Theoretical Economics 2 (2007), 1–13 1555-7561/20070001

# Contracts and uncertainty

WOJCIECH OLSZEWSKI
Department of Economics, Northwestern University

ALVARO SANDRONI
Department of Economics, University of Pennsylvania, and Department of Managerial Economics and Decision Sciences, Kellogg School of Management, Northwestern University

A decision maker, named Alice, wants to know if an expert has significant information about payoff-relevant probabilities of future events. The expert, named Bob, either knows this probability almost perfectly or knows nothing about it. Hence, both Alice and the uninformed expert face uncertainty: they do not know the payoff-relevant probability. Alice offers a contract to Bob. If he accepts this contract then he must announce the probability distribution before any data are observed. Once the data unfold, transfers between Alice and Bob occur. It is demonstrated that if the informed expert accepts some contract then the uninformed expert also accepts this contract. Hence, Alice's adverse selection problem cannot be mitigated by screening contracts that separate informed from uninformed experts. This result stands in contrast with the analysis of contracts under risk, where separation is often feasible.

KEYWORDS. Contracts, uncertainty, experts, minmax theorems.

JEL CLASSIFICATION. C44, C73, D81, D83, D86.

## 1. INTRODUCTION

Contracts may be able to screen agents facing different levels of risk. In a typical model, an agent knows privately the risks he faces and a principal offers a menu of contracts to the agent anticipating that some contracts will be accepted only by agents facing specific risks. This basic idea has been investigated in different markets, but relatively little attention has been dedicated to the possibly widespread case in which the parties involved in the contract face uncertainty: the information available to them is so vague that it cannot be properly summarized by probabilities.

Consider a principal, named Alice, who must make a decision without knowing the payoff-relevant probabilities. A potential expert, named Bob, claims that he knows the probabilities of interest. Bob may be an informed expert who knows these probabilities almost perfectly. Bob may alternatively be a false expert who does not know anything about the payoff-relevant probabilities.

Wojciech Olszewski: wo@northwestern.edu
Alvaro Sandroni: sandroni@econ.upenn.edu

Alice offers a contract to Bob. If Bob accepts the contract he must deliver to Alice all probabilities at period zero (i.e., before any data are revealed). Then, as the data unfold, transfers occur between Alice and Bob. These transfers depend upon the probabilities Bob announced at period zero and the data observed in the subsequent periods.

The question addressed in this paper is whether there exists a screening contract that separates informed and uninformed experts. If informed, the expert must accept the contract and reveal the probabilities truthfully. If uninformed, the expert must not accept the contract. If such a contract exists, then Alice knows that she can rely on the probabilities announced to her, and make her decision under rational expectations.

We make several assumptions that simplify the task of finding such a screening contract. The expert, if informed, is assumed to announce the probabilities truthfully. We assume also that Alice is willing to pay large amounts to learn the relevant probabilities. Finally, we assume the uninformed expert to be extremely averse to uncertainty. He accepts the contract only when it is beneficial to him in the worse-case scenario. That is, he accepts the contract when his expected utility with the contract (calculated under *any* probability measure over the future realization of the data) exceeds his utility without the contract.

We show that even under these rather extreme assumptions, a screening contract that separates informed and uninformed experts does not exist. If the informed expert accepts a contract then the uninformed expert also accepts the contract. Hence, Alice faces an adverse selection problem that cannot be eased by screening contracts and the discipline that data could potentially deliver. Alice may not be able to find out the quality of the announced probabilities. This result stands in contrast with the analysis of contracts under risk where separation is often feasible.

This paper is organized as follows. The next section gives a brief review of related literatures. In Section 2, the model and result is shown when Alice has one data point. Section 3 extends the result to the case in which multiple data points are available. Section 4 concludes. Proofs are in the Appendix.

### *Related literatures*

As is well known, Ellsberg (1961) demonstrated that the distinction between risk (where the perceived chances can be represented by probabilities) and uncertainty (where the perceived chances cannot be represented by probabilities) has empirical meaning. However, this distinction cannot be properly made under Savage's (1954) axiomatic foundation. Recently, considerable effort has been dedicated to provide an alternative axiomatic foundation where this distinction can be made.

The literature on uncertainty is quite large. An important idea in this literature is maxmin expected utility representation, according to which the value of an alternative is determined as if the decision maker calculated his minimal expected utility within a class of possible probability measures (see Gilboa and Schmeidler 1989). Bob, if uninformed, faces uncertainty, and his decision (to accept or reject Alice's contract) is consistent with this representation. However, our main objective, unlike that of most of the literature on uncertainty, is not to deliver a representation theorem, but rather to show strategies that effectively reduce uncertainty.

A recent branch of the literature on games against nature (which can be traced back to Milnor 1954 and Wald 1950) shows that it is possible, without knowing the data generating process, to strategically produce empirical models that will not be rejected by future realizations of the data (see, among others, Cesa-Bianchi and Lugosi 2006, Dekel and Feinberg 2006, Foster and Vohra 1998, Fudenberg and Levine 1999, Hart 2005, Hart and Mas-Colell 2001, Lehrer 2001, Lehrer and Solan 2003, Olszewski and Sandroni 2006, Rustichini 1999, Sandroni 2003, Sandroni et al. 2003, and Vovk and Shafer 2005). This paper shares with this literature a basic tool: the minmax theorem. Moreover, an empirical test can be seen as a special contract that delivers high payoffs (to the expert) if his empirical model is not rejected and low payoffs otherwise.

## 2. Model and result

A decision maker, named Alice, does not know the probability of an outcome that can either be 0 or 1. Alice has two available choices, also labelled 0 or 1. Alice prefers choice $j \in \{0,1\}$ when the outcome is $j$. She is averse to uncertainty and is willing to pay for a credible announcement of the probability of outcome 1.

A potential expert, named Bob, claims to know almost perfectly the probability of outcome 1. Bob may be either an informed or an uninformed expert. If informed, Bob has a signal $p \in [0,1]$ that says that the probability of outcome 1 is in a neighborhood $B_\varepsilon(p)$ of $p$, where

$$B_\varepsilon(p) \equiv [p-\varepsilon, p+\varepsilon] \cap [0,1] \quad \text{for some } \varepsilon > 0.$$

If uninformed, Bob knows nothing about the probability of outcome 1.

Alice offers a contract to Bob. If Bob accepts the contract then he must deliver to Alice a probability $p \in [0,1]$ of outcome 1. The contract specifies wealth transfers $c(0,p)$ and $c(1,p)$ that can be either positive or negative. So, if Bob announces $p \in [0,1]$ and $i \in \{0,1\}$ occurs then Alice transfers $c(i,p) \in \mathbb{R}$ to Bob and Bob's wealth is $w + c(i,p)$. Without a contract, Bob's wealth is $w > 0$.

Alice can find out a close estimate of the probability of outcome 1 if she can write a contract such that Bob, if informed, accepts it and truthfully reveals his signal. In addition, Bob, if uninformed, must not accept the contract. We assume that Bob, if informed, always reveals his signal $p$ truthfully. This assumption makes Alice's task significantly easier. The remaining question is the existence of a screening contract that separates the informed and the uninformed expert (i.e., the contract is accepted by an informed expert, but is not accepted by an uninformed expert).

Assumption 1. There exists $\eta > 0$ such that for every $p \in [0,1]$ and $i \in \{0,1\}$, if $c(i,p) < 0$ then $c(i,p) < -\eta$.

This assumption says that there exists a minimum cost $\eta$ to Bob if he must pay Alice. This cost can be understood as the value of the minimum time required to make any payment.[1]

[1]It is natural to assume also that $w + c(i,p) \geq 0$ for all $p \in [0,1]$ and $i = \{0,1\}$ to insure that Bob is always solvent. This restriction, however, is not necessary for the results.

ASSUMPTION 2. For every $p \in [0,1]$ there is an outcome $i \in \{0,1\}$ such that $c(i,p) < 0$ and $c(j,p) \geq 0$ for $j \neq i$.

By this assumption, for one outcome Bob makes a strictly positive payment to Alice and for the other outcome Alice makes a (positive or zero) payment to Bob. This assumption is made for simplicity and is not necessary for Proposition 1 and Corollary 1. If it does not hold then there exists $\bar{p} \in [0,1]$ such that either $c(i,\bar{p}) \geq 0$ for both outcomes or $c(i,\bar{p}) < 0$ for both outcomes. In the former case, an uninformed expert could accept the contract and report $\bar{p}$ (because then Bob never makes a strictly positive payment to Alice). In the latter case, an informed expert who receives a signal $\bar{p}$ does not accept the contract and reveal $\bar{p}$ (because then Bob makes a strictly positive payment to Alice for both outcomes).

DEFINITION 1. An *informed expert accepts a contract c* if there exists $\tilde{\varepsilon} > 0$ such that for every $p \in [0,1]$,

$$\min_{p' \in B_{\tilde{\varepsilon}}(p)} p' u(w + c(1,p)) + (1-p')u(w + c(0,p)) \geq u(w),$$

where $u : \mathbb{R} \to \mathbb{R}$ is a strictly increasing utility function.

This definition should be interpreted as follows. The informed expert knows that the probability of outcome 1 is in $B_\varepsilon(p)$. He faces some residual uncertainty because he does not know the exact probability of outcome 1. According to Gilboa and Schmeidler's 1989 model of decision-making under uncertainty, an individual evaluates an uncertain prospect by the minimum of the expected utilities over a (subjectively determined) range of probabilities. However, this range of probabilities (for the informed expert who accepts the contract) should be a subset of $B_\varepsilon(p)$. Indeed, recent research in decision theory (see, for example, Gajdos et al. 2006) suggests models in which the subjective range of probabilities is a subset of the set of all objectively possible probabilities.

Assume therefore that the informed expert evaluates the contract $c$ by the minimum of expected utilities over a set $B \subset B_\varepsilon(p)$. Alice does not know Bob's set $B$ as it is subjectively determined. Definition 1 requires that the informed expert accept the contract $c$ when he is sufficiently precisely informed for any subjective set of probabilities $B \subset B_\varepsilon(p)$.

DEFINITION 2. An *informed expert strongly accepts a contract c* if for every $p \in [0,1]$ and some $\delta > 0$,

$$p u(w + c(1,p)) + (1-p)u(w + c(0,p)) \geq u(w) + \delta.$$

Bob, if informed, strongly accepts contract $c$ if his expected utility with $c$ is bounded above his utility without a contract. The probability used in the computation of the expected utility is Bob's signal $p$.

As long as the utilities $\{u(w + c(i,p)) : p \in [0,1],\ i \in \{0,1\}\}$ remain bounded, if an expert strongly accepts a contract then he accepts the contract. The following result is a converse.

LEMMA 1. *Under Assumptions 1 and 2, if an informed expert accepts a contract $c$ then he strongly accepts contract $c$.*

Here is an example of a contract that is strongly accepted by an informed expert.

EXAMPLE 1. Assume that $w=10$ and $u(x)=x$. Let $\bar{c}$ be such that

$$\bar{c}(1,p)=\begin{cases}(1-p)^2-1.5+\delta & \text{if } p<0.5\\ (1-p)^2+1+\delta & \text{if } p\geq 0.5\end{cases}$$

$$\bar{c}(0,p)=\begin{cases}p^2+1+\delta & \text{if } p<0.5\\ p^2-1.5+\delta & \text{if } p\geq 0.5\end{cases}$$

where $\delta>0$ is small—let's say 0.1.

Contract $\bar{c}$ satisfies Assumptions 1 and 2. In addition, for $\varepsilon$ small enough, Bob, if informed, accepts contract $\bar{c}$.[2]

Now consider Bob's decision when he is uninformed. Consider first the option of accepting contract $\bar{c}$ and announcing a probability $q\in[0,1]$. Bob prefers this option to rejecting the contract if

$$\min_{p'\in C} p'c(1,q)+(1-p')c(0,q)\geq 0, \tag{1}$$

where $C$ is some closed interval of $[0,1]$.

If $C$ is large enough (e.g., the interior of $C$ contains $0.4/1.5$ and $1.1/1.5$) then (1) does not hold for any $q\in[0,1]$.[3] Thus Bob, if uninformed and sufficiently averse to uncertainty, prefers to not accept the contract rather than to accept it and deterministically announce any probability of outcome 1. ◊

The key element missing in the analysis of Example 1 is that Bob need not deliver a probability of outcome 1 deterministically. By selecting the probability of outcome 1 at random, Bob produces a new source of risk, but not a new source of uncertainty because, by definition, he knows the odds that each probability is selected.

Let $\zeta\in\Lambda[0,1]$ be a probability measure over $[0,1]$ with finite support. By selecting the probability of outcome 1 according to $\zeta=(p_1,\gamma_1;\ldots;p_n,\gamma_n)$, where $p_1,\ldots,p_n,\gamma_1,\ldots,\gamma_n\in[0,1]$ and $\sum_{i=1}^n\gamma_i=1$, Bob picks $p_i$ with probability $\gamma_i$.

---

[2]This follows because if $p<0.5$ then $\bar{c}(1,p)\leq -0.4$ and $\bar{c}(0,p)\geq 1.1$, and if $p\geq 0.5$ then $\bar{c}(1,p)\geq 1.1$ and $\bar{c}(0,p)\leq -0.4$. Moreover, for all $p\in[0,1]$,

$$pc(1,p)+(1-p)c(0,p)\geq\delta>0$$

and so for a small $\tilde{\varepsilon}$,

$$p'c(1,p)+(1-p')c(0,p)\geq\delta \quad \text{for all } p'\in B_{\tilde{\varepsilon}}(p).$$

[3]This follows because if $p'$ is slightly smaller than $0.4/1.5$ then $p'c(1,q)+(1-p')c(0,q)$ is an increasing function for $q\geq 0.5$ and $0.4/1.5c(1,1)+(1-0.4/1.5)c(0,1)=0$. Hence, for any $p'$ slightly smaller than $0.4/1.5$ and $q\geq 0.5$,

$$p'c(1,q)+(1-p')c(0,q)<0.$$

By an analogous argument, this inequality holds for $p'$ slightly greater than $1.1/1.5$ and $q<0.5$.

Let

$$\zeta(i) \equiv \sum_{i=1}^{n} \gamma_i u(w + c(i, p_i))$$

be the expected utility of the uninformed expert if the outcome is $i \in \{0,1\}$.

DEFINITION 3. An *uninformed expert accepts a contract* $c$ if there exists $\bar{\zeta} \in \Lambda[0,1]$ such that

$$\bar{\zeta}(0) \geq u(w) \quad \text{and} \quad \bar{\zeta}(1) \geq u(w). \tag{2}$$

If (2) holds, then no matter which outcome is realized, the expected utility of the uninformed expert is greater with the contract than without it. Hence, it follows immediately that

$$\min_{p' \in [0,1]} p'\bar{\zeta}(1) + (1-p')\bar{\zeta}(0) \geq u(w).$$

So, Bob accepts the contract no matter how averse to uncertainty he might be. We now show the main result of this paper.[4]

PROPOSITION 1. *If an informed expert strongly accepts a contract* $c$ *then an uninformed expert also accepts contract* $c$.

Bob, if uniformed, faces uncertainty because he does not know the probabilities of 0 and 1. However, he can virtually eliminate his uncertainty as far as the decision regarding the contract is concerned by selecting his announcements strategically. This follows because no matter whether 0 or 1 occurs, Bob's expected utility, calculated under the known odds of his own randomization, is greater with the contract than without it, provided that the same is true for the informed expert.

An intuition for Proposition 1 is as follows. Consider a zero-sum game between Bob and Nature. Nature chooses a probability $p' \in [0,1]$. Bob chooses $\zeta \in \Lambda[0,1]$. Bob's payoff is

$$p'\zeta(1) + (1-p')\zeta(0).$$

For every strategy $p'$ of Nature, there is a strategy for Bob (to select $p'$ with probability one) that delivers payoff

$$p'u(w + c(1,p')) + (1-p')u(w + c(0,p')).$$

By assumption, this payoff is greater than $u(w)$. Hence, by Fan's 1953 minmax theorem, there is a strategy $\bar{\zeta}$ that gives Bob a payoff greater than $u(w)$, no matter which strategy Nature chooses. Corollary 1 follows from Lemma 1 and Proposition 1.[5]

[4]In fact, a stronger result can be obtained, in which (2) is replaced with $\bar{\zeta}(0) \geq u(w) + \delta'$ and $\bar{\zeta}(1) \geq u(w) + \delta'$ for some $\delta' > 0$.

[5]Notice that Assumption 2 is redundant here; see the discussion that follows Assumption 2.

COROLLARY 1. *Under Assumption 1, if an informed expert accepts a contract $c$, then an uninformed expert also accepts contract $c$.*

That is, Alice cannot write a contract that screens informed and uninformed experts. It is not feasible for Alice to determine whether Bob is an honest and informed expert, or a strategic and uninformed expert. The nonexistence of screening contracts shows that Alice's adverse selection problem cannot be mitigated by contractual arrangements. Assume that Bob is informed, but Alice is not willing to accept Bob's advice without the security that a choice of contracts could provide (if a screening contract did exist). Then, Alice cannot learn from Bob the relevant probabilities.

In Proposition 1, Alice has only one data point at her disposal. However, the basic idea still holds if Alice has several data points. In the next section, a result analogous to Proposition 1 is presented in a model with multiple data points.

## 3. A MULTI-PERIOD MODEL

Each period one outcome, 0 or 1, is observed.[6] Let $N$ be the set of natural numbers. Let $\{0,1\}^t$, $t \in N$, be the $t$-fold Cartesian product of $\{0,1\}$. Let $\Omega = \{0,1\}^\infty$ be the set of all *paths*, i.e., infinite sequences of outcomes. Let $\overline{\Omega} = \cup_{t\in N}\{0,1\}^t \cup \{s_0\}$ be union of the null history $s_0$ with the set of all finite histories.

For every finite history $s_t \in \{0,1\}^t$, $t \in N$, a *cylinder* with base on $s_t$ is the set $C(s_t) = \{s \in \Omega \mid s = (s_t, \ldots)\}$ of all paths whose $t$ initial elements coincide with $s_t$. Let $\mathfrak{I}_t$ be the $\sigma$-algebra that consists of all finite unions of cylinders with base on $\{0,1\}^t$. Let $\mathfrak{I}$ be the $\sigma$-algebra generated by the algebra $\mathfrak{I}^0 \equiv \cup_{t\in N}\mathfrak{I}_t$, i.e., $\mathfrak{I}$ is the smallest $\sigma$-algebra that contains $\mathfrak{I}^0$.

Let $(\Omega, \mathfrak{I}, P)$ be a measure space, where $P$ is a probability measure. Let $\Delta(\Omega)$ be the set of probability measures on $(\Omega, \mathfrak{I})$. Let $\Lambda\Delta(\Omega)$ be the set of probability measures on $\Delta(\Omega)$ with finite support.

At period zero, a principal, named Alice, offers a contract to a potential expert named Bob. If Bob accepts the contract then he must deliver a probability measure $P \in \Delta(\Omega)$ to Alice at period zero. The contract specifies monetary transfers depending upon Bob's report and the data observed by Alice.

DEFINITION 4. A *contract* is a function $C : \overline{\Omega} \times \Delta(\Omega) \to \mathbb{R}$.

A contract $C$ specifies a transfer of $C(s_0, P)$ at period zero and a transfer of $C(s_t, P)$ at period $t$ if $s_t$ is observed. Transfers can be positive or negative. If Bob rejects Alice's contract then Bob's endowment is exogenously given by a function $w : \overline{\Omega} \to \mathbb{R}$ that is assumed to be bounded (i.e., there exists $M > 0$ such that $|w(s_t)| \le M$ for all $s_t \in \overline{\Omega}$). So without a contract, Bob's wealth at period $t$ is $w(s_t)$. If Bob accepts a contract $C$ and announces $P$, then his endowment, at period $t$, is $w(s_t) + C(s_t, P)$.

[6] The results may immediately be extended to the case in which there are finitely many possible outcomes in each period.

ASSUMPTION 3. A contract $C$ is bounded. That is, for every $P \in \Delta(\Omega)$ there exists $M(P)$ such that

$$|C(s_t, P)| \leq M(P) \text{ for every } s_t \in \overline{\Omega}.$$

A contract is bounded if it does not specify arbitrarily large transfers. Hence, bounded contracts ensure limited liability.

Given a path $s \in \Omega$, where $s = (s_t, \ldots)$, and a probability measure $P \in \Delta(\Omega)$, let

$$U(s, P) \equiv \sum_{t=0}^{\infty} \beta^t u(w(s_t) + C(s_t, P)) - \sum_{t=0}^{\infty} \beta^t u(w(s_t))$$

be the difference in Bob's discounted sum of utilities with and without a contract, where $\beta \in [0, 1)$ is a discount factor and $u : \mathbb{R} \to \mathbb{R}$ is a continuous utility function.

Let $U_s : \Delta(\Omega) \to \mathbb{R}$ and $U_P : \Omega \to \mathbb{R}$ be functions such that $U_s(P) = U_P(s) = U(s, P)$.[7] Let $E^P$ be the expectation operator associated with $P \in \Delta(\Omega)$. Let $E^\zeta$ be the expectation operator associated with a random generator of probability measures $\zeta \in \Lambda\Delta(\Omega)$.[8]

DEFINITION 5. A contract $C$ is *strongly accepted by an informed expert* if for some $\delta > 0$,

$$E^P\{U_P\} \geqslant \delta \quad \text{for every } P \in \Delta(\Omega). \tag{3}$$

Assume that Bob knows the data generating process. If (3) is satisfied, then Bob's expected discounted utility is higher (and bounded above) with the contract than without it. In a manner similar to Section 2, one can show that under some mild assumptions, a contract is strongly accepted if and only if it is accepted by almost-perfectly informed experts who face some residual uncertainty.

DEFINITION 6. A contract $C$ is *strongly accepted by an uninformed expert* if there exists a random generator of theories $\zeta_C \in \Delta(\Omega)$ and some $\delta' > 0$ such that

$$E^{\zeta_C}\{U_s\} > \delta' \quad \text{for every path } s \in \Omega. \tag{4}$$

Assume that Bob does not know the data generating process. If (4) is satisfied then his expected utility (calculated under $\zeta_C$) is higher (and bounded above) with the contract than without it, no matter which sequence of outcomes is realized.

PROPOSITION 2. *Let $C$ be a bounded contract. If $C$ is strongly accepted by an informed expert then it is strongly accepted by an uninformed expert.*

By Proposition 2, there is no contract that separates informed and uninformed experts. With limited liability, it is not feasible for Alice to write down a screening contract that is accepted only by the informed experts.

---

[7]The function $U_P$ is assumed to be $\mathfrak{J}$-measurable.

[8]Given that $\zeta$ has finite support, $E^\zeta\{U_s\}$ is well defined.

## 4. Conclusion

The nonexistence of a contract that separates informed and uninformed experts shows that contracts cannot mitigate the adverse selection problem faced by a principal, who wants to know payoff-relevant probabilities, and an agent who may or may not know these probabilities. In particular, even if the agent knows the payoff-relevant probabilities and the principal is willing to pay large amounts to learn these probabilities, she will not learn them perfectly because she cannot be certain that the expert is informed.

## Appendix

Proof of Lemma 1. Let $\tilde{\eta} \equiv u(w-\eta)-u(w)$, and let $\delta := -\tilde{\eta}\tilde{\varepsilon} > 0$. Assume that $c(1,p) < 0$ (the proof is completely analogous if $c(0,p) < 0$). Then, $p \leq 1-\tilde{\varepsilon}$. Otherwise,

$$\min_{p' \in B_{\tilde{\varepsilon}}(p)} p' u(w+c(1,p))+(1-p')u(w+c(0,p)) = u(w+c(1,p)) < u(w).$$

Given that $p \leq 1-\tilde{\varepsilon}$, $u(w)$ is less than or equal to

$$\begin{aligned}
&\min_{p' \in B_{\tilde{\varepsilon}}(p)} p' u(w+c(1,p))+(1-p')u(w+c(0,p)) \\
&\quad = \left(p+\tilde{\varepsilon}\right) u(w+c(1,p))+(1-p-\tilde{\varepsilon})u(w+c(0,p)) \\
&\quad = p u(w+c(1,p))+(1-p)u(w+c(0,p))+\tilde{\varepsilon}\left(u(w+c(1,p))-u(w+c(0,p))\right) \\
&\quad \leq p u(w+c(1,p))+(1-p)u(w+c(0,p))+\tilde{\varepsilon}\tilde{\eta}.
\end{aligned}$$

Hence

$$p u(w+c(1,p))+(1-p)u(w+c(0,p)) \geq u(w)+\delta. \qquad \square$$

### 4.1 *Proof of Propositions 1 and 2*

Theorem (Fan 1953, Theorem 2). *Let $X$ be a compact Hausdorff space and let $Y$ be a convex subset of a linear space (not topologized). Let $f$ be a real-valued function on $X \times Y$ such that for every $y \in Y$, the function $f(x,y)$ is lower semicontinuous with respect to $x$. If $f$ is convex with respect to $x$ and concave with respect to $y$, then $f$ satisfies the minmax property:*

$$\min_{x\in X}\sup_{y\in Y} f(x,y) = \sup_{y\in Y}\min_{x\in X} f(x,y).$$

Proof of Proposition 1. Let $H : [0,1] \times \Lambda[0,1] \to [0,1]$ be defined by

$$H(p,\zeta) \equiv p\zeta(1)+(1-p)\zeta(0).$$

By definition,

$$\zeta(i) = E^{\zeta}\{u(w+c(i,p))\}$$

where $E^{\zeta}$ is the expectation operator associated with $\zeta \in \Lambda[0,1]$.

By the linearity of the expectation operator, $H$ is linear in both $p$ and $\zeta$. Moreover, $H$ is continuous on $p$ that belong to the compact set $[0,1]$. Hence, it follows from Fan's 1953 minmax theorem that

$$\min_{p\in[0,1]} \sup_{\zeta\in\Lambda[0,1]} H(p,\zeta) = \sup_{\zeta\in\Lambda[0,1]} \min_{p\in[0,1]} H(p,\zeta).$$

Given any $p \in [0,1]$, let $\zeta_p \in \Lambda[0,1]$ be such that $p$ is selected with probability one. Then

$$H(p,\zeta_p) = pu(w+c(1,p)) + (1-p)u(w+c(0,p)) \geq u(w)+\delta,$$

so that

$$\min_{p\in[0,1]} \sup_{\zeta\in\Lambda[0,1]} H(p,\zeta) \geq u(w)+\delta.$$

It follows that

$$\sup_{\zeta\in\Lambda[0,1]} \min_{p\in[0,1]} H(p,\zeta) \geq u(w)+\delta.$$

Therefore, for some $\bar{\zeta} \in \Lambda[0,1]$,

$$H(p,\bar{\zeta}) \geq u(w)+\delta/2 \text{ for all } p \in [0,1].$$

So

$$\bar{\zeta}(0) = H(0,\bar{\zeta}) \geq u(w)+\delta/2 \quad \text{and} \quad \bar{\zeta}(1) = H(1,\bar{\zeta}) \geq u(w)+\delta/2. \qquad \square$$

Lemma 2. *Let the set $\Omega$ be endowed with the product topology (i.e., the topology that comprises unions of cylinders with finite base). Let $C$ be a bounded contract. Then, the function $U_P$ is continuous.*

Proof. Assume, by contradiction, that $s(n) \to \bar{s}$ as $n \to \infty$ and $U(s(n),P)$ does not converge to $U(\bar{s},P)$ as $n \to \infty$; assume further, passing to a subsequence of $s(n)$ if necessary, that no subsequence of $U(s(n),P)$ converges to $U(\bar{s},P)$. Take a subsequence of $s(n)$ (also indexed by $n$) such that the first $n$ coordinates of $s(n)$ and $\bar{s}$ are identical. Given that $U(s,P)$ belongs to a compact subset of the real line, there is a convergent subsequence of $U(s(n),P)$ (still indexed by $n$). Let $\overline{U}$ be the limit of this convergent subsequence, i.e., $U(s(n),P) \to \overline{U}$ as $n \to \infty$. Notice that $|\overline{U} - U(\bar{s},P)| > 0$. Let

$$U_n(s,P) \equiv \sum_{t=0}^{n} \beta^t u(w(s_t)+C(s_t,P)) - \sum_{t=0}^{n} \beta^t u(w(s_t))$$

and

$$U^n(s,P) \equiv \sum_{t=n+1}^{\infty} \beta^t u(w(s_t)+C(s_t,P)) - \sum_{t=n+1}^{\infty} \beta^t u(w(s_t)).$$

Then

$$U(s,P) = U_n(s,P) + U^n(s,P)$$

and

$$|\overline{U}-U(\bar{s},P)|=|\overline{U}-U(s(n),P)+U_n(s(n),P)+U^n(s(n),P)-U_n(\bar{s},P)-U^n(\bar{s},P)|.$$

By definition, $U_n(s(n),P)=U_n(\bar{s},P)$ (because the first $n$ coordinates of $s(n)$ and $\bar{s}$ are identical). So,

$$|\overline{U}-U(\bar{s},P)|=|\overline{U}-U(s(n),P)+U^n(s(n),P)-U^n(\bar{s},P)|.$$

Both $|\overline{U}-U(s(n),P)|$ and $|U^n(s(n),P)-U^n(\bar{s},P)|\leq|U^n(s(n),P)|+|U^n(\bar{s},P)|$ converge to zero as $n$ goes to infinity. Hence, $|\overline{U}-U(\bar{s},P)|$ converges to zero as $n\to\infty$, which is a contradiction. □

Proof of Proposition 2. Let $H:\Delta(\Omega)\times\Lambda\Delta(\Omega)\to[0,1]$ be defined by

$$H(P,\zeta)=E^PE^\zeta U(s,Q).$$

By the linearity of the expectation operator, $H$ is linear in both $P$ and $\zeta$. By Lemma 2, $U(s,Q)$ is continuous in $s$ for every $Q$. Given that $E^\zeta U(s,Q)$ can be written as a finite linear combination of the form

$$\sum_{i=1}^{n}\gamma_i U(s,P_i)$$

it follows that $E^\zeta U(s,Q)$ is continuous in $s$.

Let the set $\Delta(\Omega)$ be endowed with the weak*-topology. The weak*-topology on $\Delta(\Omega)$ is the weakest topology that makes the expectation operator $E^P h$ a continuous function when $h:\Omega\to\mathbb{R}$ is continuous.[9] So, $H(P,\zeta)$ is continuous in $P$. Moreover, $\Delta(\Omega)$ is a compact and metric (and so Hausdorff) space (see Rudin 1973, Theorems 3.15 and 3.17).

The proof now proceeds as the proof of Proposition 1. It follows from Fan's 1953 minmax theorem that

$$\min_{P\in\Delta(\Omega)}\sup_{\zeta\in\Lambda\Delta(\Omega)}H(P,\zeta)=\sup_{\zeta\in\Lambda\Delta(\Omega)}\min_{P\in\Delta(\Omega)}H(P,\zeta).$$

Given any $P\in[0,1]$, let $\zeta_P\in\Lambda\Delta(\Omega)$ be such that $P$ is selected with probability one. Then

$$H(P,\zeta_P)=E^P\{U_P\}\geqslant\delta,$$

so

$$\min_{P\in\Delta(\Omega)}\sup_{\zeta\in\Lambda\Delta(\Omega)}H(P,\zeta)\geq\delta.$$

It follows that

$$\sup_{\zeta\in\Lambda\Delta(\Omega)}\min_{P\in\Delta(\Omega)}H(P,\zeta)\geq\delta.$$

Therefore, for some $\zeta_C\in\Lambda\Delta(\Omega)$,

$$H(P,\zeta_C)\geq\delta/2\text{ for all }P\in\Delta(\Omega).$$

Given any $s\in\Omega$, let $P_s\in\Delta(\Omega)$ be the probability measure that assigns probability one to $s$. Then, for any $s\in\Omega$,

$$E^{\zeta_C}\{U_s\}=H(P_s,\zeta_C)\geq\delta/2.$$ □

[9]See Rudin (1973, Chapter 3) for a definition and basic results regarding the weak*-topology.

References

Cesa-Bianchi, Nicolò and Gábor Lugosi (2006), *Prediction, Learning, and Games*. Cambridge University Press, Cambridge. [3]

Dekel, Eddie and Yossi Feinberg (2006), "Non-bayesian testing of a stochastic prediction." *Review of Economic Studies*, 73, 893–906. [3]

Ellsberg, Daniel (1961), "Risk, ambiguity, and the Savage axioms." *Quarterly Journal of Economics*, 75, 643–669. [2]

Fan, Ky (1953), "Minimax theorems." *Proceedings of the National Academy of Science of the U.S.A.*, 39, 42–47. [6, 9, 10, 11]

Foster, Dean P. and Rakesh V. Vohra (1998), "Asymptotic calibration." *Biometrika*, 85, 379–390. [3]

Fudenberg, Drew and David K. Levine (1999), "An easier way to calibrate." *Games and Economic Behavior*, 29, 131–137. [3]

Gajdos, Thibault, Takashi Hayashi, Jean-Marc Tallon, and Jean-Christophe Vergnaud (2006), "Attitude toward imprecise information." Unpublished paper, University of Texas at Austin. [4]

Gilboa, Itzhak and David Schmeidler (1989), "Maxmin expected utility with a non-unique prior." *Journal of Mathematical Economics*, 18, 141–153. [2, 4]

Hart, Sergiu (2005), "Adaptive heuristics." *Econometrica*, 73, 1401–1430. [3]

Hart, Sergiu and Andreu Mas-Colell (2001), "A general class of adaptive strategies." *Journal of Economic Theory*, 98, 26–54. [3]

Lehrer, Ehud (2001), "Any inspection is manipulable." *Econometrica*, 69, 1333–1347. [3]

Lehrer, Ehud and Eilon Solan (2003), "No-regret with bounded computational capacity." Unpublished paper, School of Mathematical Sciences, Tel Aviv University. [3]

Milnor, John (1954), "Games against nature." In *Decision Processes* (Robert M. Thrall, Clyde H. Coombs, and Robert L. Davis, eds.), 49–59, Wiley, New York. [3]

Olszewski, Wojciech and Alvaro Sandroni (2006), "Strategic manipulation of empirical tests." Unpublished paper, Department of Economics, Northwestern University. [3]

Rudin, Walter (1973), *Functional Analysis*. McGraw-Hill, New York. [11]

Rustichini, Aldo (1999), "Optimal properties of stimulus-response learning models." *Games and Economic Behavior*, 29, 244–273. [3]

Sandroni, Alvaro (2003), "The reproducible properties of correct forecasts." *International Journal of Game Theory*, 32, 151–159. [3]

Sandroni, Alvaro, Rann Smorodinsky, and Rakesh V. Vohra (2003), "Calibration with many checking rules." *Mathematics of Operations Research*, 28, 141–153. [3]

Vovk, Vladimir and Glenn Shafer (2005), "Good randomized sequential probability forecasting is always possible." *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 67, 747–763. [3]

Wald, Abraham (1950), *Statistical Decision Functions*. Wiley, New York. [3]

---

Submitted 2005-7-12. Final version accepted 2006-11-19. Available online 2006-11-19.