Local stability under evolutionary game dynamics

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We prove that any regular evolutionarily stable strategy (ESS) is asymptotically stable under any impartial pairwise comparison dynamic, including the Smith dynamic; under any separable excess payoff dynamic, including the BNN dynamic; and under the best response dynamic. Combined with existing results for imitative dynamics, our analysis validates the use of regular ESS as a blanket sufficient condition for local stability under evolutionary game dynamics.

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A basic task of evolutionary game theory is to find conditions under which equilibrium play is dynamically stable. Much of the literature focuses on global convergence to equilibrium, identifying global payoff structures that ensure eventual equilibrium play regardless of agents' initial behavior.[1] Inevitably, these requirements on payoff structure are quite demanding: although each condition is satisfied in some applications, a "typical" game with many strategies does not satisfy any known condition for global convergence.

To obtain stability results relevant to a wider range of games, one can turn instead to local stability results, seeking conditions under which an equilibrium is robust to small changes in a population's behavior. Because conditions for local stability need only constrain payoffs near the equilibrium in question, they are far easier to satisfy, and so are more likely to be applicable.

The natural starting point for discussions of local stability is the notion of an *evolutionarily stable strategy* (*ESS*), introduced for single-population random matching models by Maynard Smith and Price (1973). These authors envision a monomorphic population whose members all play the same mixed strategy, and they call this mixed strategy an ESS if the incumbent population is capable of resisting invasions by any small mutant group whose members all play some alternative mixed strategy.[2]

　　[1]Classes of games admitting global convergence results include potential games, supermodular games, dominance-solvable games, and stable games. Sandholm (2009) offers a survey of these results.

　　[2]This interpretation is formalized by Cressman (1992, 2006) and Cressman et al. (2001)—see Section 5.

Most economic applications of evolutionary game theory do not proceed from this monomorphic, mixed-strategist model, but from a polymorphic model in which agents choose among the available pure strategies. This, for example, is the setting of the replicator dynamic of Taylor and Jonker (1978). However, Maynard Smith and Price's (1973) conditions retain their importance in this new context: Taylor and Jonker (1978), Hofbauer et al. (1979), Zeeman (1980), and Hofbauer and Sigmund (1988) show that ESS provides a sufficient condition for local stability under the replicator dynamic. Because in this setting Maynard Smith and Price's (1973) conditions are applied to population states—that is, to distributions of the agents over the pure strategies—we refer to a strategy distribution that satisfies these conditions as an *evolutionarily stable state*.[3]

Subsequent studies of local stability under evolutionary dynamics have tended to focus on interior equilibria—that is, equilibria in which every pure strategy is played by a positive mass of players.[4] Analyses that do not impose this restriction are less common. The most noteworthy results in this direction are due to Cressman (1997), who studies a general class of dynamics that captures imitative behavior. Cressman (1997) shows that near rest points, the linearization of any imitative dynamic preserves the basic structure of the linearization of the replicator dynamic, which itself is the fundamental example of an imitative dynamic. Combining this insight with results from Taylor and Jonker (1978) and Hines (1980), Cressman (1997) proves that in nondegenerate cases, regular ESSs are locally stable under all imitative dynamics.[5]

Imitation is undoubtedly a basic component of human decision making, and most early analyses of evolutionary game dynamics in the economics literature focus on models that can be interpreted as imitative.[6] However, the direct evaluation of strategic alternatives, rather than indirect evaluation through the experiences of others, is also a basic mode of choice; indeed, it is the approach implicitly followed in most economic and game-theoretic modeling. Direct evaluation has a number of important consequences for evolutionary game dynamics: strategies' growth rates no longer need to be tied to their current levels of utilization, and unused strategies may be chosen if agents deem them worthwhile.

The first evolutionary dynamic based on direct evaluation of alternative strategies to gain wide currency in economics is the best response dynamic of Gilboa and Matsui (1991). By having revising agents always switch to a strategy that is currently optimal, this dynamic offers a natural amalgam of inertial evolutionary modeling and traditional, rational game-theoretic analysis. At the same time, optimization requires exact knowledge of all strategies' payoffs. Because evolutionary analysis is employed in settings where agents intermittently and myopically update their strategies, the assumption that

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[3]Thomas (1984) is an early reference that emphasizes the distinction between evolutionarily stable strategies and evolutionarily stable states.

[4]See Hofbauer (1995b, 2001), Hopkins (1999), Hofbauer and Hopkins (2005), and Sandholm (2007).

[5]The formal definitions of ESS and regular ESS are presented in Section 3.

[6]The reason for this is in part historical. Evolutionary game dynamics were introduced in biology to model natural selection, with game payoffs representing fitnesses, and dynamics describing relative rates of births and deaths in animal populations. Early work by economists on deterministic evolutionary game dynamics retained this biological approach. Only later, with the work of Björnerstedt and Weibull (1996), Weibull (1995), Schlag (1998), and Hofbauer (1995a), was it recognized that these explicitly biological models could be reinterpreted as models of imitation by economic agents.

this myopic updating is done in an optimal way seems somewhat incongruous. To retain the assumption of direct evaluation of strategies, one can suppose instead that each revising agent considers only a single, randomly chosen alternative to his current strategy. Protocols of this sort form the basis for two families of evolutionary dynamics from the literature. When agents compare the candidate strategy's payoff to the population's average payoff, aggregate behavior is described by an *excess payoff dynamic*. If instead each agent compares the candidate strategy's payoff to the payoff of his current strategy, aggregate behavior is described by a *pairwise comparison dynamic*. The *Brown–von Neumann–Nash* (*BNN*) *dynamic* (Brown and von Neumann 1950) and the *Smith dynamic* (Smith 1984) are early examples from these two classes of dynamics; only more recently have excess payoff dynamics and pairwise comparison dynamics been studied systematically.[7]

All of the direct evaluation dynamics described above differ from imitative dynamics in crucial ways: they allow agents to switch to unused strategies and they do not exhibit the percentage-growth-rate functional form that is characteristic of imitative dynamics. Given these basic differences, it is unclear whether conditions on payoffs that imply local stability under imitative dynamics do the same for dynamics based on other choice principles.[8]

In this paper, we study the local stability of rest points under impartial pairwise comparison dynamics, which include the Smith dynamic as a special case, under separable excess payoff dynamics, which include the BNN dynamic, and under the best response dynamic. By modifying Lyapunov functions used by Hofbauer and Sandholm (2009) to study evolution in stable games, we prove that any regular ESS is locally asymptotically stable under all of the dynamics in the classes noted above. Combined with Cressman's (1997) results for imitative dynamics, our analysis validates the use of regular ESS as a blanket sufficient condition for local stability under evolutionary game dynamics.

The paper proceeds as follows. Section 2 introduces population games and evolutionary dynamics, and presents the classes of dynamics under study. Section 3 defines the notions of ESS and regular ESS. Section 4 presents our local stability theorem, describes the intuition behind its proof, and compares this analysis with local stability analyses for imitative dynamics. Section 5 presents extensions to multipopulation models. Section 6 offers concluding discussion. All proofs are presented in the Appendix.

2. The model

2.1 *Population games*

To keep the notation manageable, we focus first on games played by a single population of agents. Analogous results for multipopulation models are presented in Section 5.

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[7]See Weibull (1996), Hofbauer (2001), and Sandholm (2005, 2010).

[8]Indeed, Friedman (1991, p. 656) suggests that ESS should not be viewed as a general sufficient condition for local stability. Furthermore, imitative dynamics are known to have special properties in related contexts. Samuelson and Zhang (1992) prove that under imitative dynamics, a strictly dominated strategy must vanish along any interior solution trajectory, but Hofbauer and Sandholm (2007) show that any continuous evolutionary dynamic that is not purely imitative allows dominated strategies to survive in some games.

We suppose that there is a unit mass of agents, each of whom chooses a pure strategy from the set $S = \{1, \dots, n\}$. The aggregate behavior of these agents is described by a *population state* $x \in X$, where $X = \{x \in \mathbb{R}_+^n : \sum_{j \in S} x_j = 1\}$ is the simplex and x_j represents the proportion of agents who choose pure strategy j.

We identify a *population game* with a continuously differentiable payoff function $F : X \to \mathbb{R}^n$. The scalar $F_i(x)$ represents the payoff to strategy i when the population state is x, and the matrix $DF(x) \in \mathbb{R}^{n \times n}$ denotes the derivative of F at x. We let $\bar{F}(x) = \sum_{j \in S} x_j F_j(x)$ denote the *average payoff* obtained by the members of the population, and we let $\hat{F}_i(x) = F_i(x) - \bar{F}(x)$ denote the *excess payoff* to strategy i over the population's average payoff.

The simplest population games are those with linear payoffs, for which there is a matrix $A \in \mathbb{R}^{n \times n}$ such that $F_i(x) = \sum_{j \in S} A_{ij} x_j$.[9] In this case, the payoffs to all strategies can be expressed concisely as $F(x) = Ax$, implying that $DF(x) = A$. More generally, payoffs may depend nonlinearly on the population state. For example, if we use a population game to model network congestion, then the payoff from using a given link is constant at low levels of traffic, but falls steeply as the link nears its capacity.[10] None of the results to follow requires payoffs to be linear: the continuous differentiability of F is all that is needed for our analysis to hold.

2.2 *Revision protocols and evolutionary dynamics*

To derive evolutionary dynamics from a model of individual decision making, we introduce *revision protocols*, which describe how agents adjust their choices of strategies during recurrent play of the game at hand. Formally, a *revision protocol* is a Lipschitz continuous map $\rho : \mathbb{R}^n \times X \to \mathbb{R}_+^{n \times n}$ that takes payoff vectors π and population states x as arguments, and returns nonnegative matrices as outputs. The scalar $\rho_{ij}(\pi, x)$ is called the *conditional switch rate* from strategy i to strategy j. If we imagine that agents receive revision opportunities independently according to rate R Poisson processes, then $\rho_{ij}(\pi, x)/R$ represents the probability that an i player who receives a revision opportunity switches to strategy $j \neq i$.[11] This framework allows for the possibility that agents observe all strategies' current payoffs and utilization levels, but in many cases the information requirements are much weaker than this; see the examples below.

A revision protocol ρ and a population game F together define an ordinary differential equation on the state space X:

$$\dot{x}_i = \sum_{j \in S} x_j \rho_{ji}(F(x), x) - x_i \sum_{j \in S} \rho_{ij}(F(x), x). \tag{M}$$

Equation (M) is called the *mean dynamic* generated by ρ and F. The first term in (M)

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[9] One could interpret $F_i(x)$ here as representing the expected payoff of a random match in the symmetric normal form game A. This is not our preferred interpretation, because in the subsequent model of strategy revision, $F_i(x)$ is best viewed as a realized payoff. To retain a matching interpretation, one can view $F_i(x)$ as a realized payoff if each agent is matched once with every opponent. See Section 6 for further discussion.

[10] For more on large-population congestion games, see Beckmann et al. (1956) and Sandholm (2001).

[11] Note that the rate R must be large enough that $\sum_{j \neq i} \rho_{ij}(\pi, x)/R$ never exceeds 1, and that the diagonal elements of $\rho(\pi, x)$ play no formal role in the model.

captures the inflow of agents to strategy i from other strategies, whereas the second term captures the outflow of agents to other strategies from strategy i.

Deriving evolutionary dynamics from revision protocols not only allows us to study the aggregate implications of different rules for individual choice, but also lets us find microfoundations for existing dynamics. For instance, the replicator dynamic

$$\dot{x}_i = x_i \hat{F}_i(x) \tag{1}$$

is defined by the property that the percentage growth rate of each strategy in use is equal to its excess payoff. Although Taylor and Jonker (1978) introduced this dynamic to model natural selection in animal populations, for the purposes of economic analysis the replicator dynamic can be viewed as a model of payoff-dependent imitation. One revision protocol that generates this dynamic is *pairwise proportional imitation,*[12]

$$\rho_{ij}(\pi, x) = x_j [\pi_j - \pi_i]_+, \tag{2}$$

where $[d]_+ = \max\{d, 0\}$ is the positive part of d. The x_j term in this protocol reflects the fact that a revising agent picks a candidate strategy by observing the current strategy of a randomly chosen opponent. The agent switches to the opponent's strategy if its payoff is higher than that of his own current strategy, doing so with probability proportional to the payoff difference. Substituting protocol (2) into equation (M) and simplifying the result yields the replicator dynamic (1).

2.3 *Families of evolutionary dynamics*

This approach to dynamics via revision protocols also allows us to define families of evolutionary dynamics, where members of the same family are derived from qualitatively similar protocols.

EXAMPLE 1. An *imitative dynamic* is an evolutionary dynamic of the form

$$\dot{x}_i = x_i G_i(x), \tag{3}$$

where the C^1 map $G : \mathbb{R}^n \to \mathbb{R}^n$ satisfies *monotone percentage growth rates*:

$$G_i(x) \geq G_j(x) \text{ if and only if } F_i(x) \geq F_j(x), \tag{4}$$

as well as the forward invariance condition $x'G(x) = 0$ (see equation (13) below). The simplest and best-known imitative dynamic, the replicator dynamic (1), is obtained when $G(x)$ is the excess payoff function $\hat{F}(x) = F(x) - \mathbf{1}\bar{F}(x)$, where $\mathbf{1} \in \mathbb{R}^n$ is the vector of ones.

Imitative dynamics can be derived from revision protocols of the imitative form

$\rho_{ij}(\pi, x) = x_j r_{ij}(\pi, x),$ where for all $i, j, k \in S,$

$$\pi_j \geq \pi_i \text{ if and only if } r_{kj}(\pi, x) - r_{jk}(\pi, x) \geq r_{ki}(\pi, x) - r_{ik}(\pi, x).$$

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[12]This protocol is due to Schlag (1998). Other imitative protocols that generate the replicator dynamic can be found in Björnerstedt and Weibull (1996), Weibull (1995), and Hofbauer (1995a).

The imitative aspect of these protocols is manifested by the x_j term, the presence of which leads the dynamic (3) to take a simple percentage-growth-rate form. For more on the foundations of imitative dynamics, see Björnerstedt and Weibull (1996), Weibull (1995), Hofbauer (1995a), and Sandholm et al. (2008). ◇

Under imitative protocols, agents select candidate strategies by observing their opponents' behavior.[13] Under the protocols to follow, agents instead choose candidate strategies directly, allowing them to switch to unused strategies whose payoffs are sufficiently appealing.

EXAMPLE 2. Suppose that the rate at which current strategy i players switch to the candidate strategy j depends only on the payoff difference between them, and that switches to better-performing strategies are the only ones that have positive probability. Such rules are described by revision protocols of the form

$$\rho_{ij}(\pi, x) = \phi_{ij}(\pi_j - \pi_i),$$

where each function ϕ_{ij} is *sign-preserving*, in the sense that $\mathrm{sgn}(\phi_{ij}(d)) = \mathrm{sgn}([d]_+)$.[14] Dynamics (M) generated by protocols of this form are called *pairwise comparison dynamics* (Sandholm 2010). We call such a dynamic *impartial* if $\phi_{ij} \equiv \phi_j$, so that the function of the payoff difference that describes the conditional switch rate from i to j does not depend on an agent's current strategy i. The simplest example of a pairwise comparison dynamic is obtained by setting $\phi_{ij}(c) = [c]_+$, which yields the *Smith dynamic* (Smith 1984):

$$\dot{x}_i = \sum_{j \in S} x_j [F_i(x) - F_j(x)]_+ - x_i \sum_{j \in S} [F_j(x) - F_i(x)]_+.$$

Even in this simple case, we obtain an equation of motion that is noticeably more complicated than the general imitative dynamic (3). ◇

EXAMPLE 3. Suppose that the rate at which agents switch from strategy i to strategy j is solely a function of strategy j's excess payoff, and that agents switch only to strategies whose payoffs exceed the population average. Such rules are represented by protocols of the form

$$\rho_{ij}(\pi, x) = \phi_j \left(\pi_j - \sum_{k \in S} x_k \pi_k \right),$$

where each function ϕ_j is sign-preserving: $\mathrm{sgn}(\phi_j(d)) = \mathrm{sgn}([d]_+)$. We call the dynamics (M) generated by protocols of this form *separable excess payoff dynamics*. Choosing $\phi_j(d) = [d]_+$ yields the *BNN dynamic* (Brown and von Neumann 1950):

$$\dot{x}_i = [\hat{F}_i(x)]_+ - x_i \sum_{j \in S} [\hat{F}_j(x)]_+.$$

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[13]Of course, the decision about whether to actually switch to the candidate strategy will depend on the current strategy's payoff, the candidate strategy's payoff, or both.

[14]More explicitly, $\phi_{ij}(d) > 0$ if $d > 0$ and $\phi_{ij}(d) = 0$ if $d \leq 0$.

For more on this and other examples of excess payoff dynamics, see Skyrms (1990), Swinkels (1993), Weibull (1996), Hofbauer (2001), and Sandholm (2005).   ◇

EXAMPLE 4. To obtain a dynamic that reflects myopic optimization, one can suppose that revising agents always switch to a best response: $\rho_{i\bullet}(\pi, x) = \arg\max_{y \in X} y'\pi$. Then aggregate behavior is described by the *best response dynamic* (Gilboa and Matsui 1991; also see Hofbauer 1995b):

$$\dot{x} \in \arg\max_{y \in X} y'F(x) - x.$$

The best response dynamic is not a (single-valued) differential equation, but a (multivalued) differential inclusion: there may be multiple feasible directions of motion from states at which more than one strategy is optimal. Therefore, our derivation here is somewhat informal, taking a multivalued revision protocol as its basis. It is possible, though, to provide a precise account of evolutionary processes based on multivalued protocols; see Benaïm et al. (2005) and Gorodeisky (2008, 2009).   ◇

3. EVOLUTIONARILY STABLE STATES

We now introduce our sufficient condition for local stability. To begin, we call $x^* \in X$ an *evolutionarily stable state* if

$$(y - x^*)'F(x^*) \leq 0 \text{ for all } y \in X \tag{5}$$

there is a neighborhood $O \subset X$ of x^* such that for all $y \in O - \{x^*\}$,

$$(y - x^*)'F(x^*) = 0 \text{ implies that } (y - x^*)'F(y) < 0. \tag{6}$$

Condition (5) says that x^* is a Nash equilibrium. Condition (6) requires that if a state y near x^* is an alternative best response to x^*, then an infinitesimal group of invaders whose aggregate behavior is described by x^* can invade an incumbent population playing y.[15] If $F(x) = Ax$ is linear, then the force of condition (6) does not change if we require the implication to hold for all $y \in X - \{x^*\}$, as originally specified by Maynard Smith and Price (1973) in this linear setting.[16]

To prove their local stability results for the replicator dynamic, Taylor and Jonker (1978) introduce the slightly stronger notion of a *regular ESS*, which is defined by the following two conditions:

$$F_i(x^*) = \bar{F}(x^*) > F_j(x^*) \text{ whenever } x_i^* > 0 \text{ and } x_j^* = 0 \tag{7}$$

for all $y \in X - \{x^*\}$, $(y - x^*)'F(x^*) = 0$ implies that $(y - x^*)'DF(x^*)(y - x^*) < 0$. $\quad$ (8)

Condition (7) says that x^* is a *quasistrict equilibrium*: a Nash equilibrium at which each unused strategy performs strictly worse than each strategy in use. Condition (8) says

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[15]To see this, notice that the inequality in condition (6) can be rewritten as $(x^*)'F(y) > y'F(y)$. Other equivalent characterizations of ESS can be found in Bomze and Weibull (1995).

[16]This follows from the fact that every $y \in X - \{x^*\}$ satisfies $y - x^* = c(w - x^*)$ for some $w \in O - \{x^*\}$ and $c > 0$.

that as the state moves from x^* toward an alternate best response y (that is, in direction $y - x^*$), the payoff advantage of x^* over y grows at a linear rate. Condition (8) is equivalent to condition (6) when F is linear (e.g., when F is generated by matching in a normal form game), but is slightly stronger than condition (6) when nonlinear payoffs are allowed.

EXAMPLE 5. The conditions for regular ESS are only slightly more demanding than those for ESS, as can be seen by looking at instances of games with nonregular ESSs. First, consider the two-strategy game defined by $F_1(x) = 1$ and $F_2(x) = x_1 - \varepsilon$. As long as $\varepsilon \geq 0$, state $e_1 = (1, 0)$ is the unique Nash equilibrium of F and an ESS of F. However, for e_1 to be a regular ESS, it must be that $\varepsilon > 0$, so that e_1 is a quasistrict (and even a strict) equilibrium.

Next, consider the two-strategy game defined by $F_1(x) = (x_2 - \frac{1}{2})^3 + \varepsilon(x_2 - \frac{1}{2})$ and $F_2(x) = 0$. For any $\varepsilon \geq 0$, state $x^* = (\frac{1}{2}, \frac{1}{2})$ is the unique Nash equilibrium of F and an ESS of F. However, if $\varepsilon = 0$, then $DF(x^*) = \mathbf{0}$, so condition (8) clearly fails. For instance, letting $y = e_2$, we have that $(e_2 - x^*)'DF(x^*)(e_2 - x^*) = 0$. In words, this equality says that if the population begins at equilibrium x^*, and if some agents switch from strategy 1 to strategy 2, then although the payoff to strategy 2 falls below the payoff to mixed strategy x^*, strategy 2's payoff disadvantage initially does not grow at a linear rate. If instead $\varepsilon > 0$, then

$$DF(x^*) = \begin{pmatrix} 0 & \varepsilon \\ 0 & 0 \end{pmatrix},$$

so the payoff disadvantage of strategy 2 relative to x^* does grow at a linear rate: $(e_2 - x^*)'DF(x^*)(e_2 - x^*) = -\frac{1}{4}(\partial F_1/\partial x_2)(x^*) = -\frac{1}{4}\varepsilon < 0$. Thus, when $\varepsilon > 0$, condition (8) is satisfied and x^* is a regular ESS. The presence of the nonlinear payoff function in this example is no accident: as we noted above, nonlinear payoffs are necessary for discrepancies between conditions (6) and (8) to occur.   ◊

It will be useful to have a more concise formulation of regular ESS. To this end, let $TX = \{z \in \mathbb{R}^n : \sum_{j \in S} z_j = 0\}$ denote the tangent space of X, let $S(x) = \{j \in S : x_j > 0\}$ denote the support of state x, and let $\mathbb{R}^n_{S(x)} = \{y \in \mathbb{R}^n : y_j = 0 \text{ whenever } j \notin S(x)\}$ denote the set of vectors in $\mathbb{R}^n$ whose nonzero components correspond to strategies in the support of x.

OBSERVATION 1. *State x^* is a regular ESS if and only if it is a quasistrict equilibrium* (7) *that satisfies*

$$z'DF(x^*)z < 0 \text{ for all nonzero } z \in TX \cap \mathbb{R}^n_{S(x^*)}. \tag{9}$$

In words, state x^* is a regular ESS if it is a quasistrict equilibrium, and if the payoff derivative matrix $DF(x^*)$ is negative definite with respect to $TX \cap \mathbb{R}^n_{S(x^*)}$, the set of vectors tangent to the face of X containing x^*.

4. Local stability

4.1 *The main result*

To prepare for our main result, we review the relevant notions of stability for dynamical systems. Let x^* be a rest point of the dynamic (M). We say that x^* is *Lyapunov stable* if for every neighborhood O of x^*, there exists a neighborhood O' of x^* such that every solution $\{x_t\}_{t\geq 0}$ of (M) that starts in O' is contained in O: that is, $x_0 \in O'$ implies that $x_t \in O$ for all $t \geq 0$. The rest point x^* is *attracting* if there is a neighborhood $\hat{O}$ of x^* such that every solution that starts in $\hat{O}$ converges to x^*. Finally, x^* is *asymptotically stable* if it is Lyapunov stable and attracting.

Taylor and Jonker (1978) and Hines (1980) prove that any regular ESS is asymptotically stable under the replicator dynamic (1) by showing that the eigenvalues of the relevant linearized system have negative real parts. Cressman (1997) extends this linearization analysis to general imitative dynamics. We describe these analyses in detail in Section 4.3.

Members of the other classes of dynamics introduced in Section 2.3 often fail to be differentiable, so we cannot use linearization to obtain local stability results. Instead, by constructing suitable Lyapunov functions, we prove that any regular ESS is locally stable under the classes of dynamics from Examples 2, 3, and 4.

THEOREM 1. *Let x^* be a regular ESS of F. Then x^* is asymptotically stable under*

(i) *any impartial pairwise comparison dynamic for F;*

(ii) *any separable excess payoff dynamic for F;*

(iii) *the best response dynamic for F.*

4.2 *Intuition for the main result*

We now provide the intuition behind the proof of Theorem 1. The complete proof is presented in the Appendix.

The proof of the theorem builds on analyses from Hofbauer and Sandholm (2009) of evolutionary dynamics in stable games. A population game F is a *strictly stable game* if

$$z'DF(x)z < 0 \text{ for all nonzero } z \in TX \text{ and all } x \in X.$$

That is, F is a strictly stable game if at all states $x \in X$, the payoff derivative matrix $DF(x)$ is negative definite with respect to directions tangent to X.

By constructing suitable global Lyapunov functions, Hofbauer and Sandholm (2009) show that the Nash equilibrium of any strictly stable game is unique and that it is globally asymptotically stable under the dynamics considered in Theorem 1. The Lyapunov functions for the three classes of dynamics considered in the theorem are

$$\Psi(x) = \sum_{i \in S} \sum_{j \in S} x_i \int_0^{F_j(x)-F_i(x)} \phi_j(s)\, ds \tag{10}$$

$$\Gamma(x) = \sum_{j \in S} \int_0^{\hat{F}_j(x)} \phi_j(s)\,ds \tag{11}$$

$$M(x) = \max_{y \in X}(y - x)'F(x),$$

respectively. In the first two cases, the Lyapunov functions depend not only on the game F, but also on the exact specification of the dynamic: the latter dependence is through the appearance of the conditional switch rate functions ϕ_j in the formulas for Ψ and Γ.

To see why this analysis is relevant here, recall from Observation 1 that a regular ESS x^* is a quasistrict equilibrium whose payoff derivative matrix $DF(x^*)$ is negative definite with respect to directions in $TX \cap \mathbb{R}^n_{S(x^*)}$. If all strategies are in use at x^*, this set of directions is just TX, so that F resembles a stable game near x^*. Thus, if $x^* \in \text{int}(X)$, the global Lyapunov functions for stable games given above also serve as local Lyapunov functions for the ESS x^*.

Suppose instead that $x^* \notin \text{int}(X)$, so that at least one pure strategy is unused at x^*. Because x^* is a quasistrict equilibrium, all unused strategies earn strictly lower payoffs than strategies in the support of x^*; because payoffs are continuous, this is true not only at x^* itself, but also at states near x^*. Under any dynamic that respects these payoff differences, solutions from initial conditions near x^* should approach $X_{x^*} = \{x \in X : S(x) = S(x^*)\}$, the face of X that contains x^*. If x^* is a pure state (i.e., a vertex of X), and thus a strict equilibrium, then X_{x^*} is simply the singleton $\{x^*\}$, and the foregoing argument suffices to prove local stability.[17]

Now, assume that x^* is neither in the interior nor at a vertex of X, and suppose that we confine our attention to the behavior of the dynamic on face X_{x^*} in the vicinity of x^*. One way to do so is to consider a restricted game in which only the strategies in the support of x^* are available. Condition (9) can be interpreted as saying that this restricted game resembles a stable game near x^*. By this logic, the Lyapunov functions from Hofbauer and Sandholm (2009) can be used to establish convergence to x^* from nearby initial conditions on face X_{x^*}.

To prove Theorem 1, we need to construct local Lyapunov functions whose value decreases not only along solutions on face X_{x^*}, but also along solutions in the interior of X starting near x^*. To do so, we augment each of the Lyapunov functions listed above by an additional term. For impartial pairwise comparison dynamics, we replace the original Lyapunov function Ψ with the function

$$\Psi_{x^*}(x) = \Psi(x) + C\Omega_{x^*}(x),$$

where $\Omega_{x^*}(x) = \sum_{j \notin S(x^*)} x_j$ is the mass placed on strategies outside the support of x^*, and $C > 0$ is a constant. For the other classes of dynamics, the term $C\Omega_{x^*}(x)$ is added to the functions Γ and M above.

When the current state x is in the interior of X, the value of Ψ need not decrease. However, if x is close to x^*, then the value of Ω_{x^*} does decrease, because agents playing

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[17]In particular, the function Ω_{x^*} introduced below is a local Lyapunov function at x^* in this case: see Lemmas 1(ii), 2(ii), and 3(ii) in the Appendix.

strategies outside of $S(x^*)$ switch to strategies in $S(x^*)$, though the rate of decrease of Ω_{x^*} approaches zero as the state approaches the boundary of X. To prove the theorem, we must show that if the constant C is large enough, then the value of $C\Omega_{x^*}$ always falls fast enough to compensate for any growth in the value of Ψ, so that all told, the value of Ψ_{x^*} falls. The arguments required to accomplish this are somewhat different for each of the three classes of dynamics considered in the theorem: see Lemmas 1, 2, and 3 in the Appendix.

The argument just described requires precise estimates of the behavior of Ψ, Γ, M, and Ω_{x^*} in a neighborhood of x^*, and these estimates take full advantage of the regular ESS conditions (7) and (8). Still, it seems possible that the conclusions of Theorem 1 could be extended to cases where x^* only satisfies the ESS conditions, (5) and (6). One can prove by elementary arguments that when F has just two strategies, any ESS is locally stable under the dynamics considered in Theorem 1. Moreover, a close examination of the proof of Theorem 1(iii) reveals that in the case of the best response dynamic, the quasistrictness condition (7) from the definition of regular ESS can be replaced with the weaker requirement

there is a neighborhood $O \subset X$ of x^* such that for all $y \in O - \{x^*\}$,

$$F_i(y) \geq F_j(y) \text{ whenever } x_i^* > 0 \text{ and } x_j^* = 0, \text{ with } F_i(y) > F_j(y) \text{ if } y_j > 0,$$

which allows for unused optimal strategies both at the equilibrium x^* and elsewhere on faces of X that contain x^*. Whether the conclusions of Theorem 1 can be extended to all ESSs of arbitrary population games is an open question.

4.3 *Comparison to stability analysis for imitative dynamics*

The proofs of asymptotic stability of regular ESS under pairwise comparison dynamics, excess payoff dynamics, and the best response dynamic, though differing in their details, all follow the same basic path, using augmented versions of Lyapunov functions for stable games. In contrast, Cressman's (1997) proof of asymptotic stability of regular ESS under imitative dynamics employs a different approach based on linearization. Here we present a version of Cressman's (1997) analysis, and contrast it with the analysis of direct evaluation dynamics proposed here.

Given a dynamic $\dot{x} = V(x)$ defined on a full-dimensional set in $\mathbb{R}^n$, the asymptotic stability of the rest point x^* can be established by showing that all eigenvalues of the derivative matrix $DV(x^*)$ have negative real parts. Because the state space for evolutionary game dynamics is the simplex X, the local stability of the rest point x^* can be established by showing that the "relevant eigenvalues" of $DV(x^*)$ have negative real parts, where by "relevant eigenvalues" we mean those that correspond to eigenvectors in (the complexification of) the $n-1$-dimensional tangent space $TX = \{z \in \mathbb{R}^n : z'\mathbf{1} = 0\}$. If we let $\Phi = I - (1/n)\mathbf{1}\mathbf{1}' \in \mathbb{R}^{n \times n}$ denote the orthogonal projection of $\mathbb{R}^n$ onto TX, then these relevant eigenvalues are also the eigenvalues of $DV(x^*)\Phi$ that correspond to eigenvectors in TX.

Let us write the general imitative dynamic (3) as

$$\dot{x} = V(x) = \text{diag}(x)G(x), \tag{12}$$

where the C^1 function G satisfies monotone percentage growth rates (4). To ensure that the simplex is forward invariant under (12), we must have $V(x) \in TX$ or, equivalently,

$$x'G(x) = 0. \tag{13}$$

Notice that if we set $G = \hat{F}$, then (12) becomes the replicator dynamic (1).

We can compute the derivative matrix of V at state x as

$$DV(x) = Q(x)DG(x) - xG(x)' + \text{diag}(G(x)), \tag{14}$$

where $Q(x) = \text{diag}(x) - xx' \in \mathbb{R}^{n \times n}$. If $x^* \in \text{int}(X)$ is an interior Nash equilibrium, then the monotonicity condition (4) and the identity (13) imply that $G(x^*) = \mathbf{0}$, and hence that

$$DV(x^*)\Phi = Q(x^*)DG(x^*)\Phi. \tag{15}$$

Because $Q(x)$ is symmetric, positive definite with respect to $TX \times TX$, and maps $\mathbf{1}$ to $\mathbf{0}$, a lemma of Hines (1980) implies that the relevant eigenvalues of $DV(x^*)$ will have negative real parts if $DG(x^*)$ is negative definite with respect to $TX \times TX$.

Now suppose that $x^* \in \text{int}(X)$ is a regular ESS. Then by Observation 1, $DF(x^*)$ is negative definite with respect to $TX \times TX$. To take advantage of this fact, Cressman (1997) proves that

$$\Phi DG(x^*)\Phi = c\Phi DF(x^*)\Phi \text{ for some } c \geq 0. \tag{16}$$

That is, at any interior Nash equilibrium, the linearization of any monotone percentage-growth-rate function G is a multiple of the linearization of the payoff function F. Together, (15), (16), and the lemma of Hines (1980) imply that in nondegenerate cases ($c \neq 0$), an interior regular ESS is asymptotically stable.

Suppose next that x^* is a boundary Nash equilibrium with support $\{1, \ldots, n^*\}$, where $n^* < n$. In this case, the upper left $n^* \times n^*$ block of $DV(x^*)$ is an $n^* \times n^*$ analogue of equation (15). Thus, if x^* is a regular ESS, a version of the argument above shows that in nondegenerate cases, this block generates $n^* - 1$ relevant eigenvalues of $DV(x^*)$ with a negative real part. The lower right block of $DV(x^*)$, whose source is the third summand in (14), is a diagonal matrix with diagonal entries $\{G_j(x^*)\}_{j=n^*+1}^{n}$; that each of these $n - n^*$ entries can be shown to be a relevant eigenvalue of $DV(x^*)$. However, if x^* is a regular ESS, and thus a quasistrict equilibrium (7), then monotonicity condition (4) implies that $G_j(x^*) < 0$ for all $j > n^*$. Therefore, in nondegenerate cases, all $(n^* - 1) + (n - n^*) = n - 1$ relevant eigenvalues of $DV(x^*)$ have negative real parts, implying that the regular ESS x^* is asymptotically stable.

Cressman's (1997) analysis of imitative dynamics and our analysis of direct evaluation dynamics have some broad features in common. In both cases, the negative definiteness of $DF(x^*)$ is used to control the growth of strategies in the support of x^*, and the quasistrictness of x^* is used to show that small incursions by unused strategies are

eliminated. Thus, in both cases, the full strength of regular ESS is called upon to establish local stability.

In most other respects, though, the analyses are quite different. Cressman's (1997) analysis takes advantage of the simple functional form (12) of imitative dynamics, and relies on the surprising fact that all of these dynamics behave similarly in the neighborhood of an equilibrium. Moreover, this analysis requires the assumption that the dynamic is nondegenerately differentiable: when condition (16) holds with $c \neq 0$, linearization allows us to separate the analyses of strategies inside and outside the support of x^*.

Our analysis of direct evaluation dynamics differs on each of these points. The dependence of the Lyapunov functions (10) and (11) on the revision protocol ϕ suggests that different dynamics exhibit different behavior in the vicinity of x^*. Moreover, because all of the basic examples of direct evaluation dynamics—Smith, BNN, best response—are not differentiable, our stability analysis must be based on Lyapunov functions. Because we do not use linearization, we cannot cleanly separate the analyses of strategies inside and outside of the support of x^*. Indeed, the role of Lemmas 1, 2, and 3 is to show that the forces that equilibrate the use of strategies in the support of x^* are not too badly compromised as the strategies outside the support of x^* are being eliminated.

An alternate possibility for unifying the analyses of imitative and direct evaluation dynamics is to construct Lyapunov functions for the former dynamics. In the case of the replicator dynamic, it is well known from the work of Hofbauer et al. (1979), Zeeman (1980), and Hofbauer and Sigmund (1988) that the function

$$H_{x^*}(x) = \sum_{i \in S(x^*)} x_i^* \log \frac{x_i^*}{x_i}$$

serves as a local Lyapunov function for the ESS x^*. However, this function seems unrelated to the Lyapunov functions considered in this paper, and no generalizations of this function for other imitative dynamics are known.

5. Multipopulation games

In this section, we explain how our local stability results extend to multipopulation settings.

Before proceeding, we should briefly discuss a well-known result of Selten (1980) (see also Weibull 1995) that shows that in multipopulation games defined by random matching, various formulations of ESS are equivalent to strict equilibrium. More broadly, this equivalence between ESS and strict equilibrium holds in any multipopulation game in which each agent's payoffs are independent of the choices of other agents in his own population. Although this form of payoff independence is always present in random matching contexts, it seems rather exceptional in "playing the field" models, in

which agents' payoffs depend directly on opponents' aggregate behavior.[18] In such contexts, which are common in applications of population games, the notions of ESS introduced below are not especially more restrictive than those for the single-population setting.

To define a multipopulation game, we suppose that there are $p > 1$ *populations* of agents, with population $p \in \mathcal{P} = \{1, \ldots, p\}$ having mass $m^p > 0$. Agents in population p choose pure strategies from the set $S^p = \{1, \ldots, n^p\}$, and the total number of pure strategies available in all populations is $n = \sum_{p \in \mathcal{P}} n^p$. Aggregate behavior in population p is represented by a *population state* in $X^p = \{x^p \in \mathbb{R}_+^{n^p} : \sum_{i \in S^p} x_i^p = m^p\}$, where $x_i^p \in \mathbb{R}_+$ represents the mass of players in population p choosing strategy $i \in S^p$. Elements of $X = \prod_{p \in \mathcal{P}} X^p = \{x = (x^1, \ldots, x^p) \in \mathbb{R}_+^n : x^p \in X^p\}$, the set of *social states*, describe behavior in all p populations at once. The tangent spaces of X and X^p are $TX^p = \{z^p \in \mathbb{R}^{n^p} : \sum_{i \in S^p} z_i^p = 0\}$ and $TX = \prod_{p \in \mathcal{P}} TX^p$, respectively.

We identify a multipopulation game with its C^1 *payoff function* $F : X \to \mathbb{R}^n$. The component $F_i^p : X \to \mathbb{R}$ denotes the payoff function for strategy $i \in S^p$, whereas $F^p : X \to \mathbb{R}^{n^p}$ denotes the payoff functions for all strategies in S^p. To interpret the definitions below, note that inner products of elements of $\mathbb{R}^n$ are defined via sums that range over all strategies in all populations: for instance, the aggregate payoff over all p populations at social state $x \in X$ is given by $x'F(x) = \sum_{p \in \mathcal{P}} \sum_{i \in S^p} x_i^p F_i^p(x)$.

In this context, a strictly stable game is a game whose derivative matrices that satisfy the (multipopulation) negative definiteness condition

$$z'DF(x)z < 0 \text{ for all nonzero } z \in TX \text{ and all } x \in X. \tag{17}$$

The Lyapunov functions introduced in Hofbauer and Sandholm (2009) continue to serve in multipopulation games in which this condition holds.

To use these functions as the starting point for a local stability analysis, we follow Taylor (1979) and use a definition of ESS that, like condition (17), makes use of payoff comparisons that aggregate over all populations. In particular, we call $x^* \in X$ a *Taylor ESS* if it is a Nash equilibrium that satisfies

there is a neighborhood $O \subset X$ of x^* such that for all $y \in O - \{x^*\}$,
$$(y - x^*)'F(x^*) = 0 \text{ implies that } (y - x^*)'F(y) < 0. \tag{18}$$

Strengthening this definition slightly, we call x^* a *regular Taylor ESS* if it is a quasistrict equilibrium that satisfies

for all $y \in X - \{x^*\}$, $(y - x^*)'F(x^*) = 0$ implies that $(y - x^*)'DF(x^*)(y - x^*) < 0$. (19)

By extending arguments from Taylor and Jonker (1978), Taylor (1979) shows that any regular Taylor ESS is asymptotically stable under the (standard) multipopulation repli-

---

[18]For instance, in multipopulation models of traffic congestion, each population corresponds to an origin/destination pair. Clearly, the delays a driver experiences should depend not only on the behavior of drivers with other origin/destination pairs, but also on the behavior of other drivers with the same origin/destination pair.

cator dynamic. Similarly, a simple extension of our analysis shows that a regular Taylor ESS is asymptotically stable under multipopulation versions of all of the dynamics considered in Theorem 1.[19]

The inequalities in conditions (18) and (19) require that the aggregate payoffs of the $\wp$ incumbent populations exceed the aggregate payoffs of the $\wp$ invading populations. A less demanding solution concept can be obtained by requiring only that there be at least one population $p \in \mathscr{P}$ in which incumbents outperform invaders:

there is a neighborhood $O \subset X$ of x^* such that for all $y \in O - \{x^*\}$, there is a $p \in \mathscr{P}$

(20)

such that $(y^p - x^{*p})'F^p(x^*) = 0$ implies that $(y^p - x^{*p})'F^p(y) < 0$.

We call a Nash equilibrium that satisfies condition (20) a *Cressman ESS*.[20]

As we noted at the beginning of the paper, Maynard Smith and Price's (1973) definition of ESS serves as a stability condition in two distinct single-population contexts: the monomorphic, mixed-strategist framework these authors envisioned, and the polymorhphic, pure-strategist framework first studied by Taylor and Jonker (1978). In multipopulation settings, both of these roles cannot be played by a single definition. Cressman (1992, 2006) and Cressman et al. (2001) argue convincingly that to capture stability in the multipopulation analogue of Maynard Smith and Price's (1973) mixed-strategist framework, the appropriate solution concept is Cressman ESS.[21] However, following Taylor (1979), we have argued here that in the context of multipopulation pure-strategist dynamics, it is the more demanding notion of Taylor ESS that provides a general sufficient condition for local stability.

6. Discussion

We study the evolution of behavior in population games, assuming that agents employ revision protocols based on the direct evaluation of alternative strategies. We show that under three classes of dynamics generated by such protocols—impartial pairwise comparison dynamics, separable excess payoff dynamics, and the best response dynamic—any regular ESS is locally stable. To conclude the paper, we now discuss some of the assumptions maintained throughout the analysis and suggest directions for future research.

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[19]In defining these multipopulation dynamics, we can allow different populations to employ different revision protocols from the same class (e.g., from the class of protocols that defines impartial pairwise comparison dynamics). Among other things, this flexibility can be used to offset differences in payoff scales across populations.

[20]This concept is called *monomorphic ESS* in Cressman (1992), and *N-species ESS* in Cressman et al. (2001) and Cressman (2006). A related notion for two-population games is that of a *Nash–Pareto pair*; see Hofbauer and Sigmund (1988).

[21]Cressman (1992, 2006) and Cressman et al. (2001) consider a collection of $\wp$-dimensional replicator systems, with one system for each strategy profile y other than the candidate for stability, x^*. The pth component of the state variable in the $\wp$-dimensional system describes the fraction of the pth population using the invading mixed strategy y^p; the remainder of the population uses the incumbent mixed strategy x^{*p}. It is shown that the origin (i.e., the state at which all members of each population p choose the incumbent mixed strategy x^{*p}) is asymptotically stable in each such system if and only if x^* is a Cressman ESS.

All of our analysis is conducted in a population game framework in which the payoff $F_i(x)$, which is used as an input of the agents' revision protocols, represents a realized payoff. This framework includes congestion games and similar models of multilateral externalities, the "playing the field" models of Maynard Smith (1982), and contests generated by the deterministic matching of all pairs of agents in normal form games. If F were generated instead by a single round of random matching in the normal form game A, then $F_i(x) = (Ax)_i$ would represent the expected payoff to strategy i. In this case, agents would need to observe the population state and compute expected payoffs in order to employ the revision protocols studied here. Because these requirements seem rather stringent for evolutionary modeling, it would be more natural in this context to assume that each agent employs a revision protocol that conditions directly on the outcomes of his random matches. Although Schlag (1998) derives the replicator dynamic in such an environment, little work has been done to formulate direct evaluation dynamics in this setting, leaving a promising avenue for future research.

The present paper studies local stability in recurrent play of simultaneous-move games. Understanding local stability in recurring sequential-move games is a topic of clear importance, but it introduces a number of complications to the analysis. Deriving evolutionary dynamics for extensive form games from a specification of individual behavior necessitates the use of revision protocols that incorporate agents' assessments of the consequences of diverting play to unreached information sets. Because dynamics developed for simultaneous move games do not account for such possibilities, it is not surprising that these dynamics can exhibit unexpected properties in extensive form games, or that few general stability results exist.[22] Finally, because unreached information sets generate components of observationally equivalent equilibria, set-valued analogues of the ESS concept can be expected to play a basic role in studies of local stability for extensive form games.[23] Whether the analyses developed in Cressman (1997) and in the present paper can be adapted to dynamics for extensive form games is thus a difficult open question.

Appendix

Proof of Theorem 1(i). Let $\dot{x} = V_F(x)$ be an impartial pairwise comparison dynamic for F defined by revision protocol $\rho_{ij}(\pi, x) = \phi_j(\pi_j - \pi_i)$, and let $\psi_k(d) = \int_0^d \phi_k(s)\, ds$ be the definite integral of ϕ_k. Define the C^1 function $\Psi_{x^*} : X \to \mathbb{R}$ by

$$\Psi_{x^*}(x) = \Psi(x) + C\Omega_{x^*}(x)$$
$$\equiv \sum_{i \in S} \sum_{j \in S} x_i \psi_j(F_j(x) - F_i(x)) + C \sum_{j \notin S(x^*)} x_j,$$

---

[22]Cressman (2003) provides a thorough account of these issues and takes the first steps toward defining deterministic dynamics that respect extensive form structure.

[23]For set-valued versions of ESS, see Thomas (1985), Swinkels (1992), and Balkenborg and Schlag (2001, 2007).

where the constant $C > 0$ is determined later. Hofbauer and Sandholm (2009) observe that the function Ψ is nonnegative, with $\Psi(x) = 0$ if and only if $x \in NE(F)$. It follows that Ψ_{x^*} too is nonnegative, with $\Psi_{x^*}(x) = 0$ if and only if x is a Nash equilibrium of F with support$(x) \subseteq$ support(x^*). Because x^* is an ESS, it is isolated in the set of Nash equilibria (see Bomze and Weibull 1995), so there is a neighborhood O of x^* on which x^* is the unique zero of Ψ_{x^*}. If we can show that there is also a neighborhood O' of x^* such that $\dot\Psi_{x^*}(x) < 0$ for all $x \in O' - \{x^*\}$, then Ψ_{x^*} is a strict local Lyapunov function for x^*, implying that x^* is asymptotically stable (see Weibull 1995).

To reduce the amount of notation in the analysis to come, let $\mathbf{1}^0 \in \mathbb{R}^n$ be the vector whose jth component equals 0 if $j \in$ support(x^*) and equals 1 otherwise, so that $(\mathbf{1}^0)'x$ is the mass of agents who use strategies outside the support of x^* at state x. Then we can write $\Psi_{x^*}(x) = \Psi(x) + C(\mathbf{1}^0)'x$, and so can express the time derivative of Ψ_{x^*} as

$$\dot\Psi_{x^*}(x) = \dot\Psi(x) + C(\mathbf{1}^0)'\dot x.$$

In the course of establishing a global stability result for stable games, Hofbauer and Sandholm (2009) show that the time derivative of Ψ satisfies

$$\dot\Psi(x) \le \dot x'DF(x)\dot x,$$

with equality holding precisely at the Nash equilibria of F. To finish the proof of Theorem 1(i), it is enough to show that

$$\dot x'DF(x)\dot x + C(\mathbf{1}^0)'\dot x \le 0$$

for all $x \in O' - \{x^*\}$. This follows directly from the following lemma and choosing $C \ge M/N$.

LEMMA 1. *Let $\dot x = V_F(x)$ be a pairwise comparison dynamic for F and let x^* be a regular ESS of F. Then there is a neighborhood O' of x^* and constants $M, N > 0$ such that for all $x \in O'$,*

  (i) $\dot x'DF(x)\dot x \le M(\mathbf{1}^0)'x$

  (ii) $(\mathbf{1}^0)'\dot x \le -N(\mathbf{1}^0)'x$.

PROOF. Suppose without loss of generality that $S(x^*) =$ support(x^*) is given by $\{1, \ldots, n^*\}$. Then to complement $\mathbf{1}^0 \in \mathbb{R}^n$, let $\mathbf{1}^* \in \mathbb{R}^n$ be the vector whose first n^* components equal 1 and whose remaining components equal 0, so that $\mathbf{1}^* + \mathbf{1}^0 = \mathbf{1} \equiv (1, \ldots, 1)'$. Next, decompose the identity matrix I as $I^* + I^0$, where $I^* = \text{diag}(\mathbf{1}^*)$ and $I^0 = \text{diag}(\mathbf{1}^0)$, and, finally, decompose I^* as $\Phi^* + \Xi^*$, where $\Xi^* = (1/n^*)\mathbf{1}^*(\mathbf{1}^*)'$ and $\Phi^* = I^* - \Xi^*$. Notice that Φ^* is the orthogonal projection of $\mathbb{R}^n$ onto $TX \cap \mathbb{R}^n_{S(x^*)} = \{z \in \mathbb{R}^n_0 : \sum_{j \in S} z_j = 0, \text{ and } z_j = 0 \text{ whenever } j \notin S(x^*)\}$ and that $I = \Phi^* + \Xi^* + I^0$. Note also that if $x^* \in \text{int}(X)$, then this decomposition becomes $I = \Phi + \Xi + \mathbf{0}$, where $\Phi = I - (1/n)\mathbf{1}\mathbf{1}'$ and $\Xi = (1/n)\mathbf{1}\mathbf{1}'$ are the orthogonal projections of $\mathbb{R}^n$ onto $TX = \{z \in \mathbb{R}^n : \sum_{j \in S} z_j = 0\}$ and span$(\{\mathbf{1}\})$, respectively.

Using this decomposition of the identity matrix, we can write

$$
\begin{aligned}
\dot{x}' DF(x)\dot{x} &= ((\Phi^* + \Xi^* + I^0)\dot{x})' DF(x)((\Phi^* + \Xi^* + I^0)\dot{x}) \\
&= (\Phi^*\dot{x})' DF(x)(\Phi^*\dot{x}) + ((\Xi^* + I^0)\dot{x})' DF(x)\dot{x} \qquad (21) \\
&\quad + (\Phi^*\dot{x})' DF(x)((\Xi^* + I^0)\dot{x}).
\end{aligned}
$$

Because x^* is a regular ESS, we know that $z' DF(x^*)z < 0$ for all nonzero $z \in TX \cup \mathbb{R}^n_{S(x^*)}$. Thus, because $DF(x)$ is continuous in x, there is a neighborhood $\hat{O}$ of x^* on which the first term of (21) is nonpositive.

Turning to the second term, note that because $\mathbf{1}'\dot{x} = 0$ and $(\mathbf{1}^0)' = \mathbf{1}'I^0$, we have that

$$
(\Xi^* + I^0)\dot{x} = \left(\frac{1}{n^*}\mathbf{1}^*(\mathbf{1}^*)' + I^0\right)\dot{x} = \left(-\frac{1}{n^*}\mathbf{1}^*(\mathbf{1}^0)' + I^0\right)\dot{x} = \left(\left(I - \frac{1}{n^*}\mathbf{1}^*\mathbf{1}'\right)I^0\right)\dot{x}.
$$

Let $\|A\|$ denote the spectral norm of the matrix A (see Horn and Johnson 1985, Example 5.6.6). Then applying the spectral norm inequalities $|Ax| \le \|A\||x|$ and $\|AB\| \le \|A\|\|B\|$, and the Cauchy–Schwarz inequality $|x'y| \le |x||y|$, we find that

$$
\begin{aligned}
((\Xi^* + I^0)\dot{x})' DF(x)\dot{x} &= \left(\left(I - \frac{1}{n^*}\mathbf{1}^*\mathbf{1}'\right)I^0\dot{x}\right)' DF(x)\dot{x} \\
&\le |I^0\dot{x}| \left\|I - \frac{1}{n^*}\mathbf{1}(\mathbf{1}^*)'\right\| \|DF(x)\||\dot{x}|.
\end{aligned}
$$

Because $DF(x)$, $V_F(x)$, and $\rho_{ij}(F(x), x)$ are continuous in x, and hence bounded, on the compact set X, we can find constants K and R such that

$$
\left\|I - \frac{1}{n^*}\mathbf{1}(\mathbf{1}^*)'\right\| \|DF(x)\||\dot{x}| \le K \quad \text{and} \quad \max_{i,j \in S} \rho_{ij}(F(x), x) \le R \quad \text{for all } x \in X. \qquad (22)
$$

Now because x^* is a quasistrict equilibrium, we have that $F_i(x^*) = \bar{F}(x^*) > F_j(x^*)$ for all $i \in \text{support}(x^*) = \{1, \dots, n^*\}$ and all $j \notin \text{support}(x^*)$. Thus, because the protocol $\rho_{ij}(\pi, x) = \phi_j(\pi_j - \pi_i)$ is sign preserving, we have $\rho_{ij}(F(x^*), x^*) = 0$ for such i and j, and because F is continuous, there is a neighborhood $O' \subseteq \hat{O}$ of x^* on which for such i and j we have $F_i(x) > F_j(x)$, and hence $\rho_{ij}(F(x), x) = 0$. From this argument and the bound on ρ_{ij} in (22), it follows that for $x \in O'$, we have

$$
\begin{aligned}
|I^0\dot{x}| &= \sqrt{\sum_{j>n^*} |\dot{x}_j|^2} \\
&\le \sum_{j>n^*} |\dot{x}_j| \\
&= \sum_{j>n^*} \left| \sum_{k \in S} x_k \rho_{kj}(F(x), x) - x_j \sum_{k \in S} \rho_{jk}(F(x), x) \right| \\
&\le \sum_{j>n^*} \left(\sum_{k \in S} x_k \rho_{kj}(F(x), x) + x_j \sum_{k \in S} \rho_{jk}(F(x), x) \right) \qquad (23)
\end{aligned}
$$

$$= \sum_{j>n^*} \sum_{k>n^*} x_k \rho_{kj}(F(x), x) + \sum_{j>n^*} x_j \sum_{k \in S} \rho_{jk}(F(x), x)$$

$$\leq 2Rn \sum_{j>n^*} x_j$$

$$= 2Rn(\mathbf{1}^0)'x.$$

We therefore conclude that at all $x \in O'$,

$$((\Xi^* + I^0)\dot{x})'DF(x)\dot{x} \leq 2KRn(\mathbf{1}^0)'x.$$

Essentially the same argument provides a similar bound on the third term of (21), completing the proof of part (i) of the lemma.

We proceed with the proof of part (ii) of the lemma. Following the line of argument after equation (22) above, we note that because x^* is quasistrict and because the pairwise comparison dynamic satisfies sign preservation, we have $\rho_{ji}(F(x^*), x^*) > 0$ and $\rho_{ij}(F(x^*), x^*) = 0$ whenever $i \in \text{support}(x^*) = \{1, \ldots, n^*\}$ and $j \notin \text{support}(x^*)$. So, because F and ρ are continuous, sign preservation implies that there is a neighborhood O' of x^* and an $r > 0$ such that $\rho_{ji}(F(x), x) > r$ and $\rho_{ij}(F(x), x) = 0$ for all $i \leq n^*$, $j > n^*$, and $x \in O'$. Applying this observation and canceling like terms when both j and k are greater than n^* in the sums below, we find that for all $x \in O'$,

$$(\mathbf{1}^0)'\dot{x} = \sum_{j>n^*} \dot{x}_j$$

$$= \sum_{j>n^*} \left(\sum_{k \in S} x_k \rho_{kj}(F(x), x) - x_j \sum_{k \in S} \rho_{jk}(F(x), x) \right)$$

$$= \sum_{j>n^*} \left(\sum_{k>n^*} x_k \rho_{kj}(F(x), x) - x_j \sum_{k \in S} \rho_{jk}(F(x), x) \right) \tag{24}$$

$$= -\sum_{j>n^*} x_j \sum_{i \leq n^*} \rho_{ji}(F(x), x)$$

$$\leq -rn^*(\mathbf{1}^0)'x.$$

This completes the proof of the lemma. ◁

The lemma completes the proof of Theorem 1(i). □

PROOF OF THEOREM 1(ii). Let $\dot{x} = V_F(x)$ be a separable excess payoff dynamic for F defined by revision protocol $\rho_{ij}(\pi, x) = \phi_j(\pi_j - x'\pi)$. Define the C^1 function $\Psi_{x^*}: X \to \mathbb{R}$ by

$$\Gamma_{x^*}(x) = \Gamma(x) + C\Omega_{x^*}(x) \equiv \sum_{i \in S} \int_0^{\hat{F}_i(x)} \phi_i(s)\, ds + C \sum_{j \notin S(x^*)} x_j,$$

where the constant $C > 0$ is determined later. Hofbauer and Sandholm (2009) show that the function Γ is nonnegative, with $\Gamma(x) = 0$ if and only if $x \in NE(F)$, and that

$$\dot{\Gamma}(x) \le \dot{x}'DF(x)\dot{x},$$

with equality holding precisely at the Nash equilibria of F. Therefore, the result follows if we can show that

$$\dot{x}'DF(x)\dot{x} + C(\mathbf{1}^0)'\dot{x} \le 0$$

for all $x \in O' - \{x^*\}$. This inequality follows directly from the following lemma, choosing $C \ge K$.

LEMMA 2. *Let $\dot{x} = V_F(x)$ be the separable excess payoff dynamic for F defined by revision protocol $\rho_{ij}(\pi, x) = \phi_j(\pi_j - x'\pi)$, and let x^* be a regular ESS of F. Then there is a neighborhood O' of x^* and a $K > 0$ such that for all $x \in O'$, we have*

(i) $\dot{x}'DF(x)\dot{x} \le KT(x)(\mathbf{1}^0)'x$

(ii) $(\mathbf{1}^0)'\dot{x} = -T(x)(\mathbf{1}^0)'x$,

where $T(x) = \sum_{j \in S} \phi_j(\hat{F}_j(x))$.

PROOF. The proof of this lemma follows the same lines as that of Lemma 1, but with inequalities (23) and (24) replaced by (26) and (25) below.

Suppose again that support$(x^*) = \{1, \ldots, n^*\}$. Because x^* is quasistrict, we have that $F_i(x^*) = \bar{F}(x^*) > F_j(x^*)$ for all $i \in$ support$(x^*) = \{1, \ldots, n^*\}$ and all $j \notin$ support(x^*). Thus, because the protocol $\rho_{ij}(\pi, x) = \phi_j(\pi_j - x'\pi)$ is sign preserving, we have $\phi_j(\hat{F}(x_j^*)) = 0$ whenever $j \notin$ support(x^*), and because F is continuous, there is a neighborhood $O' \subseteq \hat{O}$ of x^* on which for such j we have that $\bar{F}(x^*) > F_j(x^*)$, and hence that $\phi_j(\hat{F}_j(x)) = 0$. This implies in turn that for $x \in O'$, we have

$$
\begin{aligned}
(\mathbf{1}^0)'\dot{x} &= \sum_{j > n^*} \dot{x}_j \\
&= \sum_{j > n^*} \left(\sum_{k \in S} x_k \rho_{kj}(F(x), x) - x_j \sum_{k \in S} \rho_{jk}(F(x), x) \right) \\
&= \sum_{j > n^*} \left(\phi_j(\hat{F}_j(x)) - x_j \sum_{k \in S} \phi_k(\hat{F}(x)) \right) \\
&= -\sum_{j > n^*} x_j \sum_{k \in S} \phi_k(\hat{F}(x)) \\
&= -T(x)(\mathbf{1}^0)'x,
\end{aligned}
\tag{25}
$$

and so

$$
|I^0\dot{x}| = \sqrt{\sum_{j > n^*} |\dot{x}_j|^2} \le \sum_{j > n^*} |\dot{x}_j| = -\sum_{j > n^*} \dot{x}_j = T(x)(\mathbf{1}^0)'x.
\tag{26}
$$

The proof otherwise follows that of Lemma 1.   ◁

    The proof of Theorem 1(ii) is now complete.   □

PROOF OF THEOREM 1(iii). Let $\dot{x} \in V_F(x) \equiv \arg\max_{y \in X} y'F(x) - x$ be the best response dynamic for F. Define the Lipschitz continuous function

$$M_{x^*}(x) = M(x) + C\Omega_{x^*}(x) \equiv \max_{y \in X}(y-x)'F(x) + C \sum_{j \notin S(x^*)} x_j.$$

Hofbauer (2001) and Hofbauer and Sandholm (2009) show that M is nonnegative, with $M(x) = 0$ if and only if $x \in NE(F)$, and that along each solution trajectory $\{x_t\}_{t \geq 0}$, we have

$$\dot{M}(x_t) = (\dot{x}_t)'DF(x_t)\dot{x}_t - M(x_t)$$

for almost all $t \geq 0$. Therefore, by standard results on Lyapunov functions for differential inclusions (see Theorems A.2 and A.3 of Hofbauer and Sandholm 2009), the result will follow if we show that

$$\dot{x}'DF(x)\dot{x} + C(\mathbf{1}^0)'\dot{x} \leq 0$$

for all $x \in O' - \{x^*\}$ and all $\dot{x} \in V_F(x)$. This inequality follows from the following lemma, setting $C \geq K$.

LEMMA 3. *Let* $\dot{x} \in V_F(x) \equiv \arg\max_{y \in X} y'F(x) - x$ *be the best response dynamic for F, and let* x^* *be a regular ESS of F. Then there is a neighborhood* O' *of* x^* *and a* $K > 0$ *such that for all* $x \in O'$ *and* $\dot{x} \in V_F(x)$, *we have*

   (i) $\dot{x}'DF(x)\dot{x} \leq K(\mathbf{1}^0)'x$

   (ii) $(\mathbf{1}^0)'\dot{x} = -(\mathbf{1}^0)'x.$

PROOF. The proof of this lemma follows the same lines as that of Lemma 1, but with inequalities (23) and (24) replaced by (28) and (27) below.

    Suppose once again that support$(x^*) = \{1, \ldots, n^*\}$. If we choose the neighborhood O' as we did in the proof of Lemma 2, then no $j > n^*$ is optimal at any $x \in O'$. Thus, for such x and any $\dot{x} \in V_F(x)$, we have

$$(\mathbf{1}^0)'\dot{x} = \sum_{j>n^*} \dot{x}_j = -\sum_{j>n^*} x_j = -(\mathbf{1}^0)'x \tag{27}$$

and hence

$$|I^0\dot{x}| = \sqrt{\sum_{j>n^*} |\dot{x}_j|^2} \leq \sum_{j>n^*} |\dot{x}_j| = -\sum_{j>n^*} \dot{x}_j = (\mathbf{1}^0)'x. \tag{28}$$

The proof otherwise follows that of Lemma 1.   ◁

    The proof of Theorem 1(iii) is now complete.   □

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