *Random interruptions*

Assume that the number of exogenous interruptions in the circle is a random variable $\tilde{O}$ with bounded support whose realization is not known by partisan citizens at the moment of deciding whether to become leaders. For any given numbers of leaders for parties A and B, the probability of party A winning the election is the same regardless of the realization of $\tilde{O}$ and is given by Lemma 2. Thus, the decisiveness in favor of party A is as given by Lemma 3 and therefore the equilibrium number of leaders is as given by Theorem 1. Expected voter turnout is now the expectation (conditional on the realization of $\tilde{O}$) of the expression for voter turnout in Theorem 1, and similarly for the expected winning margin.

5.2 *Heterogeneous leadership costs*

Assume that the cost of activism (or equivalently, the importance of the election) is heterogeneous across partisan voters and distributed according to some continuous probability density with the lower endpoint of the support given by $\underline{c} > 0$. Recall that the decision to become a leader depends on the decisiveness cost-benefit calculation. The equilibrium number of leaders L^* is given by the largest integer solution to the inequality

$$P_A(L,L) > \underline{c}/G.$$

There is really no difference with the homogeneous case. There may be some inefficiency, though, since the leaders in equilibrium are not necessarily the partisan citizens with lower costs. Only the citizens with a cost/benefit ratio smaller than $P_A(L^*, L^*)$ may become leaders.

5.3 *Different effectiveness of leaders*

We have assumed so far that the interval of influence of a leader is random but the effectiveness of the leaders of each party is the same. That is, we have assumed a leader could persuade all the voters in the leader's interval of influence to vote for the leader's preferred party. More generally, we can assume that leaders of party A and party B can attract only the fractions $\alpha \in (0,1]$ and $\beta \in (0,1]$, respectively, of the potential voters in their interval of influence. This setup allows for the proportions α and β to depend on a predisposition of impressionable voters in favor of either party. For $\alpha = \beta$, the equilibrium number of leaders of the model is the one we previously obtained; only the voter turnout should be scaled down accordingly if $\alpha = \beta < 1$. If $\alpha \neq \beta$, then the equilibrium is no longer necessarily symmetric. To extend the model we need to recalculate the decisiveness in favor of party A and party B.

LEMMA 5. *Let $\gamma = \alpha/(\alpha+\beta)$. If L_A and L_B are positive, the decisiveness in favor of party A is*

$$P_A(L_A, L_B) = (1-\gamma)^{L_A-1}\gamma^{L_B}\frac{(L_A+L_B-2)!}{(L_A-1)!(L_B-1)!}$$

and the decisiveness in favor of party B is

$$P_B(L_A, L_B) = ((1-\gamma)^{L_A}\gamma^{L_B-1}\frac{(L_A+L_B-2)!}{(L_A-1)!(L_B-1)!}.$$

(See the proof in the Appendix.)

The constant ratio of decisiveness

$$P_B(L_A, L_B)/P_A(L_A, L_B) = (1-\gamma)/\gamma = \beta/\alpha$$

is an indicator of the advantage of party B relative to party A. We can easily check

$$P_A(L_A, L_B) \gtreqless P_A(L_A-1, L_B) \iff (L_A-1) \lesseqgtr (\beta/\alpha)(L_B-1)$$

and

$$P_B(L_A, L_B) \gtreqless P_B(L_A, L_B-1) \iff (L_A-1) \gtreqless (\beta/\alpha)(L_B-1).$$

That is, both the decisiveness in favor of party A and the decisiveness in favor of party B peak when the number of leaders is at or near the ratio

$$\frac{L_A-1}{L_B-1} = \frac{\beta}{\alpha}.$$

As in the case of symmetric effectiveness, the equilibrium conditions

$$P_A(L_A, L_B) > \frac{c}{G} \geq P_A(L_A + 1, L_B)$$
$$P_B(L_A, L_B) > \frac{c}{G} \geq P_B(L_A, L_B + 1)$$

must be satisfied simultaneously, which implies that decisiveness must be at its peak. Note that for decisiveness to be at its peak, the party at a disadvantage must have more leaders than the other party. That is, decisiveness peaks when the probabilities of winning the election are near 50%.

6. Final remarks

There is no generally accepted model of the common social phenomenon of massive voluntary participation in large elections. A recent survey article by Feddersen (2004) has signalled what appears to be a growing interest in group-based models of voter turnout in which group members participate in elections because they are directly coordinated and rewarded by leaders. This paper is intended to be a contribution to the literature about groups in elections. At the substantive level, by having leaders self-select endogenously out of the population, our model sheds some light on how groups of voters can be formed and how voting behavior is affected by the underlying parameters of the model. These parameters include the cost of political activism, the importance of the election, and the strength or weakness of social interactions, even if the vast majority of voters do not behave strategically at the ballot box.

We carry out the analysis in a highly stylized environment. At the technical level, we are able to obtain an attractive closed form expression for the equilibrium of the model and the decisiveness of a leader. The equilibrium uniquely pins down the number of leaders for each party. The uniqueness is essential to construct a theory of voter turnout and the closeness of elections. We obtain intuitive comparative statics results with respect to the expected voter turnout and some unexpected results with respect to the expected winning margin. The fact that we deal successfully with the issues of existence and uniqueness of the pure strategy equilibrium is, we believe, an encouraging step in the direction of a satisfactory group-based model of elections.

We believe our model is flexible enough to be extended in a number of ways beyond those discussed in this paper. A version of the model that takes account of the different effectiveness of leaders may be useful as a building block to incorporate the variable effort of leaders. Though we focus on policy motivated leaders, incorporating a private interest for leaders is clearly feasible (e.g. a reward from the party proportional to the fraction of voters brought to the voting booth). Other extensions that may be of interest are allowing leaders to have overlapping intervals of influence and considering elections with three or more candidates.

Further afield, it would be interesting to consider a group-based model of voting in which voters were arranged in a one-dimensional policy space, and in which leaders were aware of their location in the policy space and were able to mobilize some random fraction of nearby voters. In such a model, we would expect there to be two different situations for a leader: leaders closer to the ideological extremes would draw voters (mostly) away from abstention, while leaders closer to moderate voters would draw voters away from the other party. As in the model we have discussed in this paper, there would be potentially a nonmonotonic relationship between expected turnout and winning margin, depending on which of the two effects of increasing the number of leaders (drawing voters away from abstention and drawing voters away from the other party) is stronger. The equilibrium location of leaders, and therefore which of the two effects is stronger, would be influenced by the objective function of leaders, e.g. whether they maximize the size of their support or are policy-motivated. It is hard to guess under which circumstances one of the two effects dominates the other without actually setting up and solving this alternative model. In the model we present in this paper we get a clear-cut result: drawing voters away from abstention dominates when turnout is low, and drawing voters away from the other party dominates when turnout is high.

We have provided in this paper what we believe to be an interesting starting point for considering the role of endogenous groups in a turnout game. Whether this or an alternative model provides a more fruitful avenue for research is an open question.

Appendix: Proofs

Before proving Lemma 1 we need some statistical results. Let $n+1 = L_A + L_B + O$ be the total number of leaders and exogenous interruptions that are distributed uniformly on the circle. Pick any leader or exogenous interruption and call that point 0 (when moving counterclockwise) and 1 (when moving clockwise). From 0 to 1 (moving clockwise) the remaining n leaders and exogenous interruptions are distributed uniformly. Let $y_1, \ldots, y_n$ (with $y_1 \leq \cdots \leq y_n$) represent the (random) location of these points. Then the interval of influence of each leader or interruption is $x_1 = y_1, \ldots, x_k = y_k - y_{k-1}, \ldots, x_{n+1} = 1 - y_n$.

Theorem A.1. *The joint distribution of the intervals*

$$(x_1 = y_1, \ldots, x_k = y_k - y_{k-1}, \ldots, x_{n+1} = 1 - y_n)$$

of the uniform order statistic $0 \leq y_1 < y_2 < \cdots < y_n \leq 1$ *is invariant under any permutation of its components.*

Proof. See Reiss (1989, p. 40). □

This implies, in particular,

Corollary A.1. *All marginal distributions of* $(x_1, \ldots, x_k, \ldots, x_{n+1})$ *of equal dimension are equal.*

Proof of Lemma 1. Given a sample size n, the joint density function of two order statistics for a uniform underlying distribution on the unit interval is

$$f(a_i, a_j) = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}(a_i)^{i-1}(a_j - a_i)^{j-i-1}(1-a_j)^{n-j}$$

for $0 \leq a_i < a_j \leq 1$. Reordering the intervals of influence (see Corollary A.1) so that all party A leaders are followed by all party B leaders and letting $L_A = i$, $L_A + L_B = j$, $L_A + L_B + O = n+1$, $a = a_i$, and $b = a_j - a_i$, we get the bivariate distribution of a and b when $O \geq 1$.

Similarly, the density function of an order statistic for a uniform underlying distribution on the unit interval is

$$f(a_i) = \frac{n!}{(i-1)!(n-i)!}(a_i)^{i-1}(1-a_i)^{n-i}$$

for $0 \leq a_i \leq 1$. Reordering the intervals of influence so that all party A leaders are followed by all party B leaders and letting $L_A = i$, $L_A + L_B = n+1$, and $a = a_i$, we get the distribution of a when $O = 0$. □

Before proving Lemmas 2 to 5, we need a couple of hypergeometric identities.[10]

LEMMA A.1. *For all $M \in \mathbb{N}$, $N \in \mathbb{N}$, and $d \in (0,1)$,*

$$\sum_{k=1}^{N}(1-d)^{M+N-k}\frac{(M+N-k-1)!}{(M-1)!(N-k)!} = \sum_{k=1}^{N}(1-d)^{M+k-1}d^{N-k}\frac{(M+N-1)!}{(M+k-1)!(N-k)!}.$$

PROOF. We proceed by induction. For any positive M, the equality is easily shown to be satisfied for $N = 1$. We claim that for any $N \geq 2$, if the equality is satisfied for $N-1$, then it is satisfied for N. To see this, evaluate the above expression on the left-hand side at N and at $N-1$. The difference is

$$\begin{aligned}\sum_{k=1}^{N}(1-d)^{M+N-k}\frac{(M+N-k-1)!}{(M-1)!(N-k)!}&\\ -\sum_{k=1}^{N-1}(1-d)^{M+N-1-k}\frac{(M+N-k-2)!}{(M-1)!(N-1-k)!}&\\ =(1-d)^{M+N-1}\frac{(M+N-2)!}{(M-1)!(N-1)!}&.\end{aligned}$$

Similarly, evaluate the above expression on the right-hand side at N and at $N-1$. The difference is

$$\begin{aligned}\sum_{k=1}^{N}(1-d)^{M+k-1}d^{N-k}\frac{(M+N-1)!}{(M+k-1)!(N-k)!}&\\ -\sum_{k=1}^{N-1}(1-d)^{M+k-1}d^{N-k-1}\frac{(M+N-2)!}{(M+k-1)!(N-1-k)!}&\\ =(1-d)^{M+N-1}(M+N-2)!\times\sum_{k=1}^{N}H(k)&,\end{aligned}$$

[10] We are very grateful to Aaron Robertson for explaining to us the Wilf–Zeilberger method (Petkovšek et al. 1997) which is used to solve hypergeometric identities.

where

$$
\begin{aligned}
H(k) &= \frac{(1-d)^{k-N} d^{N-k}(M+N-1)}{(M+k-1)!(N-k)!} - \frac{(1-d)^{k-N} d^{N-k-1}}{(M+k-1)!(N-k-1)!} \\
&= \frac{(1-d)^{k-N} d^{N-k}(M+N-1) - (1-d)^{k-N} d^{N-k-1}(N-k)}{(M+k-1)!(N-k)!} \\
&= \frac{(1-d)^{k-N} d^{N-k}(M+k-1) - (1-d)^{k+1-N} d^{N-k-1}(N-k)}{(M+k-1)!(N-k)!} \\
&= \frac{(1-d)^{k-N} d^{N-k}}{(M+k-2)!(N-k)!} - \frac{(1-d)^{k+1-N} d^{N-k-1}}{(M+k-1)!(N-k-1)!}
\end{aligned}
$$

for $k=1,\dots,N-1$ and

$$H(N) = \frac{1}{(M+N-2)!}.$$

Now define

$$J(k) = \frac{(1-d)^{k-N} d^{N-k}}{(M+k-2)!(N-k)!}.$$

Since $H(k) = J(k) - J(k+1)$ for $k=1,\dots,N-1$ and $H(N) = J(N)$, we get

$$\sum_{k=1}^{N} H(k) = J(1) = \frac{1}{(M-1)!(N-1)!}.$$

It follows that the difference between the above expression on the right-hand side evaluated at N and at $N-1$ is also equal to

$$(1-d)^{M+N-1} \frac{(M+N-2)!}{(M-1)!(N-1)!}. \qquad \square$$

LEMMA A.2. *For all $M \in \mathbb{N}$, $N \in \mathbb{N}$, and $d \in (0,1)$,*

$$
\begin{aligned}
&\sum_{k=1}^{N} \frac{(M+N-2)!}{(M+k-2)!(N-k)!}(1-d)^{M+k-2} d^{N-k} \\
&\qquad - \sum_{k=1}^{N} \frac{(M+N-1)!}{(M+k-1)!(N-k)!}(1-d)^{M+k-1} d^{N-k} \\
&\qquad\qquad = (1-d)^{M-1} d^{N} \frac{(M+N-2)!}{(M-1)!(N-1)!}.
\end{aligned}
$$

PROOF. Note that

$$
\begin{aligned}
&\sum_{k=1}^{N} \frac{(M+N-2)!}{(M+k-2)!(N-k)!}(1-d)^{M+k-2} d^{N-k} \\
&\qquad - \sum_{k=1}^{N} \frac{(M+N-1)!}{(M+k-1)!(N-k)!}(1-d)^{M+k-1} d^{N-k} \\
&\qquad\qquad = (1-d)^{M+N-1}(M+N-2)! \sum_{k=1}^{N} \hat{H}(k),
\end{aligned}
$$

where

$$\begin{aligned}\hat{H}(k) &= \frac{(1-d)^{k-N-1}d^{N-k}}{(M+k-2)!(N-k)!} - \frac{(1-d)^{k-N}d^{N-k}(M+N-1)}{(M+k-1)!(N-k)!}\\ &= \frac{(1-d)^{k-N-1}d^{N-k}(M+k-1)-(1-d)^{k-N}d^{N-k}(M+N-1)}{(M+k-1)!(N-k)!}\\ &= \frac{(1-d)^{k-N-1}d^{N-k+1}(M+k-1)-(1-d)^{k-N}d^{N-k}(N-k)}{(M+k-1)!(N-k)!}\\ &= \frac{(1-d)^{k-N-1}d^{N-k+1}}{(M+k-2)!(N-k)!} - \frac{(1-d)^{k-N}d^{N-k}}{(M+k-1)!(N-k-1)!}\end{aligned}$$

for $k=1,\dots,N-1$, and

$$\begin{aligned}\hat{H}(N) &= \frac{(1-d)^{-1}}{(M+N-2)!} - \frac{1}{(M+N-2)!}\\ &= \frac{(1-d)^{-1}d}{(M+N-2)!}.\end{aligned}$$

Now define

$$\hat{J}(k) = \frac{(1-d)^{k-N-1}d^{N-k+1}}{(M+k-2)!(N-k)!}$$

for $k=1,\dots,N$, and note that

$$\hat{H}(k) = \hat{J}(k) - \hat{J}(k+1)$$

for $k=1,\dots,N-1$, and

$$\hat{H}(N) = \hat{J}(N).$$

Thus,

$$\sum_{k=1}^{N}\hat{H}(k) = \hat{J}(1) = \frac{(1-d)^{-N}d^{N}}{(M-1)!(N-1)!}.$$

The statement in the lemma follows. □

PROOF OF LEMMA 2. Suppose first that $O=0$ (the easier case). We have

$$\begin{aligned}P(L_A,L_B) &= 1-\int_0^{1/2} h^0_{L_A,L_B}(a)\,da\\ &= 1-\int_0^{1/2}\frac{(L_A+L_B-1)!}{(L_A-1)!(L_B-1)!}\,a^{L_A-1}(1-a)^{L_B-1}\,da.\end{aligned}$$

Integrating by parts we obtain

$$P(L_A,L_B) = 1-\frac{(L_A+L_B-1)!}{L_A!(L_B-1)!}\left(\frac{1}{2}\right)^{L_A+L_B-1} - \frac{(L_A+L_B-1)!}{L_A!(L_B-2)!}\int_0^{1/2} a^{L_A}(1-a)^{L_B-2}\,db.$$

Proceeding iteratively we obtain for $O=0$,

$$P(L_A,L_B)=1-\sum_{k=1}^{L_B}\left(\frac{1}{2}\right)^{L_A+L_B-1}\frac{(L_A+L_B-1)!}{(L_A+k-1)!(L_B-k)!}.$$

Now suppose that $O\geq 1$. We have

$$P(L_A,L_B)=\int_0^{1/2}\int_0^a h_{L_A,L_B}(a,b)\,db\,da+\int_{1/2}^1\int_0^{1-a} h_{L_A,L_B}(a,b)\,db\,da.$$

Using Lemma 1,

$$\begin{aligned}P(L_A,L_B)=\frac{(L_A+L_B+O-1)!}{(L_A-1)!(L_B-1)!(O-1)!}\times&\\ \Bigg(\int_0^{1/2}\int_0^a a^{L_A-1}b^{L_B-1}(1-a-b)^{O-1}\,db\,da&\\ +\int_{1/2}^1\int_0^{1-a} a^{L_A-1}b^{L_B-1}(1-a-b)^{O-1}\,db\,da\Bigg)&.\end{aligned}$$

Consider the first inner integral. Integrating by parts we obtain

$$\int_0^a b^{L_B-1}(1-a-b)^{O-1}\,db=\frac{-a^{L_B-1}(1-2a)^O}{O}+\frac{L_B-1}{O}\int_0^a b^{L_B-2}(1-a-b)^O\,db.$$

Proceeding iteratively we obtain

$$\begin{aligned}&\int_0^a b^{L_B-1}(1-a-b)^{O-1}\,db\\ &=-\sum_{k=1}^{L_B}\frac{(L_B-1)!(O-1)!}{(L_B-k)!(O+k-1)!}a^{L_B-k}(1-2a)^{O+k-1}+\frac{(L_B-1)!(O-1)!}{(O+L_B-1)!}(1-a)^{O+L_B-1}.\end{aligned}$$

Consider the second inner integral. Integrating by parts we obtain

$$\int_0^{1-a} b^{L_B-1}(1-a-b)^{O-1}\,db=\frac{L_B-1}{O}\int_0^{1-a} b^{L_B-2}(1-a-b)^O\,db.$$

Proceeding iteratively we obtain

$$\int_0^{1-a} b^{L_B-1}(1-a-b)^{O-1}\,db=\frac{(L_B-1)!(O-1)!}{(O+L_B-1)!}(1-a)^{O+L_B-1}.$$

Substituting both inner integrals into the previous expression for $P(L_A,L_B)$, we get

$$\begin{aligned}P(L_A,L_B)=\int_0^1\frac{(L_A+L_B+O-1)!}{(L_A-1)!(L_B+O-1)!}a^{L_A-1}(1-a)^{L_B+O-1}\,da&\\ -\frac{(L_A+L_B+O-1)!}{(L_A-1)!}\sum_{k=1}^{L_B}\int_0^{1/2}\frac{a^{L_A+L_B-k-1}(1-2a)^{O+k-1}}{(L_B-k)!(O+k-1)!}\,da&.\end{aligned}$$

The first term in this expression for $P(L_A, L_B)$ is equal to

$$\int_0^1 h^0_{L_A,L_B+O}(a)\,da,$$

which is equal to 1 because $h^0_{L_A,L_B+O}$ is a probability density with support $(0,1)$. With respect to the second term, integrating by parts we get

$$\int_0^{1/2} a^{L_A+L_B-k-1}(1-2a)^{O+k-1}\,da = 2\frac{O+k-1}{L_A+L_B-k}\int_0^{1/2} a^{L_A+L_B-k}(1-2a)^{O+k-2}\,da.$$

Proceeding iteratively we obtain

$$\int_0^{1/2} a^{L_A+L_B-k-1}(1-2a)^{O+k-1}\,da = \left(\frac{1}{2}\right)^{L_A+L_B-k}\frac{(O+k-1)!(L_A+L_B-k-1)!}{(L_A+L_B+O-1)!}.$$

Thus, for $O \geq 1$,

$$P(L_A, L_B) = 1 - \sum_{k=1}^{L_B}\left(\frac{1}{2}\right)^{L_A+L_B-k}\frac{(L_A+L_B-k-1)!}{(L_A-1)!(L_B-k)!}.$$

To show that the expressions for $P(L_A, L_B)$ for the cases $O=0$ and $O\geq 1$ are equivalent, we need to verify the identity

$$\sum_{k=1}^{L_B}\left(\frac{1}{2}\right)^{L_A+L_B-k}\frac{(L_A+L_B-k-1)!}{(L_A-1)!(L_B-k)!} = \sum_{k=1}^{L_B}\left(\frac{1}{2}\right)^{L_A+L_B-1}\frac{(L_A+L_B-1)!}{(L_A+k-1)!(L_B-k)!}.$$

This identity follows from Lemma A.1 letting $M = L_A$, $N = L_B$, and $d = \frac{1}{2}$. □

Proof of Lemma 3. Using the definition of $P_A(L_A, L_B)$ and Lemma 2,

$$\begin{aligned} P_A(L_A, L_B) &= P(L_A, L_B) - P(L_A-1, L_B) \\ &= \sum_{k=1}^{L_B}\left(\frac{1}{2}\right)^{L_A+L_B-2}\frac{(L_A+L_B-2)!}{(L_A+k-2)!(L_B-k)!} \\ &\qquad - \sum_{k=1}^{L_B}\left(\frac{1}{2}\right)^{L_A+L_B-1}\frac{(L_A+L_B-1)!}{(L_A+k-1)!(L_B-k)!}. \end{aligned}$$

Thus, we need to verify

$$\begin{aligned} &\sum_{k=1}^{L_B}\left(\frac{1}{2}\right)^{L_A+L_B-2}\frac{(L_A+L_B-2)!}{(L_A+k-2)!(L_B-k)!} \\ &\qquad - \sum_{k=1}^{L_B}\left(\frac{1}{2}\right)^{L_A+L_B-1}\frac{(L_A+L_B-1)!}{(L_A+k-1)!(L_B-k)!} \\ &\qquad\qquad = \frac{1}{2^{L_A+L_B-1}}\frac{(L_A+L_B-2)!}{(L_A-1)!(L_B-1)!}. \end{aligned}$$

This identity follows from Lemma A.2 letting $M = L_A$, $N = L_B$, and $d = \frac{1}{2}$. □

Proof of Lemma 4. With respect to near single-peakedness from Lemma 3, we obtain

$$\frac{P_A(L_A+1,L_B)}{P_A(L_A,L_B)}=\frac{L_A+L_B-1}{2L_A}.$$

Thus,

$$\frac{P_A(L_A+1,L_B)}{P_A(L_A,L_B)}\gtreqless 1 \qquad \Longleftrightarrow \qquad L_B\gtreqless L_A+1.$$

With respect to symmetry from the definition of decisiveness, we obtain

$$P_A(L_A,L_B)=P_B(L_B,L_A),$$

and from Lemma 3 we obtain

$$P_A(L_A,L_B)=P_A(L_B,L_A).$$ □

Proof of Lemma 5. Since parties A and B attract the fractions α and β, respectively, of the voters in the intervals corresponding to the leaders for party A and party B, party B wins the election if αa is smaller than βb.

Suppose first that $O=0$. In this case, B wins the election if $a<\beta/(\alpha+\beta)$. Thus

$$P(L_A,L_B)=1-\frac{(L_A+L_B-1)!}{(L_A-1)!(L_B-1)!}\int_0^{1-\gamma}a^{L_A-1}(1-a)^{L_B-1}\,da,$$

where $\gamma=\alpha/(\alpha+\beta)$. Integrating by parts iteratively, as in the proof of Lemma 2, we obtain

$$P(L_A,L_B)=1-\sum_{k=1}^{L_B}\frac{(L_A+L_B-1)!}{(L_A+k-1)!(L_B-k)!}(1-\gamma)^{L_A+k-1}\gamma^{L_B-k}.$$

Now suppose that $O\geq 1$. Note that party B loses the election if $b<\alpha a/\beta$. Note also that $b\leq 1-a$, and $1-a\leq \alpha a/\beta$ if and only if $a\geq \beta/(\alpha+\beta)=1-\gamma$. Thus,

$$P(L_A,L_B)=\int_0^{1-\gamma}\int_0^{\alpha a/\beta}h_{L_A,L_B}(a,b)\,db\,da+\int_{1-\gamma}^{1}\int_0^{1-a}h_{L_A,L_B}(a,b)\,db\,da.$$

Using Lemma 1,

$$\begin{aligned}P(L_A,L_B)=\frac{(L_A+L_B+O-1)!}{(L_A-1)!(L_B-1)!(O-1)!}\times\\ \left(\int_0^{1-\gamma}\int_0^{\alpha a/\beta}a^{L_A-1}b^{L_B-1}(1-a-b)^{O-1}\,db\,da\right.\\ \left.+\int_{1-\gamma}^{1}\int_0^{1-a}a^{L_A-1}b^{L_B-1}(1-a-b)^{O-1}\,db\,da\right).\end{aligned}$$

Consider the inner integrals. Integrating by parts iteratively, as in the proof of Lemma 2, we obtain

$$\int_0^{\alpha a/\beta} b^{L_B-1}(1-a-b)^{O-1}\,db$$
$$= -\sum_{k=1}^{L_B} \frac{(L_B-1)!(O-1)!}{(L_B-k)!(O+k-1)!}(\alpha a/\beta)^{L_B-k}(1-a-\alpha a/\beta)^{O+k-1} + \frac{(L_B-1)!(O-1)!}{(O+L_B-1)!}(1-a)^{O+L_B-1}.$$

As shown in the proof of Lemma 2,

$$\int_0^{1-a} b^{L_B-1}(1-a-b)^{O-1}\,db = \frac{(L_B-1)!(O-1)!}{(O+L_B-1)!}(1-a)^{O+L_B-1}.$$

Substituting both inner integrals into the previous expression for $P(L_A, L_B)$, we get

$$P(L_A, L_B) = 1 - \frac{(L_A+L_B+O-1)!}{(L_A-1)!}\sum_{k=1}^{L_B}\int_0^{1-\gamma} \frac{(\alpha a/\beta)^{L_A+L_B-k-1}(1-a-\alpha a/\beta)^{O+k-1}}{(L_B-k)!(O+k-1)!}\,da.$$

With respect to the second term, integrating by parts iteratively

$$\int_0^{1/2} a^{L_A+L_B-k-1}(1-2a)^{O+k-1}\,da = (1-\gamma)^{L_A+L_B-k}\frac{(O+k-1)!(L_A+L_B-k-1)!}{(L_A+L_B+O-1)!}.$$

Thus, for $O \geq 1$,

$$P(L_A, L_B) = 1 - \sum_{k=1}^{L_B}\frac{(L_A+L_B-k-1)!}{(L_A-1)!(L_B-k)!}(1-\gamma)^{L_A+L_B-k}.$$

To show that the expressions for $P(L_A, L_B)$ for the cases $O = 0$ and $O \geq 1$ are equivalent, we need to verify the identity

$$\sum_{k=1}^{L_B}(1-\gamma)^{L_A+L_B-k}\frac{(L_A+L_B-k-1)!}{(L_A-1)!(L_B-k)!} = \sum_{k=1}^{L_B}(1-\gamma)^{L_A+k-1}\gamma^{L_B-k}\frac{(L_A+L_B-1)!}{(L_A+k-1)!(L_B-k)!}.$$

This identity follows from Lemma A.1 letting $M = L_A$, $N = L_B$, and $d = \gamma$.

Finally, using the definition of $P_A(L_A, L_B)$ and one of the (equivalent) expressions for $P(L_A, L_B)$, we obtain

$$P_A(L_A, L_B) = P(L_A, L_B) - P(L_A-1, L_B)$$
$$= \sum_{k=1}^{L_B}\frac{(L_A+L_B-2)!}{(L_A+k-2)!(L_B-k)!}(1-\gamma)^{L_A+k-2}\gamma^{L_B-k} - \sum_{k=1}^{L_B}\frac{(L_A+L_B-1)!}{(L_A+k-1)!(L_B-k)!}(1-\gamma)^{L_A+k-1}\gamma^{L_B-k}.$$

Thus, to prove the statement of the Lemma with respect to $P_A(L_A, L_B)$ we need to verify the identity

$$\sum_{k=1}^{L_B} \frac{(L_A+L_B-2)!}{(L_A+k-2)!(L_B-k)!}(1-\gamma)^{L_A+k-2}\gamma^{L_B-k} \\ -\sum_{k=1}^{L_B} \frac{(L_A+L_B-1)!}{(L_A+k-1)!(L_B-k)!}(1-\gamma)^{L_A+k-1}\gamma^{L_B-k} \\ =(1-\gamma)^{L_A-1}\gamma^{L_B}\frac{(L_A+L_B-2)!}{(L_A-1)!(L_B-1)!}.$$

This identity follows from Lemma A.1 letting $M=L_A$, $N=L_B$, and $d=\gamma$. The proof of the statement with respect to $P_B(L_A, L_B)$ proceeds along similar lines. □

PROOF OF COROLLARY 3. Using the expression for L^* in Theorem 1 we get

$$\frac{1}{2^{2L^*+1}}\frac{(2L^*)!}{L^*!L^*!} \leq c/G < \frac{1}{2^{2L^*-1}}\frac{(2L^*-2)!}{(L^*-1)!(L^*-1)!}.$$

Since

$$\frac{1}{2^{2L+1}}\frac{(2L)!}{L!L!}$$

is positive and converges monotonically to zero as L approaches infinity, we get that $L^*\to+\infty$ as $c/G\to 0$. Moreover, since

$$\frac{1}{2^{2L+1}}\frac{(2L)!}{L!L!} - \frac{1}{2^{2L-1}}\frac{(2L-2)!}{(L-1)!(L-1)!} \to 0 \quad \text{as } L\to+\infty,$$

we get

$$\frac{c/G}{\frac{1}{2^{2L^*+1}}\frac{(2L^*)!}{L^*!L^*!}} \to 1 \quad \text{as } c/G\to 0.$$

Using the expressions for the expected voter turnout and the expected winning margin in Theorem 1, we obtain

$$W/T = \frac{1}{2^{2L^*}}\frac{(2L^*)!}{L^*!L^*!}.$$

Thus

$$\frac{c/G}{W/(2T)} \to 1 \quad \text{as } c/G\to 0,$$

as stated in the corollary.

Now, using the Stirling formula,

$$\frac{1/\sqrt{4\pi L^*}}{\frac{1}{2^{2L^*+1}}\frac{(2L^*)!}{L^*!L^*!}} \to 1.$$

Thus,

$$\frac{1/\sqrt{4\pi L^*}}{W/(2T)} \to 1$$

or equivalently

$$\frac{L^*}{(T/W)^2/\pi} \to 1 \quad \text{as } c/G \to 0.$$

The approximation result for $1/O$ follows from this and the expression for T in Theorem 1. □

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