Theoretical Economics 3 (2008), 29–84 1555-7561/20080029

Repeated games with incomplete information on one side

Marcin Pęski
Department of Economics, University of Chicago

This paper studies repeated games with incomplete information on one side and equal discount factors for both players. The payoffs of the informed player I depend on one of two possible states of the world, which is known to her. The payoffs of the uninformed player U do not depend on the state of the world (that is, U knows his payoffs), but player I's behavior makes knowledge of the state of interest to player U. We define a finitely revealing equilibrium as a Bayesian perfect equilibrium where player I reveals information in a bounded number of periods. We define an ICR profile as a strategy profile in which (a) after each history the players have individually rational payoffs and (b) no type of player I wants to mimic the behavior of the other type. We show that when the players are patient, all Nash equilibrium payoffs in the repeated game can be approximated by payoffs in finitely revealing equilibria, which themselves approximate the set of all ICR payoffs. We provide a geometric characterization of the set of equilibrium payoffs, which can be used for computations.

Keywords. Repeated games, incomplete information, discounting.
JEL classification. C73.

1. Introduction

Many strategic situations involve long-run interactions in which there is uncertainty about payoffs. Aumann and Maschler (1995) (written 1966–68) introduce repeated games with incomplete information to model such situations. There are two players. The informed player (player I, she) knows which of two states of the world is true; the uninformed player (U, he) starts the repeated interaction with prior beliefs. Player I's, but not player U's, stage-game payoffs depend on the state of the world. It was understood very early that this model leads to novel strategic issues that cannot be adequately analyzed by focusing separately on either the uncertainty or the long-run aspect. These issues include questions of learning, strategic revelation of information, and reputation effects.

Marcin Pęski: mpeski@uchicago.edu
This paper is based on a chapter of my dissertation "Essays on Game Theory" defended at Northwestern Universtity, 2005. This project grew out of discussion with Gregory Pavlov. I am especially grateful to Martin W. Cripps, John Conlon, Jeff Ely, Wojciech Olszewski, Eddie Dekel, Eduardo Faingold, Sergiu Hart, seminar participants at Hebrew University and Washington University in St. Louis, Bart Lipman in his role as the editor, and two anonymous referees. All remaining errors are mine.

As a first step toward a complete analysis of equilibrium behavior, one may ask about equilibrium payoffs. This is the goal of the current paper. Suppose that the two players discount the future with the same discount factor δ, and that the stage-game payoffs satisfy a certain full-dimensionality condition. Then, if the players are sufficiently patient, all feasible payoffs that satisfy appropriate individual rationality and incentive compatibility conditions can be approximated by payoffs in sequential equilibria. The main result provides a geometric characterization for *the whole correspondence* of payoff sets for each initial prior.

The analysis is divided into three parts. The first part characterizes a lower bound on the set of payoffs in a simple subclass of finitely revealing equilibria. The second part constructs an upper bound on the set of payoffs in profiles that satisfy both incentive compatibility and individual rationality. The third part shows that these bounds are equal. Specifically, when players are patient, any payoff in a profile that satisfies the incentive compatibility and individual rationality conditions can be approximated by payoffs in Nash equilibria; furthermore, any payoff in a Nash equilibrium can be approximated by payoffs in finitely revealing equilibria. As a by-product of the proof, I obtain a geometric characterization of the equilibrium correspondence that can be used in applications.

The full-dimensionality condition implies that there is an open set of player I's payoffs such that for any degenerate prior $p \in \{0,1\}$, any payoff of I is attained in some equilibrium. The assumption allows for flexibility in choosing continuation payoffs, and its role and strength are comparable to the standard requirement in the folk theorem literature of a feasible payoff set with a non-empty interior (for example, Fudenberg et al. 1994).

I now describe the characterization in more detail. The major difficulty with repeated games with incomplete information is their lack of stationarity. The stage payoffs of player U depend on his beliefs, which change throughout the game. Some stationarity can be restored by focusing on *finitely revealing equilibria*, i.e. equilibria in which player I reveals information in finitely many periods. During periods when player I does not reveal information, the prior belief of player U does not change, and payoffs can be analyzed through the methods of dynamic programming from the literature on games with imperfect monitoring (see Abreu et al. 1990 and Fudenberg et al. 1994). Section 3 describes the lower bound on payoffs in finitely revealing equilibria, assuming full dimensionality; this in turn forms a lower bound on the set of payoffs in all equilibria. The main result shows that these two bounds are equal.

The reader might find such a result intuitive. In any equilibrium, the belief of player U is a martingale, and thus converges. This means that, with high probability, substantial amounts of information are revealed only finitely many times, and thus any equilibrium is "approximately" finitely revealing. However, this intuition does not easily turn into a proof, and in fact fails utterly in the no-discounting case, in which examples of equilibrium payoffs that cannot be approximated by finitely revealing profiles are well-known (see Aumann and Hart 1986).

An *ICR profile* is any (potentially mixed) strategy profile that satisfies two conditions:

IR The continuation payoffs of each player after each history are individually rational.

IC Each type of player I is indifferent between playing any pure strategy in the support of her own mixed strategy and weakly prefers any such strategy to any pure strategy in the support of the other type's mixed strategy.

The incentive compatibility condition *IC* ensures that neither type of player I wants to mimic the other. Any equilibrium profile is necessarily an ICR profile, but not the reverse; in an ICR profile, player U is not required to best-respond to player I, and there might be profitable deviations for some type of player I not in the support of I's strategy.

Section 4 characterizes the upper bound on the set of ICR profiles. This characterization is related to an idea in Fudenberg and Levine (1994), who show that the equilibrium payoffs in a game with complete information (but imperfect monitoring) cannot lie beyond a certain hyperplane. Here one could try to use hyperplanes to bound payoffs in a game that starts with some fixed prior p. However, in order to obtain a tight bound, one needs to control payoffs across all games starting with any prior $p \in [0,1]$ at the same time. For this purpose, I use biaffine functions, i.e., functions that are affine in the prior and payoffs separately. Biaffine functions are introduced in Aumann and Hart (1986) to study bimartingales, which are useful in games with no discounting.

The lower and upper bounds are stated as correspondences that assign payoff sets to prior beliefs $p \in [0,1]$. Characterizations in Sections 3 and 4 derive two classes of geometric constraints on the infinitesimal changes in payoffs with respect to the infinitesimal changes in prior beliefs. These constraints are different, but related. Using a certain "differential technique," I show that they can be satisfied by only one correspondence.

1.1 *Related literature*

The characterization of equilibrium payoffs is the goal of a large part of the repeated games literature. This field originated with Aumann et al. (1966–68) and initially concentrated on the no-discounting criterion. In that model, Hart (1985) shows that all feasible and individually rational payoffs that satisfy the incentive compatibility condition can be obtained in an equilibrium. Shalev (1994) and Koren (1992) present sharper results in the case of *known own payoffs*, where player U's payoffs do not depend on the state of the world.

So far, there have been no analogous results for games with discounting. The most advanced analysis, found in Cripps and Thomas (2003), looks at the limit correspondence of payoffs when the probability of one of the types is close to 1.[1] It is shown there that the set of payoffs of player U and the high probability type are close to the folk theorem payoffs in a complete information game. Cripps and Thomas (1997) and Chan (2000) ask the same question within the framework of reputation games. All these results are proved by the construction of finitely revealing equilibria.

[1] Cripps and Thomas (2003) discuss also the limit of payoff sets when the two players become infinitely patient, but player I becomes patient much more quickly than does player U. Their characterization is closely related to Shalev and Koren's results for the no-discounting case.

This paper extends Hart's result to discounting in the special case of two states of the world and known own payoffs. The main result indicates a curious relationship between incomplete information and long-run payoff criteria. In games with complete information, the sets of equilibrium payoffs in the no-discounting and discounting cases are equal, i.e., the folk theorem characterizations of Rubinstein (1979) and Fudenberg and Maskin (1986) coincide. Under incomplete information, the difference between the two cases is non-trivial: the set of equilibrium payoffs in the no-discounting case is included (typically strictly) in the set of equilibrium payoffs in the discounted case. Section 2.11 explains this difference by comparing the meaning of individual rationality in each case.

Cripps et al. (2005) study reputation effects in games with strictly conflicting interests. They provide an upper bound on the payoffs of player U and the normal type of player I. Although the authors do not state it in this way, their methods are very closely related to the derivation of the upper bound on ICR payoffs in the current paper. (Note that reputation games do not satisfy the full-dimensionality assumption; however, this assumption is not necessary for the upper bound on ICR payoffs.) In fact, biaffine functions lead to a simple argument for reputational effects, and they can be used to show that these effects are continuous with respect to games that have only close to strictly conflicting interests.

2. Model and main result

This section introduces the model and definitions, then states the main result.

2.1 *Notation*

For any $v \in \mathbb{R}^K$, let $\|v\|$ be the Euclidean length of v. Let $\Phi^d \subseteq \mathbb{R}^d$ be a set of all unitary vectors in $\mathbb{R}^d$: $\Phi^d = \{\phi \in \mathbb{R}^d : \|\phi\| = 1\}$. Let $\Phi^d_+ \subseteq \Phi^d$ be the subset of vectors with nonnegative coordinates: $\Phi^d_+ = \{\phi : \phi_k \geq 0\}$. I use the following set operators. For any set A, let ΔA denote the set of probability distributions on A. For any $A \subseteq \mathbb{R}^3$,

- $\operatorname{int} A$ denotes the interior of A (the largest open set contained in A)
- $\operatorname{cl} A$ denotes the closure of A (the smallest closed set containing A)
- $\operatorname{proj} A$ denotes the projection of A on its last two coordinates:

$$\operatorname{proj} A = \{(v_0, v_1) : \text{there is } v_U \text{ such that } (v_U, v_0, v_1) \in A\}$$

- $\operatorname{con} A$ denotes the convexification of A (the smallest convex set containing A).

2.2 *Repeated game*

Two players, uninformed U and informed I, repeatedly play a stage game. There are finite sets of pure actions S_U for player U and S_I for player I. Player I knows the state of the world $k \in \{0, 1\}$. I say that k is a type of player I and write $-k = 1 - k$. Player I's payoffs in the stage game depend on the state of the world and are given by $g_k : S_U \times S_I \to \mathbb{R}$.

Player U's payoffs do not depend on the state of the world (*known-payoff case*) and are given by $g_U : S_U \times S_I \to \mathbb{R}$. Let M denote the uniform bound on payoffs:

$$M = \max_{i=U,0,1} \max_{a_U \in S_U, a_I \in S_I} |g_i(a_U, a_I)|. \tag{1}$$

Players have access to a public randomization device (this assumption is for convenience only and can be dropped using standard arguments). Let $H_t = (S_U \times S_I \times [0,1])^t$ be the space of t-period *histories* of actions and public signals. A *(behavior) strategy of player U* is a mapping $\sigma_U : \cup_t H_t \to \Delta S_U$. A *(behavior) strategy of player I of type k* is a mapping $\sigma_k : \cup_t H_t \to \Delta S_U$. A *strategy profile* is a triple $\sigma = (\sigma_U, \sigma_0, \sigma_1)$.

Future payoffs are discounted by the factor $\delta < 1$, the same for each player. Let $\Gamma(p, \delta)$ denote the game with initial prior p and discount factor δ. Let $v_k(\sigma_U, \sigma_k)$ denote the repeated game payoff of type k when player U uses the strategy σ_U and player I uses σ_k. Let $v_U(\sigma_U, \sigma_k)$ denote the payoff of player U facing type k. The expected payoff of player U in the game with initial prior p is equal to $p v_U(\sigma_U, \sigma_1) + (1-p) v_U(\sigma_U, \sigma_0)$. Denote the vector of payoffs as follows:

$$v^p(\sigma_U, \sigma_0, \sigma_1) = \big(p v_U(\sigma_U, \sigma_1) + (1-p) v_U(\sigma_U, \sigma_0), v_0(\sigma_U, \sigma_0), v_1(\sigma_U, \sigma_1)\big).$$

2.3 *Nash equilibria*

A strategy profile $\sigma = (\sigma_U, \sigma_0, \sigma_1)$ is a *Nash equilibrium* if σ_U is a best response for player U and σ_k ($k = 0, 1$) is a best response for type k of player I. Denote the set of Nash equilibrium payoffs in the game $\Gamma(p, \delta)$ by

$$NE_\delta(p) = \{v^p(\sigma) : \sigma \text{ is a Nash equilibrium}\}.$$

The correspondence $NE_\delta : [0,1] \rightrightarrows \mathbb{R}^3$ can be treated as a subset of $[0,1] \times \mathbb{R}^3$. Define $NE \subseteq [0,1] \times \mathbb{R}^3$ by

$$NE = \bigcap_{\delta < 1} \text{cl} \bigcup_{\delta \le \delta' < 1} NE_{\delta'}.$$

The correspondence NE describes the largest reasonable definition of the limit set of equilibrium payoffs as $\delta \to 1$. Notice that NE is necessarily closed and can be treated as an upper hemi-continuous correspondence $NE : [0,1] \rightrightarrows \mathbb{R}^3$.

2.4 *Finitely revealing equilibria*

An *updating rule* is a mapping $p : \cup H_t \to [0,1]$, such that $p(\varnothing) = p$. The rule p is *consistent* with σ if, given any history h_t, beliefs are updated via Bayes' formula after any action $a_t \in S_I$ such that $\sigma_k(h_t)(a_t) > 0$ for some $k = 0, 1$. The rule p is *SR-consistent (consistent with support restriction)* if, additionally, U's beliefs never change after any history along which player I has already fully revealed her type (if $p(h_t) \in \{0,1\}$ then $p(h_t, h_s) = p(h_t)$ for any continuation history (h_t, h_s)).[2] A profile σ is *sequentially rational* given the updating rule p if, after any history h_t, the continuation strategies are best responses to the strategy of the opponent and beliefs $p(h_t)$.

[2] Madrigal et al. (1987) discuss various support restrictions in equilibria of extensive form games.

Fix strategies σ_0, σ_1 of player I and an updating rule p. For any history h_t, period t is a *period of revelation* if (a) there is uncertainty about the types of player I ($p(h_t) \notin \{0,1\}$) and (b) the types of player I play different (possibly mixed) actions at period t ($\sigma_0(h_t) \neq \sigma_1(h_t)$). Say that strategy σ_I is *K-revealing* for some K if there is an SR-consistent updating rule p such that for any t, along any history h_t the number of periods of revelation $t' < t$ is not greater than K.

For example, if $p_0 \in \{0,1\}$, then the beliefs remain constant along any path ($p_t(h_t) = p_0$), and any profile is 0-revealing. If there is initial uncertainty about the type of player I ($p_0 \notin \{0,1\}$), then in any 0-revealing profile σ, two types of player I play the same strategy along any path. The payoff in a 0-revealing σ belongs to the convex hull of the stage-game payoffs when player I's types play the same action:

$$v^p(\sigma) \in V = \operatorname{con}\{(g_U(a_U,a_I), g_0(a_U,a_I), g_1(a_U,a_I)) : a_U \in S_U \text{ and } a_I \in S_I\}. \tag{2}$$

The set $V \subseteq \mathbb{R}^3$ is called a set of *feasible non-revealing payoffs*. Note for further reference that V is spanned by finitely many vertices.

If player I follows a K-revealing strategy, it *does not* mean that she will stop revealing any information after K periods. In particular, a K-revealing strategy does not put any bound on the occurrence of the last period of revelation. Also, a K-revealing strategy does not require player I to reveal her information fully.

A profile $\sigma = (\sigma_U, \sigma_0, \sigma_1)$ is a *K-revealing equilibrium* if there is an updating rule p that is SR-consistent with σ, σ is sequentially rational given p, and (σ_0, σ_1) is K-revealing given p. Any *K-revealing equilibrium* satisfies the conditions for a sequential equilibrium in Kreps and Wilson (1982).[3] Denote the correspondence of payoffs in all finitely revealing equilibria in game $\Gamma(p,\delta)$ by

$$FE_\delta(p) = \bigcup_{K=0}^{\infty} \{v^p(\sigma) : \sigma \text{ is a } K\text{-revealing equilibrium}\}.$$

Define the limit correspondence FE as

$$FE = \bigcup_{\delta<1} \operatorname{int} \bigcap_{\delta \le \delta' < 1} FE_{\delta'}.$$

The correspondence FE consists of all interior equilibrium payoffs in finitely revealing equilibria for a sufficiently high discount factor δ. It is the smallest possible reasonable definition of the limit set of finitely revealing equilibrium payoffs as $\delta \to 1$. (Note that the definitions of NE and FE interchange intersection and union to obtain the largest and the smallest reasonable definitions.)

2.5 *Individual rationality*

Let $(\sigma_U, \sigma_0, \sigma_1)$ be a Nash equilibrium profile. The expected payoff of player U is not smaller than U's minmax value; similarly, the weighted average of the expected payoffs

[3] Strictly speaking, it satisfies an appropriate extension of Kreps and Wilson's (1982) conditions to games with infinitely many stages. See also the discussion in Mailath and Samuelson (2006).

of the types of player I is not smaller than the weighted minmax of player I:

$$p v_U(\sigma_U,\sigma_1)+(1-p)v_U(\sigma_U,\sigma_0)\geq m_U=\max_{\alpha_U\in\Delta S_U}\min_{\alpha_I\in\Delta S_I} g_U(\alpha_U,\alpha_I)$$

and

$$\phi_0 v_0(\sigma_U,\sigma_0)+\phi_1 v_1(\sigma_U,\sigma_1)\geq m_I(\phi)=\max_{\alpha_I\in\Delta S_I}\min_{\alpha_U\in\Delta S_U}\sum_{k=0,1}\phi_k g_k(\alpha_U,\alpha_I) \text{ for any } \phi\in\Phi^2_+.$$

For example, let $\phi^{k*}\in\Phi^2_+$ be such that $\phi^{k*}_k=1$ and $\phi^{k*}_{-k}=0$. Then $m_I(\phi^{k*})$ is the minmax payoff of type k in the complete information repeated game between player U and type k. This is standard (see Hart 1985 or Sorin 1999). Define the set of *individually rational* payoffs as

$$IR=\{(v_U,v_0,v_1): v_U\geq m_U \text{ and } \forall\phi\in\Phi^2_+,\ \phi\cdot(v_0,v_1)\geq m_I(\phi)\}. \tag{3}$$

Say that the profile σ satisfies *ex ante* individual rationality if $v^p(\sigma)\in IR$. Let $v^{p(h_t)}(\sigma|h_t)$ denote the vector of expected continuation payoffs after the history h_t given the strategy profile σ and an SR-consistent updating rule p. The profile σ satisfies the condition *IR* if

IR: $v^{p(h_t)}(\sigma|h_t)\in IR$ after any positive probability history h_t (where positive probability is with respect to the prior beliefs of player U).

The condition *IR* requires that individual rationality hold not only ex ante but also after any positive probability history. Of course, any equilibrium profile satisfies *IR*.

2.6 *Incentive compatibility*

Say that the profile σ satisfies the condition *IC* if

IC: Given the strategy of player U, each type k of I is indifferent between the pure strategies in the support of her own mixed strategy and weakly prefers any such pure strategy to any pure strategy in the support of the mixed strategy of type $-k$.

If the profile σ satisfies *IC*, then

$$v_0(\sigma_U,\sigma_0)\geq v_0(\sigma_U,\sigma_1) \text{ and } v_1(\sigma_U,\sigma_1)\geq v_1(\sigma_U,\sigma_0). \tag{4}$$

Recall the set V of feasible non-revealing payoffs defined in (2). Consider the correspondence

$$E^{F,IC}(p)=\{(pv^1_U+(1-p)v^0_U,v^0_0,v^1_1): v^0,v^1\in V \text{ such that } v^0_0\geq v^1_0 \text{ and } v^1_1\geq v^0_1\}. \tag{5}$$

Because of (4), if σ satisfies the *IC* condition, then $v^p_U(\sigma)\in E^{F,IC}(p)$. On the other hand, one can show that each vector $v\in \operatorname{int} E^{F,IC}(p)$ is the payoff in a profile that satisfies

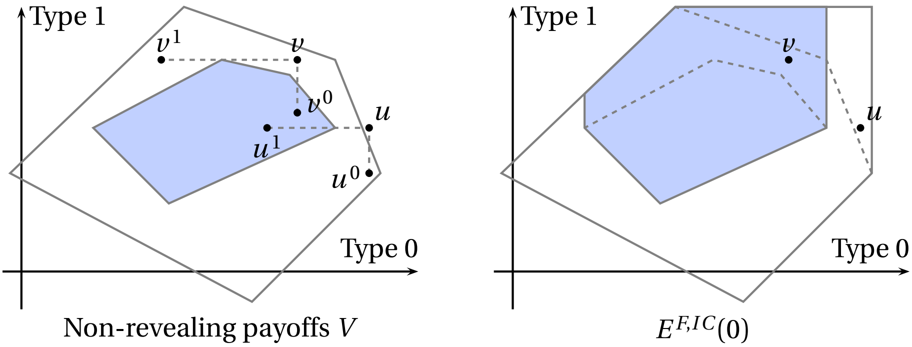

FIGURE 1. The construction of $E^{F,IC}(0)$. The shaded areas correspond to payoffs that are individually rational for player U ($v_U \geq m_U$).

the *IC* condition when δ is sufficiently high.[4] For this reason, I refer to $E^{F,IC}$ as the correspondence of *feasible and incentive compatible payoffs*.

Because the sets $E^{F,IC}(k)$, $k=0$, 1 are important later, it is helpful to develop some intuition into how they are constructed. Figure 1 presents the construction of the set $E^{F,IC}(0)$. The left-hand side shows an example of the set of non-revealing payoffs V. The solid line outlines the projection of V onto the payoffs of the player I types, proj V; the shaded area is the set of payoffs of I that are associated with individually rational payoffs of U, $\text{proj}\{v \in V : v_U \geq m_U\}$. Four payoff vectors $v^0, v^1, u^0, u^1 \in V$ are indicated. Notice that $v_1^1 \geq v_1^0$ and $v_0^0 \geq v_1^0$, hence $v = (v_U^0, v_0^0, v_1^1) \in E^{F,IC}(0)$. Similarly, $u = (u_U^0, u_0^0, u_1^1) \in E^{F,IC}(0)$. The right-hand side of Figure 1 presents $\text{proj}\, E^{F,IC}(0)$ (solid line) and $\text{proj}\{v \in E^{F,IC}(0) : v_U \geq m_U\}$ (shaded area). In particular, because (v_0^0, v_1^0) belongs to the shaded area on the left-hand side, (v_0, v_1) belongs to the shaded area on the right-hand side. On the other hand, because (u_0^0, u_1^0) does not belong to the shaded area on the left-hand side, (u_0, u_1) does not belong to the shaded area on the right-hand side.

2.7 *ICR profiles*

A strategy profile σ is an *ICR profile* if it satisfies the *IR* and *IC* conditions. Note that the continuation profile of an ICR profile after any positive probability history is also an ICR profile. Denote the set of payoffs in all ICR profiles in the game $\Gamma(p, \delta)$ and the limit

---

[4]Because $v \in \text{int}\, E^{F,IC}(p)$, there exist $v^0, v^1 \in \text{int}\, V$ such that $v = (pv_U^1 + (1-p)v_U^0, v_0^0, v_1^1)$, $v_0^0 \geq v_0^1$, and $v_1^1 \geq v_1^0$. Fix any two actions $a_0^*, a_1^* \in S_I$ such that $a_0^* \neq a_1^*$. Fix an action $a_U^* \in S_U$. Define

$$\tilde{v}^k = \frac{1}{\delta}\left(v^k - (1-\delta)(g_U(a_U^*, a_k^*), g_0(a_U^*, a_k^*), g_1(a_U^*, a_k^*))\right).$$

For sufficiently high δ, $\tilde{v}^k \in V$. Find strategy profiles σ^k such that $v^k(\sigma^k) = \tilde{v}^k$ for $k = 0, 1$. (Such profiles exist because $\tilde{v}^k \in V$; public randomization may need to be used.) Construct a strategy profile σ^* such that $\sigma^*(\varnothing) = (a_U^*, a_0^*, a_1^*)$, $\sigma^*(a_U, a_0^*) = \sigma^0$, and $\sigma^*(a_U, a_I) = \sigma^1$ for any a_U and any $a_I \neq a_0^*$. Then σ^* satisfies the *IC* condition and $v^p(\sigma^*) = v$.

correspondence by

$$ICR_\delta(p)=\{v^p(\sigma):\sigma \text{ is an ICR profile}\}$$
$$ICR=\bigcap_{\delta<1}\text{cl}\bigcup_{\delta\le\delta'<1} ICR_{\delta'}.$$

Any equilibrium profile is necessarily an ICR profile, but not the reverse. In an ICR profile, player U is not required to best-respond to player I, and there might be profitable deviations for each of the types of player I as long as they do not belong to the supports of the mixed strategies of the two types.

2.8 *Feasible, incentive compatible and ex ante individually rational payoffs*

Let

$$E_p=IR\cap E^{F,IC}(p). \tag{6}$$

Here, E_p is the set of feasible, incentive compatible, and *ex ante* individually rational payoffs. As a corollary to the previous sections, we obtain the following result.

Corollary 1. *For any* $\delta<1$ *and any* $p\in[0,1]$, $NE_\delta(p)\subseteq ICR_\delta(p)\subseteq E_p$.

In general, the last inclusion is strict. This is because the *IR* condition requires individual rationality after any positive probability history, and that is, typically, more restrictive than ex ante individual rationality. However, when the prior $p=0,1$ is degenerate, Proposition 1 in Section 3 shows that $\text{int}\,E_p\subseteq NE(p)$. In particular, E_p has a non-empty interior if and only if $NE(p)$ has a non-empty interior; in such a situation, $E_p=NE(k)$. (An analogous result holds in the no-discounting case; see Hart 1985.) [5]

Figure 2 presents the steps in the construction of the sets E_k, $k=0,1$. The left-hand side presents the projections proj $E^{F,IC}(k)$, $k=0,1$ on the sets of player I's payoffs (compare with Figure 1). The shaded areas correspond to player U's individually rational payoffs and the thick line bounds the payoffs that are individually rational for player I. By definition, E_k is the set of these payoffs in $E^{F,IC}(k)$ that are individually rational for the two players. The projections of the sets E_k are depicted as the shaded areas in the central part of the figure.

2.9 *Full-dimensionality*

Assumption 1 (Full-dimensionality). proj int $E_0\cap$ proj int $E_1\neq\emptyset$.

[5] One can also check that

$$\{(v_U,v_k):(v_U,v_k,v_{-k})\in E_k\}$$
$$=\text{con}\{(g_U(a_U,a_I),g_k(a_U,a_I)):a_U\in S_U \text{ and } a_I\in S_I\}\cap\{(v_U,v_k):v_U\geq m_U \text{ and } v_k\geq m_I(\phi^{k*})\}.$$

Thus, the set of payoffs of player U and type k in the incomplete information game with degenerate prior $p=k$ is equal to the set of feasible and individually rational payoffs of players U and I in the complete information game between player U and type k of player I. Not surprisingly, the above result implies the complete information folk theorem.

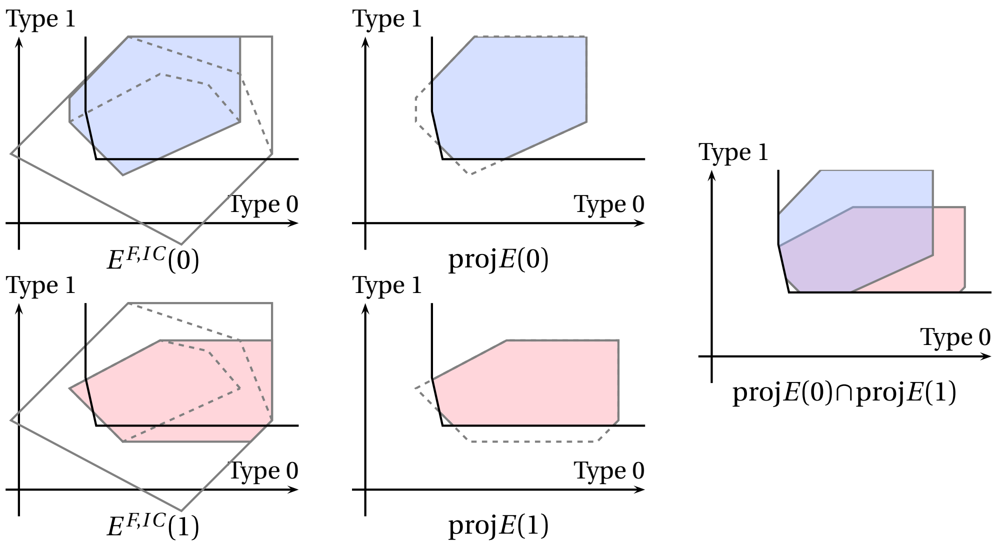

Figure 2. The construction of the sets E_0 and E_1. The thick lines delineate the sets of player I's individually rational payoffs. The shaded areas correspond to payoffs that are individually rational for player U ($v_U \geq m_U$).

See the right-hand side graph of Figure 2. Full-dimensionality says that there are two interior payoff vectors $v^0 \in \operatorname{int} E_0$ and $v^1 \in \operatorname{int} E_1$ such that the payoffs of each type of player I are equal, $v_k^0 = v_k^1$ for $k = 0, 1$. The assumption relates directly to the standard requirement of full dimensionality from the folk theorem literature (for example, Fudenberg et al. 1994). The next result says that if one drops the "int" from the assumption, then it is always satisfied. The role of the assumption is explained in Section 3.3.

Lemma 1. $\operatorname{proj} E_0 \cap \operatorname{proj} E_1 \neq \varnothing$.

Proof. For each k, choose a payoff vector $v^k \in V$ that maximizes the payoff of type k of player I among all payoff vectors in V that yield individually rational payoffs to player U:

$$v^k \in \arg \max_{v \in V, v_U \geq m_U} v_k.$$

Let $\alpha_U^* \in \Delta S_U$ be the minmax action of player U, i.e., the mixed action that guarantees him at least m_U. Then, for $k = 0, 1$,

$$v_k^k \geq \max_{\alpha_I \in \Delta S_I} g_k(\alpha_U^*, \alpha_I) \text{ and } v_k^k \geq v_k^{-k}. \tag{7}$$

For $k = 0, 1$, define $u^k = (v_U^k, v_0^0, v_1^1)$. Because of (7), $u^k \in E^{F,IC}(k)$. I now check that $u^k \in IR$. For any $\phi \in \Phi_+^2$,

$$m_I(\phi) \leq \max_{\alpha_I \in \Delta S_I} \sum_{k=0,1} \phi_k g_k(\alpha_U^*, \alpha_I) \leq \sum_{k=0,1} \max_{\alpha_I^k} \phi_k g_k(\alpha_U^*, \alpha_I) \leq \sum_{k=0,1} \phi_k v_k^k.$$

Thus $u^k \in E_k$ and $(v_0^0, v_1^1) \in \operatorname{proj} E_k$ for $k = 0, 1$. This yields the lemma. □

2.10 *Main result*

So far, it is clear that $FE \subseteq NE \subseteq ICR$. The main result of this paper shows that the inclusions can be replaced by equalities.

Theorem 1. *If Assumption 1 holds, then*

$$\operatorname{cl} FE = NE = ICR.$$

In other words, all Nash equilibrium payoffs can be approximated by payoffs in finitely revealing equilibria, and the limit set of all Nash equilibrium payoffs is equal to the limit set of payoffs in ICR profiles.

Theorem 1 is proved in parts. Section 3 characterizes a lower bound $FE^* \subseteq \operatorname{cl} FE$. Section 4 characterizes an upper bound $ICR^* \supseteq ICR$. Section 5 shows that the two bounds are equal. Assumption 1 is used in the first and the third parts, but is not necessary for the discussion of the upper bound in Section 4.

2.11 *Individual rationality and the no-discounting case*

It is useful to compare the main result to the no-discounting case. Formally, define the payoffs as Banach limits of the finite period averages of stage-game payoffs (see Hart 1985). Nash equilibrium profiles are defined in the standard way. Let $NE_{\text{nd}}(p) \subseteq \mathbb{R}^3$ denote the set of payoffs in Nash equilibria. (An interested reader is encouraged to look at the excellent survey of all related methods in Sorin 1999.)

Theorem 2 (Shalev 1994, Koren 1992). *For any* $k = 0, 1$,

$$NE_{nd}(k) = E_k.$$

For any $p \in (0,1)$,

$$NE_{nd}(p) = \{((1-p)v_U^0 + p v_U^1, v_0, v_1) : (v_U^k, v_0, v_1) \in E_k, k = 0, 1\}.$$

In particular, for any $p \in (0,1)$

$$\operatorname{proj} NE_{nd}(p) = \operatorname{proj} E_0 \cap \operatorname{proj} E_1.$$

The theorem has a simple interpretation: any equilibrium payoff can be obtained in a 1-revealing equilibrium in which player I immediately reveals all her information and subsequently the players play the complete information game. It is shown below (Proposition 4) that, if Assumption 1 holds, then

$$NE_{\text{nd}}(p) \subseteq \operatorname{cl} FE \subseteq NE.$$

Thus, the payoffs in the no-discounting case are contained in the set of payoffs in the discounted case. Typically, the inclusion is strict.

The discrepancy between the two cases can be attributed to the restrictiveness of individual rationality.[6] Consider the following condition:

[6] I am grateful to Martin W. Cripps for suggesting this connection.

IR-in-every-state: The profile σ satisfies *IR* and after any positive probability history h_t, $v_U(\sigma_U, \sigma_k | h_t) \geq m_U$ for any type k of player I.

This condition says that the continuation payoffs of player U are individually rational after all positive probability histories, conditional on each state of the world. In the no-discounting case, the conditions *IR* and *IR-in-every-state* are equivalent. The intuition behind this fact is very simple. If $v_U^{\text{nd}}(\sigma_U, \sigma_k) < m_U < v_U^{\text{nd}}(\sigma_U, \sigma_{-k})$, then the two types of player I play substantially different mixed actions in infinitely many periods. Thus, player U learns the type of player I in finite time, with a probability arbitrarily close to 1. Upon learning that player I has type k, player U should play a best response, which guarantees him a payoff of at least m_U. Since the payoffs received in finitely many periods do not matter, it must be that $v_U^{\text{nd}}(\sigma_U, \sigma_k) \geq m_U$. This yields a contradiction.

In the discounted case, *IR-in-every-state* implies *IR*, but *IR* is typically weaker. To see why *IR* does not imply *IR-in-every-state*, notice that, in the discounted case, the payoffs in each period matter. It may happen that player U agrees on a low-payoff action today only because he counts on a reward tomorrow if player I turns out to be of type k. If today's action yields a payoff lower than m_U and tomorrow's continuation payoff given type $-k$ is equal to m_U, then today's discounted payoff given type $-k$ is lower than m_U.

Because the condition *IR* is usually less restrictive than *IR-in-every-state* in the discounted case, one should expect that a larger set of profiles can be sustained as equilibria in the discounted case.

3. Lower bound—finitely revealing equilibria

In this section, I characterize the sets of payoffs in finitely revealing equilibria. It is convenient to consider separately equilibria in which player I begins with a non-revealing strategy (Section 3.1) and when she reveals some information (Section 3.2). In the last part, I construct a lower bound FE^* on the correspondence of finitely revealing payoffs *FE*.

3.1 *Non-revealing strategies*

Let v be the payoff in an equilibrium of the game $\Gamma(p, \delta)$ and let $v(a_U, a_I)$ be the continuation payoffs. Let α_U^* and α_k^* denote the first period mixed action of player U and type k of player I. Denote also the expected action of player I by $\alpha_I^* = p\alpha_1^* + (1-p)\alpha_0^*$. Then, by the definition of continuation payoffs,

$$\begin{aligned} v_U &= (1-\delta) g_U(\alpha_U^*, \alpha_I^*) + \delta v_U(\alpha_U^*, \alpha_I^*), \\ v_k &= (1-\delta) g_k(\alpha_U^*, \alpha_k^*) + \delta v_k(\alpha_U^*, \alpha_k^*) \text{ for } k = 0, 1. \end{aligned} \tag{8}$$

Because v is an equilibrium payoff, incentive compatibility must hold: for any a_U, a_0, and a_1,

$$\begin{aligned} v_U &\geq (1-\delta) g_U(a_U, \alpha_I^*) + \delta v_U(a_U, \alpha_I^*) \\ v_k &\geq (1-\delta) g_j(\alpha_U^*, a_k) + \delta v_k(\alpha_U^*, a_k) \text{ for } k = 0, 1. \end{aligned} \tag{9}$$

If player I does not reveal any information in the first period, then the posterior beliefs of player U do not change. Thus, $v(a_U, a_I)$ are equilibrium payoffs in the game with the same initial prior. Such payoffs can be analyzed using the self-generation technique of Abreu et al. (1990) that is further extended in Fudenberg et al. (1994), Fudenberg and Levine (1994), and Kandori and Matsushima (1998). It is convenient to solve the following problem. Recall that $\Phi \in \mathbb{R}^3$ is the space of unit vectors in $\mathbb{R}^3$, and fix $\phi \in \Phi$. What is the highest possible value of $\phi \cdot v$ if, after any realized action profile (a_U, a_I), the continuation payoffs lie below a hyperplane ϕ that passes through v,

$$\phi \cdot v(a_U, a_I) \leq \phi \cdot v? \tag{10}$$

It cannot be larger than

$$\begin{aligned} \eta(\phi, p) = \max_v \phi \cdot v \text{ such that there are mixed actions } \alpha_U^* \in \Delta S_U, \alpha_0^*, \alpha_1^* \in \Delta S_I \text{ and} \\ \text{continuation payoffs } v : S_U \times S_I \to \mathbb{R}^3 \text{ so that} \\ \text{(1) equations (8) hold} \\ \text{(2) the incentive compatibility inequalities (9) hold} \\ \text{(3) payoff corrections are separated from the origin by} \\ \text{the hyperplane } \phi\text{: inequalities (10) hold} \\ \text{(4) player } I \text{ plays non-revealing strategies: if } p \in (0,1), \text{ then } \alpha_0^* = \alpha_1^*. \end{aligned} \tag{11}$$

There are two differences between the way that problem (11) is formulated and the literature on games with imperfect monitoring. First, condition (4) requires that two types of player I play the same strategies when the prior p is nondegenerate. This ensures that the actions of player I are non-revealing. Second, the continuation payoffs depend only on the realized action of player I and not on the actions played by the two types separately. However, the reader should not expect these to cause any major difficulty. The first simply imposes a constraint on the set of available strategies; the second is dealt with in a way analogous to games with imperfect monitoring that fail identifiability (see, for example, Fudenberg and Levine 1994).

The next two results are proved in Section A of the Appendix.

PROPOSITION 1. *For $k = 0, 1$,*

$$\operatorname{int} E_k \subseteq \operatorname{int} \bigcap_{\phi \in \Phi} \{v : \phi \cdot v \leq \eta(\phi, k)\} \subseteq FE(k).$$

This result constructs equilibria in which player U believes that he faces type k and $\phi \cdot v \leq \eta(\phi, k)$ for each continuation payoff v and each ϕ.

PROPOSITION 2. *Fix $p \in (0,1)$ and set $A \subseteq FE(p)$ such that $A = \operatorname{cl} \operatorname{int} A$. Then*

$$\operatorname{int}[\operatorname{con}(A \cup V) \cap IR] \subseteq \operatorname{int} \bigcap_{\phi \in \Phi} \left\{v : \phi \cdot v \leq \max\left[\max_{v' \in A} \phi \cdot v', \eta(\phi, p)\right]\right\} \subseteq FE(p). \tag{12}$$

This result constructs equilibria in which player I plays non-revealing strategies until the continuation payoffs fall into the set A. After that, players follow the equilibrium strategies associated with payoffs in A. If A consists of finitely revealing payoffs, then such profiles are also finitely revealing equilibria. In any such equilibrium, the payoff vector is equal to the convex combination of some $v \in V$ and $a \in A$ where the weight on v depends on the expected time it takes to push the continuation payoff into A.

As mentioned above (Section 2.9), Proposition 1 has an exact equivalent in the no-discounting literature. The situation is only slightly different with Proposition 2. When $p \in (0,1)$, Hart (1985) shows that the set of equilibrium payoffs is convex and contained in IR. That result looks like equation (12), but with V dropped. The difference is easy to explain: in the discounted case, periods in which player I plays non-revealing actions contribute to the total discounted payoffs.

3.2 *Revelation of information*

I now show how to construct payoffs in equilibria that begin with the revelation of information. Take any $p_0 < p_1$ and subsets of finitely revealing payoffs $A_{p_j} \subseteq FE(p_j)$ for $j = 0,1$. For any $p \in (p_0, p_1)$, define the set

$$A_p = \left\{ \left(\frac{p - p_0}{p_1 - p_0} v_U^1 + \frac{p_1 - p}{p_1 - p_0} v_U^0, v_0, v_1 \right) : (v_U^j, v_0, v_1) \in A_{p_j} \text{ for } j = 0,1 \right\}.$$

If the sets A_{p_j} are open and convex, then A_p is open and convex.

Proposition 3. *If the sets A_{p_j} are open and convex, then $A_p \subseteq FE(p)$ for any $p \in (p_0, p_1)$.*

Proof. Assume that the set A_p is non-empty (otherwise there is nothing to prove). Take $v = (v_U, v_0, v_1) \in A_p$. By the definition of A_{p_j}, there are payoff vectors

$$v^j = (v_U^j, v_0, v_1) \in A_{p_j} \text{ for } j = 0,1$$

such that

$$(v_U, v_0, v_1) = \left(\frac{p - p_0}{p_1 - p_0} v_U^1 + \frac{p_1 - p}{p_1 - p_0} v_U^0, v_0, v_1 \right).$$

Because sets A_{p_0} and A_{p_1} are open and convex, and because of the definition of the correspondence FE, there exist δ_0 and $\varepsilon > 0$ such that $FE_\delta(p)$ contains a ball with center at v^j and radius 2ε: $B(v^j, 2\varepsilon) \subseteq FE_\delta(p)$ for $\delta \geq \delta_0$. Assume that δ_0 is high enough that $(1 - \delta_0)M \leq \varepsilon$. (Recall that M is defined in (1) as the uniform bound on stage game payoffs.) I show that for all $\delta \geq \delta_0$, $B(v, \varepsilon) \subseteq FE_\delta(p)$.

Find a profile of mixed actions of player I, (α_0, α_1), so that each type randomizes between all actions in such a way that the posterior after every action is equal to either p_0 or p_1. Denote by S_I^j the set of actions after which the posterior is equal to p_j. Then $S_I = S_I^0 \cup S_I^1$. Take any mixed action $\alpha_U \in \Delta S_U$ of player U. Construct continuation payoffs $v : S_U \times S_I \rightarrow \mathbb{R}^3$ as follows. For $a_I \in S_I^j$,

$$v_k(a_U, a_I) = \frac{1}{\delta}\left[v_k - (1-\delta')g_k(\alpha_U, a_I)\right] \text{ for any } k = 0, 1$$
$$v_U(a_U, a_I) = \frac{1}{\delta}\left[v_U^k - (1-\delta')g_U(a_U, p\alpha_1 + (1-p)\alpha_0)\right].$$

Then $v(a_U, a_I) \in B(v^j, 2\varepsilon)$, so there exist finitely revealing continuation equilibria with payoffs $v(a_U, a_I)$. Hence v is a payoff in a finitely revealing equilibrium. □

3.3 *Finitely revealing correspondence*

Let $\mathscr{F}$ be the collection of all correspondences $F \subseteq [0,1] \times \mathbb{R}^3$ such that $F(p) \in \mathbb{R}^3$ is closed for each $p \in [0,1]$ and F satisfies the following properties.

FE-1 For any $\alpha \in [0,1]$, $p_0 < p_1$, and $(v_U^k, v_0, v_1) \in F(p_k)$ for $k = 0, 1$,

$$\left(\alpha v_U^1 + (1-\alpha)v_U^0, v_0, v_1\right) \in F(\alpha p_1 + (1-\alpha)p_0).$$

FE-2 $E_k \subseteq F(k)$ for $k = 0, 1$.

FE-3 $\text{con}(F(p) \cup V) \cap IR \subseteq F(p)$ for $p \in (0,1)$.

All three properties correspond to the propositions above: FE-1 says that F contains all payoffs in equilibria that start with revealing information and later continue with payoffs in F; FE-2 insures that F contains all equilibrium payoffs in states 0 and 1; and FE-3 says that F contains all payoffs in profiles that start with a non-revealing action and later continue with payoffs in F.

Proposition 4. *The correspondence $FE^* = \bigcap_{F \in \mathscr{F}} F(p)$ is closed and $FE^* \in \mathscr{F}$, i.e., it satisfies properties FE-1, FE-2, and FE-3. Moreover, if Assumption 1 holds, then*

$$FE^*(p) \subseteq \text{cl}\, FE(p) \textit{ for any } p \in [0,1].$$

Proof. The fact that $FE^* \in \mathscr{F}$ is immediate. In Section A.4 of the Appendix I show that FE^* is closed. For any $p \in [0,1]$, define

$$FE'(p) = \text{cl}\,\text{int}\, FE(p).$$

In the second part of Section A.4, I show that $FE' \in \mathscr{F}$. By the definition of FE^*, $FE^* \subseteq FE'$. This completes the proof. □

The correspondence FE^* is a lower bound on the set of finitely revealing equilibrium payoffs. This bound is characterized purely in geometric terms. Notice that the set FE^* is a function of the sets V and IR. Therefore any pair of stage games that have the same minmax values and the same convex hull of non-revealing payoffs generate the same sets FE^*.

This is a good place to explain the role of Assumption 1. The assumption implies that there are two open and convex sets $A_k \subseteq E_k$, $k=0,1$, such that

$$\operatorname{proj} A_0 = \{(v_0, v_1) : (v_U^k, v_0, v_1) \in A_k \text{ for } k=0,1\} = \operatorname{proj} A_1.$$

By Proposition 1, sets A_k are contained in the sets of equilibrium payoffs for $p=k$. For each $p \in (0,1)$, let A_p be defined as in the statement of Proposition 3. Then, A_p is non-empty, open, and, by the above result, contained in $FE^*(p) \subseteq \operatorname{cl} FE(p)$. (There is a natural interpretation of the payoffs in A_p as those obtained by a single full revelation of I's type.) Hence, the assumption guarantees that for each p, the set $FE^*(p) \subseteq FE(p) \subseteq NE(p)$ has a non-empty interior. This plays the same role as the standard full-dimensionality assumption in the complete information case. Open sets give enough room to construct continuation payoffs with appropriate incentives.

4. Upper bound—ICR profiles

This section develops tools to bound the set of ICR payoffs. By Corollary 1, $ICR(k) \subseteq E_k$ for any $k=0,1$.

4.1 *Separation with biaffine functions*

The function $l : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ is *biaffine* if for any $\alpha \in \mathbb{R}$, any $p, p' \in [0,1]$, and any $(v_0, v_1), (v_0', v_1') \in \mathbb{R}^2$,

$$l(\alpha p + (1-\alpha)p', v_0, v_1) = \alpha l(p, v) + (1-\alpha) l(p', v_0, v_1)$$
$$l(p, \alpha(v_0, v_1) + (1-\alpha)(v_0', v_1')) = \alpha l(p, v_0, v_1) + (1-\alpha) l(p, v_0', v_1').$$

Proposition 5. *Take any biaffine $l : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ such that for any $k=0,1$ and any $(v_U^k, v_0^k, v_1^k) \in E_k \cup V$,*

$$v_U^k < l(k, v_0^k, v_1^k). \tag{13}$$

Then, for any $\varepsilon > 0$, there is $\delta_0 < 1$ such that for all $\delta \geq \delta_0$, for any $p \in (0,1)$ and $v = (v_U, v_0, v_1) \in ICR_\delta(p)$,

$$v_U \leq l(p, v_0, v_1) + \varepsilon.$$

Moreover, if the inequality in (13) is reversed, then the following statement continues to hold with the inequality also reversed.

Take a biaffine function l that is above all ICR payoffs for degenerate priors, $k \in \{0,1\}$ (by Corollary 1, these payoffs are contained in the set E_k) *and* above the set of non-revealing payoffs V (even those non-revealing payoffs that are not individually rational). The proposition says that it is also above all ICR payoffs for any prior $p \in [0,1]$.[7]

[7]Biaffine functions (or more generally, biconvex functions) were introduced in Aumann and Hart (1986) to analyze bimartingales, which Hart (1985) used in his characterization of the set of equilibrium payoffs in the no-discounting case. In the no-discounting case, the thesis of Proposition 5 holds for any biaffine funcion l such that for any $k=0,1$ and any $(v_U^k, v_0^k, v_1^k) \in E$, (13) is satisfied. In particular, it is not required that l lie above (below) the set of non-revealing payoffs V. The proof in the no-discounting case is shorter due to the bimartingale property.

Suppose that the thesis of the proposition does not hold, and that v is a payoff in an ICR profile such that $v_U - l(p, v_0, v_1) \geq \varepsilon > 0$. In the proof, I propose to measure the information revealed by player I's first period action by a specific function denoted here as *Info*. In particular, if, in the first period, player I plays a non-revealing mixed action, then $\mathit{Info} = 0$. The proof of the proposition shows that there exists a continuation payoff vector $v(a_U, a_I)$ such that

$$v_U(a_U, a_I) - l(p, v_0(a_U, a_I), v_1 v_0(a_U, a_I)) \geq v_U - l(p, v_0, v_1) + O(\varepsilon - \mathit{Info}). \tag{14}$$

Here, $O(\varepsilon - \mathit{Info})$ is not smaller than a term proportional to $\varepsilon - \mathit{Info}$. To see some intuition, suppose that $\mathit{Info} = 0$, that is, player I plays a non-revealing action and the vector of first period payoffs belongs to V. Recall that v is a convex combination of the first period payoffs and the payoffs in the continuation ICR profiles in a game with the same prior p. Because the first-period payoffs belong to V and, by assumption, lie below the biaffine function l, there must be at least one continuation payoff vector $v(a_U, a_I)$ that lies further from the biaffine function l than the original payoff vector. In fact, I can bound how much U's continuation payoff moves away from l by a term that is proportional to the distance between the original payoff of U and l, which is of order ε (with a coefficient of proportionality of order $(1-\delta)$.)

Because of (14), the distance between U's continuation payoff and the biaffine function l decreases only when player I reveals a substantial amount of information, and increases otherwise. However, information cannot be revealed indefinitely—at a certain moment, player U learns the type of player I, and there is nothing more to reveal. But this means that the distance between U's payoffs and the biaffine function l must grow to infinity. This leads to a contradiction. Hence, it cannot be that $v_U - l(p, v_0, v_1) \geq \varepsilon$. The proof of the proposition develops this intuition formally.

Proof of Proposition 5. Take any $(p, v) \in ICR_\delta$ such that $p \notin \{0, 1\}$. Let $\sigma = (\sigma_U, \sigma_0, \sigma_1)$ be a strategy profile supporting v as an ICR payoff in the game $\Gamma(p, \delta)$. Let $\alpha_U = \sigma_U(\emptyset)$ be the first period mixed action of player U, and $\alpha_k = \sigma_k(\emptyset)$ be the first period mixed action of type k player I. Define the average mixed action used by player I:

$$\alpha_I = p\alpha_1 + (1-p)\alpha_0.$$

Let $S_I^* = \operatorname{supp} \alpha_I = \operatorname{supp} \alpha_0 \cup \operatorname{supp} \alpha_1$ be the set of actions that player I plays with positive probability. After any action $a_I \in S_I^*$, player U updates his prior about the state of the world using Bayes' formula:

$$p(a_I) = \frac{p\alpha_1(a_I)}{\alpha_I(a_I)}.$$

For any positive probability pair of actions $a_U \in \operatorname{supp} \alpha_U$ and $a_I \in S_I^*$, the continuation payoffs $v(a_U, a_I)$ are payoffs in an ICR profile. Let $w(a_U, a_I) \in \mathbb{R}^2$ denote the continuation payoffs of the two types of I that would make them indifferent across all actions in S_I^*:

$$w_k(a_U, a_I) = v_k(a_U, a_I) + \frac{1}{\delta}[v_k - (1-\delta) g_k(\alpha_U, a_I) - \delta v_k(\alpha_U, a_I)]. \tag{15}$$

If type k of player I plays action a_I with positive probability (i.e., $a_I \in \operatorname{supp} \alpha_k$), then $v_k(\alpha_U, a_I) = w_k(\alpha_U, a_I)$; otherwise, $v_k(\alpha_U, a_I) \leq w_k(\alpha_U, a_I)$. This follows from the *IC* condition.

Suppose that the hypothesis of the proposition is satisfied, i.e., for any $k = 0, 1$ and any $(v_U^k, v_0^k, v_1^k) \in E_k \cup V$, $v_U^k < l(k, v_0^k, v_1^k)$. (The case of the reverse inequality is completely analogous.) I relegate two technical steps to the appendix. Section B.1 shows that when δ is sufficiently high, then, for any a_I such that $p(a_I) \in \{0, 1\}$,

$$v_U(\alpha_U, a_I) - l(p(a_I), w(\alpha_U, a_I)) < 0. \tag{16}$$

Section B.2 finds a constant $C < \infty$ such that, if

$$v_U - l(p, v_0, v_1) \geq \frac{\varepsilon}{2},$$

then

$$\sum_{a_I \in S_I^*} \alpha_I(a_I)\Big[v_U(\alpha_U, a_I) - \Big(l(p(a_I), w(\alpha_U, a_I)) - \frac{1-\delta}{\delta}\frac{C}{\varepsilon}(p(a_I))^2\Big)\Big] > v_U - \Big(l(p, v_0, v_1) - \frac{1-\delta}{\delta}\frac{C}{\varepsilon}p^2\Big) + \frac{1-\delta}{4\delta}\varepsilon. \tag{17}$$

The constant C depends on the payoffs in the stage game and on the biaffine function l, but not on ε or δ.

Fix $\varepsilon > 0$ and choose δ high enough that $((1-\delta)/\delta)(C/\varepsilon) \leq \frac{1}{4}\varepsilon$. Choose $(p^*, v^*) \in ICR_\delta$ so that

$$\Big[v_U^* - \Big(l(p^*, v_0^*, v_1^*) - \frac{1-\delta}{\delta}\frac{C}{\varepsilon}(p^*)^2\Big)\Big] \geq \sup_{(p,v)\in ICR_\delta} \Big[v_U - \Big(l(p, v_0, v_1) - \frac{1-\delta}{\delta}\frac{C}{\varepsilon}p^2\Big)\Big] - \frac{1-\delta}{8\delta}\varepsilon.$$

Suppose that $v_U^* - l(p^*, v_0^*, v_1^*) \geq \frac{1}{2}\varepsilon$. Because of (17), there is an action $a_I \in S_I^*$ such that

$$v_U^*(\alpha_U, a_I) - \Big(l(p(a_I), w^*(\alpha_U, a_I)) - \frac{1-\delta}{\delta}\frac{C}{\varepsilon}(p(a_I))^2\Big) > v_U^* - \Big(l(p^*, v_0^*, v_1^*) - \frac{1-\delta}{\delta}\frac{C}{\varepsilon}(p^*)^2\Big) + \frac{1-\delta}{4\delta}\varepsilon. \tag{18}$$

Because

$$\frac{1-\delta}{\delta}\frac{C}{\varepsilon}(p(a_I))^2 - \frac{1-\delta}{\delta}\frac{C}{\varepsilon}(p^*)^2 \leq \frac{1-\delta}{\delta}\frac{C}{\varepsilon} \leq \frac{\varepsilon}{4},$$

it must be that

$$v_U^*(\alpha_U, a_I) - l(p(a_I), w^*(\alpha_U, a_I)) > \frac{\varepsilon}{2} - \frac{\varepsilon}{4} > 0.$$

Hence, by (16), $p(a_I) \in (0, 1)$ and $w^*(\alpha_U, a_I) = (v_0^*(\alpha_U, a_I), v_1^*(\alpha_U, a_I))$. Together with (18), this means that there exists an action $a_U \in S_U$ such that $v_U^*(a_U, a_I)$ is a payoff in an ICR profile and

$$v_U^*(a_U, a_I) - \Big(l(p(a_I), v_0^*(a_U, a_I), v_1^*(a_U, a_I)) - \frac{1-\delta}{\delta}\frac{C}{\varepsilon}(p(a_I))^2\Big) > v_U^* - \Big(l(p, v_0^*, v_1^*) - \frac{1-\delta}{\delta}\frac{C}{\varepsilon}p^2\Big) + \frac{1-\delta}{4\delta}\varepsilon.$$

(Note that the above bound corresponds to (14) with $Info(a_I) = 4(C/\varepsilon)(p(a_I))^2 - p^2$).) This contradicts the choice of (p^*, v^*). The contradiction implies that $v_U^* - l(p^*, v_0^*, v_1^*) < \frac{1}{2}\varepsilon$, and for each $\varepsilon > 0$, each δ such that $\max(((1-\delta)/\delta)(C/\varepsilon), ((1-\delta)/(4\delta))\varepsilon) \leq \frac{1}{4}\varepsilon$, and for each $(p, v) \in ICR_\delta$,

$$v_U - l(p, v_0, v_1) \leq v_U^* - l(p^*, v_0^*, v_1^*) + \frac{1-\delta}{\delta}\frac{C}{\varepsilon}(p^*)^2 - \frac{1-\delta}{\delta}\frac{C}{\varepsilon}p^2 + \frac{1-\delta}{4\delta}\varepsilon$$
$$\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{3} + \frac{\varepsilon}{4} = \varepsilon.$$

□

4.2 *The ICR correspondence*

Due to Proposition 5, one can use information about ICR payoffs in games with degenerate priors $p \in \{0,1\}$ to derive a bound on the payoffs in games with $p \in (0,1)$. The bound is not tight, and it can be tightened through a natural generalization. Suppose that one has some information about ICR payoffs for some priors p_0, p_1, such that $p_0 < p_1$. This information can be used to bound the sets of ICR payoffs $ICR(p)$ for all priors $p \in (p_0, p_1)$. Let $\mathscr{I}$ be a collection of correspondences $I \subseteq [0,1] \times \mathbb{R}^3$ such that

ICR-1 $I(p) \subseteq IR$ for $p \in [0,1]$.

ICR-2 $I(k) \subseteq E_k$ for $k = 0, 1$.

ICR-3 Fix any $0 \leq p_0 < p_1 \leq 1$ and any biaffine function $l : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ such that for any $(v_U^j, v_0^j, v_1^j) \in I(p_j) \cup V$, $j = 0, 1$,

$$v_U^j < l(p_j, v_0^j, v_1^j) \text{ (or } v_U^j > l(p_j, v_0^j, v_1^j)).$$

Then, for any $\alpha \in (0,1)$ and any $(v_U, v_0, v_1) \in I(\alpha p_1 + (1-\alpha)p_0)$,

$$v_U \leq l(\alpha p_1 + (1-\alpha)p_0, v_0, v_1) \text{ (or } v_U \geq l(\alpha p_1 + (1-\alpha)p_0, v_0, v_1)).$$

Property ICR-1 says that all payoffs are individually rational. Property ICR-2 says that $I(k)$ is contained in the set of equilibrium payoffs for degenerate priors $k \in \{0,1\}$. Properties 1 and 2 correspond to Corollary 1. Property 3 is a separation property. It extends the thesis of Proposition 5 to cover cases when $p_0 > 0$ or $p_1 < 1$.

Proposition 6. *The correspondence $ICR^* = \cup_{I \in \mathscr{I}} I$ is closed and $ICR^* \in \mathscr{I}$. Moreover, $ICR \subseteq ICR^*$ and ICR^* satisfies properties FE-1 and FE-3.*

Proposition 6 defines a geometric upper bound ICR^* for the set of ICR payoffs. In addition to properties ICR-1, ICR-2, and ICR-3, the proposition shows that the correspondence ICR^* satisfies properties FE-1 and FE-3 (defined in Section 3.3). The proof of the proposition is based on the methods used in the proof Proposition 5 and can be found in Section B.3.

Notice that conditions ICR-1, ICR-2, and ICR-3 that define the correspondence ICR^* are stated in purely geometrical terms and that they depend only on the sets V and IR. Thus, two games with the same convex hull of non-revealing payoffs and the same sets of individually rational payoffs generate the same correspondence ICR^*. This remark corresponds to an analogous observation about the correspondence FE^*.

5. Proof of the main result

In the previous two sections, I construct correspondences FE^* and ICR^*. By Proposition 6, the correspondence ICR^* contains the correspondence ICR. If Assumption 1 is satisfied, Proposition 4 says that the correspondence FE^* is contained in the correspondence $\operatorname{cl} FE$. Hence, Theorem 1 is a consequence of Propositions 4 and 6 and the following result.

Theorem 3. *Suppose that Assumption 1 holds. Then for any $p \in [0,1]$,*

$$FE^*(p) = ICR^*(p).$$

The characterizations in Sections 3 and 4 derive two collections of geometric properties that are satisfied by the correspondences FE^* and ICR^*. These properties lead to constraints on the infinitesimal changes in $FE^*(p)$ and $ICR^*(p)$ with respect to infinitesimal changes of p. These constraints are different, but related. The idea of the proof of Theorem 3 is to make use of these constraints and show that they can be satisfied by only one correspondence. This is done through a "differential technique." The rest is divided into three parts. First, I describe the correspondences FE^* and ICR^* through their upper and lower surfaces. It is sufficient to show that the respective surfaces of the two correspondences are equal. In the second part, I demonstrate that the upper surfaces of the two correspondences are equal. The last part deals with lower surfaces.

5.1 *Description of the correspondences FE^* and ICR^**

Define functions

$$u^I, u^F, l^I, l^F : [0,1] \times \mathbb{R}^2 \to \mathbb{R} \cup \{-\infty, \infty\}$$

in the following way. Let

$$u^F(p)(v_0, v_1) = \sup\{v_U : (v_U, v_0, v_1) \in \operatorname{con}[FE^*(p) \cup V]\}$$
$$u^I(p)(v_0, v_1) = \sup\{v_U : (v_U, v_0, v_1) \in \operatorname{con}[ICR^*(p) \cup V]\}$$

if the sets are not empty and $-\infty$ otherwise. For example, $u^I(p)(v_0, v_1) > -\infty$ if and only if $(v_0, v_1) \in \operatorname{proj} \operatorname{con}[ICR^*(p) \cup V]$. The functions u^I and u^F describe the upper surfaces of the correspondences $ICR^*(p)$ and $FE^*(p)$. Next, let

$$l^F(p)(v_0, v_1) = \inf\{v_U : (v_U, v_0, v_1) \in \operatorname{con}[FE^*(p) \cup V]\}$$
$$l^I(p)(v_0, v_1) = \inf\{v_U : (v_U, v_0, v_1) \in \operatorname{con}[ICR^*(p) \cup V]\}$$

if the sets are not empty and $+\infty$ otherwise. The functions l^I, l^F describe the lower surfaces of the two correspondences. By property FE-3 of the correspondences FE^* and ICR^* (Proposition 6), for each $p \in [0,1]$,

$$FE^*(p) = \{v \in IR : l^F(p)(v_0, v_1) \leq v_U \leq u^F(p)(v_0, v_1)\}$$
$$ICR^*(p) = \{v \in IR : l^I(p)(v_0, v_1) \leq v_U \leq u^I(p)(v_0, v_1)\}.$$

Because $FE^* \subseteq ICR^*$, it must be that

$$u^F(p)(v_0,v_1) \le u^I(p)(v_0,v_1) \tag{19}$$
$$l^I(p)(v_0,v_1) \le l^F(p)(v_0,v_1). \tag{20}$$

To prove Theorem 3, it is sufficient to show that the inequalities (19) and (20) can be in fact replaced by equalities.

The next lemma collects all the properties of the functions u^I, u^F, l^I, l^F that are needed for the subsequent proofs.

Lemma 2. *The functions u^I, u^F, l^I, l^F have the following properties.*

(i) $u^F(k)(v_0,v_1) = u^I(k)(v_0,v_1)$ *and* $l^F(k)(v_0,v_1) = l^I(k)(v_0,v_1)$, *for* $k = 0,1$.

(ii) u^I *and* u^F *are concave in* (v_0,v_1) *and* l^I *and* l^F *are convex in* (v_0,v_1).

(iii) u^I *is convex in* p; u^I *is linear in* p *above the individually rational payoffs: if for some* $p_0 < p_1$ *and* $(v_0,v_1) \in \operatorname{proj} IR$,

$$u^I(p_k)(v_0,v_1) \ge m_U, \text{ for } k = 0,1,$$

then, for any $p \in [p_0,p_1]$,

$$u^I(p)(v_0,v_1) = \frac{p-p_0}{p_1-p_0} u^I(p_1)(v_0,v_1) + \frac{p_1-p}{p_1-p_0} u^I(p_0)(v_0,v_1).$$

l^I *is concave in* p.

(iv) u^F *is concave in* p *above the individually rational payoffs: if for some* $p_0 < p_1$, $(v_0,v_1) \in \operatorname{proj} IR$,

$$u^F(p_k)(v_0,v_1) \ge m_U, \text{ for } k = 0,1,$$

then, for any $p \in [p_0,p_1]$,

$$u^F(p)(v_0,v_1) \ge \frac{p-p_0}{p_1-p_0} u^F(p_1)(v_0,v_1) + \frac{p_1-p}{p_1-p_0} u^F(p_0)(v_0,v_1).$$

l^F *is convex in* p *above the individually rational payoffs: if for some* $p_0 < p_1$, $(v_0,v_1) \in \operatorname{proj} IR$,

$$l^F(p_k)(v_0,v_1) \ge m_U, \text{ for } k = 0,1,$$

then, for any $p \in [p_0,p_1]$,

$$l^F(p)(v_0,v_1) \le \frac{p-p_0}{p_1-p_0} l^F(p_1)(v_0,v_1) + \frac{p_1-p}{p_1-p_0} l^F(p_0)(v_0,v_1).$$

Proof. Part (i) is a consequence of FE-2 and ICR-2. Part (ii) comes from the definitions of the functions u^I, u^F, l^I, l^F. Part (iii) is a consequence of ICE-3 and Proposition 6 (the fact that ICR^* satisfies property FE-1). Part (iv) is implied by property FE-3 of the correspondence FE^*. □

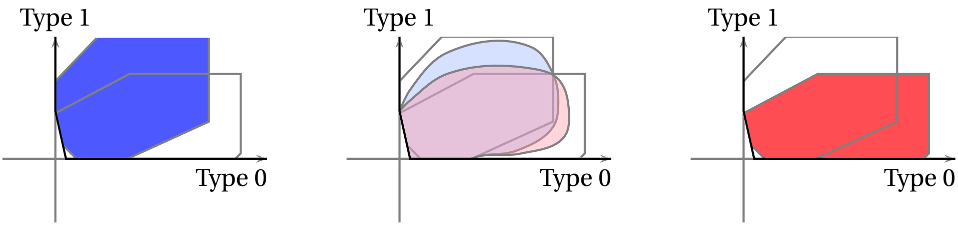

FIGURE 3. The sets proj $ICR^*(p)$ for $p \in [0,1]$.

Suppose that, instead of part (iv) of Lemma 2, its stronger version were true. More precisely, suppose that the function u^F is known to be concave in p for *all* p and (v_0, v_1). Because of (19), $u^F(p) \leq u^I(p)$ for all p, and, because of part (i) of the lemma, $u^F(p) = u^I(p)$ for $p = 0, 1$. Because of part (iii), u^I and l^I are convex in all p, which implies that $u^F(p) = u^I(p)$ for all p, and the correspondences ICR^* and FE^* are equal. Unfortunately, the stronger version of part (iv) of the lemma is not generally true; the fact that the functions u^F and l^F are known to be, respectively, concave and convex only above the individually rational payoffs is the main source of difficulty of the proof of Theorem 3.

Observe that proj $ICR^*(p)$ consists of all payoffs (v_0, v_1) at which the upper surface of ICR^* lies above the individually rational payoffs, $u^I(p)(v_0, v_1) \geq m_U$. These sets play an important role in the statement of the lemma. It is helpful to notice that they have some monotonicity properties.

LEMMA 3. *For any* $p \in (0,1)$,

$$\text{proj}\, ICR^*(0) \cap \text{proj}\, ICR^*(1) \subseteq \text{proj}\, ICR^*(p) \subseteq \text{proj}\, ICR^*(0) \cup \text{proj}\, ICR^*(1).$$

For any $p < p'$,

$$\text{proj}\, ICR^*(p) \cap \text{proj}\, ICR^*(1) \subseteq \text{proj}\, ICR^*(p') \cap \text{proj}\, ICR^*(1),$$
$$\text{proj}\, ICR^*(p') \cap \text{proj}\, ICR^*(0) \subseteq \text{proj}\, ICR^*(p) \cap \text{proj}\, ICR^*(0).$$

By Corollary 1 and Proposition 1, proj $ICR^*(k) = $ proj E_k for each $k = 0, 1$. The left and right graphs on Figure 3 present the projections of the payoffs in $ICR^*(k)$ for $k = 0, 1$ on the set of player I's payoffs (compare with Figure 2). For general $p \in [0,1]$, the sets proj $ICR^*(p)$ have the following monotonicity properties. First, proj $ICR^*(p)$ contains the intersection of, and is contained in the union of the sets proj $ICR^*(0)$ and proj $ICR^*(1)$. Next, proj $ICR^*(p) \cap$ proj $ICR^*(1)$ is (setwise) increasing in p and proj $ICR^*(p) \cap$ proj $ICR^*(0)$ is (setwise) decreasing.

PROOF OF LEMMA 3. Notice that $(v_0, v_1) \in$ proj $ICR^*(p)$ if and only if $(v_0, v_1) \in$ proj IR and $u^I(p)(v_0, v_1) \geq m_U$.

The first inclusion is implied by property FE-1 of the correspondence ICR^* (Proposition 6) and part (iii) of Lemma 2 (the linearity of u^I in p over individually rational payoffs). Take any $(v_0, v_1) \notin \operatorname{proj} ICR^*(k)$ for $k = 0, 1$. Then $u^I(k)(v_0, v_1) < m_U$ for $k = 0, 1$. By part (iii) of Lemma 2, $u^I(p)(v_0, v_1) < m_U$ and $(v_0, v_1) \notin \operatorname{proj} ICR^*(p)$. This implies the second inclusion.

The monotonicity of the sets $\operatorname{proj} ICR^*(p) \cap \operatorname{proj} ICR^*(k)$ in p follows from part (iii) of Lemma 2. □

5.2 *Upper surfaces*

The goal of this section is to show that the upper surfaces of the two correspondences are equal, or, in other words, that the inequality in (19) can be replaced by an equality. There are two steps. First, I define difference functions $\theta_k(p)$, $k = 0, 1$, that (indirectly) measure the distance between the upper surfaces of the correspondences $ICR^*(p)$ and $FE^*(p)$. If θ_k is identically equal to 0 for $k = 0, 1$, then the upper surfaces of the correspondences $ICR^*(p)$ and $FE^*(p)$ are equal. Next, I show that θ_k is identically equal to 0.

5.2.1 *Difference function* For each $k = 0, 1$, define auxiliary correspondences

$$I_k(p) = \{v \in ICR^*(p) : u^I(p)(v_0, v_1) = m_U \text{ and } (v_0, v_1) \in \operatorname{proj} ICR^*(k)\}$$
$$F^*(p) = \{v \in \operatorname{con}(FE^*(p) \cup V) : v_U \geq m_U\}.$$

The correspondence $I_k(p)$ consists of payoff vectors on the upper surface of $ICR^*(p)$, for which player U's payoff is equal to his minmax value and player I's payoffs belong to $\operatorname{proj} ICR^*(k)$ for $k = 0, 1$. The correspondence $F^*(p)$ consists of payoff vectors in the convex hull of the sets $FE^*(p)$ and V, and such that U's payoff is not lower than the minmax. Note that $FE^*(p) \subseteq F^*(p)$, and the inclusion is typically strict, because the payoffs of player I must be individually rational for any $v \in FE^*(p)$. For each $k = 0, 1$, define functions

$$\theta_k(p) = \sup_{v^I \in I_k(p)} \inf_{v^F \in F^*(p)} \frac{\left\| (v_0^I, v_1^I) - (v_0^F, v_1^F) \right\|}{(u^I(1)(v_0^I, v_1^I) - m_U)},$$

with the convention that $0/0 = 0$ and $a/0 = \infty$ for $a > 0$. The functions θ_k measure the distance between the correspondences ICR^* and FE^*. By part (i) of Lemma 2, for each $k = 0, 1$,

$$\theta_k(0) = \theta_k(1) = 0.$$

The next lemma shows that if the functions θ_k are identically equal to 0, then the inequality in (19) can be replaced by an equality.

Lemma 4. *Suppose that $\theta_k(p) = 0$ for each $p = [0, 1]$ and $k = 0, 1$. Then $u^F(p)(v_0, v_1) = u^I(p)(v_0, v_1)$ for each p and each $(v_0, v_1) \in \operatorname{proj} ICR^*(p)$.*

Proof. I show that for any $(v_0, v_1) \in \operatorname{proj} ICR^*(1) \cap \operatorname{proj} ICR^*(p)$, we have $u^F(p)(v_0, v_1) = u^I(p)(v_0, v_1)$. For any $(v_0, v_1) \in \operatorname{proj} ICR^*(1)$, define

$$p(v_0, v_1) = \inf\{p : u^I(p)(v_0, v_1) \geq m_U\}.$$

Because $u^I(1)(v_0,v_1) \geq m_U$, $p(v_0,v_1)$ is well defined. I show that for any $(v_0,v_1) \in \operatorname{proj} ICR^*(1)$,

$$u^I(p(v_0,v_1))(v_0,v_1) = u^F(p(v_0,v_1))(v_0,v_1) \geq m_U. \tag{21}$$

If $p(v_0,v_1)=1$, then (21) holds by part (i) of Lemma 2. If $p(v_0,v_1)=0$, then the equality in (21) holds by part (i) of Lemma 2 and the inequality comes from the convexity (hence, upper semi-continuity) of u^I in p. If $0 < p(v_0,v_1) < 1$, then, by part (iii) of Lemma 2, $u^I(p)(v_0,v_1)$ is convex in $p \in (0,1)$ and hence continuous in a neighborhood of $p(v_0,v_1)$. This implies that

$$(m_U,v_0,v_1) = (u^I(p(v_0,v_1))(v_0,v_1),v_0,v_1) \in I_1(p).$$

Because $\theta_1(p)=0$,

$$u^F(p(v_0,v_1),(v_0,v_1)) \geq m_U,$$

which, together with (19), implies (21).

Observe that

$$u^I(1)(v_0,v_1) = u^F(1)(v_0,v_1) \geq m_U \text{ for any } (v_0,v_1) \in \operatorname{proj} ICR^*(1).$$

Because of (21) and parts (iii) and (iv) of Lemma 2, the inequalities in (19) can be replaced by equalities for any $p \in [p(v_0,v_1),1]$.

An analogous argument shows that for any $(v_0,v_1) \in \operatorname{proj} ICR^*(0) \cap \operatorname{proj} ICR^*(p)$, $u^F(p)(v_0,v_1) = u^I(p)(v_0,v_1)$. By the first part of Lemma 3, $\operatorname{proj} ICR^*(p) \subseteq \operatorname{proj} ICR^*(0) \cup \operatorname{proj} ICR^*(1)$. This concludes the proof of the lemma. □

Notice that for any $(v_0,v_1) \in \mathbb{R}^2$, $u^I(p)(v_0,v_1)$ is either (a) equal to a convex combination of the form

$$u^I(p)(v_0,v_1) = \alpha v_U^V + (1-\alpha)u^I(p)(v_0^I,v_1^I)$$

for some $\alpha \in [0,1]$, $v^V \in V$, and $(v_0^I,v_1^I) \in \operatorname{proj} ICR^*(p)$ such that $\alpha(v_0^V,v_1^V) + (1-\alpha)(v_0^I,v_1^I) = (v_0,v_1)$, or (b) equal to $-\infty$ if such a tuple of elements cannot be found. Because

$$u^F(p)(v_0,v_1) \geq \alpha v_U^V + (1-\alpha)u^F(p)(v_0^I,v_1^I),$$

the previous lemma leads to the following result.

Corollary 2. *Suppose that $\theta_k(p)=0$ for each $p=[0,1]$ and $k=0,1$. Then we have $u^F(p)(v_0,v_1) = u^I(p)(v_0,v_1)$ for each p and each (v_0,v_1).*

5.2.2 *"Differential" step* Here I show that $\theta_k(p)=0$ for $p \in [0,1]$ and $k=0,1$. Because the arguments are exactly analogous, I assume without loss of generality that $k=1$. To save on notation, I also drop the subscript and write θ instead of θ_1.

The next result is a crucial step in the proof.

Lemma 5. *Suppose that $\theta(p^u)=0$ for some $p^u<1$. There are constants D and $d^*, \theta^*>0$, such that if $\theta(p)\leq\theta^*$, then*

$$\theta(p')\leq\theta(p)+D\big((p'-p)\theta(p)+(p'-p)^2\big) \tag{22}$$

for any $p, p'\in[p^u, \frac{1}{2}(1+p^u)]$ such that $p\leq p'$ and $p'-p<d^*$.

Hence, $\theta(p')$ is bounded by $\theta(p)$ plus a term that is of second order in $\theta(p)$ and $p'-p$. The lemma is proved in Section C (in the Appendix).

Suppose that $\theta(p^u)=0$ for some $p^u<1$. I show that $\theta(p)=0$ for any $p\in[p^u, \frac{1}{2}(1+p^u)]$. Take any

$$m\geq\max\left[\frac{1}{d^*}, \sqrt{\frac{D}{\theta^*}}, D, \frac{1}{\theta^*}e^{4D}\right]$$

and consider any increasing sequence of numbers $p^u=p_0\leq p_1\leq\cdots\leq p_{2m}\leq\frac{1}{2}(1+p^u)$ such that $\left|p_{i+1}-p_i\right|\leq\frac{1}{m}\leq d^*$ for all i. Applying Lemma 5 once shows that $\theta(p_1)\leq D(\frac{1}{m})^2\leq\theta^*$. Inductive application of Lemma 5 leads to

$$\begin{aligned}
\theta(p_i)&\leq\theta(p_{i-1})+D((p_i-p_{i-1})\theta(p_{i-1})+(p_i-p_{i-1})^2)\\
&\leq\max(\theta(p_{i-1}), p_i-p_{i-1})(1+2D(p_i-p_{i-1}))\\
&\leq\max\left(\theta(p_{i-1}), \frac{1}{m}\right)\left(1+2D\frac{1}{m}\right)\\
&\leq\max\left(\theta(p_1), \frac{1}{m}\right)\left(1+2D\frac{1}{m}\right)^{i-1}\\
&\leq\frac{1}{m}\max\left(\frac{D}{m}, 1\right)\left(1+2D\frac{1}{m}\right)^{2m}\\
&\leq\frac{1}{m}e^{4D}.
\end{aligned}$$

Lemma 5 applies at each step because the choice of m implies that $\theta(p_i)\leq\theta^*$. Since any sequence of p_i's could have been chosen, it follows that for any $p\in[p^u, \frac{1}{2}(1+p^u)]$, $\theta(p)$ is not larger than $1/m$ times a constant. But, in turn, any large m could have been chosen. This shows that $\theta(p)=0$ for any $p\in[p^u, \frac{1}{2}(1+p^u)]$.

I use the lemma to prove that $\theta(p)=0$ for all p. Recall that $\theta(p)=0$ for $p=0,1$. Construct a sequence of prior beliefs: $p_0^u=0$ and

$$p_{n+1}^u=\frac{1}{2}(1+p_n^u).$$

Then $\lim_n p_n^u=1$, and the above argument shows that $\theta(p)=0$ for any $p\in[0,1]$.

Together with Corollary 2, this means that the upper surfaces of ICR^* and FE^* are equal.

5.3 *Lower surfaces*

The goal of this section is to show that the lower surfaces of the two correspondences are equal (that is, the inequality in (20) can be replaced by an equality).

For any $p \in [0,1]$ and any (v_0, v_1), define

$$\theta(p, v_0, v_1) = l^F(p)(v_0, v_1) - l^I(p)(v_0, v_1),$$

where, as a convention, I take $\infty - \infty = 0$. The function θ measures the distance between the lower surfaces of the two correspondences. Because of (20), $\theta(p, v_0, v_1) \geq 0$. Notice that $l^F(p)(v_0, v_1) < \infty$ whenever $u^F(p)(v_0, v_1) > -\infty$. This is equivalent to $u^I(p)(v_0, v_1) > -\infty$, which, in turn, implies that $l^I(p)(v_0, v_1) < \infty$. Hence, $\theta(p, v_0, v_1)$ is always finite and bounded by $2M$.

The proofs of the two lemmas below can be found in Section D (in the Appendix).

LEMMA 6. *Suppose that $\theta(p, v_0, v_1) > 0$ for some p, v_0, v_1. Then there is $p' \in [0,1]$ such that*

$$\theta(p', v_0, v_1) \geq \theta(p, v_0, v_1)$$
$$l^I(p')(v_0, v_1) \leq m_U - \tfrac{1}{2}\theta(p', v_0, v_1).$$

LEMMA 7. *Suppose that*

$$l^I(p)(v_0, v_1) \leq m_U - \tfrac{1}{2}\theta(p, v_0, v_1)$$

for some p, v_0, v_1. Then, there exist v_0' and v_1' such that

$$\theta(p, v_0', v_1') \geq \frac{4M}{4M - \theta(p, v_0, v_1)}\theta(p, v_0, v_1).$$

Denote

$$\theta^* = \sup_{p \in [0,1], (v_0, v_1)} \theta(p, v_0, v_1)$$

and suppose that $\theta^* > 0$. I show that this leads to a contradiction. Choose $p \in [0,1]$ and (v_0, v_1) such that $\theta(p, v_0, v_1) \geq \frac{1}{2}\theta^*$. Notice that $p \in (0,1)$. Alternating between the two lemmas, I construct a sequence (p^n, v_0^n, v_1^n) such that

$$\theta(p^n, v_0^n, v_1^n) \geq \frac{1}{2}\left(\frac{4M}{4M - \theta^*}\right)^n \theta^* \to \infty.$$

This yields the contradiction.

6. COMMENTS AND CONCLUSION

An interesting application of the methods developed in this paper is present in unpublished work by Gregory Pavlov, who studies repeated bargaining between a firm and a union. The union is uncertain about the firm's commitment to aggressive bargaining. It turns out that the best equilibrium for the union involves a screening phase during which the union resorts to strikes and the firm reveals information about its type finitely many times. When one interprets the discount factor converging to 1 as a division of periods into smaller and smaller units, the length of the screening phase converges to a positive constant. Pavlov uses differential equations to describe the set of equilibrium payoffs.

The model of this paper has obvious limitations. For example, it is stated for only two states of the world. When there are more than two states of the world, it is still possible to define finitely revealing equilibria and construct the correspondence FE^* as in Proposition 4. One can also show an analog of Proposition 5. However, it is unclear how to proceed further and prove that the two bounds are equal. In yet another extension, one can relax the known-own-payoffs assumption.

More importantly, this paper is concerned only with one-sided incomplete information. I believe that it can serve as a step toward the characterization of payoffs in games with multisided incomplete information. So far, that problem remains open.[8]

Appendix

A. Proofs for Section 3

A.1 *Linear problem (11)*

In order to solve the linear problem (11), it is convenient to introduce payoff correction functions[9] $x : S_U \times S_I \to \mathbb{R}^3$. Let

$$x(a_U, a_I) = \delta(g(a_U, a_I) - v(a_U, a_I)).$$

Then (8) and (9) take the form

$$\begin{aligned} v_U &\geq g_U(a_U, \alpha_I^*) + x_U(a_U, \alpha_I^*) \\ v_k &\geq g_j(\alpha_U^*, a_k) + x_k(\alpha_U^*, a_k) \text{ for } k = 0, 1, \end{aligned}$$

with equalities when $a_U \in \operatorname{supp} \alpha_U^*$ and $a_k \in \operatorname{supp} \alpha_k^*$. Similarly, (10) corresponds to $\phi \cdot x(a_U, a_I) \leq \mathbf{0}$. The payoff corrections are used to characterize the function $\eta(.)$.

Lemma 8. *(i) If either (a) $\phi_k > 0$ for some $k = U, 0, 1$ or (b) $\phi_U < 0$ and $\phi_k < 0$ for some $k = 0, 1$, then $\eta(p, \phi) \geq \max_{v' \in V} \phi \cdot v'$. If $p \in (0, 1)$, then $\eta(p, \phi) = \max_{v' \in V} \phi \cdot v'$.*

(ii) If $\phi_U = -1$ and $\phi_k = 0$, for $k = 0, 1$, then $\eta(p, \phi) = -m_U$.

(iii) If $\phi_U = 0$ and for $k = 0, 1$, $\phi_k \leq 0$, then $\eta(p, \phi) = -m_I(-\phi_0, -\phi_1)$.

[8] In a recent paper, Athey and Bagwell (forthcoming) study a repeated Bertrand duopoly where each firm has private information about its own costs. A specific structure of the game allows the authors to use mechanism design tools to describe the optimal equilibria. It would be very interesting to check whether the methods of this paper extend to their model.

[9] This representation is introduced in Kandori and Matsushima (1998). They refer to $x(.,.)$ as a "sidepayment contract."

(iv) If either (a) $\phi_0, \phi_1 > 0$ or (b) $\phi_U \neq 0$ and for some $k = 0, 1$, $\phi_{-k} > 0$, then $\eta(k, \phi) = \infty$.

Proof. The proof consists of solving the linear problem (11) for each case separately. The first three parts are relatively standard and therefore omitted.

Case (iv)(a). Suppose that $k = 0$ and that $\phi_0, \phi_1 > 0$. Take any pure actions $a_U^* \in S_U$, $a_0^*, a_1^* \in S_I$, such that $a_0^* \neq a_1^*$. I enforce the profile (a_U^*, a_0^*, a_1^*) with a payoff correction

$$x(a_U^*, a_0^*) = (0, 0, 0) \text{ and } x_1(a_U^*, a_1^*) = X_1 \text{ for any large } X_1.$$

- Choose the payoff correction $x_1(a_U, a_I)$, $(a_U, a_I) \neq (a_U^*, a_0^*)$ so that type 1's incentive compatibility holds.
- Choose $x_0(a_U^*, a_1^*)$ small enough so that type 0's incentive compatibility holds and condition (10) is satisfied at the profile (a_U^*, a_1^*) (this can be done because $\phi_0 > 0$).
- Choose $x_0(a_U^*, a_0)$, $a_0 \neq a_0^*, a_1^*$ small enough so that type 0's incentive compatibility holds.
- Choose $x_1(a_U^*, a_1)$, $a_1 \neq a_1^*$, so that type 1's incentive compatibility holds and condition (10) is satisfied at the profile (a_U^*, a_1) (this can be done because $\phi_1 > 0$).
- Choose all other payoff corrections to make (10) satisfied for any other pair of actions.

Case (iv)(b). Suppose that $k = 0$, $\phi_U \neq 0$, and $\phi_1 > 0$. Take any pure actions $a_U^* \in S_U$, $a_0^*, a_1^* \in S_I$, such that $a_0^* \neq a_1^*$. I enforce the profile (a_U^*, a_0^*, a_1^*) with a payoff correction

$$x(a_U^*, a_0^*) = (0, 0, 0) \text{ and } x_1(a_U^*, a_1^*) = X_1 \text{ for any large } X_1.$$

- Choose $x_U(a_U, a_0^*)$, $a_U \neq a_U^*$, so that player U's incentive compatibility holds.
- Choose $x_0(a_U^*, a_0)$, $a_0 \neq a_0^*$, so that 0's incentive compatibility holds.
- Choose $x_U(a_U^*, a_1^*)$, to make condition (10) satisfied at the profile (a_U^*, a_1^*) (this can be done because $\phi_U \neq 0$).
- Choose $x_1(a_U^*, a_1)$, $a_1 \neq a_1^*$ small enough so that 1's incentive compatibility holds and condition (10) is satisfied at the profile (a_U^*, a_1) (this can be done because $\phi_1 > 0$).
- Choose all other payoff corrections for players U and type 0.
- Choose $x_1(a_U, a_I)$, $a_U \neq a_U^*$, to satisfy condition (10) at profiles (a_U, a_I) (this can be done because $\phi_1 \neq 0$). □

A.2 *Proof of Proposition 1*

Lemma 9. *For any* $k=0,1$,

$$E_k \subseteq \bigcap_{\phi} \{v : \phi \cdot v \leq \eta(k,\phi)\}.$$

Proof. Suppose that $k=0$. Observe that

$$IR = \bigcap_{\substack{\phi \text{ satisfies cases (ii) and (iii)} \\ \text{from Lemma 8}}} \{v : \phi \cdot v \leq \eta(k,\phi)\}.$$

I show that

$$E^{F,IC}(0) \subseteq \bigcap_{\substack{\phi \text{ satisfies case (i) but not cases (ii)–(iv)} \\ \text{from Lemma 8}}} \{v : \phi \cdot v \leq \eta(0,\phi)\}.$$

Indeed, take any $(v_U^0, v_0^0, v_1^1) \in E^{F,IC}(0)$ where $v^0, v^1 \in V$ and $v_1^0 \leq v_1^1, v_0^1 \leq v_0^0$. Suppose that ϕ satisfies case (1) and $\phi_1 \leq 0$; then

$$\phi \cdot (v_U^0, v_0^0, v_1^1) \leq \phi \cdot (v_U^0, v_0^0, v_1^0) \leq \max_{v' \in V} \phi \cdot v' \leq \eta(0,\phi).$$

Suppose that ϕ satisfies case (i) but not cases (ii)–(iv) and that $\phi_1 > 0$. Then it must be that $\phi_U = 0$ and $\phi_0 \leq 0$. Hence,

$$\phi \cdot (v_U^0, v_0^0, v_1^1) \leq \phi \cdot (v_U^1, v_0^1, v_1^1) \leq \max_{v' \in V} \phi \cdot v' \leq \eta(0,\phi).$$ □

Take a closed convex set with a smooth boundary

$$E^* \subseteq \operatorname{int} E_k \subseteq \bigcap_{\phi \in \Phi} \{v : \phi \cdot v \leq \eta(k,\phi)\}.$$

(Note that the second inclusion holds by Lemma 9.) I show that there exists δ_0 such that E^* is a set of equilibrium payoffs for any $\delta \geq \delta_0$. Take any boundary vector of payoffs $v \in \operatorname{bd} E^*$ and a vector ϕ normal to E^*. For a high enough δ_v, there is a mixed action profile $(\alpha_U, \alpha_0, \alpha_1) \in \Delta S_U \times \Delta S_I \times \Delta S_I$ and a continuation payoff function $v : S_U \times S_I \rightarrow \operatorname{int} E^*$ so that the payoffs v are supported by the profile and the continuation payoffs (equations (8)) and incentive compatibility hold (inequalities (9)). By the argument in Fudenberg et al. (1994), for each $v \in E^*$ there is an open set $U \ni v$ such that each $\delta \geq \delta_v$, each $v' \in E^* \cap v$ can be supported with an action profile and continuation payoffs inside E^* so that incentive compatibility holds. In the terminology of Fudenberg et al. (1994), E^* is locally self-decomposable. Lemma 4.2 of Fudenberg et al. (1994) shows that $E^* \subseteq FE(k)$.

A.3 *Proof of Proposition 2*

Assume without the loss of generality that A is convex (if not, then it is easy to use public randomization to construct non-revealing equilibria with payoffs in $\operatorname{con} A$). By Parts (i)–(iii) of Lemma 8, if $p \in (0,1)$ then

$$\eta(p,\phi) = \begin{cases} -m_U & \text{if } \phi_U = -1, \phi_0 = \phi_1 = 0 \\ -m_I(-\phi_0, -\phi_1) & \text{if } \phi_0, \phi_1 \leq 0, \phi_U = 0 \\ \max_{v' \in V} \phi \cdot v' & \text{otherwise.} \end{cases}$$

Together with the definition of individually rational payoffs IR in (3), this implies that the first inclusion in (12) holds as a equality:

$$\operatorname{int}[\operatorname{con}(A \cup V) \cap IR] = \operatorname{int} \bigcap_{\phi \in \Phi} \left\{ v : \phi \cdot v \leq \max \left[\max_{v' \in A} \phi \cdot v', \eta(\phi, p) \right] \right\}.$$

In order to show the second inclusion, an intermediate result is needed.

Lemma 10. *Suppose that $A' \subseteq A$ is closed and convex with a nonempty interior and a smooth boundary. Then for each $v^* \in \operatorname{int}[\operatorname{con}(A' \cup V) \cap IR]$ there is $\varepsilon > 0$ and a closed, convex set $W \subseteq IR$ with a nonempty interior and a smooth boundary such that $v^* \in W$, and for each $v \in \operatorname{bd} W$ and $\phi \in \Phi$ such that ϕ is normal to W at v, either $v \in A'$ or $\phi \cdot v \leq \eta(\phi, p) - \varepsilon$.*

Proof. Instead of a complete argument, I present only the construction of the set W. For each $\varepsilon, \gamma > 0$, define

$$W_\varepsilon = \bigcap_{\phi \in \Phi} \left\{ v : \phi \cdot v \leq \max \left[\max_{v' \in A'} \phi \cdot v', \eta(\phi, p) - \varepsilon \right] \right\}$$

$$W_{\varepsilon,\gamma} = \left\{ v \in W_\varepsilon : \inf_{v' \notin W_\varepsilon} \left\| v - v' \right\| \geq \gamma \right\}$$

$$\overline{W}_{\varepsilon,\gamma} = \left\{ v : \inf_{v' \in W_{\varepsilon,\gamma}} \left\| v - v' \right\| \leq \gamma \right\}.$$

The set $W_{\varepsilon,\gamma}$ consists of the elements of W_ε that are at least γ-far from the complement of W_ε; the set $\overline{W}_{\varepsilon,\gamma}$ consists of the vectors that are at most γ-far from $W_{\varepsilon,\gamma}$. Observe that $\overline{W}_{\varepsilon,\gamma} \subseteq W_\varepsilon$, but the inclusion might be strict. It is easy to check that, for sufficiently small ε and γ, $\overline{W}_{\varepsilon,\gamma}$ is closed and convex, and has a nonempty interior and a smooth boundary. It is easy to see that $\cup_{\varepsilon,\gg 0} W_{\varepsilon,\gamma} = \operatorname{int}[\operatorname{con}(A' \cup V) \cap IR]$. Because A' has a smooth boundary, when γ is sufficiently small, $A' \subseteq \overline{W}_{\varepsilon,\gamma}$. One can check that, for such ε and γ and any $v \in \operatorname{bd} \overline{W}_{\varepsilon,\gamma} \setminus A'$, if ϕ is normal to $\overline{W}_{\varepsilon,\gamma}$ at v, then $\phi \cdot v \leq \eta(\phi, p) - \varepsilon$. □

Take any $v^* \in \operatorname{int}[\operatorname{con}(A \cup V) \cap IR]$. There exist $\delta_0 < 1$ and closed, convex sets A' and A'' such that $A' \subseteq \operatorname{int} A'' \subseteq A'' \subseteq \operatorname{int} A$, $v^* \in \operatorname{int}[\operatorname{con}(A' \cup V) \cap IR]$, and $A'' \subseteq FE_\delta(p)$ for $\delta \geq \delta_0$. By the previous lemma, there exists $\varepsilon > 0$ and a closed, convex set W with a nonempty interior and a smooth boundary such that $v^* \in W$ and for each $v \in \operatorname{bd} W$ and $\phi \in \Phi$ such that ϕ is normal to W at v, either $v \in A'$ or $\phi \cdot v \leq \eta(\phi, p) - \varepsilon$.

I show that there exists $\delta^* \geq \delta_0$ such that $W \subseteq FE_\delta(p)$ for all $\delta \geq \delta^*$. More precisely, I show that for each $v \in W$, either (a) $v \in \operatorname{int} A''$, hence v is a payoff in a finitely revealing equilibrium, or (b) v is supported by a profile of mixed actions α_U^*, α_k^*, $k=0,1$, and continuation payoffs $v(a_U, a_I) \in W$ (equations (8)) such that incentive compatibility (inequalities (9)) holds. Using the argument for Lemma 4.2 of Fudenberg et al. (1994), one can construct a strategy that makes $v \in W$ a payoff in a finitely revealing equilibrium.

I show that (b) holds for any $v \in W \setminus \operatorname{int} A''$. By the definition of the function $\eta(\phi, p)$, there exists $\delta_v < 1$ such that for each $\delta \geq \delta_v$ there is a profile α_U^*, α_k^*, $k=0,1$ and continuation payoffs $v(a_U, a_I) \in \operatorname{int} W$ so that equations (8) and inequalities (9) hold. Because the continuation payoffs $v(a_U, a_I)$ belong to the interior of W, for each v, one can find a neighborhood $U_v \ni v$ such that $\delta_v \geq \delta_{v'}$ for each $v' \in U_v$. Because $W \setminus \operatorname{int} A''$ is compact, one can find $\delta^* \geq \delta_0$ such that for each $\delta \geq \delta^*$ and each $v \in W \setminus \operatorname{int} A''$, there are a mixed profile and continuation payoffs so that (8) and (9) hold.

A.4 *Proof of Proposition 4*

This section contains the steps missing from the proof of Proposition 4.

A.4.1 *FE^* is closed* Suppose not and there is $(p^n, v^n) \to (p^*, v^*)$ with $(p^n, v^n) \in FE^*$ and $v^* \notin FE^*(p^*)$. Without loss of generality I assume that $p^n > p^*$ for all n (taking a subsequence might be necessary). Since $FE^*(p^*)$ is closed and bounded (notice that $[-M, M]^3 \times [0,1] \in \mathscr{F}$) and satisfies property FE-3, there is a vector $\phi = (\phi_U, \phi_0, \phi_1) \in \Phi$ such that $\phi_U \neq 0$ and

$$\phi \cdot v^* > \sup_{v \in FE^*(p^*) \cup V} \phi \cdot v =: x^*.$$

I assume without loss of generality that $\phi_U > 0$ (the argument in the other case is analogous). Let $x_1 = \sup_{v \in FE^*(1) \cup V} \phi \cdot v$. Define the biaffine function $l_\phi : [0,1] \times \mathbb{R}^2 \to \mathbb{R}$ by

$$l_\phi(p, v_0, v_1) = \frac{1}{\phi_U}\Big(\frac{p - p^*}{1 - p^*} x_1 + \frac{1-p}{1-p^*} x^*\Big) - \frac{1}{\phi_U}(\phi_0, \phi_1) \cdot (v_0, v_1).$$

Then $v_U \leq l_\phi(p, v_0, v_1)$ for each $p \in \{p^*, 1\}$ and for each $v \in FE^*(p) \cup V$ and for high enough n, $v_U^n > l_\phi(p^n, v_0^n, v_1^n)$.

Define

$$FE^{**} = \{(p, v) \in FE^* : \text{if } p^* \leq p \leq 1, \text{ then } v_U \leq l_\phi(p, v_0, v_1)\}.$$

Then for a high enough n, $v^n \notin FE^{**}(p^n)$. I show that FE^{**} satisfies properties FE-1, FE-2, and FE-3. Because $v^n \in FE^*(p^n)$ and $FE^{**} \subseteq FE^*$, this implies that FE^* is not the smallest set in the collection $\mathscr{F}$, which contradicts the definition of FE^*.

It is immediate to check that FE^{**} has properties FE-2 and FE-3. I check that FE^{**} satisfies property FE-1.

Lemma 11. *Suppose $F \subseteq [0,1] \times \mathbb{R}^3$ satisfies property FE-1. Suppose that there are $p_0 < p_1$ and a biaffine function l such that*

$$v_U \leq l(p_j, v_0, v_1) \text{ for each } v \in F(p_j) \text{ and } j = 0, 1.$$

Define

$$F^{**}=\{(p,v)\in F: \text{if } p^*\leq p\leq 1, \text{ then } v_U\leq l(p,v_0,v_1)\}.$$

Then F^{**} satisfies property FE-1.

One can apply the lemma to the correspondence FE^*, priors $p^*, 1$, and biaffine function l_ϕ to conclude that FE^{**} satisfies FE-1.

Proof of Lemma 11. Take any $p'^0 < p'^1$ and $(v_U^j, v_0, v_1)\in I'(p'^j)$, $j=0,1$. For any p, define $v^p = ((p-p'^0)/(p'^1-p'^0))v^1+((p'^1-p)/(p'^1-p'^0))v^0$. Notice that by the biaffinity of l,

$$v_U^p - l(p,v_0,v_1) \tag{A.1}$$

is affine in p. Because F satisfies FE-1, then $v^p\in FE(p)$ for any $p\in[p'^0,p'^1]$.

If $p'^0, p'^1\in[p_0,p_1]$, then $v_U^j\leq l(p'^j,v_0,v_1)$. The biaffinity of l implies that $v_U^p\leq l(p,v_0,v_1)$ for each $p\in(p'^0,p'^1)$.

If $p'^0<p_0<p'^1\leq p_1$, then $v_U^1\leq l(p'^1,v_0,v_1)$. Suppose that there is $p\in(p_0,p'^1)$ such that $v_U^p > l(p,v_0,v_1)$. Then it must be that $v_U^0 > l(p'^0,v_0,v_1)$. Thus (A.1) is decreasing in p. In particular, $v_U^{p_0} > l(p_0,v_0,v_1)$. But this contradicts the thesis of the lemma.

If $p'^0<p_0<p_1<p'^1$ and there is $p\in(p_0,p_1)$ such that $v_U^p > l(p,v_0,v_1)$, then, for at least one j, $v_U^j > l(p'^j,v_0,v_1)$. Suppose that $j=0$. We have $v_U^1 > l(p'^1,v_0,v_1)$, so $v^{p_j} > l(p_j,v_0,v_1)$ for any $j=0,1$ (because (A.1) is affine). If $v_U^1\leq l(p'^1,v_0,v_1)$, then (A.1) is decreasing in p and $v_U^{p_0} > l(p_0,v_0,v_1)$. In each case, we get a contradiction to the thesis of the lemma. □

A.4.2 $FE'\in\mathscr{F}$ I need to check that FE' satisfies properties FE-1, FE-2, and FE-3. Assumption 1 implies that for $k=0,1$, the sets E_k have a non-empty interior. Because E_k is convex, we have $E_k=\operatorname{cl}\operatorname{int}E_k$ and $E_k\subseteq FE'(p)$ by Proposition 1. This establishes FE-2. Proposition 2 implies FE-3:

$$FE'(p)=\operatorname{cl}\operatorname{int}FE(p)\subseteq\operatorname{cl}\operatorname{int}[\operatorname{con}(\operatorname{int}FE'(p)\cup V)\cap IR]\subseteq FE'(p).$$

Thus only property FE-1 needs to be verified. Recall that Assumption 1 together with Proposition 3 imply that there are non-empty, open, and convex sets $A_p\subseteq\operatorname{int}FE(p)$, $p\in[0,1]$, such that $\operatorname{proj}A_p=\operatorname{proj}A_{p'}$ for any $p,p'\in[0,1]$.

Take $p^0<p^1$ and $v^j=(v_U^j,v_0,v_1)\in FE'(p^j)$ for $j=0,1$. I show that for any $p\in(p^0,p^1)$,

$$v^p=\frac{p-p^0}{p^1-p^0}v^1+\frac{p^1-p}{p^1-p^0}v^0\in FE'(p). \tag{A.2}$$

There are sequences $v^{j,n}=(v_U^{j,n},v_0^{j,n},v_1^{j,n})\in\operatorname{int}FE(p^j)$ such that $v^{j,n}\to v^j$ for $j=0,1$. I have already established property FE-3; hence

$$\operatorname{con}[\{v^{j,n}\}\cup A_{p^j}]\subseteq\operatorname{int}FE(p^j) \text{ for each } j,n.$$

A simple continuity argument shows that

$$\operatorname{int}\operatorname{con}[\{v^j\}\cup A_{p^j}]\subseteq FE'(p^j),$$

where $\operatorname{int}\operatorname{con}[\{v^j\}\cup A_{p^j}]$ is non-empty, open, and convex. But this implies that sequences $v^{j,n}\to v^j$ can be chosen in such a way that $(v_0^{0,n},v_1^{0,n})=(v_0^{1,n},v_1^{1,n})$. Then, for each $p\in(p^0,p^1)$,

$$v^{p,n}=\frac{p-p^0}{p^1-p^0}v^{1,n}+\frac{p^1-p}{p^1-p^0}v^{0,n}\in\operatorname{int}FE(p).$$

Because $v^{p,n}\overset{n\to\infty}{\longrightarrow}v^p$, this demonstrates (A.2).

B. Proofs for Section 4

B.1 *Proof of property (16)*

Suppose that $p(a_I)=0$. There are open neighborhoods $U_{IR}\supseteq IR$ and $U_E\supseteq E^{F,IC}(0)$ such that for any $(u_U,u_0,u_1)\in U_{IR}\cap U_E$, $u_U<l(0,u_0,u_1)$. I show that for a high enough δ, $(v_U(\alpha_U,a_I),w(\alpha_U,a_I))\in U_{IR}\cap U_E$.

Because of the *IR* condition, $v_U(\alpha_U,a_I)\geq m_U$ and $(v_U(\alpha_U,a_I),v_0,v_1)\in IR$. By (15),

$$\|w(\alpha_U,a_I)-(v_0,v_1)\|\leq 2\frac{1-\delta}{\delta}M. \tag{B.1}$$

Hence for δ high enough, $(v_U(\alpha_U,a_I),w(\alpha_U,a_I))\in U_{IR}$.

Let σ' denote the strategy of player I that takes action a_I in the first period and then follows with continuation strategy $\sigma_0(a_U,a_I)$; let $v_U(\sigma_U,\sigma',\sigma')$ denote the expected payoff of player U when the two types of player I use the strategy σ'. Recall that after observing a_I, player U is certain that he faces type 0. Hence the difference between the expected payoffs $v_U(\sigma_U,\sigma',\sigma')$ and the expected continuation payoffs $v_U(\alpha_U,a_I)$ is not larger than the difference between the first-period payoffs and

$$\left\|v_U(\alpha_U,a_I)-v_U(\sigma_U,\sigma',\sigma')\right\|\leq\frac{1-\delta}{\delta}M. \tag{B.2}$$

Clearly,

$$(v_U(\sigma_U,\sigma',\sigma'),v_0(\sigma_U,\sigma',\sigma'),v_1(\sigma_U,\sigma',\sigma'))\in V.$$

Type 0 is indifferent between the strategies σ' and σ_0 because of condition *IC*. Type 1 prefers to play σ_1; hence we have $v(\sigma_U,\sigma',\sigma')\leq v_1$. This implies that $(v_U(\sigma_U,\sigma',\sigma'),v_0,v_1)\in E_0^{F,IC}$. Together with (B.1) and (B.2), this implies that for δ high enough, $(v_U(\alpha_U,a_I),w(\alpha_U,a_I))\in U_E$.

B.2 *Proof of inequality (17)*

Observe that

$$\begin{aligned}\sum_{a_I\in S_I^*}&\alpha_I(a_I)(v_U(\alpha_U,a_I)-l(p(a_I),w(\alpha_U,a_I))-(v_U-l(p,v_0,v_1)))\\ &=\sum_{a_I\in S_I^*}\alpha_I(a_I)([v_U(\alpha_U,a_I)-v_U]+[l(p,v_0,v_1)-l(p,w(\alpha_U,a_I))])\\ &\qquad+\sum_{a_I\in S_I^*}\alpha_I(a_I)[l(p,w(\alpha_U,a_I))-l(p(a_I),w(\alpha_U,a_I))].\end{aligned} \tag{B.3}$$

I bound each of the terms in (B.3). Observe that

$$\sum_{a_I\in S_I^*}\alpha_I(a_I)v_U(\alpha_U,a_I)=\frac{1}{\delta}v_U-\frac{1-\delta}{\delta}\sum_{a_I\in S_I^*}\alpha_I(a_I)g_U(\alpha_U,a_I)$$
$$\sum_{a_I\in S_I^*}\alpha_I(a_I)w(\alpha_U,a_I)=\frac{1}{\delta}(v_0,v_1)-\frac{1-\delta}{\delta}\sum_{a_I\in S_I^*}\alpha_I(a_I)(g_0(\alpha_U,a_I),g_1(\alpha_U,a_I))$$

and $v_U\geq l(p,v_0,v_1)+\frac{1}{2}\varepsilon$. Hence

$$\begin{aligned}&\sum_{a_I\in S_I^*}\alpha_I(a_I)([v_U(\alpha_U,a_I)-v_U]+[l(p,v_0,v_1)-l(p,w(\alpha_U,a_I))])\\&\quad=\frac{1-\delta}{\delta}\sum_{a_I\in S_I^*}\alpha_I(a_I)(v_U-g_U(\alpha_U,a_I)+l(p,g_0(\alpha_U,a_I),g_1(\alpha_U,a_I))-l(p,v_0,v_1))\\&\quad\geq\frac{1-\delta}{\delta}\frac{\varepsilon}{2}-\frac{1-\delta}{\delta}[g_U(\alpha_U,\alpha_I)-l(p,g_0(\alpha_U,\alpha_I),g_1(\alpha_U,\alpha_I))]\geq\frac{1-\delta}{\delta}\frac{\varepsilon}{2}.\end{aligned}$$

(The last inequality holds because, by the hypothesis, for any $(v_U',v_0',v_1')\in V$,

$$v'-l(p,v_0',v_1')=p(v'-l(1,v_0',v_1'))+(1-p)(v'-l(0,v_0',v_1'))\leq 0.)$$

The second term in (B.3) is bounded by

$$\begin{aligned}&\sum_{a_I\in S_I^*}\alpha_I(a_I)[l(p,w(\alpha_U,a_I))-l(p(a_I),w(\alpha_U,a_I))]\\&\qquad=\sum_{a_I\in S_I^*}\alpha_I(a_I)[l(p,(v_0,v_1))-l(p(a_I),(v_0,v_1))]\\&\qquad\qquad+\sum_{a_I\in S_I^*}\alpha_I(a_I)\big[(l(p,w(\alpha_U,a_I))-l(p,(v_0,v_1)))\\&\qquad\qquad\qquad-(l(p(a_I),w(\alpha_U,a_I))-l(p(a_I),(v_0,v_1)))\big].\end{aligned}$$

The first term above is equal to 0 because l is affine in p and $\sum_{a_I\in S_I^*}\alpha_I(a_I)p(a_I)=p$. By biaffinity, the second term is not smaller than

$$\begin{aligned}&0-2C'\sum_{a_I\in S_I^*}\alpha_I(a_I)\|w(\alpha_U,a_I)-(v_0,v_1)\|\big|p-p(a_I)\big|\\&\qquad\geq-4C'M\frac{1-\delta}{\delta}\sum_{a_I\in S_I^*}\alpha_I(a_I)\big|p-p(a_I)\big|,\end{aligned}$$

where C' is defined by

$$C'=\sup_{\|v\|\leq 1}\max(|l(0,v)|,|l(1,v)|).$$

Also, observe that $\|w(\alpha_U,a_I)-(v_0,v_1)\|\leq 2((1-\delta)/\delta)M$.

Putting the two bounds together leads to

$$\sum_{a_I\in S_I^*}\alpha_I(a_I)(v_U(\alpha_U,a_I)-l(p(a_I),w(\alpha_U,a_I))-(v_U-l(p,(v_0,v_1))))$$
$$\geq \frac{1-\delta}{\delta}\Big(\frac{\varepsilon}{2}-4C'M\sum_{a_I\in S_I^*}\alpha_I(a_I)|p-p(a_I)|\Big).$$

Let

$$C=32C'^2M^2. \tag{B.4}$$

Observe that for any $x\geq 0$, $\frac{1}{4}\varepsilon-4CMx>-(C'/\varepsilon)x^2$, and

$$\frac{\varepsilon}{4}-4C'M\sum_{a_I\in S_I^*}\alpha_I(a_I)|p-p(a_I)|=\sum_{a_I\in S_I^*}\alpha_I(a_I)\Big(\frac{\varepsilon}{4}-4C'M|p-p(a_I)|\Big)$$
$$>-\frac{C}{\varepsilon}\sum_{a_I\in S_I^*}\alpha_I(a_I)(p(a_I)-p)^2$$

Then, for any for any $a_I\in S_I^*$,

$$\frac{1-\delta}{\delta}\Big(\frac{\varepsilon}{2}-4C'M\sum_{a_I\in S_I^*}\alpha_I(a_I)|p-p(a_I)|\Big)>-\frac{1-\delta}{\delta}\frac{C}{\varepsilon}\sum_{a_I\in S_I^*}\alpha_I(a_I)(p(a_I)-p)^2+\frac{1-\delta}{4\delta}\varepsilon$$
$$=\frac{1-\delta}{\delta}\frac{C}{\varepsilon}\Big(p^2-\sum_{a_I\in S_I^*}\alpha_I(a_I)(p(a_I))^2\Big)+\frac{1-\delta}{4\delta}\varepsilon.$$

B.3 *Proof of Proposition 6*

It is easy to see that $ICR^*\in\mathcal{I}$. The proof is divided into a few steps.

B.3.1 *ICR* satisfies FE-1*

Define $I\subseteq[0,1]\times\mathbb{R}^3$:

$$I=\Big\{\Big(p,\frac{p-p_0}{p_1-p_0}v_U^1+\frac{p_1-p}{p_1-p_0}v_U^0,v_0,v_1\Big):$$
$$(v_U^j,v_0,v_1)\in ICR^*(p_j)\text{ for }j=0,1\text{ and }p_0<p<p_1\Big\}.$$

Then I satisfies FE-1. One easily checks that I satisfies properties ICR-1, ICR-2, and ICR-3 and $I\in\mathcal{I}$. Since $I\supseteq ICR^*$, it must be that $I=ICR^*$.

B.3.2 *ICR* satisfies FE-3*

Define $I\subseteq[0,1]\times\mathbb{R}^3$: for each p,

$$I(p)=\operatorname{con}[ICR^*(p)\cup V]\cap IR.$$

Then $I\supseteq ICR^*$ and I satisfies FE-3. I show that $I\in\mathcal{I}$. Conditions ICR-1 and ICR-2 hold by definition (notice that $\operatorname{con}(E_k\cup V)\cap IR=E_k$ for each $k=0,1$). To see that the separation property ICR-3 holds as well, suppose that $p_0<p_1$ and there is a biaffine

function $l:[0,1]\times\mathbb{R}^2\to\mathbb{R}$ such that $v_U^j\leq l(p_j,v_0^j,v_1^j)$ for any $(v_U^j,v_0^j,v_1^j)\in I(p_j)\cup V$, $j=0,1$. Because l is biaffine, for any $v\in V$ and any $p\in(p_0,p_1)$,

$$v_U\leq \frac{p-p_0}{p_1-p_0}l(p_1,v_0,v_1)+\frac{p_1-p}{p_1-p_0}l(p_0,v_0,v_1)=l(p,v_0,v_1).$$

By property ICR-3 applied to the set ICR^*, for any $p\in(p_0,p_1)$ and any $(v_U,v_0,v_1)\in ICR^*(p)$,

$$v_U\leq l(p,v_0,v_1). \tag{B.5}$$

Therefore, inequality (B.5) holds also for any $(v_U,v_0,v_1)\in \mathrm{con}[ICR^*(p)\cup V]=I(p)$.

B.3.3 *ICR^* is closed* I show that $\mathrm{cl}\,I\in\mathscr{I}$ for any $I\in\mathscr{I}$. A simple continuity argument shows that ICR-1 and ICR-3 hold for $\mathrm{cl}\,I$. I show that $(\mathrm{cl}\,I)(1)\subseteq E_1$ (the argument for $(\mathrm{cl}\,I)(0)\subseteq E_0$ is analogous). Suppose that there is $(p^n,v_U^n,v_0^n,v_1^n)\in\mathscr{I}$ such that $(p^n,v_U^n,v_0^n,v_1^n)\to(1,v_U^*,v_0^*,v_1^*)$ and $(v_U^*,v_0^*,v_1^*)\notin E_1$. Because $(v_U^*,v_0^*,v_1^*)\in IR$, it must be that $(v_U^*,v_0^*,v_1^*)\notin \mathrm{con}(E_1\cup V)$. Because the sets E_1 and V are bounded, there is $\phi\in\Phi^3$ and $\phi_U>0$ such that either

$$\begin{aligned}\phi\cdot(v_U^*,v_0^*,v_1^*)&>\max_{(v_U,v_0,v_1)\in\mathrm{con}(E_1\cup V)}\phi\cdot(v_U,v_0,v_1)\\ \text{or }\phi\cdot(v_U^*,v_0^*,v_1^*)&<\min_{(v_U,v_0,v_1)\in\mathrm{con}(E_1\cup V)}\phi\cdot(v_U,v_0,v_1).\end{aligned}$$

Assume without loss of generality that the first inequality holds, and define

$$\begin{aligned}c^0&=\frac{1}{\phi_U}\max_{(v_U,v_0,v_1)\in\mathrm{con}(E_1\cup V)}\phi\cdot(v_U,v_0,v_1)\\ c^1&=\frac{1}{\phi_U}\max_{(v_U,v_0,v_1)\in\mathrm{con}(E_1\cup V)}\phi\cdot(v_U,v_0,v_1)\\ l(p,v_0,v_1)&=(1-p)c^0+pc^1-\frac{1}{\phi_U}(\phi_0,\phi_1)\cdot(v_0,v_1).\end{aligned}$$

Then, for $k=0,1$, any $(v_U,v_0,v_1)\in\mathrm{con}(E_1\cup V)$,

$$v_U=\frac{1}{\phi_U}\phi\cdot(v_U,v_0,v_1)-\frac{1}{\phi_U}(\phi_0,\phi_1)\cdot(v_0,v_1)\leq l(k,v_0,v_1).$$

Property ICR-3 applied to the set ICR^* implies that $v_U^n\leq l(p^n,v_0^n,v_1^n)$ for all n. But this leads to a contradiction:

$$\lim_{n\to\infty}v_U^n-l(p^n,v_0^n,v_1^n)=v_U^*-l(1,v_0^*,v_1^*)=\frac{1}{\phi_U}\phi\cdot(v_U^*,v_0^*,v_1^*)-c^1>0.$$

B.3.4 *ICR^* contains ICR payoffs* Define the set $I\subseteq[0,1]\times\mathbb{R}^3$ as *the smallest* closed set such that

(a) $ICR\subseteq I$

(b) for any $k\in\{0,1\}$, $E_k\subseteq I(k)$

(c) I satisfies FE-1 and FE-3.

The set I is well-defined as the intersection of all closed sets with properties (a)–(c). I show that $I \in \mathscr{I}$, which implies that $ICR \subseteq I \subseteq ICR^*$.

Property ICR-1 holds by Corollary 1. The next two subsections establish properties ICR-2 and ICR-3.

B.3.5 *I satisfies property ICR-2.* For each $\phi \in \Phi^3$ and $\phi_U \neq 0$, for $k = 0, 1$ define $c^k_\phi = \sup_{v \in E_k \cup V} \phi \cdot v$. Define $I' \subseteq [0,1] \times \mathbb{R}^3$ by

$$I'(p) = \bigcap_{\phi \in \Phi^3} \{\phi \cdot v \leq (1-p)c^0_\phi + p c^1_\phi\}.$$

Then, I' is closed. Clearly, it satisfies properties (a), (b), and FE-3. Property FE-1 follows from Proposition 5. For each $k = 0, 1$, $I'(k) \subseteq E_k$. Also, by definition, $I \subseteq I'$. Hence, for $k = 0, 1$, $I(k) \subseteq I'(k) \subseteq E_k$.

B.3.6 *I satisfies property ICR-3.* I show that the correspondence I satisfies ICR-3 with inequality "$<$" (the other case is analogous). I start with a lemma.

Lemma 12. *Suppose that for each $p_0 < p_1$ and biaffine function l, if*

$$v_U \leq l(p_j, v_0, v_1) \text{ for each } v \in I(p_j) \cup V \text{ and } j = 0, 1,$$

then $v_U \leq l(p, v_0, v_1)$ for each $v \in ICR(p)$ and $p \in (p_0, p_1)$. Then the correspondence I satisfies property ICR-3.

Proof. Suppose to the contrary that there are $p_0 < p_1$, a biaffine function l, $p \in (p_0, p_1)$, and $v \in I(p)$ such that $v^j_U \leq l(p_j, v^j_0, v^j_1)$ for each $v^j \in I(p_j) \cup V$ and $v_U > l(p, v_0, v_1)$. Define $I' \subseteq [0,1] \times \mathbb{R}^3$ by

$$I' = \{(p, v_U, v_0, v_1) \in I : \text{if } p_0 \leq p \leq p_1, \text{ then } v_U \leq l(p, v_0, v_1)\}.$$

Then, $I' \subsetneq I$ and I' is closed. By the hypothesis of the lemma, I' satisfies property (a). Clearly, I' satisfies property (b) and FE-3. An application of Lemma 11 shows that I' satisfies property FE-1. But this contradicts the definition of I as the smallest closed set that satisfies properties (a)–(c). □

Therefore, if I violates ICR-3, then there are $\varepsilon > 0$, $p_0 < p^* < p_1$, a biaffine function l, and $v \in ICR(p^*)$ such that $v^j_U \leq l(p_j, v^j_0, v^j_1)$ for each $v^j \in I(p_j) \cup V$ and $j = 0, 1$, and $v_U \geq l(p^*, v_0, v_1) + 11\varepsilon$. Choose sequences $\delta_n \to 1$ and $(p^n, v^n_U, v^n_0, v^n_1) \in ICR_{\delta_n}$ such that

$$(p^n, v^n_U, v^n_0, v^n_1) \to (p^*, v^*_U, v^*_0, v^*_1) \in ICR$$
$$v^*_U \geq l(p^*, v^*_0, v^*_1) + 10\varepsilon$$

and, for any n,

$$(p^n, v^n_U, v^n_0, v^n_1) \in \operatorname*{arg\,max}_{p_0 \leq p \leq p_1, (p, v_U, v_0, v_1) \in ICR_{\delta_n}} v_U - \left(l(p, v_0, v_1) - \frac{1-\delta_n}{\delta_n} \frac{D}{\varepsilon} p^2 \right), \quad \text{(B.6)}$$

where D is the constant defined in equation (B.4). For each n, let $(\alpha_U^n, \alpha_0^n, \alpha_1^n)$ be the profile of first-period mixed actions, let $v^n(a_U, a_I)$ be the continuation payoffs, and let $p^n(a_I)$ be the posterior beliefs (posteriors are well-defined only after the action is played with positive probability). For any $a_U \in S_U$, $a_I \in S_I$, and any type $k = 0, 1$, define

$$w_k^n(a_U, a_I) = v_k^n(a_U, a_I) + \frac{1}{\delta}[v_k - (1-\delta_n) g_k(a_U, a_I) - \delta_n v_k^n(\alpha_U^n, a_I)]. \tag{B.7}$$

These continuation payoffs would make the two types of player I indifferent between any action a_I. Observe that, for any a_I,

$$\begin{aligned} & w_k^n(\alpha_U^n, a_I) \geq v_k^n(\alpha_U^n, a_I) \\ & w_k^n(\alpha_U^n, a_I) = v_k^n(\alpha_U^n, a_I), \text{ if } p^n(a_I) \in (0,1) \cup \{k\}. \end{aligned} \tag{B.8}$$

I show below (Section B.3.7) that for all sufficiently high n, if a_I is played with positive probability and $p^n(a_I) \notin (p_0, p_1)$, then

$$v_U^n(\alpha_U^n, a_I) - l(p^n(a_I), w_0^n(\alpha_U^n, a_I), w_1^n(\alpha_U^n, a_I)) < \varepsilon. \tag{B.9}$$

The proof is concluded in the same way as the proof of Proposition 5. As in Section B.2, one can show that for a high enough δ, inequality (17) holds for any ICR payoff $(v_U, v_0, v_1) \in ICR_\delta(p)$, $p_0 < p < p_1$, such that $v_U - l(p, v_0, v_1) \geq \varepsilon$. In particular, there is an a_I such that

$$\begin{aligned} v_U^n(\alpha_U^n, a_I) - \Big(l(p^n(a_I), w_0^n(\alpha_U^n, a_I), w_1^n(\alpha_U^n, a_I)) - \frac{1-\delta_n}{\delta_n}\frac{D}{\varepsilon}(p^n(a_I))^2 \Big) \\ > v_U^n - \Big(l(p^n, v_0^n, v_1^n) - \frac{1-\delta_n}{\delta_n}\frac{C}{\varepsilon}(p^n)^2 \Big). \end{aligned}$$

Because of (B.9), $p^n(a_I) \in (p_0, p_1)$. Then, by (B.8),

$$(v_U^n(a_U, a_I), w_0^n(a_U, a_I), w_1^n(a_U, a_I)) = (v_U^n(a_U, a_I), v_0^n(a_U, a_I), v_1^n(a_U, a_I))$$

is a continuation ICR payoff for all a_U played by U with positive probability. Hence there is a profile (a_U, a_I) after which ICR continuation payoffs satisfy

$$\begin{aligned} v_U^n(a_U, a_I) - \Big(l(p^n(a_I), v_0^n(a_U, a_I), v_1^n(a_U, a_I)) - \frac{1-\delta_n}{\delta_n}\frac{C}{\varepsilon}(p^n(a_I))^2 \Big) \\ > v_U^n - \Big(l(p^n, v_0^n, v_1^n) - \frac{1-\delta_n}{\delta_n}\frac{C}{\varepsilon}(p^n)^2 \Big) \end{aligned}$$

and $p^n(a_I) \in (p_0, p_1)$. But this contradicts the choice of $(p^n, v_U^n, v_0^n, v_1^n)$ as the maximizers of expression (B.6).

B.3.7 *(B.9) is satisfied for a high enough n* Suppose not. Then one can find an action a_I and subsequences (denoted further as a sequence of ns) such that a_I is played with positive probability for all n, $p^n(a_I) \to p^*(a_I) \notin (p_0, p_1)$, $\alpha_U^n \to \alpha_U^*$, $v_U^n(a_U, a_I) \to v_U^*(a_U, a_I)$, and $w_k^n(a_U, a_I) \to w_k^*(a_U, a_I)$, and such that

$$\varepsilon \leq v_U^*(\alpha_U, a_I) - l(p^*(a_I), w_0^*(\alpha_U^*, a_I), w_1^*(\alpha_U^*, a_I)).$$

Assume without loss of generality that $p^*(a_I) \leq p_0 < 1$ (the other case, $p^*(a_I) \geq p_1$, is treated analogously). By definition (B.7), $\|w_k^n(\alpha_U^n, a_I) - (v_0^n, v_1^n)\| \leq 2M(1-\delta_n)/\delta_n$ for each n. This implies that $w(\alpha_U^*, a_I) = (v_0^*, v_1^*)$ and

$$\varepsilon \leq v_U^*(\alpha_U, a_I) - l(p^*(a_I), v_0^*, v_1^*).$$

Lemma 13. $(v_U^*(\alpha_U^*, a_I), v_0^*, v_1^*) \in I(p^*(a_I))$.

Proof. Suppose first that $p^*(a_I) > 0$. Because $p^*(a_I) < 1$, this means that for sufficiently large n, a_I is played with positive probability by each type of player I. Thus $(v_U^n(\alpha_U^n, a_I), v_0^n(\alpha_U^n, a_I), v_1^n(\alpha_U^n, a_I))$ is a convex combination of payoffs in ICR profiles. Moreover, each type of player I has to be indifferent between a_I and any other equilibrium action. Because the first-period payoffs from playing a_I converge to 0 when $n \to \infty$, this means that $\lim_{n\to\infty} v_k^n(\alpha_U^n, a_I) = v_k^*$ for each $k = 0, 1$. Hence $(v_U^*(\alpha_U^*, a_I), v_0^*, v_1^*) \in ICR(p^*(a_I)) \subseteq I(p^*(a_I))$.

Next, suppose that $p^*(a_I) = 0$. By property (b) of the set I, it is sufficient to show that

$$(v_U^*(\alpha_U^*, a_I), v_0^*, v_1^*) \in E_0 = IR \cap E^{F,IC}(0).$$

Clearly, $v_U^*(\alpha_U^*, a_I) = \lim_{n\to\infty} v_U^n(\alpha_U^n, a_I) \geq m_U$ and $(v_0^*, v_1^*) = \lim_{n\to\infty}(v_0^n, v_1^n) \in \operatorname{proj} IR$.

Let $v^n(\alpha_U^n, (a_I, \sigma_0))$ denote the vector of expected payoffs if player I chose a_I in the first period and then mimicked the strategy σ_0 of type 0 (the expectation is taken with respect to the mixed action α_U^n of player U). Observe that $v^n(\alpha_U^n, (a_I, \sigma_0)) \in V$. Because the difference between $v_U^n(\alpha_U^n, (a_I, \sigma_0))$ and $v_U^n(\alpha_U^n, a_I)$ is not higher than the first-period payoffs,

$$\left\| v_U^n(\alpha_U^n, (a_I, \sigma_0)) - v_U^n(\alpha_U^n, a_I) \right\| \leq 2M \frac{1-\delta_n}{\delta_n} \text{ for each } n.$$

Because a_I is played with positive probability by type 0, for each n,

$$v_0^n(\alpha_U^n, (a_I, \sigma_0)) = v_0^n \text{ and } v_1^n(\alpha_U^n, (a_I, \sigma_0)) \leq v_1^n.$$

Let $\bar{v}^* = \lim_{n\to\infty} v^n(\alpha_U^n, (a_I, \sigma_0)) \in V$ (note that the limit exists, possibly after taking subsequences). Hence $\bar{v}_U^* = v_U^*(\alpha_U^*, a_I)$, $\bar{v}_0^* = v_0^*$, and $\bar{v}_1^* \leq v_1^*$. This implies that we have $(v_U^*(\alpha_U^*, a_I), v_0^*, v_1^*) \in E^{F,IC}(0)$. □

Define

$$\bar{v} = \frac{p_0 - p^*(a_I)}{p^* - p^*(a_I)} (v_U^*(\alpha_U^*, a_I), v_0^*, v_1^*) + \frac{p^* - p_0}{p^* - p'(a_I)} v^*.$$

Observe that $\bar{v} \in I(p_0)$ because $p^*(a_I) \leq p_0 < p^*$ and because of the above lemma, the fact that $v^* \in I(p^*)$, $p^*(a_I) \leq p_0 < p^*$, and the fact that the correspondence I satisfies property FE-1. By the choice of a biaffine function l, $\bar{v}_U \leq l(p_0, \bar{v}_0, \bar{v}_1)$. However, notice that

$$\begin{aligned}0 &\geq \bar{v}_U - l(p_0, \bar{v}_0, \bar{v}_1)\\ &= \frac{p_0 - p^*(a_I)}{p^* - p^*(a_I)}(v_U^*(\alpha_U, a_I) - l(p^*(a_I), v_0^*, v_1^*)) + \frac{p^* - p_0}{p^* - p^*(a_I)}(v_U^* - l(p^*, v_0^*, v_1^*))\\ &\geq \frac{p_0 - p^*(a_I)}{p^* - p^*(a_I)}\varepsilon + \frac{p^* - p_0}{p^* - p^*(a_I)}10\varepsilon \geq \varepsilon > 0.\end{aligned}$$

The contradiction shows that for a high enough n, if $p^n(a_I) \leq p_0$ or $p^n(a_I) \geq p_1$, then (B.9) must hold. This ends the proof of the proposition.

C. Proof of Lemma 5

This section is devoted to the proof of Lemma 5. Section C.1 proves some geometrical results. Section C.2 defines all the constants and states a helpful assumption. Section C.3 proves the lemma given the assumption. Sections C.4 and C.5 fill in some missing steps. Section C.6 shows how to extend Theorem 3 to games that do not satisfy the assumption in Section C.2.

C.1 *Geometrical results*

Suppose that $A \subseteq \mathbb{R}^2$ is a (not necessarily bounded) set with finitely many extreme points $A_{\text{extr}} \subseteq A$. Then there exists a finite set of unitary vectors $\Phi_A \subseteq \Phi^2$ such that $A = \bigcap_{\phi \in \Phi_A} \{a \in \mathbb{R}^2 : \phi \cdot a \leq \sup_{a' \in A} \phi \cdot a'\}$.

Lemma 14. *Suppose that $A \subseteq \mathbb{R}^2$ is a (not necessarily bounded) set with finitely many extreme points $A_{extr} \subseteq A$. Suppose that $B \subseteq \mathbb{R}^2$ is a finite set such that $A \cap \operatorname{con} B = \varnothing$. Then there is a constant $C_B < \infty$ such that for any $a^* \in A$, if*

$$A \cap \operatorname{con}(B \cup \{a^*\}) = \{a^*\}, \tag{C.1}$$

then, for any $b \in \operatorname{con} B$ and $\alpha \in [0,1]$,

$$\alpha \leq C_B \inf_{a' \in A} \left\| (1-\alpha)a^* + \alpha b - a' \right\|.$$

Proof. Given the assumptions about the sets A and B, there is a finite set $\Phi_B^* \subseteq \Phi^2$ of unit vectors so that, whenever $a^* \in A$ and (C.1) holds, there is $\phi \in \Phi_B^*$ so that

$$\sup_{a' \in A} \phi \cdot a' = \phi \cdot a^* < \min_{b \in B} \phi \cdot b.$$

Indeed, for any $a \in A_{\text{extr}}$, either $A \cap \operatorname{con}(B \cup \{a\}) \neq \{a\}$ or there is ϕ_a such that

$$\phi_a \cdot a = \max_{a' \in A} \phi_a \cdot a' < \min_{b \in B} \phi_a \cdot b.$$

Define

$$\Phi_B^* = \{\phi_a : A \cap \operatorname{con}(B \cup \{a\}) = \{a\}\} \cup \{\phi \in \Phi_A : \max_{a' \in A} \phi \cdot a' < \min_{b \in B} \phi \cdot b\}.$$

It is easy to check that Φ_B^* has the required property.

Suppose now that (C.1) holds for some $a^* \in A$. Then, for any $b \in \operatorname{con} B$, $\alpha \in [0,1]$, and any ϕ,

$$\inf_{a' \in A} \left\| (1-\alpha)a^* + \alpha b - a' \right\| \geq \phi \cdot ((1-\alpha)a^* + \alpha b) - \sup_{a' \in A} \phi \cdot a'$$
$$\geq \alpha [\inf_{b' \in B} \phi \cdot b' - \sup_{a' \in A} \phi \cdot a'] + (1-\alpha)[\phi \cdot a^* - \sup_{a' \in A} \phi \cdot a].$$

By the remark above, there exists $\phi \in \Phi_B^*$ so that the first term in the last line of the inequality is strictly positive and the second term is equal to 0. For such ϕ,

$$\alpha \leq \frac{1}{\min_{b' \in B} \phi \cdot b' - \sup_{a' \in A} \phi \cdot a'} \sup_{a' \in A} \left\| (1-\alpha)a^* + \alpha b - a' \right\|.$$

Define

$$C_B = \max_{\phi \in \Phi_B^* : \min_{b \in B'} \phi \cdot b' > \sup_{a' \in A} \phi \cdot a'} \frac{1}{\min_{b' \in B} \phi \cdot b' - \sup_{a' \in A} \phi \cdot a'}.$$

This constant is well-defined, as a maximum over finitely many finite constants. □

Say that $v^* \in \mathbb{R}^3$ *is represented by* tuple (v, α, ρ), $v \in \mathbb{R}^3, \alpha \in [0,1]$, $\rho \in \Delta B$, where $B \subseteq \mathbb{R}^3$ is a finite set, if

$$v^* = \alpha \sum_{b \in B} b \rho(b) + (1-\alpha) v.$$

Lemma 15. *Suppose that v^* is represented by $(\bar{v}, \alpha, \rho)$ for some $\alpha \in (0,1)$. Then there is an open neighborhood $U \ni \bar{v}$ such that for any $\bar{v}' \in U \cap \operatorname{con}(\operatorname{supp} \rho \cup \{\bar{v}\})$, v^* is represented by $(\bar{v}', \alpha', \rho')$ for some $\alpha' < \alpha$ and $\rho' \in \Delta(\operatorname{supp} \rho)$.*

Proof. Let $U_V \subseteq \operatorname{con}(\operatorname{supp} \rho \cup \{\bar{v}\})$ be defined by

$$U_V = \left\{ \alpha'' \sum_{v \in \operatorname{supp} \rho'} v \rho''(v) + (1-\alpha'')\bar{v} : \alpha'' \in [0,\alpha), \rho'' \in \Delta \operatorname{supp} \rho, \right.$$
$$\left. \text{and } \rho(v) > \frac{(1-\alpha)\alpha''}{\alpha(1-\alpha'')} \rho''(v) \text{ for each } v \in \operatorname{supp} \rho. \right\}.$$

Then $\bar{v} \in U_V$ and U_V is open relative to $\operatorname{con}(\operatorname{supp} \rho \cup \{\bar{v}\})$. Hence, there exists an open set $U \subseteq \mathbb{R}^3$ such that $U_V = U \cap \operatorname{con}(\operatorname{supp} \rho \cup \{\bar{v}\})$.

Take any $\bar{v}' \in U_V$ and $\bar{v}' \neq \bar{v}$ and find $\alpha'' \in [0,\alpha)$ and $\rho'' \in \Delta \operatorname{supp} \rho$ such that

$$\bar{v}' = \alpha'' \sum_{v \in \operatorname{supp} \rho'} v \rho''(v) + (1-\alpha'')\bar{v}$$
$$\rho(v) > \frac{(1-\alpha)\alpha''}{\alpha(1-\alpha'')} \rho''(v) \text{ for each } v \in \operatorname{supp} \rho.$$

Let

$$\alpha' = \frac{\alpha - \alpha''}{1-\alpha''},$$

which is less than α because $\alpha<1$, and

$$\rho'(v)=\frac{1}{\alpha-\alpha''}[\alpha(1-\alpha'')\rho(v)-(1-\alpha)\alpha''\rho''(v)]>0$$

Then $\rho'\in\Delta(\operatorname{supp}\rho)$ is a probability measure with support contained in $\operatorname{supp}\rho$ and $(\bar{v}',\alpha',\rho')$ represents v^*:

$$\alpha'\sum_{v\in\operatorname{supp}\rho'}v\rho'(v)+(1-\alpha')\bar{v}'=\alpha'\frac{1}{\alpha-\alpha''}\sum_{v\in\operatorname{supp}\rho'}v[\alpha(1-\alpha'')\rho(v)-(1-\alpha)\alpha''\rho''(v)]$$
$$+\frac{1-\alpha}{1-\alpha''}\left(\alpha''\sum_{v\in\operatorname{supp}\rho'}v\rho''(v)+(1-\alpha'')\bar{v}\right)=v^*.\quad\square$$

C.2 *Assumption and constants*

Throughout Sections C.3, C.4, and C.5 of the Appendix I make the following assumption.

Assumption 2. proj IR has finitely many extreme points.

This assumption implies that there is a finite set of unitary vectors $\Phi_{IR}\subseteq\Phi^2$ such that

$$\operatorname{proj}IR=\bigcap_{\phi\in\Phi_{IR}}\{(v_0,v_1)\in\mathbb{R}^2:\phi\cdot(v_0,v_1)\geq m_I(\phi)\}.$$

The assumption is restrictive. Initially, I prove that Lemma 5 holds for all games that satisfy the assumption. Together with the analysis of Section 5, this demonstrates Theorem 3 for all games satisfying the assumption. In Section C.6 I extend Theorem 3 to all games.

Define the closed sets

$$V^+=V\cap\{v:v_U\geq m_U\}\text{ and }V^-=V\cap\{v:v_U\leq m_U\}.$$

Both sets are convex and spanned by finitely many vertices. Let V^+_{vert} and V^-_{vert} consist of the vertices of, respectively, V^+ and V^-. Let

$$V^*=V^+_{\text{vert}}\cup V^-_{\text{vert}}.$$

Then V^* is finite. Let $\mathscr{B}$ be the set of all finite subsets of proj $V^*\subseteq\mathbb{R}^2$ such that $\operatorname{con}B\cap\operatorname{proj}IR=\varnothing$. For any $B\in\mathscr{B}$, let C_B be the constant from Lemma 14 applied to the set $A=\operatorname{proj}IR$. (Notice that proj IR satisfies the hypotheses of the lemma because of the assumption.) Let

$$C_F=\max_{B\in\mathscr{B}}C_B<\infty. \tag{C.2}$$

Consider a finite class of closed convex sets indexed by subsets $\Phi\subseteq\Phi_{IR}$:

$$S(\Phi)=\bigcap_{\phi\in\Phi}\{v\in V:\phi\cdot(v_0,v_1)\leq m_I(\phi)\}.$$

Observe that for each $\Phi \subseteq \Phi_{IR}$, $S(\Phi)$ is spanned by a finite set of vertices $S_{\text{vert}}(\Phi)$. Define

$$C_I = m_U - \max_{\Phi \subseteq \Phi_{IR}} \{v_U : (v_U, v_0, v_1) \in S_{\text{vert}}(\Phi), v_U < m_U\} > 0. \tag{C.3}$$

The number C_I is strictly positive because Φ_{IR} is finite and each $S_{\text{vert}}(\Phi)$ is finite.

Define

$$L = \sup_{(v_0,v_1),(v_0',v_1') \in \text{proj}\, ICR^*(1)} \frac{\left|u^I(v_0,v_1)(1) - u^I(v_0',v_1')(1)\right|}{\left\|(v_0,v_1)-(v_0',v_1')\right\|} < \infty. \tag{C.4}$$

The constant L is finite because the set $ICR^*(1) = E_1$ is the convex hull of finitely many vertices.

Recall that M is the uniform bound on the stage-game payoffs defined in (1). Let

$$C_0 = \max\left(\frac{2}{1-p^u}[L(1+MC_F)+2MC_F], 2\frac{LM}{\frac{1}{4}(1-p^u)C_I}\right) \tag{C.5}$$

$$\theta^* = \min\left(\frac{1-p^u}{2C_0}, \frac{1}{L(1+MC_F)}, \frac{1}{C_0}\right)$$

$$d^* = \min\left(\frac{1}{2(C_I LM+1)}, \frac{(1-p^u)C_I}{2LM}, \frac{1}{C_0}\right)$$

$$D = \frac{4M^2(2C_0+2+L)}{(1-p^u)^2 C_I^2} + 2C_0. \tag{C.6}$$

C.3 *Proof of Lemma 5 (given Assumption 2)*

In order to shorten the notation, I write $v_{0,1} = (v_0, v_1) \in \mathbb{R}^2$ for any $v = (v_U, v_0, v_1) \in \mathbb{R}^3$. Assume that the hypothesis of Lemma 5 is satisfied and, in particular,

$$p' - p \leq d^* \text{ and } \theta(p) \leq \theta^*.$$

Denote

$$\theta = \theta(p).$$

Take any $y^I \in I_1(p')$ for some $p' > p$. I show that there is $y^F \in F(p')$ such that

$$\left\|y_{0,1}^I - y_{0,1}^F\right\| \leq \left(\theta + D(\theta^2 + (p'-p)\theta + (p'-p)^2)\right)(u^I(1)(y_{0,1}^I) - m_U). \tag{C.7}$$

This establishes (22) and the lemma.

The main objective of the proof is to bound the distances between a number of payoff vectors. It is helpful to list all the variables involved, together with the relationships between them. There are two types of payoff vectors: for those associated with prior beliefs p, I write $x^I, x^{I,a}, x^{F,a} \in \mathbb{R}^3$, and for those associated with p', I write $y^I, y^{I,a}, y^F, y^{F,a} \in \mathbb{R}^3$. The following hold:

$$x_{0,1}^A = y_{0,1}^A \text{ for any } A = I, (I,a), F, (F,a)$$
$$x^I = \alpha^I v^I + (1-\alpha^I) x^{I,a}$$
$$y^I = \alpha^I v^I + (1-\alpha^I) y^{I,a}$$
$$y^F = \alpha^F v^I + (1-\alpha^F) y^{F,a}$$

for some $v^I \in V$ and $\alpha^I, \alpha^F \in [0,1]$.

The rest of the proof is divided into six steps. All constants are defined in Section C.2. Lemmas 16 and 17 are proved in Sections C.4 and C.5, respectively.

Step 1. Define $x^I = (u^I(p)(y^I_{0,1}), y^I_{0,1})$. Because $y^I \in I_1(p')$, it must be that $u^I(1)(y^I_{0,1}) \geq m_U$. Then

$$x^I_U = u^I(p)(y^I_{0,1}) \leq m_U; \tag{C.8}$$

otherwise, by part (iii) of Lemma 2, $u^I(p')(y^I_{0,1}) > m_U$, which contradicts $y^I \in I_1(p')$.

Lemma 16. *There exist $x^{I,a} \in I_1(p)$, $\alpha^I \in [0,1]$, and $v^I \in V$ such that $v^I \leq m_U - C_I$ and*

$$x^I = \alpha^I v^I + (1-\alpha^I)x^{I,a}.$$

For future reference, note that

$$\left\|x^{I,a}_{0,1} - x^I_{0,1}\right\| \leq \alpha^I M. \tag{C.9}$$

Step 2. Due to the concavity of the function u^I in $v_{0,1}$,

$$\begin{aligned}\alpha^I v^I_U + (1-\alpha^I)x^{I,a}_U = x^I_U = u^I(p)(x^I_{0,1}) &\geq \alpha^I u^I(p)(v^I_{0,1}) + (1-\alpha^I)u^I(p)(x^{I,a}_{0,1})\\ &\geq \alpha^I v^I_U + (1-\alpha^I)x^{I,a}_U,\end{aligned}$$

and all inequalities can be turned into equalities. Note that

$$x^{I,a}_U = u^I(p)(x^{I,a}_{0,1}) = m_U,$$

because $x^{I,a} \in I_1(p)$. Therefore, part (iii) of Lemma 2 implies that

$$u^I(p')(x^{I,a}_{0,1}) = m_U + \frac{p'-p}{1-p}\left[u^I(1)(x^{I,a}_{0,1}) - m_U\right]. \tag{C.10}$$

By the definition of the constant L in (C.4) and by (C.9),

$$u^I(1)(x^I_{0,1}) \leq u^I(1)(x^{I,a}_{0,1}) + L\left\|x^{I,a}_{0,1} - x^I_{0,1}\right\| \leq u^I(1)(x^{I,a}_{0,1}) + \alpha^I LM. \tag{C.11}$$

The above conditions and the convexity of u^I in p imply that

$$\begin{aligned}m_U = y^I_U = u^I(p')(x^I_{0,1}) &\leq \frac{p'-p}{1-p}u^I(1)(x^I_{0,1}) + \frac{1-p'}{1-p}u^I(p)(x^I_{0,1})\\ &\leq \frac{p'-p}{1-p}u^I(1)(x^{I,a}_{0,1}) + \frac{p'-p}{1-p}\alpha^I LM + \frac{1-p'}{1-p}(1-\alpha^I)x^{I,a}_U + \frac{1-p'}{1-p}\alpha^I v^I_U\\ &= m_U + \frac{p'-p}{1-p}\left[u^I(1)(x^{I,a}_{0,1}) - m_U\right] - \alpha^I\left[\frac{1-p'}{1-p}(m_U - v^I_U) - \frac{1-p'}{1-p}LM\right]\\ &= u^I(p')(x^{I,a}_{0,1}) - \alpha^I\left[\frac{1-p'}{1-p}(m_U - v^I_U) - \frac{1-p'}{1-p}LM\right].\end{aligned}$$

This leads to a bound on α^I:

$$\alpha^I \leq \alpha^{I*} = \frac{u^I(p')(x_{0,1}^{I,a}) - m_U}{\frac{1-p'}{1-p}(m_U - v_U^I) - \frac{p'-p}{1-p} LM} \tag{C.12}$$

$$= \frac{u^I(1)(x_{0,1}^{I,a}) - m_U}{(1-p')(m_U - v_U^I) - (p'-p)LM}(p'-p),$$

where the equality in the second line comes from (C.10).

Step 3. The bound (C.12) implies that

$$\alpha^I \leq \frac{u^I(1)(x_{0,1}^{I,a}) - m_U}{\frac{1}{4}(1-p^u)C_I}(p'-p)$$

for $p'-p \leq d^* \leq (1-p^u)C_I/(2LM)$. Because of (C.11) and the equality $y_{0,1}^I = x_{0,1}^I$,

$$u^I(1)(y_{0,1}^I) - m_U \geq \left(1 - \tfrac{1}{2}C_0(p'-p)\right)\left(u^I(1)(x_{0,1}^{I,a}) - m_U\right).$$

(Recall the definition of the constant C_0 in (C.5).) For $p'-p \leq d^* \leq 1/C_0$, we have $(1 - \frac{1}{2}C_0(p'-p))^{-1} \leq (1 + C_0(p'-p))$. Hence

$$u^I(1)(x_{0,1}^{I,a}) - m_U \leq (1 + C_0(p'-p))(u^I(1)(y_{0,1}^I) - m_U). \tag{C.13}$$

Step 4. By the hypothesis of the lemma, there is $x^{F,a} \in F^*(p)$ such that

$$\left\| x_{0,1}^{I,a} - x_{0,1}^{F,a} \right\| \leq (u^I(1)(x_{0,1}^I) - m_U)\theta. \tag{C.14}$$

Lemma 17. $u^I(p')(x_{0,1}^{I,a}) - u^F(p')(x_{0,1}^{F,a}) \leq C_0(p'-p)(u^I(1)(x_{0,1}^I) - m_U)\theta$.

By (C.10)

$$u^F(p')(x_{0,1}^{F,a}) \geq m_U + \left(\frac{1}{1-p} - C_0\theta\right)(u^I(1)(x_{0,1}^{I,a}) - m_U)(p'-p) \geq m_U, \tag{C.15}$$

where the last inequality holds for $\theta \leq \theta^* \leq (1-p^u)/C_0$.

Define payoff vectors

$$y^{F,a} = (u^F(p')(x_{0,1}^{F,a}), x_{0,1}^{F,a})$$
$$y^F(\alpha) := \alpha v^I + (1-\alpha)y^{F,a}.$$

Clearly, $y^{F,a}, y^F(\alpha) \in \mathrm{con}(FE^*(p) \cup V)$. Define

$$\alpha^{F*} = \frac{(1 - C_0(1-p)\theta)(u^I(1)(x_{0,1}^{I,a}) - m_U)(p'-p)}{(1 - C_0(1-p)\theta)(u^I(1)(x_{0,1}^{I,a}) - m_U)(p'-p) + (1-p)(m_U - v_U^I)}. \tag{C.16}$$

Then (C.15) implies that

$$\alpha^{F*} \leq \frac{y_U^{F,a} - m_U}{y_U^{F,a} - v_U^I}.$$

(Notice that $(x-a)/(x-b)$ is increasing in x for $a < b$.) Therefore,

$$y^F(\alpha) \in F^*(p) \text{ for each } \alpha \leq \alpha^{F*}.$$

Step 5. Use (C.12) and (C.16) to bound the difference:

$$\frac{\alpha^{I*}-\alpha^{F*}}{p'-p} \leq \frac{u^I(1)(x^{I,a}_{0,1})-m_U}{(1-p')(m_U-v^I_U)-(p'-p)LM} - \frac{(1-C_0(1-p)\theta)(u^I(1)(x^{I,a}_{0,1})-m_U)}{M(p'-p)+(1-p)(m_U-v^I_U)}.$$

For $p'-p \leq d^* \leq (1-p^u)C_I/(2LM)$,

$$(M(p'-p)+(1-p)(m_U-v^I_U)) \leq (1-p^u)C_I$$
$$(1-p')(m_U-v^I_U)-(p'-p)LM \leq (1-p^u)C_I.$$

Hence

$$\begin{aligned}\frac{\alpha^{I*}-\alpha^{F*}}{p'-p} &\leq \frac{2(u^I(1)(x^{I,a}_{0,1})-m_U)}{(1-p^u)^2C_I^2}[(M(p'-p)+(1-p)(m_U-v^I_U)) \\ &\qquad -(1-C_0(1-p)\theta)((1-p')(m_U-v^I_U)-(p'-p)LM)] \\ &\leq \frac{2(u^I(1)(x^{I,a}_{0,1})-m_U)}{C_I^2}(1+2C_0\theta)[C_0(1-p)\theta(M(p'-p) \\ &\qquad +(1-p)M)+(p'-p)(2M+LM)] \\ &\leq \frac{2[C_0(1-p)M\theta+(2+L)M(p'-p)](1+2C_0\theta)}{C_I^2}(u^I(1)(x^{I,a}_{0,1})-m_U) \\ &\leq \frac{D-2C_0}{2M}(\theta+p'-p)(u^I(1)(x^I_{0,1})-m_U),\end{aligned}$$

where the constant D is defined in equation (C.6).

Step 6. Compute the distance between $y^I_{0,1}$ and $y^F_{0,1}(\alpha)$:

$$\begin{aligned}\left\|y^I_{0,1}-y^F_{0,1}(\alpha)\right\| &= \left\|\alpha^I v^I_{0,1}+(1-\alpha^I)x^{I,a}_{0,1}-\alpha v^I_{0,1}+(1-\alpha)y^{I,a}_{0,1}\right\| \\ &= \left\|(\alpha^I-\alpha)v^I_{0,1}+(1-\alpha^I)(x^{I,a}_{0,1}-y^{I,a}_{0,1})-(\alpha^I-\alpha)y^{I,a}_{0,1}\right\| \\ &\leq \left|\alpha^I-\alpha\right|2M+\left\|x^{I,a}_{0,1}-y^{I,a}_{0,1}\right\|.\end{aligned}$$

Steps 4 and 5, inequality (C.14), and the last inequality imply that one can find α^F so that $y^F(\alpha^F)\in F(p)$ and

$$\left\|y^I_{0,1}-y^F_{0,1}(\alpha^F)\right\| \leq \left(\theta+(D-2C_0)(\theta+(p'-p)\theta+(p'-p)^2)\right)\left(u^I(1)(x^{I,a}_{0,1})-m_U\right).$$

Finally, use inequality (C.13) to obtain

$$\begin{aligned}&\left\|y^I_{0,1}-y^F_{0,1}(\alpha^F)\right\| \\ &\quad\leq \left(\theta+(D-2C_0)(\theta+(p'-p)\theta+(p'-p)^2)\right)\left(1+C_0(p-p')\right)\left(u^I(1)(y^I_{0,1})-m_U\right) \\ &\quad\leq \left(\theta+D((p'-p)\theta+(p'-p)^2)\right)\left(u^I(1)(y^I_{0,1})-m_U\right).\end{aligned}$$

This ends the proof of (C.7) and the lemma.

C.4 *Proof of Lemma 16*

The proof of the lemma is divided into two parts.

Lemma 18. *There exist $x^{I,a} \in ICR^*(p)$, $\alpha^I \in [0,1]$, and $\rho^I \in \Delta V$ such that*

$$x^I = \alpha^I \sum_{v \in V} v\rho^I(v) + (1-\alpha^I)x^{I,a}, \tag{C.17}$$

$x_U^{I,a} = m_U$, and $v_U \leq m_U - C_I$ for each $v \in \operatorname{supp} \rho^I$.

Any tuple $(x^{I,a}, \alpha^I, v^I)$ such that $x^{I,a} \in ICR^*(p)$, $\alpha^I \in [0,1]$, $\rho^I \in \Delta V$, and (C.17) holds is called a representation of x^I.

Proof. There exists at least one representation (x, α, ρ) of $x^I \in \operatorname{con}(ICR^*(p) \cup V)$. Among all such representations, choose $(x^{I,a}, \alpha^I, \rho)$ to minimize α^I. Such a representation exists by a simple compactness argument.

I show that $x_U^{I,a} = m_U$. Consider the following three cases.

1. If $\alpha^I = 0$, then $m_U \leq x_U^{I,a} = x_U^I \leq m_U$ by (C.8).

2. Suppose that $0 < \alpha^I < 1$. If $x_U^{I,a} > m_U$, then construct

$$x^{I,a\prime} = x^{I,a} + \frac{x_U^{I,a} - m_U}{x_U^{I,a} - x_U^I}(x^I - x^{I,a}).$$

By construction, $x_U^{I,a\prime} \geq m_U$. Because $x_{0,1}^I, x_{0,1}^{I,a} \in \operatorname{proj} IR$, we have $x_{0,1}^{I,a\prime} \in \operatorname{proj} IR$. Hence $x^{I,a\prime} \in IR$ and there is $\alpha' < \alpha^I$ such that $(x^{I,a\prime}, \alpha', \rho)$ is a representation of x^I. This contradicts the choice of $(x^{I,a}, \alpha^I, \rho)$ as the representation that minimizes α^I.

3. If $\alpha^I = 1$, then any $x^{I,a} \in I_1(p)$ satisfies (C.17).

Recall the definition of C_I and the set $S_{\text{vert}}(\Phi')$, where $\Phi' \subseteq \Phi_{IR}$. (These definitions may be found in Section C.2.) Define

$$\begin{aligned} v^\rho &= \sum_{v \in V} v\rho(v) \\ \Phi^I &= \{\phi \in \Phi_{IR} : \phi \cdot (v_0^\rho, v_1^\rho) \leq m_I(\phi)\} \\ S &= S_{\text{vert}}(\Phi^I). \end{aligned}$$

Then $v^\rho \in \operatorname{con} S$ and there exists $\rho^I \in \Delta S$ such that (C.17) holds.

I show that for each $v \in \operatorname{supp} \rho^I$, $v_U < m_U$. By the definition of the constant C_I in (C.3), this implies that for each $v \in \operatorname{supp} \rho^I$, $v_U < m_U - C_I$. On the contrary, suppose that there is $v \in \operatorname{supp} \rho^I$ with $v_U \geq m_U$. There are two cases to be considered.

1. If there is $\phi \in \Phi_{IR}$ such that $\phi \cdot x_{0,1}^{I,a} = m_I(\phi)$ and $\phi \cdot v_{0,1} < m_I(\phi)$, then, by the definition of the set $S(\Phi^I)$, $\phi \cdot v_{0,1}^{\rho} < m_I(\phi)$. But then, for any $\alpha \in [0,1]$,

$$\phi \cdot [(1-\alpha)x_{0,1}^{I,a} + \alpha v_{0,1}^{\rho}] < m_I(\phi) \tag{C.18}$$
$$(1-\alpha)x_{0,1}^{I,a} + \alpha v_{0,1}^{\rho} \notin \operatorname{proj} IR.$$

Recall that $x_{0,1}^I = y_{0,1}^I \in \operatorname{proj} IR$. Hence (C.18) contradicts the fact that $(x^{I,a}, \alpha^I, v^{\rho})$ is a representation of x^I.

2. Next, suppose that $\phi \cdot v_{0,1} \leq m_I(\phi)$ for any $\phi \in \Phi_{IR}$ such that $\phi \cdot x_{0,1}^{I,a} = m_I(\phi)$. This implies that for small $\varepsilon > 0$,

$$x_{0,1}^{I,a} + \varepsilon(v_{0,1} - x_{0,1}^{I,a}) \in \operatorname{proj} IR.$$

Because $v_U, x_U^{I,a} \geq m_U$, it must be that $x_U^{I,a} + \varepsilon(v_U - x_U^{I,a}) \geq m_U$. One concludes that for small $\varepsilon > 0$,

$$x^{I,a\prime} = x^{I,a} + \varepsilon(v - x^{I,a}) \in IR.$$

But then, by Lemma 15, there is a representation $(x^{I,a'}, \alpha^{I\prime}, v^{\rho})$ such that $\alpha^{I\prime} < \alpha^I$. This contradicts the choice of $(x^{I,a}, \alpha^I, v^{\rho})$ as the representation that minimizes α^I. □

Lemma 19. $x^{I,a} \in I_1(p)$.

Proof. I need to show that

$$u^I(p)(x_{0,1}^{I,a}) = m_U, \text{ and } x_{0,1}^{I,a} \in \operatorname{proj} ICR^*(1). \tag{C.19}$$

First, suppose that the first part of (C.19) is not true. Because $x_U^{I,a} = m_U$, it must be that $u^I(p)(x_{0,1}^{I,a}) > m_U$. Define

$$x = (1-\alpha^I)(u^I(p)(x_{0,1}^{I,a}), x_{0,1}^{I,a}) + \alpha^I \sum_{v \in V} v\rho^I(v).$$

Because $(x^{I,a}, \alpha, \rho)$ is a representation of x^I, $x_{0,1} = x_{0,1}^I$. By the above equation, we have $u^I(p)(x_{0,1}^I) \geq x_U > x_U^I$. But this contradicts (C.8).

Next, I show that the second part of (C.19) holds. By (C.8), $u^I(p)(x_{0,1}^I) = x_U^I$. Hence

$$\begin{aligned} u^I(p)(x_{0,1}^I) = x_U^I &= (1-\alpha^I)x_U^{I,a} + \alpha^I \sum_{v \in V} v_U \rho^I(v) \\ &= (1-\alpha^I)m_U + \alpha^I \sum_{v \in V} v_U \rho^I(v) \\ &\leq m_U - \alpha^I C_I, \end{aligned}$$

where the last inequality follows from the fact that $v_U \leq m_U - C_I$ for each $v \in \operatorname{supp} \rho^I$. By the definition of the constant L in (C.4),

$$\begin{aligned} u^I(1)(x_{0,1}^I) &\leq u^I(1)(x_{0,1}^{I,a}) + L\left\|x_{0,1}^I - x_{0,1}^{I,a}\right\| \\ &\leq u^I(1)(x_{0,1}^{I,a}) + \alpha^I LM. \end{aligned}$$

The function $u^I(.)(x^I_{0,1})$ is convex in p (part (iii) of Lemma 2). Hence

$$\begin{aligned} m_U &= u^I(p')(x^I_{0,1}) \\ &\le \frac{1-p'}{1-p} u^I(p)(x^I_{0,1}) + \frac{p'-p}{1-p} u^I(1)(x^I_{0,1}) \\ &\le \frac{1-p'}{1-p} m_U - \alpha^I \frac{1-p'}{1-p} C_I + \frac{p'-p}{1-p} u^I(1)(x^{I,a}_{0,1}) + \frac{p'-p}{1-p}\alpha^I LM. \end{aligned} \tag{C.20}$$

Suppose that $x^{I,a}_{0,1} \notin \operatorname{proj} ICR^*(1)$, or, because $x^{I,a}_{0,1} \in \operatorname{proj} IR$, $u^I(1)(x^I_{0,1}) < m_U$. Then

$$m_U > \frac{1-p'}{1-p} m_U + \frac{p'-p}{1-p} u^I(1)(x^{I,a}_{0,1}). \tag{C.21}$$

Subtract both sides of (C.21) from the corresponding sides of (C.20) and multiply by $(1-p)/\alpha^I$ to get $(1-p')C_I < (p'-p)LM$ and

$$p'-p > \frac{LM}{C_I(1-p)}.$$

Therefore, if $p'-p < LM/(C_I(1-p^u))$, then

$$x^{I,a}_{0,1} \in \operatorname{proj} ICR^*(1) \text{ and } x^{I,a} \in I_1(p). \qquad \square$$

C.5 *Proof of Lemma 17*

Lemma 20. *There exist $x \in FE^*(p)$, $\alpha^F \in [0,1]$, and $\rho^F \in \Delta V$ such that*

$$x^{F,a} = \alpha^F \sum_{v\in V} v\rho^F(v) + (1-\alpha^F)x \tag{C.22}$$

and

$$\alpha^F \le C_F \inf_{v^{IR}_{0,1} \in \operatorname{proj} IR.} \left\| v_{0,1} - v^{IR}_{0,1} \right\|$$

A tuple (x, α^F, ρ^F) such that $x \in FE^*(p)$, $\alpha^F \in [0,1]$, $\rho^F \in \Delta V$, and (C.22) holds is called an F-representation of $x^{F,a}$.

Proof. I show that there is a representation (x, α^F, ρ^F) of $x^{F,a}$ such that

$$\begin{gathered} \operatorname{proj} \operatorname{con}(\operatorname{supp} \rho^F \cup \{x\}) \cap \operatorname{proj} IR = \{x_{0,1}\} \\ \operatorname{proj} \operatorname{con}(\operatorname{supp} \rho^F) \cap \operatorname{proj} IR = \varnothing \\ \operatorname{supp} \rho^F \subseteq V^+_{\text{vert}} \cup V^-_{\text{vert}}. \end{gathered} \tag{C.23}$$

(The sets $V^+_{\text{vert}} \cup V^-_{\text{vert}}$ are defined in Section C.2.) By Lemma 14,

$$\alpha^F \le C_{\operatorname{supp}\rho^F} \inf_{v^{IR}_{0,1} \in \operatorname{proj} IR.} \left\| v_{0,1} - v^{IR}_{0,1} \right\|.$$

The lemma is a consequence of the definition of the constant C_F in equation (C.2).

Recall that $x^{F,a} \in F^*(p)$, which implies that $x_U^{F,a} \geq m_U$. If $x^{F,a} \in IR$, then $x^{F,a} \in FE^*(p)$, and the lemma is trivially true. From now on, assume that $x^{F,a} \notin IR$. There exists at least one representation (x', α^F, ρ) of $x^{F,a}$ such that (C.22) holds. Among all representations, choose the one with the lowest value of α^F and denote it by (x, α^F, ρ). Such a representation exists by a simple continuity argument, and $\alpha^F > 0$ because $x^{F,a} \notin FE^*(p)$. Let

$$v^\rho = \sum_{v \in V} v\rho(v).$$

If $v_U^\rho \leq m_U$, then $v^\rho \in V^- = \operatorname{con} V_{\text{vert}}^-$ and there is a representation (x, α^F, ρ^F) so that $\operatorname{supp} \rho^F \subseteq V_{\text{vert}}^-$. Similarly, if $v_U^\rho \geq m_U$, then there is a representation (x, α^F, ρ^F) so that $\operatorname{supp} \rho^F \subseteq V_{\text{vert}}^+$.

I discuss separately the different cases.

Case $\alpha^F = 1$. There is no representation such that $\alpha^F < 1$. I can assume that $\operatorname{supp} \rho^F \subseteq V_{\text{vert}}^+$ because $x^{F,a} = v^\rho$ and $x_U^{F,a} = v_U^\rho \geq m_U$.

I show that

$$\operatorname{con}\left(\operatorname{supp} \rho^F\right) \cap IR = \varnothing. \tag{C.24}$$

If not, then there exists $v \in \operatorname{con}(\operatorname{supp} \rho^F) \cap IR \subseteq V \cap IR$. Because $\operatorname{supp} \rho^F \subseteq V_{\text{vert}}^+$, it must be that $v_U \geq m_U$ and $v \in FE^*(p)$. A simple geometric argument shows that for any $v' \in \operatorname{int} \operatorname{con}(\operatorname{supp} \rho^F)$, there is a representation $(v, \alpha^{F\prime}, \rho')$ with $\alpha^{F\prime} < 1$. Because $x^{F,a} \in \operatorname{int} \operatorname{con}(\operatorname{supp} \rho^F)$, this leads to a contradiction.

Because $\alpha^F = 1$, equation (C.22) is satisfied by any any $x \in FE^*(p)$. Choose x so that $\operatorname{con}(\operatorname{supp} \rho^F \cup \{x\}) \cap IR = \{x\}$. Such a value of x exists by (C.24).

Because for each $v \in \operatorname{supp} \rho^F$, $v_U \geq m_U$, the above argument implies that (C.23) holds.

Case $0 < \alpha^F < 1$ *and* $x_U > m_U$. By Lemma 15 and the choice of α^F, there is an open neighborhood $U \ni x$ such that

$$U \cap \operatorname{con}\left(\operatorname{supp} \rho^F \cup \{x\}\right) \cap FE^*(p) = \varnothing.$$

There is an open neighborhood $U' \ni x$ such that $x'_U > m_U$ for any $x' \in U'$. Hence

$$U' \cap \{x' : x'_{0,1} \in \operatorname{proj} IR\} \subseteq FE^*(p).$$

These two observations imply that

$$U \cap U' \cap \operatorname{con}\left(\operatorname{supp} \rho^F \cup \{x\}\right) \cap \{x' : x'_{0,1} \in \operatorname{proj} IR\} = \varnothing. \tag{C.25}$$

Suppose that condition (C.23) does not hold and there is $v \in \mathbb{R}^3$ such that $(v_0, v_1) \in \operatorname{proj} IR$ and either $v \in \operatorname{con} \operatorname{supp} \rho^F$ or $v \in \operatorname{con}(\operatorname{supp} \rho^F \cup \{x\}) \setminus \{x\}$. Then, for small γ,

$$\gamma v + (1-\gamma)x \in U \cap U' \cap \operatorname{con}\left(\operatorname{supp} \rho^F \cup \{x\}\right)$$
$$\gamma v_{0,1} + (1-\gamma)x_{0,1} \in \operatorname{proj} IR,$$

which contradicts (C.25).

Case $0<\alpha^F<1$ *and* $x_U=m_U$. By Lemma 15 and the choice of α^F, there is an open neighborhood $U\ni x$ such that

$$U\cap \operatorname{con}(\operatorname{supp}\rho^F\cup\{x\})\cap FE^*(p)=\varnothing.$$

Because $x_U^{F,a}\geq m_U$, it must be that $v^\rho\geq m_U$ and $\operatorname{supp}\rho^F\subseteq V^+_{\text{vert}}$. This implies that

$$\operatorname{con}(\operatorname{supp}\rho^F\cup\{x\})\cap\{x':x'_{0,1}\in\operatorname{proj} IR\}\subseteq FE^*(p).$$

These two observations imply that

$$U\cap\operatorname{con}(\operatorname{supp}\rho^F\cup\{x\})\cap\{x':x'_{0,1}\in\operatorname{proj} IR\}=\varnothing.$$

The same argument as in the previous case shows that condition (C.23) must hold. □

Next, I prove Lemma 17. Use Lemma 20 to find an F-representation (x,α^F,ρ^F) of $x^{F,a}$. Then

$$\alpha^F\leq C_F\left\|x_{0,1}^{F,a}-x_{0,1}^{I,a}\right\|\leq C_F(u^I(1)(x_{0,1}^{I,a})-m_U)\theta, \tag{C.26}$$

where in the second inequality I use (C.14). Denote $v^\rho=\sum_{v\in V}v\rho(v)$.

By the definition of L in equation (C.4),

$$\begin{aligned}u^F(1)(x_{0,1}^{I,a})-u^F(1)(x_{0,1})&\leq L\left\|x_{0,1}^{I,a}-x_{0,1}\right\|\\&\leq L(\left\|x_{0,1}^{F,a}-x_{0,1}\right\|+\left\|x_{0,1}^{I,a}-x_{0,1}^{F,a}\right\|)\\&\leq L\big(\alpha^FM+\left\|x_{0,1}^{I,a}-x_{0,1}^{F,a}\right\|\big)\\&\leq L(1+MC_F)(u^I(1)(x_{0,1}^{I,a})-m_U)\theta,\end{aligned}$$

where in the last inequality I use (C.14). For sufficiently small $\theta\leq\theta^*=1/L(1+MC_F)$,

$$u^F(1)(x_{0,1})-m_U\geq(u^I(1)(x_{0,1}^{I,a})-m_U)(1-L(1+MC_F)\theta)\geq 0.$$

By part (iii) of Lemma 2, the function $u^I(p')(x_{0,1}^{I,a})$ is convex in p'. Hence

$$\begin{aligned}u^I(p')(x_{0,1}^{I,a})&\leq\frac{1-p'}{1-p}u^I(p)(x_{0,1}^{I,a})+\frac{p'-p}{1-p}u^I(1)(x_{0,1}^{I,a})\\&=\frac{1-p'}{1-p}m_U+\frac{p'-p}{1-p}u^I(1)(x_{0,1}^{I,a}),\end{aligned}\tag{C.27}$$

where I use the fact that $x^{I,a}\in I_1(p)$ and $u^I(p)(x_{0,1}^{I,a})=m_U$. By Lemma 2, the function u^F is concave in p and $x_{0,1}$. Hence,

$$\begin{aligned}u^F(p')(x_{0,1}^{F,a})&\geq\alpha^Fv_U^\rho+(1-\alpha)u^F(p')(x_{0,1})\\&\geq\alpha^Fv_U^\rho+(1-\alpha^F)\Big(\frac{1-p'}{1-p}u^F(p)(x_{0,1})+\frac{p'-p}{1-p}u^F(1)(x_{0,1})\Big)\\&\geq\frac{1-p'}{1-p}((1-\alpha^F)u^F(p)(x_{0,1})+\alpha^Fv_U^\rho)+\frac{p'-p}{1-p}u^F(1)(x_{0,1})\\&\qquad-\alpha^F\frac{p'-p}{1-p}(u^F(1)(x_{0,1})-\alpha^Fv_U^\rho).\end{aligned}$$

Recall that $x^{F,a} \in F^{*}(p)$, hence $m_U \leq x_U^{F,a} \leq (1-\alpha^F)u^F(p)(x_{0,1}) + \alpha^F v_U^{\rho}$. Also, $\left|(u^F(1)(x_{0,1}) - \alpha^F v_U^{\rho})\right| \leq 2M$. Hence,

$$u^F(p')(x_{0,1}^{F,a}) \geq \frac{1-p'}{1-p} m_U + \frac{p'-p}{1-p} u^F(1)(x_{0,1}) - \alpha^F \frac{p'-p}{1-p} 2M. \tag{C.28}$$

One can put (C.27) and (C.28) together. Using the second inequality in (C.26), one gets

$$\begin{aligned} u^I(p')(x_{0,1}^{I,a}) - u^F(p')(x_{0,1}^{F,a}) &\leq \frac{p'-p}{1-p}(u^I(1)(x_{0,1}^{I,a}) - u^F(1)(x_{0,1})) + \frac{p'-p}{1-p}\alpha^F 2M \\ &\leq \frac{2}{1-p^u}[L(1+MC_F)+2MC_F](u^I(1)(x_{0,1}^{I,a}) - m_U)(p'-p)\theta. \end{aligned}$$

The lemma follows from the definition of C_0 in equation (C.5).

C.6 *Approximation argument*

So far, I have shown that Theorem 3 is true if Assumption 2 holds. Here I argue that Assumption 2 is unnecessary. The idea is that any game can be approximated by games that satisfy Assumption 2.

Recall that the correspondences FE^* and ICR^* are defined as functions of the sets V of non-revealing payoffs and IR of individually rational payoffs. Precisely, consider two collections of closed subsets of $[-M,M]^3$: $\mathscr{V}$ consisting of all convex sets and $\mathscr{IR}$ consisting of all convex sets with the property that for any $IR \in \mathscr{IR}$ there is $m_U \in \mathbb{R}$ such that for any $(v_U, v_0, v_1) \in IR$, $(v'_U, v'_0, v'_1) \in IR$ if $v'_k \geq v_k$ and $v'_U \geq m_U$. For any $V \in \mathscr{V}$ and $IR \in \mathscr{IR}$, define sets $E_k(V, IR)$ as in equations (6) and (5). Using the sets E_k, I may restate Assumption 1: it holds if there are two open sets $A_k \subseteq E_k$ such that $\operatorname{proj} A_0 = \operatorname{proj} A_1$. This allows me to define correspondences $FE^*(V, IR)$ and $ICR^*(V, IR)$ as in Sections 3.3 and 4.2.

Theorem 3 can be interpreted as follows. Suppose that Assumption 1 (as stated in this section) holds for some $V \in \mathscr{V}$ and $IR \in \mathscr{IR}$. Then $FE^*(V, IR) = ICR^*(V, IR)$. Till now, I have shown that Theorem 1 is true, if, in addition, Assumption 2 is satisfied.

Suppose now that Assumption 1 holds for some $V \in \mathscr{V}$ and $IR \in \mathscr{IR}$. Consider an approximating sequence of closed sets $IR_n \in \mathscr{IR}$, such that Assumption 2 is satisfied for all sets in the sequence, $IR_n \subseteq IR$, and IR_n converges to the set IR in the sense of Haussdorf distance: $\lim_{n\to\infty} IR_n = IR$. Such a sequence clearly exists. Then Assumption 1 holds for a high enough n and

$$FE^*(V, IR_n) = ICR^*(V, IR_n).$$

By monotonicity, $FE^*(V, IR_n) \subseteq FE^*(V, IR)$ for all n. Two simple lemmas end the proof.

Lemma 21. $\lim_{n\to\infty} FE^*(V, IR_n) = FE^*(V, IR)$.

Proof. This is because I can take

$$FE^{**} = \operatorname{cl} \bigcup_n FE^*(V, IR_n)$$

and show that it satisfies all the properties in Section 3.3. □

LEMMA 22. $\lim_{n\to\infty} ICR^*(V, IR_n) = ICR^*(V, IR)$.

PROOF. Indeed, define the correspondence ICR_n^{**} for any $p \in [0,1]$ by

$$ICR_n^{**}(p) = IR_n \cap ICR^*(V, IR)(p)$$

and show that it satisfies all three properties in Section 4.2. As a consequence, $ICR_n^{**} \subseteq ICR^*(V, IR_n)$. □

D. PROOFS FOR SECTION 5.3

I need the following auxiliary result.

LEMMA 23. *For any* (v_0, v_1) *and any sequence* $p_n \to p$ *such that* $\lim_{n\to\infty} l^F(p_n)(v_0, v_1) > m_U$,

$$l^F(p)(v_0, v_1) = \lim_{n\to\infty} l^F(p_n)(v_0, v_1).$$

PROOF. Because the set FE^* is closed, $(\lim_{n\to\infty} l^F(p_n)(v_0, v_1), v_0, v_1) \in FE^*(p)$. Hence

$$l^F(p)(v_0, v_1) \leq \lim_{n\to\infty} l^F(p_n)(v_0, v_1).$$

Suppose that the inequality is strict, and define

$$v_U^* = \max[m_U, l^F(p)(v_0, v_1)] < \lim_{n\to\infty} l^F(p_n)(v_0, v_1).$$

Note that $(v_U^*, v_0, v_1) \in FE^*(p)$. By part (iv) of Lemma 2, for any n and any $p_m \in [p, p_n]$ (assuming without loss of generality that $p_n > p$),

$$l^F(p_m)(v_0, v_1) \leq \frac{p_m - p}{p_n - p} \max l^F(p_n)(v_0, v_1) + \frac{p_n - p_m}{p_n - p} v_U^*.$$

But then, keeping n fixed and letting $p_m \to p$,

$$\lim \inf_{m\to\infty} l^F(p_m)(v_0, v_1) \leq v_U^* < \lim_{n\to\infty} l^F(p_n)(v_0, v_1).$$

This creates a contradiction. □

D.1 *Proof of Lemma 6*

If $l^F(p)(v_0, v_1) \leq m_U + \frac{1}{2}\theta(p, v_0, v_1)$, then there is nothing to prove. Suppose not and define

$$p_0 = \sup_{p'<p} \{l^F(p')(v_0, v_1) \leq m_U + \tfrac{1}{2}\theta(p', v_0, v_1)\}$$
$$p_1 = \inf_{p'>p} \{l^F(p')(v_0, v_1) \leq m_U + \tfrac{1}{2}\theta(p', v_0, v_1)\}$$

with the convention that $p_0 = 0$ or $p_1 = 1$ if the respective sets are empty. Then $p_0 < p < p_1$, and because of Lemma 23,

$$l^F(p_i)(v_0, v_1) \geq m_U + \tfrac{1}{2}\theta(p_i, v_0, v_1) \text{ for } i = 0, 1.$$

By part (iv) of Lemma 2, l^F is convex (hence continuous for interior ps) above m_U. Thus the inequality turns into an equality whenever $p_i \neq 0, 1$. Using Lemma 2 again, for any $p' \in [p_0, p_1]$,

$$\theta(p', v_0, v_1) \leq \frac{p' - p_0}{p_1 - p_0}\theta(p_1, v_0, v_1) + \frac{p_1 - p'}{p_1 - p_0}\theta(p_0, v_0, v_1).$$

This means that there is $p' \in \{p_0, p_1\}$ such that

$$\theta(p', v_0, v_1) \geq \theta(p, v_0, v_1) > 0.$$

Because $\theta(k, v_0, v_1) = 0$ for $k = 0, 1$, it must be that $p' \notin \{0, 1\}$. By the above argument,

$$l^F(p')(v_0, v_1) = m_U + \tfrac{1}{2}\theta(p', v_0, v_1),$$

which implies that

$$l^I(p')(v_0, v_1) \leq m_U - \tfrac{1}{2}\theta(p', v_0, v_1).$$

D.2 *Proof of Lemma 7*

By the definition of the function l^I, there is $v' = (v'_U, v'_0, v'_1) \in ICR^*(p)$, $\alpha \in [0, 1]$, and $\bar{v} \in V$ such that

$$l^I(p)(v'_0, v'_1) = v'_U \text{ and } l^I(p)(v_0, v_1) = \alpha\bar{v}_U + (1-\alpha)v'_U. \tag{D.1}$$

Because $v'_U \geq m_U$, it must be that $v'_U - l^I(p)(v_0, v_1) \geq \frac{1}{2}\theta(p, v_0, v_1)$. Because $v'_U - \bar{v}_U \leq 2M$, it must be that

$$\alpha = \frac{v'_U - l^I(p)(v_0, v_1)}{v'_U - \bar{v}_U} \geq \frac{\theta(p, v_0, v_1)}{4M}. \tag{D.2}$$

Because $\bar{v} \in \text{con}(FE^*(p) \cup V)$,

$$l^F(p)(\bar{v}_0, \bar{v}_1) \leq \bar{v}_U = l^I(p)(\bar{v}_0, \bar{v}_1) \leq l^F(p)(\bar{v}_0, \bar{v}_1)$$

and all the inequalities can be replaced by equalities. In particular,

$$\theta(p, \bar{v}_0, \bar{v}_1) = 0.$$

By the convexity of l^F in (v_0, v_1) (part (ii) of Lemma 2),

$$\begin{aligned} l^F(p)(v_0, v_1) &\leq \alpha l^F(p)(\bar{v}_0, \bar{v}_1) + (1-\alpha)l^F(p)(v'_0, v'_1) \\ &= \alpha\bar{v} + (1-\alpha)l^F(p)(v'_0, v'_1). \end{aligned}$$

Because of (D.1),

$$\begin{aligned} \theta(p, v_0, v_1) &= l^F(p)(v_0, v_1) - l^I(p)(v_0, v_1) \\ &\leq (1-\alpha)\theta(p, v'_0, v'_1). \end{aligned} \tag{D.3}$$

Inequalities (D.2) and (D.3) lead to

$$\theta(p, v'_0, v'_1) \geq \frac{4M}{4M - \theta(p, v_0, v_1)}\theta(p, v_0, v_1).$$

References

Abreu, Dilip, David Pearce, and Ennio Stacchetti (1990), "Toward a theory of discounted repeated games with imperfect monitoring." *Econometrica*, 58, 1041–1063. [30, 41]

Athey, Susan and Kyle Bagwell (forthcoming), "Collusion with persistent shocks." *Econometrica*. [55]

Aumann, Robert J. and Sergiu Hart (1986), "Bi-convexity and bi-martingales." *Israel Journal of Mathematics*, 54, 159–180. [30, 31, 44]

Aumann, Robert J. and Michael B. Maschler (1995), *Repeated Games with Incomplete Information*. MIT Press, Cambridge, Massachusetts. With the collaboration of Richard E. Stearns. [29]

Aumann, Robert J., Michael B. Maschler, and Richard E. Stearns (1966–68), "Reports to the U.S. arms control and disarmament agency." Technical Report ST-80 (1968), ST-116 (1967), ST-140 (1968), Mathematica, Inc. [31]

Chan, Jimmy (2000), "On the nonexistence of reputation effects in two-person infinitely repeated games." Unpublished paper, Department of Economics, Johns Hopkins University. [31]

Cripps, Martin W., Eddie Dekel, and Wolfgang Pesendorfer (2005), "Reputation with equal discounting in repeated games with strictly conflicting interests." *Journal of Economic Theory*, 121, 259–272. [32]

Cripps, Martin W. and Jonathan P. Thomas (1997), "Reputation and perfection in repeated common interest games." *Games and Economic Behavior*, 18, 141–158. [31]

Cripps, Martin W. and Jonathan P. Thomas (2003), "Some asymptotic results in discounted repeated games of one-sided incomplete information." *Mathematics of Operations Research*, 28, 433–462. [31]

Fudenberg, Drew, David I. Levine, and Eric Maskin (1994), "The folk theorem with imperfect public information." *Econometrica*, 62, 997–1039. [30, 38, 41, 57, 59]

Fudenberg, Drew and David K. Levine (1994), "Efficiency and observability with long-run and short-run players." *Journal of Economic Theory*, 62, 103–135. [31, 41]

Fudenberg, Drew and Eric Maskin (1986), "The folk theorem in repeated games with discounting or with incomplete information." *Econometrica*, 54, 533–554. [32]

Hart, Sergiu (1985), "Nonzero-sum two-person repeated games with incomplete information." *Mathematics of Operations Research*, 10, 117–153. [31, 35, 37, 39, 42, 44]

Kandori, Michihiro and Hitoshi Matsushima (1998), "Private observation, communication and collusion." *Econometrica*, 66, 627–652. [41, 55]

Koren, Gilad (1992), "Two-person repeated games where players know their own payoffs." Unpublished paper, Courant Institute, New York University. [31, 39]

Kreps, David M. and Robert Wilson (1982), "Sequential equilibria." *Econometrica*, 50, 863–894. [34]

Madrigal, Vicente, Tommy C. C. Tan, and Sérgio Ribeiro da Costa Werlang (1987), "Support restrictions and sequential equilibria." *Journal of Economic Theory*, 43, 329–334. [33]

Mailath, George J. and Larry Samuelson (2006), *Repeated Games and Reputations*. Oxford University Press, New York. [34]

Rubinstein, Ariel (1979), "Equilibrium in supergames with the overtaking criterion." *Journal of Economic Theory*, 21, 1–9. [32]

Shalev, Jonathan (1994), "Nonzero-sum two-person repeated games with incomplete information and known-own payoffs." *Games and Economic Behavior*, 7, 246–259. [31, 39]

Sorin, Sylvain (1999), "Merging, reputation, and repeated games with incomplete information." *Games and Economic Behavior*, 29, 274–308. [35, 39]

Submitted 2006-5-8. Final version accepted 2008-2-3. Available online 2008-2-3.