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Caller Number Five and related timing games

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There are two varieties of timing games in economics: wars of attrition, in which having more predecessors helps, and pre-emption games, in which having more predecessors hurts. This paper introduces and explores a spanning class with *rank-order payoffs* that subsumes both varieties as special cases. We assume time is continuous, actions are unobserved, and information is complete, and explore how equilibria of the games, in which there is shifting between phases of slow and explosive (positive probability) stopping, capture many economic and social timing phenomena. Inspired by auction theory, we first show how each symmetric Nash equilibrium is equivalent to a different "potential function." By using this function, we straightforwardly obtain existence and characterization results. Descartes' Rule of Signs bounds the number of phase transitions. We describe how adjacent timing game phases interact: war of attrition phases are not played out as long as they would be in isolation, but instead are cut short by pre-emptive atoms. We bound the number of equilibria, and compute the payoff and duration of each equilibrium.

Keywords. Games of timing, war of attrition, preemption game.

JEL classification. C73, D81.

1. Introduction

Timing models in economics can be categorized into two classes. In the first, delay is exogenously costly, and each player prefers that others act before him. We tentatively categorize such a model as a war of attrition. In the second, the situation is reversed; the

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passage of time is exogenously beneficial, and players wish to pre-empt others. Such a model is usually classified as a pre-emption game. There are, however, many important strategic situations where players prefer to be neither first nor last (fixing the exogenous environment). Such situations can capture many new behavioral phenomena, like multiple periods of slow stopping interspersed with sudden rushes. The goal of this paper is to develop a foundation for a spanning class of timing games without restrictions on rank order payoffs.

We develop a comprehensive theory for complete information timing games with exogenous delay costs[1] and assume that the rewards depend on the players' *ordinal stopping ranks*. These rank rewards should be seen as a reduced form for a richer model, so that one can focus on the essence of the two strategic forces in timing games. The first force arises in a (many-player) war of attrition, where early stoppers earn less than later ones, so that players prefer a higher ordinal stopping rank. The opposing second force is found in a pre-emption game, where people prefer a lower ordinal rank. In either case, rewards are monotonic in the ordinal stopping ranks. Our formulation extends to non-monotonic rank-rewards.

We assume unobservable actions; this assumption is necessary for tractability, allowing us to use Nash equilibrium and thereby adapt 'potential functions' and borrow insights from mechanism design. "Silent timing games" capture economic environments where timing decisions must be made well before the action begins, as with high-tech market entry decisions, or the choice of release dates for movies. We also posit discounting and known delay costs. As in most timing game papers, we focus on symmetric equilibria in mixed strategies. This captures an anonymity of play natural in many contexts. We also exclude strategies explicitly depending on focal calendar times or random coordination devices like sunspots.

Our aim is to find and characterize all the equilibria in this class, and we proceed in two steps. First, we argue that the core elements of our game, the rank payoffs, are fully and uniquely encoded in a function Φ. This mapping corresponds to a thought experiment in which all players employ an identical atomless strategy.

Second, we establish that every equilibrium of the game is equivalent to a unique convexification of the function Φ, which we call a *potential function*. The key advantage of this reformulation is that important qualitative features of behavior in an equilibrium can then easily be derived from properties of the corresponding potential function. The idea behind the convexification procedure is that if later ranks secure less valuable rewards than earlier ranks, then atoms endogenously arise in equilibrium to make the costly delay worthwhile. Such atoms guarantee that a player obtains both large and small rank-payoffs with positive probability, thereby "ironing" players' equilibrium rank-order payoffs. And an ironed portion of the players' rank payoffs corresponds to a convexified portion of the representing function Φ.

Such convexifications have many uses in economics (where they were originally used in the study of auctions) and the sciences, but the spirit of all such applications

[1]Exogenous payoff growth over time, a feature often associated with pure pre-emption games, is an obvious extension that we pursue in other work.

is that their gradient yields equilibrium expected payoffs (see footnote 11).[2] One example of a potential function is the convex hull of the function Φ. Since it always exists, this yields an immediate proof that a symmetric Nash equilibrium exists (Theorem 1). Myerson (1981) was the first to adopt the notion of an "ironing board." His function is the *maximum* revenue auction and thus the convex hull of an integrated marginal revenue function. By contrast, we desire *all* Nash equilibria, and therefore explore a *local* convexification notion. The convex hull corresponds to the equilibrium with the highest payoff loss due to delay. Our equilibrium characterization reduces to analyzing all possible potential functions.

To illustrate the equilibrium behavior, consider the following example. Suppose that a radio call-in show awards Stones tickets to the seventy-seventh caller. If the number of other potential callers is known, and if waiting to call inflicts opportunity costs on listeners, when should they call? Intuitively, players initially strategically benefit from the delay, but eventually succumb to a fear of missing out. How long will the game last? What economic lessons can be gleaned from players' equilibrium timing behavior?

One might well imagine that players wait to call, and suddenly call en masse, jamming the phone lines. The prediction of our model is more subtle. Since delaying is explicitly costly, agents are initially locked in a war of attrition. Everyone adopts a mixed strategy, and the chance of winning is ever increasing. Ideally each wants to call when the *probability* that seventy-six have called is maximal. At that moment, everyone else would do likewise, triggering explosive calling, which we refer to as a stopping *atom*. But the story does not end there. Only one of the many callers can win, and thus the value of the expected prize is lower compared to a situation where one calls just before this atom and is the only caller who calls at that time. The pre-emption moment is thus pushed earlier in time until everyone is indifferent between pre-empting the atom and calling with the mass. Thus, the pre-emption atom 'prematurely' truncates the war of attrition phase: relative to the direct sum of equilibria from two timing games,[3] agents pre-empt earlier and do so with an excessively large mass. Both the time and size of explosive stopping moments are endogenous.

The motivational radio show example aside, our paper matches some other economic applications. For instance, entry into a growing potential new market is often most profitable for early firms after the leader, who struggle with neither market creation nor brand identification. The social phenomenon of fashionable lateness bespeaks a preference for a middling arrival rank. In rush hour one seeks to be early or late.

Moving on to characterizing the equilibria, we first note that a war of attrition phase obtains only for rising expected payoffs, when strategic and exogenous delay costs conflict. Pre-emptive behavior is likewise mandated when expected rank payoffs fall. So the slope-sign changes of expected payoffs are key. Theorem 2 bounds the number of phase transitions by the underlying deterministic rank reward using Descartes' Rule of Signs;

[2]Another example is a recent paper by Ostrovsky and Schwarz (2006), who use a convexification technique to describe players' synchronization behavior under uncertainty.

[3]In such a direct sum of equilibria, one would merely combine the equilibrium for the war of attrition, which would be played for as long as possible, with the pre-emptive atom that is just large enough so that decreasing rank payoffs are bunched together.

this provides a simple upper bound on the number of phase transitions and binds for some equilibria.

With ever-increasing costs, pre-emptive behavior is synonymous with a positive probability of stopping in an atom. A switch from a gradual war of attrition phase to a pre-emptive atom (or back) can occur only if expected rank payoffs before and after the atom and the payoff from stopping with the atom coincide. We then show that non-atomic rank and atomic rewards relate as do marginals and averages (Lemma 2). We build on this insight to deduce that any war of attrition phase ends before expected rank payoffs peak and, following an atom, any war of attrition starts after rank payoffs trough (Theorem 3). For this reason, we say that the war is 'truncated' and the atom is 'inflated.'

We then determine how many equilibria the game may have. Since war of attrition and pre-emption game phases alternate, the question is which consecutive pairs are played. The number of equilibria is then found by simple combinatorics: with J matched pairs of wars of attrition and pre-emption games, there are 2^J potential Nash equilibria (Theorem 4).

In the war of attrition, all rents—namely, the greatest minus the least expected rank payoff—are dissipated. This is not true when rank order payoffs are non-monotonic. As a result, the pre-emption games start when expected rank payoffs coincide with average atomic payoffs, before the former peaks; thus, the maximal expected rank payoff is not attained in equilibrium. Theorem 5 instead shows that the maximal payoff dissipation in the game is captured not by a difference of expected rank payoffs, but by a difference of the greatest backward average payoff and the least forward average payoff. Also, the game's expected payoff is at least the minimum of the forward average payoffs. This contrasts with the war of attrition, where the value is the least expected rank payoff.

Our analysis yields a separation of rank payoffs and time costs. Since costs play a key role in determining equilibrium strategies, one might think that not much can be said about the equilibrium without specifying the strategies. Yet the equilibrium is based on the potential function. And this function is derived from the primitive rank payoffs alone, thus determining an equilibrium for *any* time costs.

We conclude by briefly considering observable actions. This produces multiple information sets and vastly enriches the set of supportable equilibria (now subgame perfect). Still, we briefly argue that our main qualitative insight about atom inflation and war of attrition truncation from Theorem 3 remains applicable with a simple refinement.

Maynard Smith (1974) first formalized the war of attrition for theoretical biology. Two animals fight over a fallen prey, the first to give up loses, and fighting is costly for both. With multiple players, payoffs are increasing in the stopping rank. Hendricks et al. (1988) characterize equilibria of the continuous-time complete information war of attrition, while Bulow and Klemperer (1999) analyze a generalized N-player war of attrition with incomplete information. Fudenberg and Tirole (1986) apply wars of attrition to a duopoly exit game, Abreu and Pearce (2006) to bargaining. All-pay auctions and all-pay contests have a similar flavor, as only the last few/highest bids obtain the price; see Siegel (2007) for a recent insightful paper.

The pre-emption game has also been studied widely. Early work focused on tactical duels[4]: two-player zero-sum timing games played on a compact time-interval. Two duelists shoot at each other with accuracy increasing in proximity; they may or may not observe each other's shot. Modern economic examples are aptly captured by the 'Grab-the-Dollar' game: A player can either grab the money on the table or wait for one more period; meanwhile, the pot increases by one unit. Players want to be the first to take the money, but would rather grab a larger pot. This example was first outlined in Fudenberg and Tirole (1985) who apply the idea to analyze how firms decide when to adopt a new technology. Recent examples are Abreu and Brunnermeier (2003), who model financial bubbles (also with unobserved actions), Levin and Peck (2003, 2007), who look at market entry, and Bouis et al. (2006) and Argenziano and Schmidt-Dengler (2008) who study N-player investment dynamics.

In independent work, Sahuguet (2006) explores the equilibria of a three-player timing game with both pre-emption and attrition features. His payoffs are not rank-dependent. In a recent paper, Laraki et al. (2005) (LSV) study the existence of equilibria in general timing games; they provide a very compelling argument for the existence of an ϵ-equilibrium in two-person timing games, and an existence argument for two other classes (cumulative and symmetric, as defined in their paper); these existence results, however, do not overlap with our general existence and characterization theorems.[5] Amidst this large literature on timing games, we believe that our respective works are the first that provide a systematic treatment of classes of games that are neither just a pre-emption game nor just a war of attrition. We hope that our analysis suggests a wider and richer application of timing games in economics.[6] Our work offers insight into periodic unexpected rushes of uncertain size, followed by relative quiet.

Overview In Sections 2 and 3 we outline the model and derive the potential function notion for the equilibrium analysis. In Section 4, we bound the numbers of equilibria and phase transitions, and show how wars of attrition are truncated and pre-emptive atoms inflated in equilibrium. Section 5 bounds the payoffs and game durations of our equilibria. Section 6 discusses the results and potential extensions. Appendix A lays out the equilibrium analysis for observable actions, Appendix B discusses other Nash equilibria that we do not consider in the main text, and Appendix C contains proofs of several lemmata that are used in the main text.

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[4]In 1949, the RAND Corporation kick-started the study of duels (*silent timing games*) with a conference with leading economists, statisticians, and economists. For an extensive survey, see Karlin (1959).

[5]Our existence results are not subsumed by LSV. Their Theorem 1.2 assumes two players. LSV have other existence results for more than two players, but none applies to our model: our payoffs are not cumulative (Theorem 1.3) or symmetric (as defined by LSV, Theorem 1.4). Their Theorem 1.5, which may admit ordinal rank-payoffs, requires no time costs or discounting; also, it secures existence of a Nash equilibrium only if an ϵ-equilibrium exists (which would thus need to be proven separately) for every ϵ.

[6]Shinkai (2000) develops a three-player Stackelberg-type game that fits our rank-payoff formulation. In his framework, quantity pre-emption and learning from predecessors' choices interact to effectively form U-shaped rank rewards. Shinkai, however, does not model the timing decision explicitly.

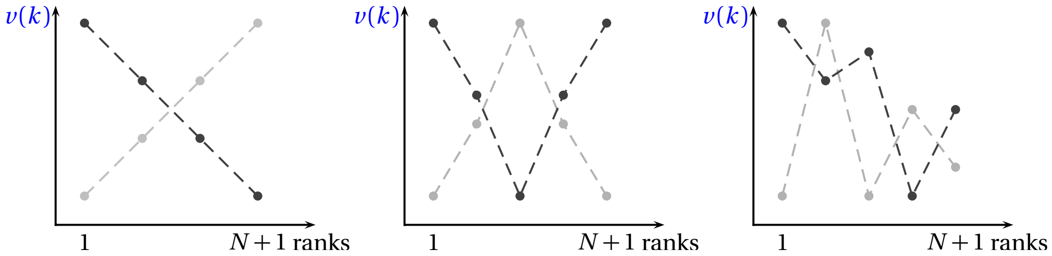

FIGURE 1. Plots of rewards structures. *Left panel:* A stylized War of Attrition reward structure (gray, higher ranks yield higher rewards), and a stylized pre-emption game reward structure (black, low ranks are better). *Middle panel:* Hill-shaped reward structure (gray, the middle rank is best), and an 'avoid-the-crowd' U-shaped reward structure (black, either a very low or very high rank is best). *Right panel:* Two general reward structures with multiple hills: there are multiple 'locally' optimal ranks.

2. A model of timing games with rank-dependent payoffs

Players There are $N+1 \geq 2$ identical players.

Strategies Play transpires in continuous time, starting at time $t=0$. Players have to decide whether 'to stop' or 'not to stop'; they may stop only once; a stopping decision is irrevocable. Actions are unobservable.

With unobservable actions, there is only one information set. A player's strategy specifies when he will stop. A mixed strategy is a non-decreasing and right-continuous cumulative distribution function (cdf) $G:[0,\infty)\to[0,1]$, whose interpretation is that a player stops with probability $G(t)$ by time t or before.

Payoffs Upon stopping, a player receives a lump-sum reward that depends on his ordinal stopping rank. This payment is captured in the *reward-function* $v:\{1,\dots,N+1\}\to\mathbb{R}_+$. For instance, in a two-player war of attrition, $v(1)=0$ and the prize is $v(2)>0$. In the Caller Number Five game, $v(k)=0$ for all $k\neq 5$, and the prize is $v(5)>0$. In general, having more predecessors helps in a war of attrition, or $v(k)<v(k+1)$ for all k. In a pre-emption game, the situation is reversed, as having more predecessors hurts, or $v(k)\geq v(k+1)$ for all k. See Figure 1 for various rank-reward structures.

Agents who stop at the same time equally share the respective rank rewards. This reflects that players are anonymous and identical and that players do not control their rank order among simultaneous stoppers.[7] Assume that $k\in\{0,\dots,N\}$ players have stopped, and $j+1\in\{1,\dots,N-k+1\}$ players stop together. Then the *atomic reward* is the average rank reward $A(k,j):=(v(k+1)+\cdots+v(k+j+1))/(j+1)$. For instance, in a war of attrition, if both agents stop immediately, then their order is randomly determined, and they share the prize equally.

There are two types of explicit costs: discounting at the interest rate $r\geq 0$, and exogenous participation costs $c(t)$, with $c(0)=0$, $\dot{c}>0$, and $\lim_{t\to\infty}c(t)=\infty$.[8]

[7] Alternatively, imagine that stoppers are randomly assigned one of the respective rank payoffs.
[8] In a related work, we also explore time benefits.

Equilibrium Players are ex ante identical and anonymous. It is then intuitive to explore symmetric strategy Nash equilibria. To avoid a continuum of arbitrary outcomes, we confine attention to equilibria whose cdfs have convex *support* starting at 0. (The support of a cdf G is the set of all t with $G(t+\varepsilon)-G(t-\varepsilon)>0$ for all $\varepsilon>0$.) To summarize:

(E1) The support of G is a connected interval $[0,T]$ or $[0,\infty)$.

This restriction is designed to preclude equilibria with explicit periods of silence due to unspecified reasons—calendar time or random holidays (sunspots). But we argue that it embodies a much stronger *stationarity* assumption. Appendix B proves that a continuum of equilibria arises absent this assumption.

3. Equilibrium analysis

In this section, we outline several tools used in equilibrium analysis: necessary conditions for mixed strategies, atomic stopping, potential functions, and general existence.

3.1 *First-order conditions for continuous strategies*

Consider a symmetric continuous strategy G. If $G(t)=g$,[9] then the expected payoff of a player who stops at time t, when all others stop according to G, is

$$\phi(g):=\sum_{k=0}^{N}\binom{N}{k}g^k(1-g)^{N-k}v(k+1).$$

The function ϕ does not depend on the equilibrium and is a primitive of the game. Specifically, not only do the rank payoffs uniquely determine ϕ, but we can uniquely deduce the rank payoffs from any degree-N polynomial ϕ. The reason is that the Bernstein polynomials $\binom{N}{k}g^k(1-g)^{N-k}$ are orthogonal and thus form a basis for the degree-N polynomials (see, for instance, Milovanović et al. 1994). In other words, given the coefficients $v(1),\dots,v(N+1)$, there is a unique expression ϕ and given ϕ, there are unique coefficients $v(1),\dots,v(N+1)$.

In any mixed strategy equilibrium, an agent must be indifferent about stopping anywhere strictly inside the support, so that expected payoffs are constant. Payoffs are discounted rewards less discounted costs,

$$e^{-rt}[\phi(G(t))-c(t)]. \tag{1}$$

Assume $\dot{G}(t)$ exists. Then in equilibrium, payoffs are constant, and equating the *marginal exogenous costs* and *marginal strategic gains* from delay, we get

$$\dot{c}+r(\phi(G)-c)=\dot{G}\phi'(G) \tag{2}$$

[9]In what follows we use g for realizations of G and $\dot{G}$ for the derivative of G (when it exists).

The number $G(t)$ is the probability that a player has stopped by time t, and so G is non-decreasing. For a continuous and increasing G, the differential equation (2) implies that $\dot{c}+r(\phi(G)-c)$ and $\phi'(G)$ have the same sign. Now, $\dot{c}>0$ and $r\geq 0$. Also, $\phi(G(t))\geq c(t)$, for otherwise, players would always be better off stopping at time 0 to get the non-negative rank rewards $v\geq 0$. So (2) is solvable only if $\phi'(G)>0$. In summary, the delay cost must be offset by a strategic delay incentive, so that advancing in the ranks yields greater payoffs and compensates for the requisite delay.

Lemma 1 (Structure of equilibria). *Any Nash equilibrium is described by a cdf G consisting solely of atomic jumps and intervals on which G is continuously differentiable.*

While a cdf is monotone, and thus almost everywhere differentiable, the jumps may be dense in $(0,1)$ (they may be the set of all rationals), and there may be non-jump points where G is not differentiable. Lemma 1, whose proof is in Section C.1, in the Appendix, rules out both possibilities.

3.2 *Analogy for atomic rewards: average vs. marginal revenue*

Consider one player and suppose that the other N players, acting independently, have stopped with probability $G(t)=g$ by time t. At this time, each of the remaining players stops with probability $h-g$, where $h>g$. We often refer to $h-g$ as an *atom* or *mass.* Then the probability that players of ranks $k+1,\dots,k+j$ stop at time t equals a trinomial coefficient $N!/k!j!(N-k-j)!$ times $g^k(h-g)^j(1-h)^{N-k-j}$. The expected payoff to a player who joins the others in stopping at this atom is then

$$\Lambda(g,h):=\sum_{k=0}^{N}\sum_{j=0}^{N-k}\frac{N!}{k!j!(N-k-j)!}g^k(h-g)^j(1-h)^{N-k-j}A(k,j).$$

Thus $\Lambda(0,h)$ is the payoff of an *initial atom* of size h, and $\Lambda(g,1)$ is the payoff of a *terminal atom* of size $1-g$. When $0<g<h<1$, $\Lambda(g,h)$ is the average payoff in the *interior atom* from g to h. Denote by $\Phi(g):=\int_0^g\phi(s)\,ds$ the anti-derivative of $\phi(g)$. This motivates the following result.

Lemma 2. $\Phi(h)-\Phi(g)=(h-g)\Lambda(g,h)$.

The algebraic details of this proof are relegated to Section C.2. The intuition is the following. Independently place each of the N other players into the stopped, atom, and remaining groups, with respective weights $(g,h-g,1-h)$. The expected average rank payoff in the 'atom' group is then $(\Phi(h)-\Phi(g))/(h-g)$, by definition of a conditional expectation. But this is how we have defined $\Lambda(g,h)$, and so these measures coincide.

This has a nice illustrative analogue in standard producer theory. When AR and MR denote average and marginal revenue, and q is quantity, then $MR-AR=qAR'(q)$. Differentiating Lemma 2 with respect to h directly yields $\phi(h)-\Lambda(g,h)=(h-g)(\partial/\partial h)\Lambda(g,h)$. This admits an analogous interpretation: $h-g$ is the mass of the atom, and corresponds to the quantity. The expectation $\Lambda(g,h)$ aggregates and averages rewards, and $\phi(h)$ is the derivative of aggregated (non-averaged) rewards. Lemma 2 thus implies that $\phi(\cdot)$ crosses $\Lambda(g,\cdot)$ from above at the local interior maxima of Λ, and from below at the minima.

Since we can deduce Φ from ϕ, and vice versa, Φ is a primitive of the game too, and suffices to uniquely identify the rank payoffs. We henceforth identify the game by Φ.

3.3 *Equilibrium, potential functions, and existence*

We have already specified that we consider only right-continuous cdfs $G:[0,\infty)\to[0,1]$ for symmetric Nash equilibria that have convex support, including 0 (labeled (E1) in Section 2). In any equilibrium, net payoffs are constant along the support of play, and there is no strict incentive to out-wait all other players. Conversely, these are sufficient conditions for a Nash equilibrium. An equilibrium is formally a cdf obeying the following conditions.

(E2) $e^{-rt}[\phi(G(t))-c(t)]$ is the same constant for all times in the support of G with $G(t)<1$.

(E3) If $G(t^*)>G(t^*-)$, then $\phi(G(t^*-))=\Lambda(G(t^*-),G(t^*))\geq\phi(G(t^*))$ (equal if $G(t^*)<1$).

Since G is a cdf, it jumps at most countably many times, and is continuous on the intervening intervals.[10] By Lemma 1, any continuous portion of G is differentiable. To find an equilibrium cdf G, we thus solve the differential equation (2) subject to the right boundary conditions, determine atomic jumps so that (E3) holds, and then ensure that the boundary conditions reflect the atomic jumps.

We now develop an alternative representation of equilibrium in a single function. This reformulation simplifies our later analysis of the timing games by affording a short proof of existence. More generally, in lieu of a potentially lengthy, complex, and *ad hoc* equilibrium analysis (like computing the number of equilibria and equilibrium payoffs), we show how it suffices to analyze a scalar function.

A C^2 function $\Gamma:[0,1]\to\mathbb{R}_+$ *induces* a strategy G for Φ if

- $\dot{G}=(\dot{c}+r[\Gamma'(G)-c])/\Gamma''(G)$ whenever $\Gamma(G(t))=\Phi(G(t))$
- if $\Gamma\neq\Phi$ on an interval (g,h), then $G(\cdot)$ jumps from g to h.

We then say that the function $\Gamma:[0,1]\to\mathbb{R}_+$ is a *potential function*[11] with respect to Φ if

(P1) $\Gamma(0)=0$, $\Gamma(1)=\Phi(1)$, and $\Gamma'(1)\geq\Phi'(1)$;

[10] We deduce later that G can have only finitely many jumps.

[11] Our phrase "potential function" is in the spirit of a harmonic function whose derivatives describe the gradient on a conservative vector field. Closest to our work, in Myerson (1981), the convex hull of integrated "virtual valuations" for the auction is a potential function; its derivatives fix the priority level for allocating the good. Hart and Mas-Colell (1989) may be the first to use the phrase "potential function" in game theory; *differences* of their potential function yielded marginal payoff contributions in a transferable utility game. Our concept bears no relation to the "potential games" literature—e.g., the potential function in Monderer and Shapley (1996) is a function of the vector of quantities in an IO game. Our potential function maps from a scalar probability.

(P2) Γ is monotonically increasing, convex, and continuously differentiable;

(P3) At each $x \in (0,1)$, either $\Gamma(x) = \Phi(x)$, or Γ is linear in an open interval around x.

The next lemma uniquely identifies potential functions Γ and the equilibria of the game Φ. Before stating the result, we provide a couple of basic identities for Φ. First, obviously $\Phi(0) = 0$. Next, since $\int_0^1 \binom{N}{k} x^k (1-x)^{N-k}\, dx = 1/(N+1)$, we see that

$$\Phi(1) = \int_0^1 \sum_{k=0}^{N} \binom{N}{k} x^k (1-x)^{N-k} v(k+1)\, dx = \frac{1}{N+1} \sum_{k=0}^{N} v(k+1).$$

In other words, $\Gamma(1)$ is the average rank payoff by (P1), while $\Gamma'(1) \geq \phi(1) = v(N+1)$.

Lemma 3 (Equivalence). *Fix Φ. Any potential function Γ induces a unique equilibrium cdf G, and any equilibrium cdf G is induced by a unique potential function Γ.*

The proof is in Section C.3. In brief, fix a potential function Γ. Differentiating Γ yields the expected rank payoffs for any probability g, which are needed for the underlying differential equation of the induced equilibrium. At $g = 0$, this determines the constant payoff for the induced equilibrium. Then, as time costs increase, rank payoffs must increase, which is ensured by convexity. Linear segments in the potential function correspond to atomic stopping, whose payoffs are given by the slope of the corresponding linear segment. Since Φ, the anti-derivative of ϕ, is a polynomial, it is arbitrarily smooth; since Γ is continuously differentiable and smooth and either coincides with Φ or is linear, at the join between a smooth and a linear segment the slopes of the smooth and linear parts coincide. Thus the payoffs from the corresponding atom and the payoffs from slow play before and after the atom coincide.

Conversely, a potential function is found by setting $\Phi(G(t)) = \Gamma(G(t))$ whenever $G(t)$ is left-continuous; increasing rank payoffs ensures the convexity of Γ. When G jumps from g to h, there is a linear segment in Γ with endpoints $(g, \Phi(g))$ and $(h, \Phi(h))$; the slope of this segment coincides with the atomic payoff, by Lemma 2. Since atomic and non-atomic payoffs coincide in equilibrium, the slopes at the end points coincide, and Γ is differentiable.

The equivalence lemma is important because it identifies which game fundamentals matter for the equilibrium analysis. For instance, costs can only speed up or slow down play. We can henceforth employ potential functions to prove theorems by alluding to geometric or graphical properties of these functions.

Example 1 (Caller Number Two of Three). Assume $N+1 = 3$ and $v = (0,1,0)$. Then $\phi(g) = 2g(1-g)$ and $\Phi(g) = g^2(1-2g/3)$. There are exactly two potential functions. First, Γ may initially equal Φ, so that $\Gamma_1(g) = \Phi(g)$ for $g \leq 1/4$ and $\Gamma_1(g) = 3g/8 - 1/24$ for $g > 1/4$. Second, Γ may be initially linear, whereupon it remains linear on $[0,1]$, by convexity, differentiability and (P3): $\Gamma_2(g) = g/3$. These potential functions obey the key properties of smoothness, convexity and boundary values: e.g. $\Gamma_2'(1) = 1/3 > \Phi'(1) = 0$.

Assume delay costs $c(t) = t$ and no discounting. This determines the speed: the first equilibrium involves smooth play described by the ODE $0 = -1 + 2\dot{G}(t)(1-2G(t))$

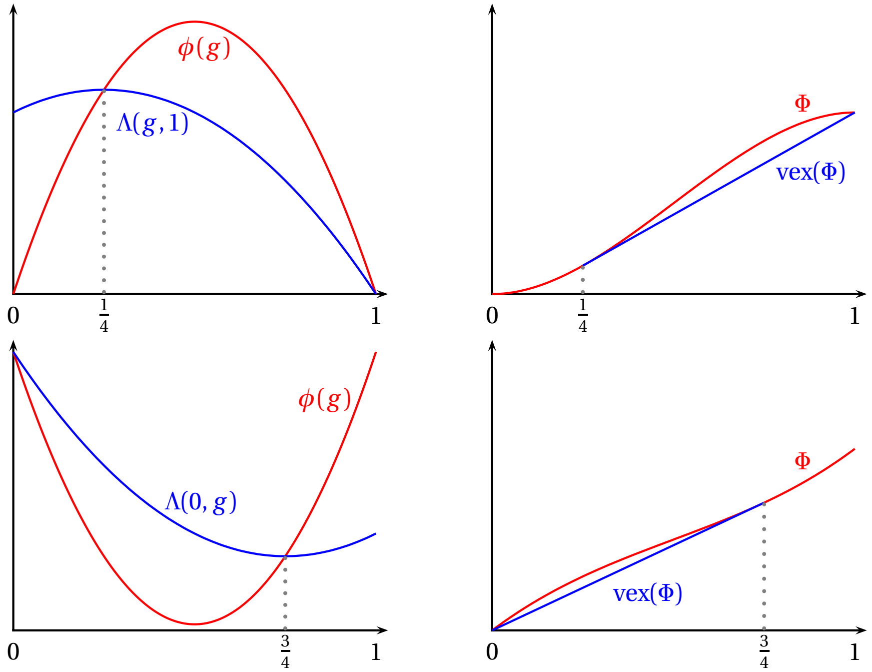

FIGURE 2. Examples 1 and 2 from Section 3: Caller Number Two of Three and U-Shaped Rank Payoffs. The top left panel depicts the expected rank and expected atomic rank payoffs $\phi(g)$ and $\Lambda(g,1)$ for the Caller Number Two of Three game. The top right panel plots the running integral of payoffs Φ and the potential function vex(Φ) identified in Theorem 1. For the U-shaped example, the bottom left panel plots ϕ and $\Lambda(0,g)$ and the bottom right panel plots Φ and the unique potential function vex(Φ). The plots also illustrate the theorems later on: both examples attain the upper bound number two of phases (Theorem 2). Consistent with Theorem 3, the war of attrition is truncated in each case. Just as in Theorem 4, there are two equilibria in the top game (the potential function for the unit jump is not drawn), and one in the bottom game.

from (2), with solution $G(t) = 1/2 - 1/2\sqrt{1-2t}$ until $G(t) = 1/4$. At that point, a jump to $G = 1$ occurs. The second equilibrium entails simply a time-0 jump to $G = 1$. $\Diamond$

EXAMPLE 2 (U-Shaped Rank Payoffs). Assume $N+1 = 3$ and $v = (1,0,1)$. Then $\phi(g) = (1-g)^2 + g^2$ and $\Phi(g) = g(1+g(2g/3-1))$. Here there is a unique potential function $\Gamma_3(g) = 5g/8$ for $g \leq 3/4$ and $\Gamma_3(g) = \Phi(g)$ for $g > 3/4$. Next, solving (2) yields $0 = -1 + 2\dot{G}(t)(2G(t)-1)$, with solution $2G(t) = 1 + \sqrt{1/4+2t}$. Continuous play begins at $t = 0$, with $G(0) = 3/4$. Figure 2 illustrates both examples. $\Diamond$

In mechanism design problems, non-monotonic payoff functions are often "ironed" to produce a monotonic function (e.g. Baron and Myerson 1982). Namely, let vex(Φ) be the convex hull of Φ, i.e. the largest convex function with vex(Φ)$(g) \leq \Phi(g)$ for every g. The "ironed" function then is the derivative vex(Φ)$'(g)$ (see Figure 3). Our potential functions follow a similar idea. Since exogenous costs are ever-increasing, the expected rank-payoffs must also be increasing. The function ϕ, however, may decline,

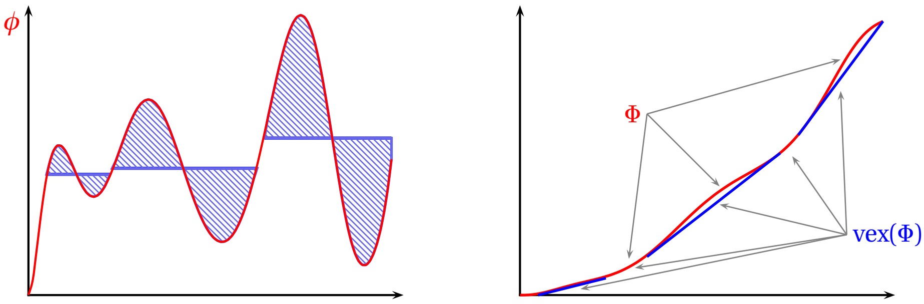

FIGURE 3. Ironing ϕ. The left panel illustrates the ironing procedure on ϕ, and the right panel depicts both Φ and the convex hull of Φ, called vex(Φ).

and these non-monotonicities must be ironed away. Our potential function describes exactly how this works: its derivative is the rank payoff and its convexity ensures that equilibrium payoffs increase. If the potential function contains a linear segment, then rank payoffs are constant, and since delay is costly, atomic stopping must occur.

THEOREM 1. *A symmetric mixed strategy equilibrium exists and ends in finite time.*

PROOF. First, vex(Φ) exists, is a potential function, and thus induces an equilibrium.

In any equilibrium, payoffs are constant on the support at $\phi(0)$. So there exists $\tilde{t} < \infty$ with $\max_g e^{-rt}[\phi(g) - c(t)] < \phi(0)$ after $\tilde{t}$. Delaying beyond $\tilde{t}$ is a dominated strategy, as rewards are discounted or eaten by exogenous delay costs, given $\lim_{t\to\infty} c(t) = \infty$. □

In the Caller Number Two of Three example, vex(Φ)$(g) = \Gamma_1(g)$. In the U-shaped example, $\Gamma_3(g)$ is the unique potential function, and therefore coincides with vex(Φ)(g).

4. BEHAVIORAL PROPERTIES OF EQUILIBRIA

4.1 *Phases and phases transitions*

We first bound the number of slope-sign changes of the expected rank rewards. Define the *sign variation* SV(γ) of the sequence $\gamma = \{\gamma_0, \gamma_1, \dots, \gamma_n\}$ as the number of sign changes left to right (zero terms being neglected). Analogously define the sign variation SV(f) of the bounded function $f : [0,1] \to \mathbb{R}$, i.e. $\mathrm{SV}(f) = \sup_n \mathrm{SV}(\{f(t_0), \dots, f(t_n)\})$ where $0 \le t_0 < t_1 < \dots < t_n \le 1$. Denote by $\Delta v(k) = v(k+1) - v(k)$ the slope of $v(k)$ at rank k and finally put $\Delta v := \{\Delta v(1), \dots, \Delta v(N)\}$.

LEMMA 4 (Variation diminishing property of expected rank rewards). *The slope-sign variations are ranked* $\mathrm{SV}(\Delta v) \ge \mathrm{SV}(\phi')$, *and* $\mathrm{SV}(\Delta v) - \mathrm{SV}(\phi')$ *is an even number. Further, the signs of the first and last slopes of* v *and* ϕ *coincide.*

PROOF. The derivative of $\phi(g)$ in g can be expressed as follows:

$$\phi'(g) = \sum_{k=1}^{N} \binom{N}{k} k g^{k-1} (1-g)^{N-k} (v(k+1) - v(k)).$$

Assume that $\mathrm{SV}(\Delta v) = m$, i.e. the first difference $\Delta v(k)$ changes its sign m times. Scale ϕ' by $g/(1-g)^N$, and let $a_k := k\binom{N}{k}(v(k+1) - v(k))$ and $z := g/(1-g)$. Then

$$\frac{g}{(1-g)^N}\phi'(g) = \sum_{k=1}^{N} k\binom{N}{k}(v(k+1) - v(k))\left(\frac{g}{1-g}\right)^k = \sum_{k=1}^{N} a_k z^k =: P(z).$$

Obviously, $P(z(g))$ and $\phi'(g)$ enjoy the same number of sign variations, i.e. positive real roots of P. By Descartes' Rule of Signs, this number is at most the number of sign changes of its coefficients $a_0, a_1, \ldots, a_N$. Also, if smaller, it is smaller by a multiple of 2. Thus $\mathrm{SV}(\Delta v) \geq \mathrm{SV}(\phi')$ and $\mathrm{SV}(\Delta v) - \mathrm{SV}(\phi')$ is even.

Finally, $\phi'(0) = \Delta v(1)$ and $\phi'(1) = \Delta v(N)$, proving the last clause. □

As noted earlier, this paper subsumes and extends two classes of standard timing games. In a *war of attrition*, an exogenous delay cost opposes a strategic incentive to outwait others. The reverse holds in a *pre-emption game*, where delay is exogenously beneficial, and players wish to pre-empt others. We now categorize game phases by their strategic incentives. There is a *war of attrition phase* if $\dot{G}(t+) > 0$ exists and $\phi'(G(t+)) > 0$ on $(\underline{t}, \bar{t})$. A *pre-emptive explosion* obtains if G jumps at t, as $G(t) > G(t-)$.

A *phase transition* occurs at some time t if below t we have one type of timing game and above t we have another. If three game phases obtain, then there are two phase transitions at t. In what follows, we shall drop the term 'phase' from the game descriptions.

Theorem 2 (Phase transitions). *(i) Equilibrium play consists solely of an alternating sequence of at most* $\mathrm{SV}(\Delta v) + 1$ *wars of attrition and pre-emptive explosions. There are no slow pre-emption games, and pre-emptive atoms always subsume the portions of the domain on which* ϕ *is decreasing.*

(ii) If ϕ *has* m *alternating slope signs, then* $\mathrm{SV}(\phi') = m-1$ *and the maximal number of phase transitions is* $m-1$*. This bound is attained in equilibrium if and only if* $\mathrm{vex}(\Phi)$ *touches every convex portion of* Φ.

This result implies that there are no slow pre-emption game phases.[12] Intuitively, we assume only exogenous costs of delay, and no benefits, and thus there can be no opposition of strategic costs of delay and exogenous benefits.

Proof. (i) Expected payoffs are constant along the support of play. Delay is exogenously costly, and so a player's expected rank reward payoff rises over time in equilibrium. If ever $\phi' < 0$ on a segment of the support $[0,1]$, then players must stop since delay is both strategically and exogenously costly. So play involves slow war of attrition phases and pre-emptive explosions. The number of times that ϕ' switches from positive to negative is bounded by $\mathrm{SV}(\Delta v)$ because ϕ cannot have more interior extrema than v, by Lemma 4. The number of alternating phases is thus the number of switches plus one.

[12]Formally, a *slow pre-emption game phase* obtains if $\dot{G}(t+) > 0$ exists and $\phi'(G(t+)) < 0$ on $(\underline{t}, \bar{t})$.

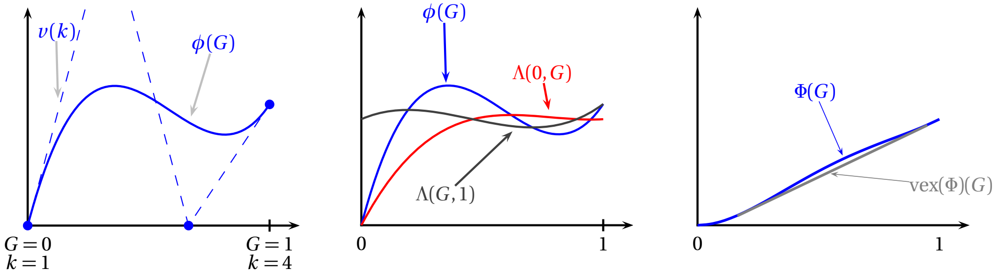

FIGURE 4. The Zick-Zack Game. In this merger of Examples 1 and 2, rank payoffs twice change slope, as $v=(0,\psi,0,1)$. If $v(2)=\psi$ (off the graph) is large enough, then the expected rank reward ϕ likewise has both a hill and a valley. Otherwise, ϕ is monotonically increasing. The middle panel plots the expected reward function ϕ and reward functions for initial atoms $\Lambda(0,g)$ and terminal atoms $\Lambda(g,1)$. The right panel plots $\Phi(g)$ and $\text{vex}(\Phi)(g)$.

(ii) A phase transition occurs if and only if Γ switches from locally linear to strictly convex or vice versa ($\Gamma''>0$). The smooth Γ changes slope only when $\Gamma=\Phi$. As a nonlinear polynomial, Φ has at most as many strictly convex portions as Γ, with equality if and only if ($\star$): Γ touches each convex portion of Φ. As vex(Φ) is a potential function, this proves sufficiency. Next, assume ($\star$). The smooth Γ includes the unique supporting tangent line between all consecutive convex portions. The unique such potential function is vex(Φ). □

One can show that the maximum number of phase transitions is attained only if both the sequence of minima of $\Lambda(0,g)$ and the sequence of maxima of $\Lambda(g,1)$ are increasing.

EXAMPLE 3 (Zick-Zack). The left panel of Figure 4 depicts the four-player game *Zick-Zack*, with rank rewards $v=(0,\psi,0,1)$, with $\psi>0$. We have

$$\phi(g)=3g(1-g)^2\cdot\psi+g^3\cdot 1 \text{ and } \phi'(g)=3\psi(2g-1)^2+3(1-\psi)g^2$$
$$\Phi(g)=\Big(1/4(1-(1-g)^4)-g(1-g)^3\Big)\cdot\psi+g^4/4.$$

Analyzing $\phi(g)$, one can see that $\phi(g)$ is monotonic for $\psi\le 1=:\underline{\psi}$, even though the underlying rank reward structure v has two slope-sign changes. This illustrates the strict inequality in Lemma 4, by a multiple of two. Then Φ is convex with the unique potential function $\Phi=\text{vex}(\Phi)$; thus there are no phase transitions (Theorem 2(i)).

If $\psi>\underline{\psi}$, then ϕ has two slope-sign changes, like v. The fourth-degree polynomial Φ thus has two points of inflection, and vex(Φ) must contain at least one linear portion. Hence, there can be at most two phase transitions (Theorem 2(i)).

Next, vex(Φ) touches both the first and second convex portions of Φ for $\underline{\psi}\le\psi\le\bar{\psi}:=(5+\sqrt{33})/4$. By Theorem 2(ii), the associated equilibrium has the maximum number of phase transitions (two): war of attrition, pre-emptive atom, and then war of attrition. ◊

We have shown that ϕ smoothes out rank payoffs relative to v, reducing the number of possible phase transitions below that suggested by a simple examination of the rank payoffs v. One could naïvely imagine that each slope-sign change of the smooth function ϕ initiates a phase transition. The naïve equilibrium would be one where a war of attrition obtains if and only $\phi' > 0$ and a pre-emption game obtains if and only if $\phi' < 0$. This is not what happens in equilibrium. First of all, while ϕ may be non-monotonic, the only equilibrium may well be a unique pre-emptive atom—for instance, with $v = (2, 0, 1)$. More subtly, the slope ϕ' does not by itself determine the current timing game, because the relation of marginal and average rewards, ϕ and Λ, is critical. Pre-emptive atoms subsume intervals when ϕ is decreasing, by Theorem 2(i); hence the atom is larger than necessary to reach a level of G, so that $\phi' > 0$; we thus say that the atom is 'inflated' relative to an atom that would be prescribed by the naïve direct-sum. The reverse, i.e. inflation of war of attrition phases, does not occur, as we now flesh out.

Theorem 3 (Truncation and atom-inflation). *Pre-emptive atoms are inflated and wars of attrition truncated: Any pre-emptive atom subsumes at least some portion of the adjacent intervals where ϕ is increasing, and where a war of attrition is played.*

Proof. A linear portion of a potential function Γ must be a common tangent to distinct convex portions of Φ, and corresponds to a pre-emptive explosion. If this tangent joins non-adjacent convex portions, then the atom is strictly inflated, as it subsumes at least one entire war of attrition phase. It therefore suffices to consider a common tangent τ of adjacent convex portions. Without inflation, such a τ must touch at consecutive points of inflection of Φ, i.e. where $\phi'(g) = 0$. This is impossible, as it would slice through Φ. □

For instance, in Example 1 (Caller Number Two of Three), at most one phase transition occurs, since ϕ' changes sign just once, from positive to negative when $g = 1/2$. Observe that the ODE defining the war of attrition is defined until time $t = 1/2$. While this may be its natural termination point, terminal atomic rewards are too small at that moment. Indeed, the atom would have size $G(1/2) = 1/2$, and $\Lambda(1/2, 1) = 1/3 < \phi(1/2) = 1/2$. Hence, the atom must be for smaller g, whence $\phi(g)$ and $\Lambda(g, 1)$ cross. This occurs when $\Lambda(g, 1)$ has a maximum at $g = 1/4$, i.e. $G(3/8) = 1/4$. This is before time $t = 1/2$, hence truncation.

4.2 *The number of equilibria*

We now find that the number of equilibria is potentially quite large—about two raised to the number of phase transitions. Specifically, let $\mathscr{E}_m$ denote the set of symmetric Nash equilibria, where m is the number of alternating slope signs of ϕ. Given the expected rank rewards ϕ, we can tie down the maximal cardinality of $\mathscr{E}_m$.[13]

Theorem 4 (Number of equilibria). *Assume ϕ has exactly m alternating slope signs. Then the maximum number of equilibria $|\mathscr{E}_m|$ is $2^{|\mathscr{I}_m|}$, where $\mathscr{I}_m$ is the set of up-slopes of ϕ followed by down-slopes.*

[13] For a recent contribution on the number of Nash equilibria in Normal form games, see McLennan (2005).

Proof. An equilibrium implies a unique set of up-slopes played (the common tangent on pairs of strictly convex portions of Φ is unique). Indeed, an initial down-slope prior to $\mathscr{J}_m$ does not affect the number of equilibria, as the down-slope is skipped in a jump. A terminal up-slope likewise does not affect the number of equilibria. It is either skipped by a pre-emptive atom or played in a war of attrition, but not both. So there is a 1–1 map from equilibria $\mathscr{E}_m$ to sets $\mathscr{J}_m$—hence, the power set enumeration for the upper bound of $|\mathscr{E}_m|$. □

The number of slopes m (up-down-... or down-up-...) is either odd or even. Suppose ϕ slopes up at $g=0$. We then have to find the number of up-slopes followed by down-slopes: if m is even, this number k satisfies $m=2k$; if m is odd then $m=2k+1$. The theorem states that the maximal number of equilibria is $|\mathscr{E}_{2k}|,|\mathscr{E}_{2k+1}|\leq 2^k$. Likewise, if ϕ slopes down at $g=0$, then when m is odd, the number of up-slopes followed by down-slopes satisfies $m=2k-1$ so that $|\mathscr{E}_{2k-1}|,|\mathscr{E}_{2k}|\leq 2^{k-1}$.

For instance, the standard war of attrition has one slope sign, and thus has $|\mathscr{E}_{2\cdot 0+1}|=2^0=1$ equilibrium. The U-shaped game (Example 2) has two slopes, but slopes down first, so that it has at most $|\mathscr{E}_{2\cdot 1}|=2^{1-1}=1$ equilibrium. Caller Number Two of Three (Example 1) has $m=2$ slopes, and exactly one up-slope followed by a down-slope, so that there are maximally $|\mathscr{E}_{2\cdot 1}|=2^1$ equilibria.

For Zick-Zack (Example 3) the theorem asserts that the terminal up-slope should not affect the maximum number of equilibria, i.e. still $|\mathscr{E}_{2\cdot 1+1}|\leq 2^1$. Why? Clearly, if $\psi\leq 1$, then the unique equilibrium is a war of attrition. If $\psi>1$, then Φ has two points of inflection, and there are three possible potential functions. The first begins with a linear segment τ_0 that touches the second convex portion of Φ and is then strictly convex. The second is strictly convex, ending with a linear portion through $(1,\Phi(1))$. This linear segment τ_1 is tangent to the first convex portion of Φ and must have slope $\Gamma'(1)\geq\Phi'(1)$. The last potential function has a linear segment τ in the interior of $[0,1]$ which is the unique common tangent to the first and second convex portions of Φ.

By construction, each of these potential functions is unique—if it exists. Observe that the tangent τ necessarily first touches Φ at some $g\in(0,1)$, because $\Phi'(0)=\phi(0)=0<\phi(g)=\Phi'(g)$ for $g>0$. However, its second touch point occurs at some interior $h<1$ only in some conditions, namely if and only if $\psi\in[\underline{\psi},\bar{\psi})$. Moreover, as is geometrically clear, the tangents τ and τ_1 coincide at the very moment that $\psi=\bar{\psi}$. The tangent τ_1 in fact exists for $\psi\geq\underline{\psi}_1:=(11+3\sqrt{17})/16$. But its slope only weakly exceeds $\Phi'(1)$ for $\psi\geq\bar{\psi}$, where $\bar{\psi}>\underline{\psi}_1$. Altogether, τ_1 is part of a potential function if and only if $\psi\geq\bar{\psi}$.

This illustrates why the terminal up-slope in Zick-Zack does not increase the number of equilibria relative to the Caller Number Two of Three game: tangent τ_1 represents a terminal atom skipping the last up-slope, while τ corresponds to an interior atom after which the terminal up-slope is played. Precisely one of the two obtains.

One can finally show that the initial tangent τ_0 exists for $9/5:=\underline{\psi}_0\leq\psi\leq 3:=\bar{\psi}_0$. For $\psi>\bar{\psi}_0$, τ_0 is no longer tangent to the second convex portion of Φ. For when $\psi=\bar{\psi}_0$, τ_0 becomes a straight line from the origin to $(1,\Phi(1))$ corresponding to a time zero unit atom. In summary, the maximum number of equilibria (two) is attained if and only if $\psi \geq \underline{\psi}_0$.[14]

When is the maximum number of equilibria attained? One may be tempted to think it sufficient that vex(Φ) touches all convex portions of Φ, as in Theorem 2(ii). But the above analysis of Zick-Zack shows that this is not enough: for $\psi \in [\underline{\psi}, \underline{\psi}_0)$, vex($\Phi$) touches both convex portions of Φ, and yet the induced equilibrium is unique.

Even when the maximal number of equilibria is attained, no equilibrium need attain the maximal number of phase transitions. In Zick-Zack, both equilibria have only one phase transition for $\psi > \bar{\psi}$, while the most phase transitions is two, by Theorem 2(i).

So how does one find all the equilibria? First, one identifies all convex portions of Φ. Next, one determines all possible pairwise connections between these convex portions; these are the lines that are tangent to two such portions. For each non-overlapping combination of these lines, one verifies whether convexity is preserved, i.e. the slopes of these lines are successively increasing. Then one combines all such feasible combinations of linear segments with the adjacent smooth, convex portions of Φ so that the combination spans the entire domain $[0,1]$. Ensuring each time that (P1) is satisfied finally yields the potential functions that induce the equilibria.

5. Equilibrium payoffs

Our analysis using potential functions allows us to see how the qualitative features of equilibrium play depend separately on time costs and rank rewards. This dichotomy is the essential reason for the simplicity of our analysis: the size and location of atoms in probability space owes to rank payoffs, while the speed of wars of attrition depends on the time cost of delay. Theorem 5 illustrates this insight in the context of total welfare.

We now ask what is each player's expected payoff, and how much "rent" is lost by delay. In the unobserved actions pure war of attrition, the (common) expected payoff is the initial rank reward $\phi(0) = v(1)$, and all rents are dissipated, namely the difference $\phi(1) - \phi(0) = v(N+1) - v(1)$ between highest and lowest rank payoffs—the total variation in rank payoffs. But with non-monotonic rank payoffs, the total variation of rank-payoffs is no longer the tightest bound on payoff dissipation.

To make simple statements about expected payoffs, we make a simple assumption about delay costs. We assume no discounting and constant marginal participation costs, $c(t) = t$, so that rent dissipation coincides with the length of the play.

Theorem 5 (Payoffs). *Assume no discounting and linear participation costs $c(t) = t$.*

(i) Fix an equilibrium corresponding to a given potential function Γ. Then the expected payoff is $\Gamma'(0)$, and the game must end after an elapse time of $\Gamma'(1) - \Gamma'(0)$.

[14] The $\underline{\psi}_0, \underline{\psi}_1$ thresholds are most easily obtained via $\Lambda(0,g)$ and $\Lambda(g,1)$. First, $\Lambda(0,g)$ has an interior maximum and minimum for all $\psi \geq \underline{\psi}_0$, and is monotonic for smaller ψ. If a potential function starts with a linear portion, then τ_0 is tangent to $\Phi(g)$ exactly when $\phi(g)$ and $\Lambda(0,g)$ intersect at an interior minimum of $\Lambda(0,g)$. The middle panel of Figure 4 illustrates this point. The threshold $\bar{\psi}_0$ for ψ allows $\phi(g)$ and $\Lambda(0,g)$ to cross at $g = 1$. The computations for $\underline{\psi}_1$ follow similar lines of reasoning using $\Lambda(g,1)$. Finally, the payoff from the interior maximum of $\Lambda(g,1)$ coincides with $\phi(1)$ at $\bar{\psi}$.

(ii) *The equilibrium with the least expected payoff and maximal length corresponds to* vex(Φ). *Thus, the least value is* vex(Φ)′(0) *and the greatest length is* vex(Φ)′(1) − vex(Φ)′(0).

For a given potential function, the equilibrium expected payoff of the game is a local minimum of the forward-looking average rewards; the game lasts until a local maximum of backward average rewards obtains. Moreover, the least expected payoff of the game is the global minimum of the forward average payoffs, and the maximal time elapse likewise occurs when the global maximum backward average rewards are reached.

Proof. (i) Fix a potential function Γ. Since the mixed strategy ensures a constant payoff along the support of play, the expected payoff of the game is the time zero payoff $\Gamma'(0)$. By Theorem 1 the game ends in finite time. The length of play depends on the payoffs dissipated—the higher the payoff they can obtain, the longer people are willing to delay. Since expected rank-payoffs must increase along the support of play, the largest rank-payoff $\Gamma'(1)$ obtains when the game ends.

(ii) Suppose, counterfactually, that $\Gamma'(0) < \text{vex}(\Phi)'(0)$ for some potential function Γ. Since $\text{vex}(\Phi) \le \Phi$ everywhere, we have $\text{vex}(\Phi)'(0) \le \Phi'(0)$, and thus $\Gamma'(0) < \Phi'(0)$. Then Γ is initially linear by (P3). But differentiability and (P3) jointly imply that Γ can change slopes only while tangent to Φ. If this happens at $g \in (0,1)$, then $\text{vex}(\Phi)(g) \le \Phi(g) = \Gamma(g) = \Gamma'(0)g < \text{vex}(\Phi)'(0)g$. This violates convexity of vex(Φ).

Similarly, at $g = 1$ we have $\Gamma'(1) \le \text{vex}(\Phi)'(1)$ for any potential function Γ. □

This result extends the standard war of attrition with monotonic rank rewards: when ϕ is monotonic, Φ is globally convex, and the only potential function is Φ itself. The expected payoff is $\Phi'(0) = \phi(0) = v(1)$ and the maximal length is $\Phi'(1) - \Phi'(0) = \phi(1) - \phi(0) = v(N+1) - v(1)$. In fact, by (P3) and Theorem 5(ii), this is the length of any unobserved actions game where vex(Φ) begins and ends on a strictly convex portion.

Since rank payoffs are smoothed in ϕ with unobserved actions, the total variation in $\phi = \Phi'$ is a tighter upper bound on payoff dissipation (e.g. Figure 4, left). But war of attrition phases are truncated, and even this measure is not tight enough. The length and expected payoff depend on the slopes of the initial and terminal tangents τ_0 and τ_1.

In Caller Number Two of Three, vex(Φ) is strictly convex for $g \le 1/4$ and linear with slope 3/8 for $g > 1/4$. The expected payoff is $\phi(0) = 0$ and the maximum length of the game is 3/8. In the U-shaped example, vex(Φ) is linear with slope 5/8 for $g < 3/4$ and strictly convex for $g \ge 3/4$. The expected payoff in the game is the first expected rank payoff in the war of attrition, $\phi(3/4) = 5/8$, and the maximum elapse time equals $\phi(1) - \phi(3/4) = 3/8$.

In Zick-Zack, with rank rewards $(0, \psi, 0, 1)$, vex(Φ) is the potential function that starts with a strictly convex portion. Thus, the minimum expected rank payoff is $\phi(0) = 0$. For $\psi \le \underline{\psi}$, Φ is strictly convex, and the unobserved actions game is equivalent to a war of attrition. If $\psi \in (\underline{\psi}, \bar{\psi})$, vex($\Phi$) touches both convex portions of Φ and hence $\text{vex}(\Phi)'(0) = \phi(0)$ and $\text{vex}(\Phi)'(1) = \phi(1)$. Thus, the maximum duration is $\phi(1) - \phi(0) = 1$, which is below the total variation ψ in rank payoffs. Finally, for $\psi > \bar{\psi}$, vex(Φ) ends with a linear portion, and the terminal payoff is governed by the slope of the tangent τ_1. The maximum duration exceeds $\phi(1)-\phi(0)$, but is still less than the total variation of ϕ.

More generally, the equilibrium payoff is unaffected by the specifics of the cost function or the discount rate—and is still $\Gamma'(0)$, as in Theorem 5. Rent dissipation $\Gamma'(1)$ determines the length of play, i.e. t solves $e^{-rt}[\Gamma'(1)-c(t)]=\Gamma'(0)$. The solution is unique since $\dot{c}>0$. In other words, Theorem 5 is immediately amenable to applications with a nonconstant cost function $c(t)$ or discounting.

What about the *most efficient* equilibrium? If $\Phi(1)\geq\Phi'(1)$, then $\Gamma^*(p)=p\Phi(1)$ is a potential function, and clearly corresponds to a time-0 complete atom. But if $\Phi(1)<\Phi'(1)$, then a time-0 jump is no longer an equilibrium. In some of these cases, we can identify the most efficient equilibrium, but we have found no clear theorem. For there are examples where the equilibrium with the greatest expected payoff is not the quickest.

Assuming that $\Phi''(1-)=\phi'(1-)>0$, for instance, if we can construct a tangent τ^* from the origin to the last convex portion of Φ, tangent at some $\bar{p}\in(0,1]$, then it is the most efficient equilibrium by both measures: shortest and greatest expected payoff. The shortest equilibrium in Zick-Zack is induced by the potential function Γ with a linear segment at the origin; such a potential function exists when $\psi\geq\underline{\psi}_0$. Since $\Gamma\neq\text{vex}(\Phi)$, its expected payoff is higher. Also, for $\psi<\bar{\psi}_0$, its terminal slope is $\Gamma'(1)=\Phi'(1)=\phi(1)=1$, which is weakly smaller than $\text{vex}(\Phi)'(1)$. Thus, it is the shortest equilibrium. For $\psi\geq\bar{\psi}_0$, the atom is complete; this equilibrium is the shortest with the maximal expected payoff.

6. Conclusion

The timing game literature has long been partitioned into wars of attrition and preemption games. The incentive structure for both varieties of timing games finds a common home in this paper. We introduce the idea of potential functions into this class of timing games, using them to characterize the symmetric Nash equilibria. This affords a short existence proof and tractable analysis of these equilibria. The resulting equilibria are rich, with interior atomic explosions that may be preceded or followed by slow wars of attrition. Further, the two types of timing games interact with each other, with anticipation of later phases influencing current play. Thus, the moments for the explosions are advanced in time relative to a naïve "direct sum".

Two extensions of our work come to mind: exogenous payoff growth over time and observed actions. We pursue the former in other work, and in Appendix A we briefly argue that our insights extend to observable actions.

Appendix

A. Lessons for observable actions

Once actions are observed, the model grows substantially more complex. Subgame perfect equilibrium (SPE) is the mandated solution concept. Since players can see the game unfolding, there are now multiple information sets, one for each number of remaining players. There are therefore far more equilibria, since the number of remaining players itself can serve as a coordination device. We thus confine attention to symmetric SPE for which players engage in a war of attrition whenever possible, and a pre-emption game only when necessary. This substitutes for the stationarity condition for Nash equilibrium. For intuitively, a pre-emptive atom requires a high degree of coordination, and a war of attrition needs no coordination at all. Despite this refinement which seeks to minimize the role of pre-emption games, we now argue that our main qualitative finding still obtains: wars of attrition are truncated, and pre-emption atoms inflated.

Let $w(k+1)$ be the expected SPE payoff from the subgame after k have stopped.

LEMMA 5. *A war of attrition obtains if $v(k+1) < w(k+2)$ while a pre-emptive atom of some size $p \in (0,1]$ occurs if $v(k+1) \geq w(k+2)$.*

PROOF. Any $p < 1$ must equate the expected rank payoffs from the continuation game and the (shared) atomic payoffs:

$$\sum_{i=0}^{N-k} \binom{N-k}{i} p^i (1-p)^{N-k-i} w(k+1+i) = \sum_{i=0}^{N-k} \binom{N-k}{i} p^i (1-p)^{N-k-i} A(k,i). \qquad (3)$$

Now, the left-hand side of (3) is flatter than its right-hand side at $p = 0$, for comparing slopes yields

$$(N-k)(v(k+2)-v(k+1))/2 + (N-k)(w(k+2)-v(k+1)) < (N-k)(v(k+2)-v(k+1))/2$$

since $w(k+2) - v(k+1) < 0$. Both sides are continuous in p and coincide for $p = 0$. Thus, they either intersect again for some $p \leq 1$, or, if not, the right-hand side atomic payoff dominates the left-hand side continuation payoff for all p, and a complete atom must obtain. □

Assuming again a constant cost of delay $c(t) = t$, the expected length of the war is $w(k+2) - v(k+1)$, while its expected payoff is $v(k+1) =: w(k+1)$. Assume that rank payoffs rise from j to k. We say that a war of attrition is *truncated in time* if its expected duration is less than $v(k) - v(j)$. Call a war of attrition *weakly truncated* (i.e. in ranks) if it nowhere obtains in $\{j, \ldots, k\}$, or if it obtains from j' to k' for some $j \leq j' < k' \leq k$. Likewise, if rank payoffs fall from j to k, the pre-emption game is *weakly inflated* (in ranks) if it obtains from j' to k' for some $j' \leq j$ and $k' \geq k$. Once an atom occurs, there is further atomic stopping until a war of attrition subgame is reached.

($\diamondsuit$) All rank payoffs on down-slopes are more valuable than the overall average remaining payoff, or $v(k+1) > A(k, N-k)$ whenever $v(k+1) < v(k)$, for any k.

THEOREM 6. *Assume ($\diamondsuit$). Wars of attrition are truncated in time and weakly truncated in ranks, and pre-emptive atoms are weakly inflated.*

PROOF. As players are symmetric, they cannot expect to gain more than the average remaining rank payoff, $w(k+1) \leq A(k, N-k)$. So ($\diamondsuit$) implies $v(k+1) > w(k+1)$.

A war of attrition along an up-slope from a minimum rank $\underline{k}$ to $\bar{k}$ lasts at most time $w(\bar{k}) - v(\underline{k})$; it is thus truncated in the *time* dimension from the naïve length $v(\bar{k}) - v(\underline{k})$.

Atomic stopping obtains whenever $v(k) > w(k+1)$. Assume that there are subsequent up-slopes of rank-rewards. If the atom is complete, then it is clearly inflated. If the atom is incomplete, then with positive probability play continues on the same down-slope. But then $v(k) > v(k+1) > w(k+1)$, and another atom follows immediately. So once atomic stopping starts, it stops only when play begins weakly on an up-slope. □

Corollary 1. *Assume* ($\diamondsuit$). *The lowest expected equilibrium payoff with unobservable actions,* $\mathrm{vex}'(\Phi)(0)$*, is a lower bound for the expected payoff in an observable actions setting.*

This corollary can, of course, be applied also to every subgame of the observable actions setting, where $\mathrm{vex}'(\Phi)(0)$ is computed for the unobservable actions game with $N+1-k$ players. The corollary is a direct consequence of $\mathrm{vex}'(\Phi)(0)$ constituting a global minimum of the right-hand side of (3). It is not true, however, that $\mathrm{vex}(\Phi)'(1) - \mathrm{vex}(\Phi)'(0)$ is a bound on the elapse time: this is due to the fact that rank payoffs become left truncated as people stop. Hence $\mathrm{vex}(\Phi)'(1)$ has no direct counterpart relation in a setting with observable actions.

Corollary 2. *Assume* ($\diamondsuit$). *There are at most as many phases as slope signs of* $v(k)$.

B. Other Nash equilibria

Assumption (E1) restricts the set of equilibria we consider. We now argue that relaxing either of the restrictions of (E1) introduces a continuum of other equilibria.

If we drop the assumption that 0 belongs to the support, then a continuum of equilibria may arise as follows.[15] Suppose that in the set of equilibria that we identify there is one with an atom at time zero and $\Gamma'(0) > \phi(0)$, as occurs in Caller Number Two of Three. Then there is a maximum time t such that $e^{-rt}\,[\Gamma'(0) - c(t)] = \phi(0)$. And for every $s \in (0, t]$, play according to Γ starting at time s is an equilibrium.

Similarly, if we abandon the requirement of a convex support, then a continuum of equilibria can be constructed. The idea behind such a construction is to have an atom from g to h that pays more than expected rank payoffs $\phi(g)$. To make this an equilibrium, the benefits of the atom must be destroyed (because payoffs in a mixed strategy must be constant on the support). In our setting this payoff destruction is achieved by prescribing sufficient delay (which is costly) until the atom occurs. (And pre-empting such an atom does not pay precisely because the rank payoff from pre-emption, $\phi(g)$, is smaller than the atomic payoff.) Economically, these kinds of atom require implicit sunspot coordination, and for this reason we believe that these equilibria are very unappealing.

More elaborately, one way to construct a continuum of equilibria in absence of the convex support assumption goes as follows. Suppose that Γ prescribes an atom at time t_i from ξ_i to ξ_{i+1}, and $G(t_i) = \xi_i$. Also assume $\xi_{i+1} < 1$, i.e. the atom is not terminal. Now pick a small $\epsilon > 0$, and compute $G(t^* - \epsilon) =: \xi_i^{-\epsilon} < \xi_i$. Let $\xi_{i+1}^{-\epsilon} < \xi_{i+2}$ be a solution so that $\Lambda(\xi_i^{-\epsilon}, \xi_{i+1}^{+\epsilon}) = \phi(\xi_{i+1}^{+\epsilon}) > \phi(\xi_{i+1})$ (since the atom at ξ_i is not terminal, for small enough ϵ such a $\xi_{i+1}^{+\epsilon} \in (\xi_{i+1}, \xi_{i+2})$ exists by Lemma 4). Finally let δ

[15] A related problem arises in all-pay auctions; for details see Baye et al. (1996).

solve $e^{-r(t_i-\epsilon)}\left[\phi(\xi_i^{-\epsilon})-c(t_i-\epsilon)\right] = e^{-r(t_i+\delta)}\left[\phi(\xi_{i+1}^{+\epsilon})-c(t_i+\delta)\right]$. Such a δ exists because $\phi(\xi_i^{-\epsilon})<\phi(\xi_{i+1}^{+\epsilon})$ and because $e^{-r(t)}\left[\phi(\xi_i^{-\epsilon})-c(t)\right]$ declines monotonically in t. Then the following is an equilibrium: play Γ until time $t_i-\epsilon$; be inactive from $t_i-\epsilon$ until $t_i+\delta$; stop in an atom of size $\xi_{i+1}^{+\epsilon}-\xi_i^{-\epsilon}$ at time $t_i+\delta$; play the portion of Γ that is defined for $\xi\in[\xi_{i+1}^{+\epsilon},1]$ thereafter for $t>t_i+\delta$. Conceptually this equilibrium without a convex support is quite similar to the equilibrium that we specify with convex support.

C. Other proofs

C.1 *Equilibrium structure: Proof of Lemma 1*

Fix t in the interior of the support of G, and assume that G does not jump at t. Since payoffs (1) are constant on the support, payoff $e^{-rt}(\phi(G(t))-c(t))=:\psi>0$ is a constant when $G(t)<1$. But this forces G to be differentiable at t. Since ϕ is a degree N polynomial, $\phi'=0$ at most $N-1$ times, between which ϕ' is positive or negative. There are three cases to consider.

Case 1 If $\phi'(G(t))>0$ at t, then G is differentiable at t, with

$$\dot G(t)=\frac{\dot c+r(\phi(G(t))-c)}{\phi'(G(t))}.$$

Case 2 If $\phi'(G(t))<0$ at t, then $\phi(G(t)-\epsilon)>\phi(G(t))>\phi(G(t)+\epsilon)$ for all small enough $\epsilon>0$. Since t is inside the support of G, but is not in an atom, there exists $\delta>0$ with $\phi(G(t)-\epsilon)>\phi(G(t-\delta))>\phi(G(t))>\phi(G(t+\delta))>\phi(G(t)+\epsilon)$. Since $e^{-r(t-\delta)}>e^{-rt}$ and $c(t-\delta)<c(t)$, a constant payoff is impossible because

$$e^{-r(t-\delta)}(\phi(G(t-\delta))-c(t-\delta))>e^{-rt}(\phi(G(t))-c(t)).$$

In other words, $\phi'(G(t))<0$ cannot obtain in equilibrium.

Case 3 Suppose $\phi'(G(t))=0$ at t. If this is a saddle point with $\phi'<0$ left and right of $G(t)$, then $G(t)=\phi^{-1}(c(t)+e^{rt}\psi)$ locally. But this is decreasing, and so not a solution. Otherwise, $\phi'>0$ is increasing on at least one side of $G(t)$, where $G(t)=\phi^{-1}(c(t)+e^{rt}\psi)$ locally describes the unique smooth solution of the ODE.

Finally, if $t=0$ or if the support interval of G is $[0,t]$, then the argument that G is differentiable (right or left, respectively) is a slight modification of the above analysis.

C.2 *Relation of atomic and expected rewards: Proof of Lemma 2*

We show that $\phi(h)-\Lambda(g,h)=(h-g)\partial\Lambda(g,h)/\partial h$; the result from the lemma follows by integrating this relation from g to h. We show the claim directly by algebraic manipulations. First

$$\begin{aligned}&\frac{\partial}{\partial h}\binom{N-k}{j}\left(\frac{h-g}{1-g}\right)^{j}\left(\frac{1-h}{1-g}\right)^{N-k-j}\\&=\underbrace{\frac{j}{(1-g)^2}\binom{N-k}{j}\left(\frac{h-g}{1-g}\right)^{j-1}\left(\frac{1-h}{1-g}\right)^{N-k-j}}_{:=\delta(k,j,g,h)}-\underbrace{\frac{N-k-j}{(1-g)^2}\binom{N-k}{j}\left(\frac{h-g}{1-g}\right)^{j-1}\left(\frac{1-h}{1-g}\right)^{N-k-j-1}}_{\delta(k+1,j,g,h)}.\end{aligned}$$

We use the notation $\rho(k,j,g,h) := \binom{N-k}{j} j\left((h-g)/(1-g)\right)^{j-1}\left((1-h)/(1-g)\right)^{N-k-j}$ and write $\mu(k,g)=\binom{N}{k}g^k(1-g)^{N-k}$. With straightforward manipulations we obtain

$$\begin{aligned}
&(h-g)\frac{\partial}{\partial h}\Lambda(g,h)\\
&\quad=(h-g)\sum_{k=0}^{N}\mu(k,g)\sum_{j=1}^{N-k}\delta(k,j,g,h)\frac{1}{1-g}\left(\frac{1}{j+1}\sum_{i=0}^{j}v(k+1+i)-\frac{1}{j}\sum_{i=0}^{j-1}v(k+1+i)\right)\\
&\quad=\sum_{k=0}^{N}\mu(k,g)\sum_{j=1}^{N-k}\rho(k,j,g,h)\left(v(k+1+j)-\frac{1}{j+1}\sum_{i=0}^{j}v(k+1+i)\right)\\
&\quad=\sum_{k=0}^{N}\mu(k,g)\sum_{j=1}^{N-k}\rho(k,j,g,h)v(k+1+j)+\sum_{k=0}^{N}\mu(k,g)\rho(k,0,g,h)v(k+1)-\Lambda(g,h)\\
&\quad=\sum_{k=0}^{N}\mu(k,g)\sum_{j=0}^{N-k}\rho(k,j,g,h)v(k+1+j)-\Lambda(g,h).
\end{aligned}$$

We now have to show that the first term coincides with $\phi(h)$,

$$\sum_{k=0}^{N}\underbrace{\binom{N}{k}h^k(1-h)^{N-k}}_{=\mu(k,h)}v(k+1)=\sum_{k=0}^{N}\mu(k,g)\sum_{j=0}^{N-k}\rho(k,j,g,h)v(k+1+j).$$

Fix $v(k+1)$ and collect all terms on the right hand side that contain $v(k+1)$. These are for i from 0 to k and $j=k-i$. Thus

$$\sum_{i=0}^{k}\mu(i,g)\rho(i,k-i,h,g)=\sum_{i=0}^{k}\binom{N}{i}g^i(1-g)^{N-i}\binom{N-i}{k-i}\left(\frac{h-g}{1-g}\right)^{k-i}\left(\frac{1-h}{1-g}\right)^{N-i-(k-i)}.$$

Further simplification and rearranging leads to

$$\begin{aligned}
&\sum_{i=0}^{k}\frac{N!}{(N-i)!i!}\frac{(N-i)!}{(N-k)!(k-i)!}g^i\frac{(1-g)^{N-i}}{(1-g)^{k-i}}(h-g)^{k-i}\left(\frac{1-h}{1-g}\right)^{N-k}\\
&\qquad=\binom{N}{k}(1-h)^{N-k}\sum_{i=0}^{k}\binom{k}{i}g^i(h-g)^{k-i}=\binom{N}{k}(1-h)^{N-k}\cdot h^k=\mu(k,h). \quad\square
\end{aligned}$$

C.3 *Potential function equivalence: Proof of Lemma 3*

Fix a potential function Γ. By (P1) it is convex, and so there are at most countably many intervals on which it is linear. Consider any such interval $[\nu,\xi]$. First, $\Gamma(\nu)=\Phi(\nu)$, by (P1) or (P3). Further, by smoothness (P2) and Lemma 2,

$$\Lambda(\nu,\xi)\equiv\frac{\Phi(\xi)-\Phi(\nu)}{\xi-\nu}=\frac{\Gamma(\xi)-\Gamma(\nu)}{\xi-\nu}=\begin{cases}\Gamma'(\xi)=\Phi'(\xi) & \text{if } \xi<1\\ \Gamma'(\nu)=\Phi'(\nu) & \text{if } \nu>0.\end{cases}$$

So (E2) obtains: stoppers earn identical payoffs just *before* atomic stopping if $\nu > 0$, for then $\Lambda(\nu,\xi)$ equals $\Phi'(\nu) = \phi(\nu)$, and *after* atomic stopping if $\xi < 1$, since $\Phi'(\xi) = \phi(\xi)$. Also, (E1) holds, as expected rank payoffs are positive, by $\Gamma(\xi) > \Gamma(\nu)$. If $\xi = 1$, then $\phi(1) = \Phi'(1) \le \Gamma'(1) = \Lambda(\nu,1)$ by (P1); so (E3) holds.

Assume $\Gamma = \Phi$ on $[\nu,\xi]$, so that $\phi = \Phi' = \Gamma'$ (which exists by (P2)). We then need not worry about (E3). Since Γ is convex by (P2) and Φ is smooth, we have $\phi' = \Gamma'' \ge 0$. Also, ϕ is strictly increasing inside the interval, being a nonconstant polynomial; thus (E1) holds, as ϕ' can only initially vanish. Assume that $G(\underline{t}) = \nu$ for some $\underline{t} \ge 0$. Thus, the ODE $\dot{G} = (\dot{c} + r[\phi(G) - c])/\phi'(G)$ in (2) admits the "constant payoff" solution $e^{-rt}[\phi(G(t)) - c(t)] = \phi(0) = \Gamma'(0)$, which is the initial payoff—recalling that the support of G must include 0. This gives (E2). Let $C(t) := c(t) + e^{rt}\Gamma'(0)$. Since ϕ is strictly increasing on (ν,ξ), $G(t) = \phi^{-1}(C(t))$ obtains on the domain $(\underline{t},\bar{t})$, where $\bar{t} = C^{-1}(\xi)$.

Next, fix an equilibrium G. Define the potential function Γ as follows. First, $\Gamma(g) = \Phi(g)$ whenever G is continuous at $G^{-1}(g)$. Next, at any jump from g to h, Γ is the linear function through $(g,\Phi(g))$ and $(h,\Phi(h))$. Using constant payoffs (E2), (E3), and $\phi(g) = \Phi'(g)$ (by Lemma 1), the slope of this line satisfies

$$\frac{\Gamma(h)-\Gamma(g)}{h-g} = \frac{\Phi(h)-\Phi(g)}{h-g} \equiv \Lambda(g,h) \begin{cases} \ge \phi(h) & \text{with equality if } h < 1 \\ = \phi(g) = \Gamma'(g) & \text{if } g > 0. \end{cases} \tag{4}$$

This gives (P3) and also (P2): Γ is increasing since $\Gamma' = \phi > 0$ by (4), and convex (Γ is linear, or has slope ϕ, which is increasing by (E2)).

Finally, we show (P1). If $\Gamma = \Phi$ near 1, then $\Gamma(1) = \Phi(1)$ and $\Gamma'(1) = \Phi'(1)$. If $\Gamma = \Phi$ near 0, then $\Gamma(0) = \Phi(0)$. If $G(t)$ starts with a jump from 0 to h, then Γ has a linear segment with slope $\Phi(h)/h$ through $(h,\Phi(h))$. This forces $\Gamma(0) = 0$. If G ends with a jump to 1, then $\Gamma'(1)$ is the final linear slope, i.e. $\Gamma'(1) \ge \phi(1) = \Phi'(1)$ by (4).

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