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A non-differentiable approach to revenue equivalence

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We give a sufficient condition on the type space for revenue equivalence when the set of social alternatives consists of probability distributions over a finite set. Types are identified with real-valued functions that assign valuations to elements of this finite set, and the type space is equipped with the Euclidean topology. Our sufficient condition is stronger than connectedness but weaker than smooth arcwise connectedness. Our result generalizes all existing revenue equivalence theorems when the set of social alternatives consists of probability distributions over a finite set. When the set of social alternatives is finite, we provide a necessary and sufficient condition. This condition is similar to, but slightly weaker than, connectedness.

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1. Introduction

Revenue equivalence has long intrigued economists. In certain settings, a broad range of mechanisms, including first- and second-price auctions, collect exactly the same expected revenue. This fact has led economists to try to delineate the conditions under which revenue equivalence holds.

In a number of recent papers, several authors provide sufficient conditions for revenue equivalence (see, for example, Krishna and Maenner 2001, Milgrom and Segal 2002 and Ely 2001).[1] One common feature of these sufficient conditions is that they are stated with reference to some topological and linear structures on the agents' type spaces, which are not defined in terms of the basic primitives of the mechanism design problem. This feature is undesirable because revenue equivalence is stated in terms of the basic primitives of the mechanism design problem. If a social choice function satisfies revenue equivalence, it does so regardless of whether the agents' type spaces are equipped with any extra structure, and regardless of the nature of any such structure.

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[1] See Section 4 for a detailed discussion of the existing literature.

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In this paper, we look for sufficient (and sometimes necessary) conditions that can be stated without referring to any extra topological or linear structures. We take as primitives only arbitrary sets of types for the agents, an arbitrary set of social alternatives, and valuation functions that assign numbers to each type–alternative pair. Types can be identified with real-valued functions that assign valuations to social alternatives. Our main result, Theorem 1, is a sufficient condition when the set of social alternatives consists of probability distributions over a finite set; this condition is weaker than smooth arcwise connectedness, but stronger than connectedness. The result generalizes all existing revenue equivalence theorems when the set of social alternatives consists of probability distributions over a finite set. Moreover, our proof is elementary, in the sense that it does not refer to the concepts of differentiation or to any envelope theorem. However, we offer no insight for general sets of social alternatives.

When the set of social alternatives is finite, we provide a necessary and sufficient condition in Theorem 4; this condition is close to, but weaker than connectedness of the type space in the Euclidean topology on real-valued functions. It seems to be of interest because, although it has long been folk wisdom that (some definition of) connectedness is intimately related to revenue equivalence, it clarifies the exact relation between the two. We also give a (similar) sufficient condition, which is also weaker than connectedness, when the set of social alternatives has cardinality lower than continuum.

Although our motivation is rather theoretical, we demonstrate that our results are relevant for some applications. We see two ways in which our results may be applied. Firstly, they seem useful in models with a mixed continuous–discrete type space;[2] we exhibit such an application in Example 1. Secondly, the existing sufficient conditions for revenue equivalence impose some assumptions that may be violated in some applications (e.g., the convexity assumed by Krishna and Maenner). Authors studying more general classes of mechanism design problems may prefer not to impose them (e.g., the absolute continuity assumed in Milgrom and Segal is not very elegant and, typically, has to be imposed only for revenue equivalence), and our results provide an alternative for these authors.

2. Setting

There is only one agent, except in Section 5.4, in which we comment on the case of more agents. Many applications (including auctions) require more agents. The results for the many-agent case follow fairly easily from the results for the one-agent case. Let S denote the type space and let A denote the set of social pure alternatives. Let ΔA denote the set of all probability distributions over the set A with finite support. Let F be an arbitrary subset of ΔA. The agent has a type-dependent and quasi-linear utility function of the form

$$v(s,a)-t,$$

[2] Mixed type spaces seem to be of interest as some variables are typically modeled as continuous (e.g. consumption, wealth, and input prices), whereas others naturally are, or are modeled as, discrete (e.g. cohort, or ability). We thank a referee for this (and several other) arguments regarding applications.

where s is the agent's type, a is the implemented social alternative, and t is a (possibly negative) monetary transfer made by the agent. She is an expected utility maximizer. For simplicity, we denote by $v(s,\alpha)$ the expected value of the function $v(s,a)$ when the social alternative is implemented according to the probability distribution $\alpha \in \Delta A$. The triple (S,F,v) defines a mechanism design problem. We thus regard the triple as the primitive of the problem.

A *social choice function* (SCF) is a mapping $f : S \to F$ and a transfer rule is a mapping $t : S \to R$. This formulation includes the special case of $F = A$, which rules out randomization. In Section 3 we assume that $F = \Delta A$ and the set A is finite, and in Section 5.4, we assume that $F = A$ and the cardinality of A is lower than continuum.[3] A mechanism is a pair (f,t), where f is an SCF and t is a transfer rule. A mechanism is *incentive compatible* if for any types s, $s' \in S$,

$$v(s,f(s)) - t(s) \geq v(s,f(s')) - t(s').$$

An SCF f is *incentive compatible* if there exists a transfer rule t such that the mechanism (f,t) is incentive compatible.

An incentive compatible SCF f satisfies the *revenue equivalence property* if for any two incentive compatible mechanisms (f,t) and (f,t') there exists a constant $c \in R$ such that

$$\forall_{s\in S} \quad t'(s) = t(s) + c.$$

We now define a topology on the type space. We assume that there exist no types s and s' such that $v(s,a) = v(s',a)$ for every $a \in A$. We can make this assumption without loss of generality, as for every type space S there exists a subspace $S' \subset S$ such that for every $s \in S$ there exists exactly one $s' \in S'$ such that $v(s,a) = v(s',a)$ for every $a \in A$, and it is easy to check that for all spaces S and S', every incentive compatible SCF $f : S \to F$ satisfies the revenue equivalence property if and only if every incentive compatible SCF $f : S' \to F$ satisfies the revenue equivalence property.

We consider the sup-norm in the type space S, i.e. the distance between any pair of types s, $s' \in S$ is given by

$$\operatorname{dist}(s,s') = \sup_{a\in A} |v(s,a) - v(s',a)|; \tag{1}$$

one can easily check that S equipped with this distance is a metric space.[4] In other words, we first identify types with real-valued functions $f \in R^A$ that assign valuations to social alternatives, and then we equip the function space R^A with the sup-norm. We prefer not to identify types explicitly with real-valued functions $f \in R^A$, and define instead the distance between types by formula (1) in order to maintain the notational conventions of standard textbooks. For finite sets A, we could equivalently equip R^A with any other norm, as any two norms on a Euclidean space are equivalent; in Section 3 where

[3] An alternative formulation of our results is that F from Section 5.4 is an arbitrary subset of ΔA, whose cardinality is lower than continuum, and F from Section 3 is an arbitrary subset of ΔA, where A is finite.

[4] It is important here that there exist no types s and s' such that $v(s,a) = v(s',a)$ for every $a \in A$, as it guarantees that $\operatorname{dist}(s,s') = 0$ implies $s = s'$.

we study finite sets A, it proves convenient to use the equivalent L_1-norm, in which the distance between any pair of types $s, s' \in S$ is given by

$$\text{dist}(s,s') = \sum_{a\in A} |v(s,a) - v(s',a)|.$$

The following example exhibits a mechanism design problem in which the type space differs from the ones considered in the existing literature on revenue equivalence.

EXAMPLE 1. Consider a principal hiring an agent to perform one of two tasks. The cost $c_1 = p$ of performing task 1 is determined by the market price p, which takes values in the interval $(0,1)$. The cost $c_2 = d \cdot p$ of performing task 2 is determined by both the input price p and the agent's ability $d \in \{0,1,2\}$ (which takes one of three levels).

Suppose that the agent knows both d and p, while the principal does not know either. The principal can be thought of choosing between which of the two (possibly unrelated) tasks she should assign the agent to. She can assign the agent to task 1 (in which case the relevant private information is c_1), or to task 2 (in which case the relevant private information is c_2). A contract can be modeled as a revelation mechanism where the agent reports a type (d,p) and the mechanism assigns a task and specifies a wage.

Notice that the valuation of being assigned to task 1 is equal to $-c_1 = -p$, while the valuation of being assigned to task 2 is equal to $-c_2 = -d \cdot p$. Thus, we identify the type space with the set

$$S = \bigcup_{d=0}^{2} \{(v_1, v_2) : v_1 \in (-1,0) \text{ and } v_2 = d \cdot v_1\} \subset R^2.$$

This set is a "fan" consisting of three disjoint segments, and one can easily imagine similar applications in which the set S may be topologically quite complicated. ◊

3. HIGH-CARDINALITY SETS OF ALTERNATIVES

In this section, we assume $F = \Delta A$ for a finite set A. This case covers a number of important applications of mechanism design, including auctions of multiple units of indivisible objects. (This application requires more agents, but, as we show in Section 5.4, the result for many agents follows fairly easily from the result for the one-agent case.) Our main result provides a sufficient condition for the type space that guarantees that every incentive compatible SCF satisfies the revenue equivalence property.

DEFINITION 1. (i) A metric space S is *gridwise connected* between points s and s' if

$$\forall_{\varepsilon>0}\ \exists_{s=s_0,\ldots,s_n=s'}\ \forall_{i=1,\ldots,n}\quad \text{dist}(s_i, s_{i-1}) < \varepsilon.$$

(ii) If, additionally, there exists a constant M, independent of ε, such that

$$\sum_{i=1}^{n} \text{dist}(s_i, s_{i-1}) \le M,$$

then the set S is *boundedly gridwise connected* between points s and s'.

(iii) A metric space S is *gridwise connected* (respectively, *boundedly gridwise connected*) if it is gridwise connected (respectively, boundedly gridwise connected) between any points $s, s' \in S$.

Connectedness implies gridwise connectedness, and if the metric space S is compact, the two concepts are equivalent (see Engelking 1989, Chapter 6, Exercise 6.1.D). Arcwise connectedness (called also, by some authors, pathwise connectedness) does not imply bounded gridwise connectedness; for example, the graphs of some continuous but nowhere differentiable functions (see, for instance, Billingsley 1982) have the former but not the latter property. However, smooth arcwise connectedness implies bounded gridwise connectedness as any pair of points of a smooth arcwise connected space can be connected by an arc of finite length. On the other hand, bounded gridwise connectedness does not imply arcwise connectedness, even in the realm of compact spaces (see Engelking 1989, Chapter 6, Exercise 6.1.G).

Finally, it is immediate to verify that the set S in Example 1 is boundedly gridwise connected.

THEOREM 1. *Suppose that $F = \Delta A$ for a finite set A. If the type space S is boundedly gridwise connected, then every incentive compatible SCF $f : S \to F$ satisfies the revenue equivalence property.*

To prove this result we need the following notation. For any pair of vectors $x, y \in R^A$, we denote by $x \circ y$ their *inner* (or *scalar*) *product*. For a SCF $f : S \to \Delta A$, every $f(s)$ is a vector of probabilities assigned to all alternatives, and so it is an element of R^A; also, every $v(s,\cdot)$ is an element of R^A. We can therefore consider the product of the vectors $f(s)$ and $v(s,\cdot)$.

LEMMA 1. *(i) For every incentive compatible SCF f and any types $s, s' \in S$,*

$$v(s,\cdot)\circ\left[f(s)-f(s')\right] \geq t(s)-t(s') \geq v(s',\cdot)\circ\left[f(s)-f(s')\right]. \tag{2}$$

(ii) Suppose that $s = s_0, \ldots, s_n = s' \in S$. Let (f,t') and (f,t'') be incentive compatible mechanisms such that $t'(s) = t''(s)$. Then

$$\left|t'(s')-t''(s')\right| \leq \sum_{i=1}^{n}\left[v(s_i,\cdot)-v(s_{i-1},\cdot)\right]\circ\left[f(s_i)-f(s_{i-1})\right]. \tag{3}$$

PROOF. (i) By incentive compatibility,

$$v(s,\cdot)\circ f(s)-t(s) \geq v(s,\cdot)\circ f(s')-t(s')$$

and

$$v(s',\cdot)\circ f(s')-t(s') \geq v(s',\cdot)\circ f(s)-t(s).$$

Rearranging, one obtains (2).

(ii) By (i),

$$\begin{aligned}\left|\left[t'(s_i)-t''(s_i)\right]-\left[t'(s_{i-1})-t''(s_{i-1})\right]\right| &= \left|\left[t'(s_i)-t'(s_{i-1})\right]-\left[t''(s_i)-t''(s_{i-1})\right]\right| \\ &\le v(s_i,\cdot)\circ\left[f(s_i)-f(s_{i-1})\right]-v(s_{i-1},\cdot)\circ\left[f(s_i)-f(s_{i-1})\right] \\ &= \left[v(s_i,\cdot)-v(s_{i-1},\cdot)\right]\circ\left[f(s_i)-f(s_{i-1})\right].\end{aligned}$$

Thus,

$$\begin{aligned}\left|t'(s_n)-t''(s_n)\right| &\le \left|\left[t'(s_n)-t''(s_n)\right]-\left[t'(s_{n-1})-t''(s_{n-1})\right]\right| \\ &\quad+\left|\left[t'(s_{n-1})-t''(s_{n-1})\right]-\left[t'(s_{n-2})-t''(s_{n-2})\right]\right|+\cdots \\ &\quad+\left|\left[t'(s_1)-t''(s_1)\right]-\left[t'(s_0)-t''(s_0)\right]\right|+\left|t'(s_0)-t''(s_0)\right| \\ &\le \sum_{i=1}^{n}\left[v(s_i,\cdot)-v(s_{i-1},\cdot)\right]\circ\left[f(s_i)-f(s_{i-1})\right]+\left|t'(s_0)-t''(s_0)\right|.\end{aligned}$$

This yields (3) for $s=s_0$ and $s'=s_n$. □

Proof of Theorem 1. Suppose that (f,t') and (f,t'') are incentive compatible. Take a pair of points s, $s'\in S$. We show that $t'(s)=t''(s)$ implies $t'(s')=t''(s')$. That is, if two transfer rules coincide at one point, they coincide at every other point. This obviously implies that any two t and t' transfer rules differ by a constant, because if (f,t) is incentive compatible, then so is (f,t'') for t'' given by

$$\forall_{s'\in S}\quad t''(s')=t(s')+\left[t'(s)-t(s)\right]$$

for a fixed $s\in S$. Thus, $t'(s')=t''(s')$, and so $t'(s')=t(s')+c$ where $c=[t'(s)-t(s)]$, for every $s'\in S$.

The proof applies the following idea: By assumption, there exists a sequence of points $s=s_0,\ldots,s_n=s'$ with the property that each s_i is within the distance ε of its predecessor. If the number of points in this sequence were bounded by a number independent of ε, Theorem 1 would follow immediately from (3); indeed, each coordinate of the term $[s_i-s_{i-1}]$ can be made as small as we wish, and each coordinate of the term $\left[f(s_i)-f(s_{i-1})\right]$ is bounded by 1. However, the number of points in the sequence $s=s_0$, $\ldots$, $s_n=s'$ is typically not bounded by a number independent of ε. We therefore pick a subsequence from the sequence $s=s_0,\ldots,s_n=s'$ whose number of points is bounded by a number that depends only on the number of alternatives in A. By picking such a subsequence, we typically lose the property that each s_i is within the distance ε of s_{i-1}. However, we can pick our subsequence in such a way that whenever s_i is not within the distance ϵ of s_{i-1}, $f(s_i)$ is as close as we wish to $f(s_{i-1})$.

More precisely, pick any number $k=1,2,\ldots$. We show that

$$\left|t'(s')-t''(s')\right|\le\frac{M+1}{k},\tag{4}$$

and so (as k can be arbitrary large)

$$t'(s)=t''(s)\Longrightarrow t'(s')=t''(s').$$

To show (4) take $s=s_0,\ldots,s_n=s'\in S$ satisfying Definition 1 (i) for $\varepsilon=1/k^{m+1}$ where m denotes the number of elements of A. Represent the simplex $\Delta A\subset[0,1]^m$ as the union of k^m cubes P_i such that any two probability vectors $p,q\in P_i$ differ at most by $1/k$ on each coordinate, i.e. if $p=(p^1,\ldots,p^m)$ and $q=(q^1,\ldots,q^m)$, then $\left|p^j-q^j\right|\le 1/k$ for every $j=1,\ldots,m$.

We now pick a sequence s_0^{FIRST}, s_0^{LAST}, …, s_N^{FIRST}, s_N^{LAST} consisting of the elements of the sequence $s_0,\ldots,s_n$ as follows. Take $s_0^{FIRST}=s_0$, and then take any P_{i_0} such that $f(s_0^{FIRST})\in P_{i_0}$; let s_0^{LAST} be the last element of $s_0,\ldots,s_n$ with the property that $f(s_0^{LAST})\in P_{i_0}$. Next, take as s_1^{FIRST} the successor of s_0^{LAST} in the sequence $s_0,\ldots,s_n$, and take any P_{i_1} such that $f(s_1^{FIRST})\in P_{i_1}$; as in the first step, let s_1^{LAST} be the last element of $s_0,\ldots,s_n$ with the property that $f(s_1^{LAST})\in P_{i_1}$. Continue in this fashion until s_N^{FIRST}, s_N^{LAST} such that $s_N^{LAST}=s_n$ is defined.

If $t'(s)=t''(s)$, then, by Lemma 1,

$$\begin{aligned}\left|t'(s')-t''(s')\right|\le&\sum_{l=0}^{N}\left|\left[v\left(s_l^{LAST},\cdot\right)-v\left(s_l^{FIRST},\cdot\right)\right]\circ\left[f\left(s_l^{LAST}\right)-f\left(s_l^{FIRST}\right)\right]\right|\\&+\sum_{l=1}^{N}\left|\left[v\left(s_l^{FIRST},\cdot\right)-v\left(s_{l-1}^{LAST},\cdot\right)\right]\circ\left[f\left(s_l^{FIRST}\right)-f\left(s_{l-1}^{LAST}\right)\right]\right|.\end{aligned}$$

By construction, $f\left(s_l^{LAST}\right)$ and $f\left(s_l^{FIRST}\right)$ differ at most by $1/k$ on each coordinate, and so

$$\begin{aligned}&\sum_{l=0}^{N}\left|\left[v\left(s_l^{LAST},\cdot\right)-v\left(s_l^{FIRST},\cdot\right)\right]\circ\left[f\left(s_l^{LAST}\right)-f\left(s_l^{FIRST}\right)\right]\right|\\&\qquad\le\frac{1}{k}\sum_{l=0}^{N}\sum_{j=1}^{m}\left|v_j\left(s_l^{LAST},\cdot\right)-v_j\left(s_l^{FIRST},\cdot\right)\right|\\&\qquad\le\frac{1}{k}\sum_{i=1}^{n}\sum_{j=1}^{m}\left|v_j\left(s_i,\cdot\right)-v_j\left(s_{i-1},\cdot\right)\right|\\&\qquad\le\frac{1}{k}\sum_{i=1}^{n}\operatorname{dist}(s_i,s_{i-1})\\&\qquad\le\frac{M}{k}.\end{aligned}$$

Since s_{j-1}^{LAST} and s_j^{FIRST} are consecutive elements of the sequence $s_0,\ldots,s_n$, the distance between them does not exceed ε. By construction, N does not exceed the number of

sets P_i. Therefore

$$\begin{aligned}\sum_{l=1}^{N}\left|\left[v\left(s_l^{FIRST},\cdot\right)-v\left(s_{l-1}^{LAST},\cdot\right)\right]\circ\left[f\left(s_l^{FIRST}\right)-f\left(s_{l-1}^{LAST}\right)\right]\right| &\\ \leq\sum_{l=1}^{N}\sum_{j=1}^{m}\left|v_j\left(s_l^{FIRST},\cdot\right)-v_j\left(s_{l-1}^{LAST},\cdot\right)\right| &\\ \leq k^m\varepsilon &\\ =\frac{1}{k}.&\end{aligned}$$

Thus,

$$\left|t'(s')-t''(s')\right|\leq\frac{M}{k}+\frac{1}{k}.$$ □

Recall that a *homeomorphism* of metric (or topological) spaces is a continuous one-to-one and onto mapping whose inverse is also continuous. A subset J of a metric space is called an *arc* if it is homeomorphic to the unit interval $[0,1]$, i.e. there exists a homeomorphism (parameterization) $j:[0,1]\to J$. The *length* of an arc is defined to be

$$\lim_{n\to\infty}\sum_{i=1}^{n}\operatorname{dist}(j(x_i^n),j(x_{i-1}^n)),\tag{5}$$

where $0=x_0^n<\cdots<x_n^n=1$ and

$$\max_{i=1,\ldots,n}\left|x_i^n-x_{i-1}^n\right|\to_{n\to\infty}0,$$

provided that the limit (5) exists and is independent of the choice of $x_0^n,\ldots,x_n^n$.

An arc is called *smooth* if there exists a differentiable parameterization $j:[0,1]\to J$ whose derivatives are continuous. The length of any smooth arc is well-defined and finite, but there exist arcs with well-defined and finite length that are not smooth. Theorem 1 implies immediately the following corollary.

COROLLARY 1. *Suppose that $F=\Delta A$ for a finite set A. Every incentive compatible SCF satisfies the revenue equivalence property whenever the type space S satisfies one of the following conditions.*

(i) Any pair of points $s,s'\in S$ can be connected by an arc of finite length.

(ii) Any pair of points $s,s'\in S$ can be connected by a smooth arc.

Since any pair of points of a connected open set $U\subset R^A$ can be connected by a smooth arc, Corollary 1 yields the following result.

COROLLARY 2. *Suppose that $F=\Delta A$ for a finite set A, and the type space $S\subset R^A$ is a connected open set, or it contains a connected, open, and dense set. Then every incentive compatible SCF satisfies the revenue equivalence property.*

Finally note that in the special case in which A consists of only two elements, connectedness of the type space S is a sufficient condition for revenue equivalence.

PROPOSITION 1. *Suppose that $F = \Delta A$ for a two-element set A. If the type space $S \subset R^A$ is connected, then every incentive compatible SCF satisfies the revenue equivalence property.*

PROOF. Let $f : S \to \Delta A$ be an incentive compatible SCF. Denote by $p(s)$ the probability assigned by $f(s)$ to the first alternative. Let $\pi : S \to R$ be given by

$$\forall_{s=(s^1,s^2)} \quad \pi(s) = s^2 - s^1.$$

Take any $s, s' \in S$. Fix $\varepsilon > 0$. Suppose that $\pi(s) \geq \pi(s')$; the opposite case is analogous. Since S is connected, so is $\pi(S)$; take $s = s_0, \ldots, s_n = s' \in S$ such that

$$\pi(s_0) \geq \pi(s_1) \geq \cdots \geq \pi(s_n) \tag{6}$$

and

$$\forall_{i=1,\ldots,n} \quad \pi(s_i) - \pi(s_{i-1}) < \varepsilon.$$

By Lemma 1 (ii), for any transfer rules t' and t'' such that (f, t') and (f, t'') are incentive compatible, if $t'(s) = t''(s)$ then

$$\begin{aligned} \left|t'(s') - t''(s')\right| &\leq \sum_{i=1}^{n} [s_i - s_{i-1}] \circ \left[f(s_i) - f(s_{i-1})\right] \\ &= \sum_{i=1}^{n} [s_i - s_{i-1}] \circ \left[\left(p(s_i) - p(s_{i-1}), p(s_{i-1}) - p(s_i)\right)\right] \\ &= \sum_{i=1}^{n} \left(p(s_i) - p(s_{i-1})\right) \left(\pi(s_{i-1}) - \pi(s_i)\right) \end{aligned} \tag{7}$$

By Lemma 1 (i),

$$[s_i - s_{i-1}] \left[f(s_i) - f(s_{i-1})\right] \geq 0.$$

Thus by (6),

$$p(s_0) \leq p(s_1) \leq \cdots \leq p(s_n).$$

Therefore the expression (7) does not exceed

$$\varepsilon \cdot \sum_{i=1}^{n} \left(p(s_i) - p(s_{i-1})\right) \leq \varepsilon \left[p(s_n) - p(s_0)\right] \leq \varepsilon.$$

Summarizing,

$$\left|t'(s') - t''(s')\right| < \varepsilon.$$

Since this inequality holds for every $\varepsilon > 0$, we obtain that $t'(s') = t''(s')$. □

Proposition 1 is of particular relevance to auctions with a single indivisible object. In these auctions, for every typical bidder, there are only two payoff relevant outcomes: whether she gets the object, or not. We show in Section 5.4 that Proposition 1 for many agents follows fairly easily from the one-agent case. Hence revenue equivalence is guaranteed as long as every bidder's type space is connected, which is weaker than the assumption of Theorem 1.

4. Comparison to existing literature

The early literature on revenue equivalence is concerned with the uniqueness of Groves mechanisms. The earliest paper we are aware of is Green and Laffont (1977), which shows that for an arbitrary set of social alternatives F equipped with a topology, if the type space S contains all continuous (or all u.s.c.) valuation functions, then all efficient SCFs satisfy the revenue equivalence property.

Walker (1978) considers a set F that is a subset of an Euclidean space. He shows that if (i) an efficient SCF f has a convex range, (ii) S contains only concave valuation functions, and (iii) S is rich enough that for any a, $a' \in A$, and for any gradient, we can find a valuation function in S that has this gradient at a, and is arbitrarily close to linear between a and a', then the efficient SCF f satisfies the revenue equivalence property. These two results take a form quite different from the results in the rest of the literature, including ours (e.g. even the set of social alternatives is equipped with a topological or linear structure that is not defined in terms of the basic primitives of the mechanism design problem), and hence a direct comparison is difficult.

Holmström (1979) shows that for an arbitrary set F, if S is (piecewise) smoothly connected with respect to an efficient SCF f, then the SCF f satisfies the revenue equivalence property. He calls S *smoothly connected with respect to* f if, for any s, $s' \in S$, there is an arc J in S, parameterized by $j : [0,1] \to J$, such that $j(0) = s$, $j(1) = s'$,

$$\frac{\partial v(j(y), a)}{\partial y}$$

exists for all a and y, and there exists some finite K such that

$$\left| \frac{\partial v(j(y), a)}{\partial y} \right| \leq K$$

for all $a \in f(J)$ and y.

Notice that Holmström's concern is the revenue equivalence property of specific SCFs, and this explains why his condition is stated in terms of both the type space and the SCF. A similar feature can be found in several subsequent results as well. For example, Williams (1999) shows that for an arbitrary set F, for any incentive compatible SCF f, if S is a connected open subset of the Euclidean space, and if $v(s, f(s'))$ as a function of (s, s') is differentiable at points that satisfy $s = s'$, then the SCF f satisfies the revenue equivalence property.[5]

The way we pose the revenue equivalence question in this paper is slightly different: we ask when it is the case that all incentive compatible SCFs satisfy the revenue equivalence property. This way of posing the question is motivated more by the mechanism design literature. If we require the assumptions of Holmström or Williams to hold for all incentive compatible SCFs simultaneously, then their results follow from the first part of our Corollary 2 (assuming, of course, that $F = \Delta A$ for a finite set A).

[5]Although Williams states his result only for efficient SCFs, his proof does not rely on efficiency, and hence we state his result in this more general way.

In the mechanism design literature, economists usually find it convenient to work with a type space where all incentive compatible SCFs satisfy the revenue equivalence property. In auction settings, Myerson (1981), and subsequently Jehiel et al. (1999), show that when $F = \Delta A$ with A finite, if S is a "rectangular" subset of R^A, then all incentive compatible SCFs satisfy the revenue equivalence property. Their results follow immediately from our Theorem 1.

Two other papers provide results similar to our Theorem 1. Krishna and Maenner show the following result (see their Proposition 1).

THEOREM 2 (Krishna and Maenner 2001). *Suppose that S is an open and convex subset of R^n, and for every social alternative a, the function v is convex with respect to s. Then every incentive compatible SCF satisfies the revenue equivalence property.*[6]

This theorem follows from our Theorem 1 when the set of social alternatives is a subset of ΔA for a finite set A. Indeed, it is easy to see that it suffices to derive their result from Theorem 1 when S is an open interval. If, however, S is an open interval, then any convex real-valued function defined on S is continuous; moreover, it satisfies the *Lipschitz condition* on every closed interval $R \subset S$, i.e.

$$\forall_{a\in F}\ \exists_{M>0}\ \forall_{s,s'\in R} \quad |v(s,a) - v(s',a)| \leq M \cdot |s - s'|.$$

Thus the sufficient condition in Theorem 1 is satisfied when S is an open interval and the function v is convex with respect to s.

Milgrom and Segal show the following result (see their Corollary 1).

THEOREM 3 (Milgrom and Segal 2002). *Suppose that $S = [0,1]$; suppose further that the function v is differentiable and absolutely continuous in s for every social alternative a, and that*

$$\sup_{a\in A} \left| \frac{\partial v}{\partial s}(s,a) \right| \tag{8}$$

is integrable on $[0,1]$. Then every incentive compatible SCF satisfies the revenue equivalence property.[7]

This theorem follows immediately from our Theorem 1 when the set of social alternatives is a subset of ΔA for a finite set A. To see this, observe that the integrability of the function given by (8) implies the integrability of $(\partial v/\partial s)(s,a)$ for every social alternative a, which in turn implies that any pair of points in S can be connected by an arc of finite length.[8]

[6] Krishna and Maenner do not explicitly assume in their Proposition 1 that S is open, but it is clear from their proof that they make this assumption. Actually, their result fails when S is not open; an example is available in a supplementary file on the journal website, http://econtheory.org/supp/277/supplement.pdf.

[7] Milgrom and Segal allow for other than quasi-linear utility specifications; here, we formulate their theorem only for quasi-linear utilities.

[8] In their footnote 10, Milgrom and Segal point out that the integrability assumption can be somewhat relaxed for quasi-linear utilities. One can easily show that their weaker assumption is also stronger than our sufficient condition in Theorem 1.

On the other hand, we assume in Theorem 1 that the set of social alternatives is a subset of ΔA for a finite set A; both Krishna and Maenner and Milgrom and Segal do not make any assumptions on the set of social alternatives. These two papers also prove a formula for indirect utilities; namely, for a fixed $s' \in S$, we have

$$\forall_{s\in S} \quad V(s) = V(s') + \int \frac{\partial v}{\partial t}(t, f(t))\, dt,$$

where the integration takes places over an arc joining s and s'.

5. Concluding comments

5.1 *A discussion of the sufficient condition in Theorem 1*

The sufficient condition in Theorem 1 is not necessary for the revenue equivalence property (even in the realm of connected type spaces). Let S_1 be the segment $\{(x,y) \in R^2 : x = 0 \text{ and } -1 \le y \le 1\}$, and S_2 be the graph of the function $h : (0,1] \to R$ given by

$$h(x) = \sin \frac{1}{x};$$

further let $S = S_1 \cup S_2$. It can be shown that S is not boundedly gridwise connected; yet every incentive compatible SCF $f : S \to \Delta A$, when $F = \Delta A$ for a finite set A, satisfies the revenue equivalence property.

We do not know if there exist connected type spaces S without the revenue equivalence property. It can be shown that if a type space S does not have the revenue equivalence property, then it contains a subspace $S' \subset S$ that is not boundedly gridwise connected between any pair of points s, $s' \in S'$; the graphs of continuous but nowhere differentiable functions (see, for instance, Billingsley 1982), and some straightforward modifications thereof, are the only examples of connected spaces with this property of which we are aware.

5.2 *Arbitrary sets of alternatives*

We know rather little about any sufficient (and "close" to necessary) conditions for revenue equivalence when A is an arbitrary set of social alternatives. On the one hand, Krishna and Maenner (2001) as well as Milgrom and Segal (2002) provide some sufficient conditions, but their conditions are driven by a particular method of proving the revenue equivalence theorem, and we have rather little sense if they are close to necessary conditions. On the other hand, Holmström (1979) and Ely (2001) give an example in which revenue equivalence fails; in their example, $S = A = [0,1]$, $v(s,a)$ is a continuous (and piecewise linear) function, and the type space equipped with the distance given by formula (1) is boundedly gridwise connected.

5.3 *A two-step approach to optimal mechanism design*

In the literature on mechanism design, the environment is often deliberately set up so that all incentive compatible SCFs satisfy the revenue equivalence property. This allows us to decompose the procedure of finding the optimal mechanism into two steps:

In step 1, we look for the cheapest (from the mechanism designer's perspective) mechanisms implementing incentive compatible SCFs by shifting the transfer rule upwards until the individual rationality constraints bind. In step 2, we maximize over all incentive compatible SCFs, taking into account the implementation costs. Revenue equivalence guarantees that step 1 can be done independently of the principal's beliefs over the type space, her risk preferences, etc.; this is so because all transfer rules are linearly ranked.

The following proposition says that the two-step approach does not lose much of its applicability even when revenue equivalence fails. To state the result we need to define individual rationality. Let $r : S \to R$ be the agent's state-dependent reservation utility function. An incentive-compatible mechanism (f,t) is *individually rational* if

$$\forall_{s\in S} \quad v(s,f(s)) - t(s) \geq r(s).$$

PROPOSITION 2. *For every incentive compatible SCF f, there exists a unique cheapest (from the mechanism designer's perspective) individually rational mechanism (f,t^*), in the sense that for any individually rational mechanism (f,t), we have $t^*(s) \geq t(s)$ for every $s \in S$.*

PROOF. Let

$$\forall_{s\in S} \quad t^*(s) = \sup\{t(s) : (f,t) \text{ is incentive compatible and individually rational}\}.$$

If (f,t^*) were not incentive compatible, then there would exist s, $s' \in S$ such that

$$v(s,f(s)) - t^*(s) < v(s,f(s')) - t^*(s').$$

Therefore, by the definition of t^*, there would exist also an incentive compatible mechanism (f,t) such that

$$v(s,f(s)) - t(s) < v(s,f(s')) - t^*(s'),$$

but, again by the definition of t^*,

$$v(s,f(s')) - t^*(s') \leq v(s,f(s')) - t(s'),$$

which contradicts the incentive compatibility of (f,t). □

5.4 *Many agents*

Throughout this paper we have considered a single agent. Notice, however, that all our results extend easily to the case of more agents, as follows. Consider n agents; each agent i learns a signal $s_i \in S_i$. Agent i has a state-dependent utility function over alternatives $v_i : S \times A \to R$, where $S = S_1 \times \cdots \times S_n$, and a quasi-linear utility with respect to transfers. Suppose that there is a single common prior probability distribution over types.[9] Given any type $s_i \in S_i$, the expected value of the function v_i over the opponents' signal profiles $s_{-i} \in S_{-i}$ maps A into R, so that each type s_i can be identified with an element of the space R^A.

[9]The assumption of a common prior is obviously restrictive.

Now, given any condition on the single-agent type space that guarantees revenue equivalence (in the case of a single agent), we obtain a condition for revenue equivalence in the case of many agents by requiring the single-agent condition to be satisfied for every agent $i = 1, \ldots, n$ when we use the expected values of v_i to identify types s_i with elements of R^A.

This extension of our results does, however, have a limitation, which may not be immediately apparent. Recall that in the single-agent case we define SCFs and transfer rules as mappings of the reported type s, and all our results should be read with these definitions in mind. For multi-agent extensions, these definitions imply that, given truthful reporting by the other agents, the action expected by player i and her expected transfer can depend only on the type reported by player i, but not on her actual type. This condition is immediately satisfied when types are independent, but is typically violated when types are correlated across agents.

We do not differ here from the existing literature, which (like Milgrom and Segal 2002) restricts attention to the single-agent case, or (like Krishna and Maenner 2001) treats the many-agent case in a similar manner to the present paper. In particular, our extension of Theorem 1 to many agents still generalizes the many-agent version of the theorem from Krishna and Maenner quoted in Section 4 when $F = \Delta A$ for a finite set A.

Appendix: Low-cardinality sets of alternatives

Consider now the case when $F = A$ is finite. Our result in this section provides a necessary and sufficient condition on the type space that guarantees that every incentive compatible SCF satisfies the revenue equivalence property. If the set A is infinite, but has cardinality lower than continuum, our theorem provides only a sufficient condition. Recall that a topological space S is *connected* if it cannot be represented as the union of two disjoint, non-empty open sets. Our sufficient (and necessary) condition says that S cannot be represented as the union of two disjoint, non-empty open sets of a particular form. To state our theorem precisely we need some notation. Given a pair B_1, B_2 of disjoint subsets of A and a function $r : B_1 \cup B_2 \to R$, let, for every $\varepsilon > 0$,

$$V_1(\varepsilon) = \bigcup_{b_1 \in B_1} \{s \in S : \forall_{b_2 \in B_2}\ v(s, b_1) - v(s, b_2) > r(b_1) - r(b_2) + \varepsilon\}$$

and

$$V_2(\varepsilon) = \bigcup_{b_2 \in B_2} \{s \in S : \forall_{b_1 \in B_1}\ v(s, b_1) - v(s, b_2) < r(b_1) - r(b_2) - \varepsilon\}.$$

Let also

$$V_1 = \bigcup_{\varepsilon > 0} V_1(\varepsilon) \qquad \text{and} \qquad V_2 = \bigcup_{\varepsilon > 0} V_2(\varepsilon).$$

Notice that, given B_1, $B_2 \subset A$, $r : B_1 \cup B_2 \to R$, and $\varepsilon > 0$, the sets $V_1(\varepsilon)$ and $V_2(\varepsilon)$, as well as V_1 and V_2, are disjoint. Notice also that the sets V_1 and V_2 are open. Indeed, if $s \in V_i(\varepsilon)$ for some $\varepsilon > 0$, then it is easy to see that $\bigcup_{\varepsilon > 0} V_i(\varepsilon)$ contains the ball around s with radius $\varepsilon/3$ in the metric given by (1). If the set A is finite, then also the sets $V_1(\varepsilon)$ and $V_2(\varepsilon)$ are open.

THEOREM 4. *(i) Suppose that the set $F = A$ is finite. Then every incentive compatible SCF satisfies the revenue equivalence property if and only if there exist no disjoint sets B_1, $B_2 \subsetneq A$, a function $r : B_1 \cup B_2 \to R$, and $\varepsilon > 0$ such that $S \subset V_1(\varepsilon) \cup V_2(\varepsilon)$ and $V_1(\varepsilon) \cap S \neq \varnothing \neq V_2(\varepsilon) \cap S$.*

(ii) Suppose that the set $F = A$ has cardinality lower than the continuum (but need not be finite). If there exist no disjoint sets B_1, $B_2 \subsetneq A$, and a function $r : B_1 \cup B_2 \to R$ such that $S \subset V_1 \cup V_2$ and $V_1 \cap S \neq \varnothing \neq V_2 \cap S$, then every incentive compatible SCF satisfies the revenue equivalence property.

PROOF. (i, $\Longrightarrow$): To the contrary suppose there exist B_1, $B_2 \subsetneq A$, $r : B_1 \cup B_2 \to R$, and $\varepsilon > 0$ with the properties specified in the theorem. Define an SCF $f : S \to A$ by

$$f(s) = \arg\max_{a \in B_1 \cup B_2} [v(s,a) - r(a)]; \tag{9}$$

take any $a \in B_1 \cup B_2$ that maximizes $[v(s,a) - r(a)]$ in the case of multiplicity. Define also a transfer rule $t : S \to R$ by

$$t(s) = r(a) \quad \text{if } f(s) = a.$$

By definition, (f,t) is an incentive compatible mechanism.

Now define $t' : S \to R$ by

$$t'(s) = \begin{cases} t(s) + \varepsilon/2 & \text{if } s \in V_1(\varepsilon) \\ t(s) - \varepsilon/2 & \text{if } s \in V_2(\varepsilon). \end{cases} \tag{10}$$

Since $S \subset V_1(\varepsilon) \cup V_2(\varepsilon)$ and $V_1(\varepsilon) \cap V_2(\varepsilon) = \varnothing$, the transfer rule t' is well-defined; since $V_1(\varepsilon) \cap S \neq \varnothing \neq V_2(\varepsilon) \cap S$, the difference $t - t'$ is not a constant function. It remains to show that (f,t') is an incentive compatible mechanism, which reduces to showing that the agent of any type $s \in V_1(\varepsilon)$ cannot profitably deviate by reporting a type $s' \in V_2(\varepsilon)$.

By definition, $s \in V_1(\varepsilon)$ implies that

$$\exists_{b_1 \in B_1} \, \forall_{b_2 \in B_2} \quad v(s,b_1) - v(s,b_2) > r(b_1) - r(b_2) + \varepsilon,$$

i.e. the agent prefers by at least ε some alternative $b_1 \in B_1$ and the transfer $r(b_1)$ to any alternative $b_2 \in B_2$ and the transfer $r(b_2)$; in particular, $f(s) \in B_1$. By a similar argument, $s' \in V_2(\varepsilon)$ implies that $f(s') \in B_2$. The two arguments together imply that the agent of any type $s \in V_1$ prefers the alternative $f(s)$ and the transfer $t'(s)$ to the alternative $f(s')$ and the transfer $t'(s')$.

(i, $\Longleftarrow$): Suppose that there exist incentive compatible mechanisms (f,t) and (f,t') such that the difference $t - t'$ is not a constant function. Let

$$S(a) = f^{-1}(a)$$

for $a \in A$. Notice that t, t', and so $t - t'$, are constant functions on every $S(a)$, as for every $s \in S(a)$ the agent has an incentive to report the element of $S(a)$ that minimizes t or t'.

Thus, $t-t'$ takes only a finite number of values, and so there exists a real number (say r) and $\varepsilon>0$ such that the values of $t-t'$ belong to the union of two intervals, $(-\infty, r-2\varepsilon)$ and $(r+\varepsilon,\infty)$, and each interval contains at least one value.

Denote by B_1 the set of all alternatives b_1 such that $(t-t')(s)\in(-\infty,r-2\varepsilon)$ for $s\in S(b_1)$, and by B_2 the set of all alternatives b_2 such that $(t-t')(s)\in(r+\varepsilon,\infty)$ for $s\in S(b_2)$. Note that $B_1\cup B_2$ is equal to the range of f, which obviously need not be equal to the entire A. Define, finally,

$$\begin{aligned}\forall_{b_1\in B_1}\quad & r(b_1)=t(s)+\varepsilon, \text{ where } s\in S(b_1)\\ \forall_{b_2\in B_2}\quad & r(b_2)=t(s)-\varepsilon, \text{ where } s\in S(b_2).\end{aligned}$$

Picking B_1, B_2, $r: B_1\cup B_2\to R$ and $\varepsilon>0$, we have defined the sets $V_1(\varepsilon)$ and $V_2(\varepsilon)$. We now show that

$$\bigcup_{b_1\in B_1} S(b_1)\subset V_1(\varepsilon) \qquad \text{and} \qquad \bigcup_{b_2\in B_2} S(b_2)\subset V_2(\varepsilon),$$

which obviously implies that $S\subset V_1(\varepsilon)\cup V_2(\varepsilon)$ and $V_1(\varepsilon)\cap S\neq\varnothing\neq V_2(\varepsilon)\cap S$.

Consider $s_1\in S(b_1)$ where $b_1\in B_1$. Then, by incentive compatibility,

$$\forall_{b_2\in B_2}\ \forall_{s_2\in S(b_2)}\quad v(s_1,b_1)-t'(s_1)\geq v(s_1,b_2)-t'(s_2),$$

or

$$v(s_1,b_2)-v(s_1,b_1)\leq t'(s_2)-t'(s_1).$$

By definition, $(t-t')(s_1)<r-2\varepsilon$ and $(t-t')(s_2)>r+\varepsilon$, and the three inequalities imply that

$$v(s_1,b_2)-v(s_1,b_1)<t(s_2)-t(s_1)-3\varepsilon.$$

Since $r(b_1)=t(s_1)+\varepsilon$ for every $s_1\in S(b_1)$, and $r(b_2)=t(s_2)-\varepsilon$ for every $s_2\in S(b_2)$ where $b_2\in B_2$, we obtain

$$v(s_1,b_2)-v(s_1,b_1)<r(b_2)-r(b_1)-\varepsilon,$$

or

$$\forall_{b_2\in B_2}\quad v(s_1,b_1)-r(b_1)>v(s_1,b_2)-r(b_2)+\varepsilon, \tag{11}$$

which means that $s_1\in V_1(\varepsilon)$.

Now consider $s_2\in S(b_2)$ where $b_2\in B_2$. Then, by incentive compatibility,

$$\forall_{b_1\in B_1}\ \forall_{s_1\in S(b_1)}\quad v(s_2,b_2)-t(s_2)\geq v(s_2,b_1)-t(s_1). \tag{12}$$

The left-hand side of this inequality is equal to

$$v(s_2,b_2)-t(s_2)=v(s_2,b_2)-r(b_2)-\varepsilon$$

and the right-hand side is equal to

$$v(s_2,b_1)-t(s_2)=v(s_2,b_1)-r(b_1)+\varepsilon>v(s_2,b_1)-r(b_1).$$

Thus, inequality (12) implies that

$$\forall_{b_1\in B_1} \quad v(s_2,b_2)-r(b_2)>v(s_2,b_1)-r(b_1)+\varepsilon, \tag{13}$$

which means that $s_2\in V_2(\varepsilon)$.

(ii) Suppose that there exist incentive compatible mechanisms (f,t) and (f,t') such that the difference $t-t'$ is not a constant function. Let

$$S(a)=f^{-1}(a)$$

for $a\in A$. By an argument similar to (i, $\Longleftarrow$), there exists a real number (say r) such that the values of $t-t'$ belong to the union of two intervals, $(-\infty,r)$ and (r,∞), and each of the two intervals contains at least one value.

Denote by B_1 the set of all alternatives b_1 such that $(t-t')(s)\in(-\infty,r)$ for $s\in S(b_1)$, and by B_2 the set of all alternatives b_2 such that $(t-t')(s)\in(r,\infty)$ for $s\in S(b_2)$. For every $b_1\in B_1$ take any $\varepsilon>0$ such that $2\varepsilon<r-(t-t')(s)$ for $s\in S(b_1)$, and define

$$r(b_1)=t(s)+\varepsilon.$$

Similarly, for every $b_2\in B_2$ take any $\varepsilon>0$ such that $\varepsilon<(t-t')(s)-r$ for $s\in S(b_2)$, and define

$$r(b_2)=t(s)-\varepsilon.$$

By an argument similar to (i, $\Longleftarrow$), if $s_1\in S(b_1)$ where $b_1\in B_1$ then (11) holds for ε corresponding to b_1; and if $s_2\in S(b_2)$ where $b_2\in B_2$ then (13) holds for ε corresponding to b_2. This yields

$$\bigcup_{b_1\in B_1} S(b_1)\subset V_1 \qquad \text{and} \qquad \bigcup_{b_2\in B_2} S(b_2)\subset V_2,$$

which in turn yields $S\subset V_1\cup V_2$ and $V_1\cap S\neq\varnothing\neq V_2\cap S$. □

Remark. The sufficient condition in part (ii) is stronger (for infinite sets A) than the necessary and sufficient condition in part (i), and part (ii) (for infinite A) does not hold under the weaker condition in (i).

There are several problems with modifying the proof of (i, $\Longrightarrow$) to obtain the converse of (ii). First, formula (9) does not define a SCF as there may be no $a\in B_1\cup B_2$ that maximizes $[v(s,a)-r(a)]$. But even if such an a exists for every s, formula (10) does not typically yield a t' such that (f,t') is incentive compatible (no matter what ε we pick), unless we assume the stronger condition in (i) instead of that in (ii). That formula does not guarantee that the agent of type $s\in V_i$ cannot profitably deviate by reporting another type s' that belongs to the same V_i.

We suspect that the necessary and sufficient condition for infinite A has to be much more complicated than the condition in Theorem 4.

The following argument demonstrates that the necessary and sufficient condition in Theorem 4 can be easily verified for the mechanism design problem described in Example 1 (and in practice, for all mechanism design problems with only two social alternatives).

EXAMPLE 1 CONTINUED. The condition in Theorem 4 is violated if there exist disjoint sets B_1, $B_2 \subsetneq A$, a function $r : B_1 \cup B_2 \to R$, and $\varepsilon > 0$ such that $S \subset V_1(\varepsilon) \cup V_2(\varepsilon)$ and $V_1(\varepsilon) \cap S \neq \varnothing \neq V_2(\varepsilon) \cap S$. Any such sets B_1, B_2 must be singletons. Therefore, letting $B_1 = \{b_1\}$, $B_2 = \{b_2\}$, and $r = r(b_1) - r(b_2)$,

$$V_1(\varepsilon) = \{(v_1, v_2) : v_1 - v_2 > r + \varepsilon\}$$

and

$$V_2(\varepsilon) = \{(v_1, v_2) : \ v_1 - v_2 < r - \varepsilon\}.$$

That is, $V_1(\varepsilon)$ consists of points lying below an ε-neighborhood of some line L whose slope is 1, and $V_2(\varepsilon)$ consists of points lying above an ε-neighborhood of the line L. It remains to observe, which is a simple geometric exercise, that such a line L does not exist for the set S in Example 1. However, if we slightly modify Example 1 so that the cost of performing task 2 is $c_2 = 2d + d \cdot p$, then such a line L exists. ◊

Theorem 4 yields the following corollary.

COROLLARY 3. *Suppose A is a set of cardinality lower than the continuum. If S is a connected subset of R^A, then every incentive compatible SCF $f : S \to A$ satisfies the revenue equivalence property.*

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