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Rationalizable voting

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When is a finite number of binary voting choices consistent with the hypothesis that the voter has preferences that admit a (quasi)concave utility representation? I derive necessary and sufficient conditions and a tractable algorithm to verify their validity. I show that the hypothesis that the voter has preferences represented by a concave utility function is observationally equivalent to the hypothesis that she has preferences represented by a quasiconcave utility function, I obtain testable restrictions on the location of voter ideal points, and I apply the conditions to the problem of predicting future voting decisions. Without knowledge of the location of the voting alternatives, voting decisions by multiple voters impose no joint testable restrictions on the location of their ideal points, even in one dimension. Furthermore, the voting records of any group of voters can always be embedded in a two-dimensional space with strictly concave utility representations and arbitrary ideal points for the voters. The analysis readily generalizes to choice situations over general finite budget sets.

Keywords. Revealed preferences, testable restrictions, voting, ideal points.

JEL classification. D01, D70.

1. Introduction

What can we learn about individual voter preferences on the basis of data consisting of a finite number of binary choices? Is it possible to deduce whether the voter has preferences represented by a utility function that rationalizes these choices? Without restrictions on the shape of the utility function, finite voting data impose minimal testable restrictions that are no stronger than the familiar cyclic consistency of choices. However, I show in this paper that concavity or its variants impose significant testable restrictions on observed voting choices.

I assume a spatial framework such that voters are confronted with a finite number of choices between two alternatives drawn from a finite-dimensional Euclidean policy space. I consider the assumptions that each choice indicates strict or weak preference separately and develop rationalizability criteria for each of the two interpretations of the data in the voting record. Accordingly, I derive necessary and sufficient conditions

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so that the voting record is rationalized by a concave utility function and by a quasiconcave utility function. I show that finite voting records do not allow us to discriminate between the hypothesis that voters have (strictly) concave utility representations and the hypothesis that the voters have (strictly) quasiconcave utility representations. Yet, unlike the classical revealed preference theory of the consumer, the testable restrictions that emerge due to concavity and its variants extend well beyond the cyclical consistency of the choices in the voting record.

The rationalizability conditions I derive can be verified on a computer using a tractable algorithm that involves a finite sequence of standard linear programming routines. I apply these conditions to test the hypothesis that voter preferences admit satiation at candidate ideal points and to predict future voting choices. The assumption that voting data arise from the maximization of a (quasi)concave utility representation generates nontrivial testable restrictions on the location of voter ideal points, yielding a promising route for the nonparametric estimation of these points.

Estimates of voter ideal points that rely on parametric restrictions on voters' utility representations are now routinely obtained using probabilistic choice models and data from past voting decisions (e.g., Poole and Rosenthal 1985, Heckman and Snyder 1997). Unlike these techniques, which simultaneously estimate both the voters' utility functions and the location of the voting alternatives, the nonparametric tests developed in this paper require knowledge of the location of the voting alternatives in Euclidean space. In fact, I show that if the location of the voting alternatives is unknown and unrestricted, then voting data impose no testable restrictions whatsoever on the joint location of voter ideal points, even if the space of alternatives is one-dimensional. For any arbitrary set of ideal points for the voters and for any record of voting decisions (yes or no) by these voters on any finite number of choice situations, it is possible to simultaneously locate the voting alternatives and find strictly concave utility functions for all voters such that all voters have the prespecified ideal points and all individual voting decisions are rationalized by these utility functions.

I also consider the possible role of intermediate assumptions regarding knowledge of the alternatives that appear in the voting agenda to identify the location of voter ideal points. For example, it may be known that the same alternative appears in two separate choice situations for the voters, even though the location of the voting alternatives in Euclidean space is unknown. I establish that such additional equality restrictions on the voting alternatives across choice situations, coupled with (quasi)concavity of individual voter preferences, do not jointly restrict voter ideal points in spaces of two dimensions or higher. In particular, the voting records of any group of voters can always be embedded in a two-dimensional space with strictly concave utility representations and arbitrarily prespecified ideal points for these voters.

The present study is connected with a branch of the literature on the revealed preference theory of the consumer pioneered by Afriat (1967). Afriat derives necessary and sufficient conditions that must be met by a finite number of observations of prices and quantity choices of commodities so that these observations are consistent with individual maximization of a monotone concave utility function. At the same time, he constructs the required utility representation. Varian (1982) builds on this approach to study the nonparametric estimation of demand. The present study follows in this tradition, but differs from the classical theory of the consumer in that I assume finite nonconvex budget sets that violate free disposal assumptions.

A number of other studies explore the testable implications of concavity and monotonicity assumptions from observations of choices over nonstandard (although not necessarily finite) budget sets (e.g., Matzkin 1991, Chavas and Cox 1993, Forges and Minelli 2009). Finite budget sets are assumed by Chambers and Echenique (2009), who consider the testable implications of supermodularity, assuming nonsatiated preferences. In the present study, I admit data that may contradict nonsatiation. In fact, as already discussed, given my interest in political environments in which voters are typically assumed to have ideal points, much of the analysis focuses on tests of the hypothesis of satiation. Nonmonotonicities (although not necessarily leading to satiation) may also arise naturally in economic models of altruism, as recently studied by Cox et al. (2008). Note that, while I do not assume it, I do not rule out monotonicity of preferences, and the present analysis can be applied to the problem of a consumer facing a finite budget set, as discussed in Section 6.

While I shed monotonicity assumptions, I do rely on convexity of voter preferences, so the analysis is intimately related with the literature on the concavifiability of individual preferences. Kannai (1977) tackles this question for the case of continuous preferences on infinite convex sets. For my purposes, the relevant question is concavifiability of preferences on finite sets—a question that was recently taken up by Richter and Wong (2004) and Kannai (2005), whose results provide a departure point for the present study. Via an application of a Theorem of the Alternative, Richter and Wong derive a necessary and sufficient condition for the existence of a (strictly) concave utility function that represents complete and transitive preferences over finite sets. Kannai (2005) discusses various alternative conditions that focus on the construction of the requisite utility function. In the present study, I consider a range of possible utility representations from strict concavity to mere quasi-concavity of the rationalizing utility function. The conditions I derive differ from those of Richter and Wong (2004) and Kannai (2005) in that they are applicable to any partial order over a finite set of alternatives.

In addition to the connection with the extensive literature on ideal point estimation using roll-call voting records that is reviewed in Kalandrakis (2006), a number of recent studies analyze the consistency of voting choices with the assumption that the voter's preferences admit specific parametric utility representations. Bogomolnaia and Laslier (2007) establish bounds on the number of policy dimensions of the policy space that are sufficient to represent any voter preferences over a fixed number of alternatives by Euclidean utility functions. They also briefly consider general convex (although not strictly concave) preferences and independently establish a theorem related to the embedding Theorem 5 of the present study. Degan and Merlo (2009) establish conditions on observable choices over multiple elections to falsify the hypothesis that voters with Euclidean preferences vote sincerely. Working in a discrete space of alternatives, Schwartz (2007) shows that observed voting histories cannot refute in either direction the hypothesis that a committee's majority rule social preference over the finite number of voting alternatives in the voting record is transitive. He also provides a sufficient condition on voting histories so that the committee's preference profile over this finite set has a single-peaked representation and a sufficient condition to refute the existence of such a representation.

I now proceed to the analysis. In the next section, I develop notation and review the question of rationalizability without convexity restrictions. In Section 3, I consider the rationalization of voting records by concave utility functions. In Section 4, I analyze how or whether the conditions derived in Section 3 can be used for the nonparametric estimation of voter ideal points. In Section 5, I study the use of the voting record for the purposes of prediction. I discuss how the analysis generalizes to arbitrary finite revealed preference data in Section 6. I conclude in Section 7. All proofs are relegated to the Appendix.

2. Rationalizable voting

A voter is confronted with a finite number m of binary choices. In keeping with the literature on roll-call voting in legislatures, I call each pairwise comparison a *voting item* and index the set of voting items by $M = \{1, \dots, m\}$. The *agenda* for voting item j is an ordered pair (p_j, q_j), with the alternatives p_j, q_j drawn from a set X and assumed distinct, i.e., $p_j, q_j \in X$, $p_j \neq q_j$. For most of the analysis I assume a spatial framework such that X corresponds to finite-dimensional Euclidean space, but I explicitly introduce this restriction only when necessary for the results. The voter's *decision* on the jth voting item is denoted by $v_j \in \{\text{yes}, \text{no}\}$, with the interpretation that the voter votes in favor of alternative p_j on the jth voting item when the voting decision is $v_j = \text{yes}$ and votes in favor of alternative q_j when the voting decision is $v_j = \text{no}$. For most of the analysis prior to my consideration of multiple voters deciding on the same agenda in Section 4.2, I represent this information more economically using a *voting record* $V = \{(y_j, z_j)\}_{j \in M}$, which is a collection of m ordered pairs of alternatives obtained from the voting agenda and corresponding decisions according to the following definition.

Definition 1. The *voting record* $V = \{(y_j, z_j)\}_{j \in M}$ is generated from the agenda and voting decisions $\{((p_j, q_j), v_j)\}_{j \in M}$ if $(y_j, z_j) = (p_j, q_j)$ for all voting items $j \in M$ with $v_j = \text{yes}$ and $(y_j, z_j) = (q_j, p_j)$ for all voting items $j \in M$ with $v_j = \text{no}$.

Thus, we can equivalently think of an individual voting record as an irreflexive revealed preference relation $V \subset X \times X$. Any subset V' of V represents a restricted voting record comprising a subset of voting items. If it is necessary to be explicit about the subset of voting items $M' \subseteq M$ included in a restricted voting record, I use the more specific notation $V_{M'} = \{(y_j, z_j)\}_{j \in M'}$. It will be convenient to keep track of subsets of the voting alternatives that are contained in the voting agenda. For that purpose, for any restricted voting record V', let $Y(V')$ represent the set of voting alternatives that the voter voted in favor of in voting record V', i.e.,

$$Y(V') = \{x \mid (x, z) \in V'\}.$$

Similarly, let $N(V')$ denote the set of voting alternatives the voter voted against in voting record V', i.e.,

$$N(V') = \{x \mid (y, x) \in V'\}.$$

Finally, let $X(V')$ denote the set of alternatives compared in the voting record V', so that

$$X(V') = N(V') \cup Y(V').$$

The following additional notation is mostly standard, but I briefly review it for completeness. For any set $K \subset \mathbb{R}^d$, I use $\mathcal{C}(K)$ to denote the convex hull of K. I denote the set of extreme points of $K \subset \mathbb{R}^d$ by $\mathcal{E}(K)$, which is the set of all the elements of K that cannot be written as a strict convex combination of alternatives in K. For finite $K \subset \mathbb{R}^d$, the set of extreme points of K, $\mathcal{E}(K)$, is nonempty and coincides with the vertexes of $\mathcal{C}(K)$. I use $|K|$ to indicate the cardinality of the set K, and let $K \setminus K'$ denote set difference between sets K and K'.

Upon observing a voting record V, a first question to address is whether there exists a utility function[1] such that every voting decision in V is consistent with the utility maximization of that function. In particular, I seek conditions for the existence of a utility function that satisfies the following rationalizability criterion.

Definition 2. A utility function $u : X \to \mathbb{R}$ *strictly rationalizes* the voting record V if $u(y) > u(z)$ for all $(y, z) \in V$.

The above definition rules out the possibility of indifference between any pair of alternatives in any voting item. The requirement that any vote indicates strict preference apparently maximizes the information on the voter's preferences that can be extracted from the voting record. It is not a particularly stringent requirement if, for example, the following conditions hold: The space of alternatives satisfies $X = \mathbb{R}^d$, voters have "thin" indifference sets, and one of the alternatives in the agenda of each voting item is drawn from a distribution that is absolutely continuous with respect to Lebesgue measure. Alternatively, voter indifference arises naturally in many equilibrium models of voting when proposals are determined endogenously by a utility-maximizing agenda setter. Thus, a less restrictive interpretation of the voting record leads to the following weaker criterion.

Definition 3. A utility function $u : X \to \mathbb{R}$ *rationalizes* the voting record V if $u(y) \geq u(z)$ for all $(y, z) \in V$.

In accordance with the above definitions, I say that a voting record is (strictly) *rationalizable*, if there exists a utility function that (strictly) rationalizes that record. Naturally, these tests are most relevant when the voter's decisions are sincere, that is, they reveal a true preference between the pair of alternatives compared in each voting item.

[1] Of course, an intermediate question is whether there exists a preference ordering that is consistent with the voter's choices. Since V is finite, such an ordering exists only if a utility function exists.

This interpretation is standard in the revealed preference analysis of the consumer and is maintained in the analysis that follows. Of course, if the voter is assumed to be a strategic participant in a voting game, then voting decisions may not reveal a true preference because, for example, the voter believes that her vote is not pivotal and hence does not alter the voting outcome, or because some voting items constitute only the early stages of a sequential voting game and the voter believes that a sophisticated voting strategy (Farquharson 1969) may lead to more favorable outcomes in subsequent voting stages. Nevertheless, the above rationalizability criteria are also applicable in certain game-theoretic voting contexts in which each voting item corresponds to a simple voting game in which the voter's payoff is determined by the alternative that wins the collective vote. If the voting record data are obtained from a sequence of observations of individual play in such games, then (strict) rationalizability tests the hypothesis that the voter employs a weakly undominated voting strategy.

Well known arguments imply that, without any additional requirements on the rationalizing utility function, even the strongest of the above two rationalizability criteria places weak testable restrictions on finite voting records.

Proposition 1. *(i) Every voting record is rationalizable.*

(ii) A voting record V is strictly rationalizable if and only if it satisfies

$$Y(V') \neq N(V') \text{ for all nonempty } V' \subseteq V. \tag{A}$$

Part (i) is trivial since a constant function rationalizes any voting record. To see part (ii), note that condition (A) is, in fact, the familiar acyclicity condition. In particular, (A) is necessary and sufficient to ensure that there does not exist a set of voting items and corresponding votes that produce a chain of comparisons between voting alternatives of the form $x \succ x' \succ \cdots \succ x$. If (A) holds, the transitive closure of the voting record V is a strict partial order in the set of alternatives $X(V)$, which can be extended to a strict linear order by Szpilrajn (1930) (e.g., Lemma 2 in Richter 1966). Since the set of alternatives $X(V)$ is finite, the construction of a rationalizing utility function u is trivial.

Condition (A) can be traced to general revealed preference analyses by Arrow (1959), Richter (1966), etc. It amounts to a finite version of Ville–Houthakker Strong Axiom of Revealed Preference (SARP) in the context of revealed preference theory of the consumer, but has significantly less bite in the context of binary voting on a finite number of voting items. For example, a sufficient condition on the voting agenda for condition (A) to be satisfied for all voting decisions is

$$|X(V_{M'})| > |M'| \text{ for all } M' \subseteq M. \tag{N}$$

Condition (N) is a mild restriction, requiring that the voting agenda is sufficiently rich so that the number of alternatives voted on in each subset of voting items exceeds the number of voting items in that subset. Figure 1 illustrates four voting records in two-dimensional space, only one of which (Figure 1(a)) violates (N) and (A). In light of the above discussion, questions about the rationalizability of voting choices become interesting only under additional restrictions on voters' preferences. I take up this analysis in the next section.

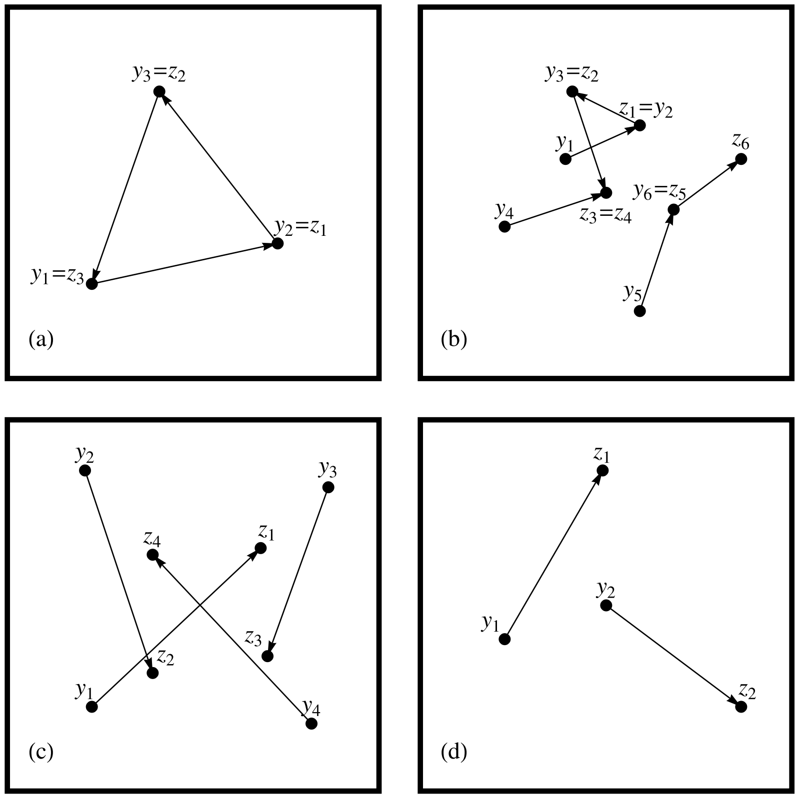

FIGURE 1. Strict rationalizability. An arrow emanating from y and pointing to z indicates $(y,z) \in V$. (a) The voting record $V = \{(y_j, z_j)\}_{j=1}^{3}$ violates (A) and (S). (b) The voting record $V = \{(y_j, z_j)\}_{j=1}^{6}$ satisfies (S′) (and (S)) with the sequence $V_1 = V$ and $x_1 = z_6$; $V_2 = V_{\{1,2,3,4,5\}}$ and $x_2 = z_5$; $V_3 = V_{\{1,2,3,4\}}$ and $x_3 = z_4$; $V_4 = V_{\{1,2\}}$ and $x_4 = z_2$; $V_5 = V_{\{1\}}$ and $x_5 = z_1$; $V_6 = \varnothing$. (c) The voting record $V = \{(y_j, z_j)\}_{j=1}^{4}$ satisfies (A) and violates (S) because there is no $z_j \in \mathcal{E}(X(V))$. (d) The voting record $V = \{(y_j, z_j)\}_{j=1}^{2}$ satisfies (S).

3. Concave rationalizations

In this section, I cast the analysis in a spatial setting by assuming that the space of alternatives corresponds to the d-dimensional Euclidean space, that is, $X = \mathbb{R}^d$, $d \geq 1$. I seek conditions on observed voting records that are consistent with the hypothesis that the voter's decisions are generated by convex preferences. I consider several variants of this hypothesis, the strongest of which is the existence of a rationalizing utility function, $u : \mathbb{R}^d \to \mathbb{R}$, that is strictly concave:

$$u(\lambda x + (1-\lambda)x') > \lambda u(x) + (1-\lambda)u(x') \text{ for all } x, x', x \neq x' \text{ and all } \lambda \in (0,1). \tag{1}$$

A weaker restriction is strict quasiconcavity:

$$u_i(\lambda x + (1-\lambda)x') > \min\{u_i(x), u_i(x')\} \text{ for all } x, x', x \neq x' \text{ and all } \lambda \in (0,1). \tag{2}$$

When relevant, I also consider mere concavity and quasiconcavity, which are obtained from (1) and (2), respectively, by allowing weak inequality. The restriction to quasiconcave utility representations has a natural place in the theory of voting. In one-dimensional space ($d = 1$), strict quasiconcavity of preferences boils down to the familiar single-peakedness condition from social choice theory. Concavity is apparently a much stronger requirement that is harder to justify in commonly constructed political scales that represent the space of alternatives because the property is not preserved by dimensionwise monotonic transformations of these scales. Of course, in cases in which there is an unambiguous numerical representation of the space of alternatives in Euclidean space, concavity has a clear interpretation in terms of the voter's attitude toward risk, which may justify such voter utility functions on substantive grounds.

The a priori merits of assuming concave versus quasiconcave voter preferences notwithstanding, it turns out that it is impossible to discriminate between these utility representations on the basis of finite voting records. Yet, not all voting records that are strictly rationalizable can be strictly rationalized by a (quasi)concave utility function. In the next theorem, I state necessary and sufficient conditions.

Theorem 1. *Assume $X = \mathbb{R}^d$ and let $V = \{(y_j, z_j)\}_{j=1}^m$ be a voting record. Then the following conditions are equivalent.*

(S) *For all nonempty $V' \subseteq V$, there exists $x \in \mathcal{E}(X(V'))$ such that $x \notin Y(V')$.*

(S′) *There exists a sequence of restricted voting records $\{V_t\}_{t=1}^{k+1}$ and alternatives $\{x_t\}_{t=1}^{k}$ such that $V_1 = V$, $V_{k+1} = \varnothing$, $V_{t+1} = \{(y, z) \in V_t \mid z \neq x_t\}$, $x_t \in \mathcal{E}(X(V_t))$, and $x_t \notin Y(V_t)$ for all $t = 1, \ldots, k$.*

(S_c) *There exists a strictly concave utility function that strictly rationalizes V.*

(S'_c) *There exists a concave utility function that strictly rationalizes V.*

(S_q) *There exists a strictly quasiconcave utility function that strictly rationalizes V.*

(S'_q) *There exists a quasiconcave utility function that strictly rationalizes V.*

Furthermore, if $d = 1$, then (S) is equivalent to the following condition.

(S_1) *For all $V' \subseteq V$ with $|V'| = 2$, there exists $x \in \mathcal{E}(X(V'))$ such that $x \notin Y(V')$.*

Of course, condition (S) implies condition (A) but it is, in fact, a significant strengthening of that condition. This is in contrast to standard neoclassical theory of the consumer where a version of SARP acyclicity is sufficient for that consumer to have a (strictly) concave utility representation as shown by Afriat (1967), Varian (1982), Matzkin and Richter (1991), etc. Note that in the standard theory of the consumer, choice is restricted to convex budget sets and the data do not directly contradict nonsatiation. Convexity of the budget sets is relaxed by Matzkin (1991), who allows for co-convex sets (i.e., sets B such that $B^c \cap \mathbb{R}^d_+$ is convex), Chambers and Echenique (2009), who study finite budget sets, and Forges and Minelli (2009), who consider more general budget sets than Matzkin (1991) but exclude finite sets due to a free disposal assumption. In the setting of Forges and Minelli (2009), concavity entails additional testable restrictions beyond those required for rationalization of observed choices.[2] In the present analysis, the voter chooses from finite, certainly nonconvex budget sets, the data may violate nonsatiation, and the additional restrictions implied by concavity are significant.

Condition (S) requires that, on the basis of any restricted voting record $V' \subseteq V$, we cannot refute the possibility that one of the extreme points of $X(V')$ is least preferred among all alternatives in $X(V')$. The necessity of (S) is straightforward, as a quasiconcave function minimized over a convex polytope must attain a minimum at one of the polytope's extreme points. Thus, as pointed out by a referee, Theorem 1 conforms with a widely used principle in revealed preference theory by establishing that if the data do not contradict the hypothesis that the voter has preferences represented by a (strictly quasi)concave utility function, then we can actually obtain such a function. Before I discuss the sufficiency of condition (S), note that the condition involves identifying an extreme point with the required property for each of the $\sum_{h=1}^{m}\binom{m}{h}$ subsets $X(V') \subseteq X(V)$ of the voting alternatives. While this task appears daunting as the number of voting items increases, the equivalent condition[3] (S′) of Theorem 1 provides a palatable remedy: It suffices to identify such extreme points for at most m restricted voting records $V' \subseteq V$. The practical value of this equivalent version of condition (S) is that the validity of (S′) can be ascertained on a computer using standard routines via the following algorithm.

Algorithm 1. *Assume $X = \mathbb{R}^d$ and let $V = \{(y_j, z_j)\}_{j=1}^m$ be a voting record.*

Step 1. *Set $V_1 = V$ and proceed to Step 2 with $t = 1$.*

Step 2. *For each $x \in N(V_t) \setminus Y(V_t)$, test whether $x \in \mathcal{E}(X(V_t))$ until $x \in \mathcal{E}(X(V_t)) \cap (N(V_t) \setminus Y(V_t))$ is found; set $x_t = x$ and proceed to Step 3. If $\mathcal{E}(X(V_t)) \cap (N(V_t) \setminus Y(V_t)) = \varnothing$, V does not satisfy (S).*

Step 3. *Set $V_{t+1} = \{(y, z) \in V_t \mid z \neq x_t\}$ and proceed to Step 4.*

Step 4. *If $V_{t+1} = \varnothing$, V satisfies (S); else, proceed to Step 2 with $t = t + 1$.*

It is straightforward to show that Algorithm 1 terminates with the correct conclusion in at most m iterations. Computationally, the most demanding part of the algorithm is the identification of an extreme point $x_t \in \mathcal{E}(X(V_t))$ in Step 2, a task that can be executed efficiently by solving at most $|N(V_t) \setminus Y(V_t)|$ linear programs. As an illustration of condition (S′), consider the voting record in the example depicted in Figure 1(b), for which it is easy to ascertain the validity of the condition with a sequence of length $k = 5 < m = 6$: First $V_1 = V$ and $x_1 = z_6$; then $V_2 = V_{\{1,2,3,4,5\}}$ and $x_2 = z_5$; $V_3 = V_{\{1,2,3,4\}}$ and $x_3 = z_4$;

[2]As discussed by Chambers and Echenique (2009), concavity coupled with supermodularity jointly imply additional testable restrictions, although neither imposes such restrictions as an individual assumption.

[3]Yet another equivalent statement of condition (S) that appears in previous versions is

$$\text{(S}''\text{)} \qquad \text{for all nonempty } V' \subseteq V, \text{ there exists } x \in N(V') \text{ such that } x \notin \mathcal{C}(Y(V')).$$

$V_4 = V_{\{1,2\}}$ and $x_4 = z_2$; and, finally, $V_5 = V_{\{1\}}$ and $x_5 = z_1$, at which point we clearly obtain $V_6 = \varnothing$.

The sufficiency of condition (S) (or (S′)) follows by an inductive proof that proceeds by reversing the order of the algorithm described in the previous paragraph. First, assuming the sequence $\{(V_t, x_t)\}_{t=1}^{k}$ is available, observe that $\{x_k\} = N(V_k)$ (because $V_{k+1} = \varnothing$) and that x_k is an extreme point of $X(V_k)$, so that it is trivial to find a concave function that assumes its minimum over $X(V_k)$ at x_k, thus strictly rationalizing the voting record V_k. This function can then be modified by moving "outward" so as to represent revealed preferences over $X(V_{k-1})$ by preserving the existing comparisons among alternatives in $X(V_k)$ and by assigning a sufficiently lower indifference contour to the extreme point x_{k-1}. Proceeding as above, at the tth step of the process, it is possible to strictly rationalize revealed preferences over the larger set $X(V_{k-t+1})$ by assigning a sufficiently lower indifference contour to the extreme point x_{k-t+1}, etc. These arguments are almost identical for the proof of Theorem 2 that deals with the case of mere rationalizability, so, to avoid duplication, I rely on Theorem 2 to prove Theorem 1 in the Appendix.

A different simplification of condition (S) obtains in the one-dimensional case ($d = 1$). Here, condition (S) is equivalent to (S_1), which requires only the existence of the requisite extreme points for pairs of voting items. In one dimension, $X(V')$ has only two extreme points for any nonempty restricted voting record $V' \subseteq V$, so that if condition (S) fails for a restricted record $V' = V_{M'}$ that comprises three or more voting items, then the condition must also fail for a further restricted voting record $V_{\{j,h\}}$ that comprises a pair of voting items $\{j, h\} \subseteq M'$ such that $Y(V_{\{j,h\}}) = \mathcal{E}(Y(V_{M'}))$. Intuition may suggest that an analogous weakening of condition (S) is possible in more than one dimension by requiring that this condition be applied only to subsets that comprise at most $d+1$ voting items when $d > 1$. Unfortunately, this is not the case, as is illustrated in Figure 1(c) in a two-dimensional setting: While condition (S) holds for all triplets ($d + 1 = 3$) of voting items, it fails when we consider all four items in the voting record. In two or more dimensions, there is no bound on the number of extreme points analogous to the one that holds in one dimension.

Theorem 1 also establishes that if there exists a quasiconcave utility function that strictly rationalizes a voting record, then there also exists a (strictly) concave function that strictly rationalizes that voting record. The finiteness of the voting record is clearly necessary for this result. It is well known since de Finetti (1949) that there exist (strictly) convex preferences that do not admit a concave utility representation, a problem studied systematically by Kannai (1977). Thus, given the asymmetric part of such preferences as a (infinite) voting record, it is not possible to recover a concave function that strictly rationalizes it, even though there exists a quasiconcave function that does. But when a quasiconcave utility function strictly rationalizes a *finite* voting record, alternatives in the record are sufficiently far apart to be able to assign inferior voting alternatives to sufficiently lower indifference contours and transform the original quasiconcave utility function to a strictly concave one.

Even finite information on preferences can bar concave but allow quasiconcave representations, and Richter and Wong (2004) provide an example of complete and transitive preferences over a set K of three alternatives with that property. That example, though, requires both strict preference and indifference, and indifference is ruled out a priori when we interpret all voting decisions to reveal strict preference. This leaves open the possibility that finite voting records may discriminate between the hypothesis that the voter's preferences are represented by a quasiconcave utility function and the hypothesis that they are represented by a concave utility function if the voting decisions may indicate indifference. Yet, if voting decisions may indicate indifference, we can always rationalize any voting record by a concave constant utility function even if the voter's actual preferences cannot be represented by a concave function. Furthermore, the example of Richter and Wong (2004) involves preferences that do not admit a strictly quasiconcave representation. As I establish in the remainder of this section, if voting decisions may indicate indifference, then finite voting records can be rationalized by a strictly concave utility function whenever they can be rationalized by a strictly quasiconcave utility function. Thus, the analysis establishes that the hypotheses of (strict) concavity or quasiconcavity of voter preferences are *observationally equivalent* for any finite voting record and either of the two rationalizability criteria I have defined.

The necessary and sufficient condition for a voting record to be rationalizable by a strictly (quasi)concave function, turns out to be only mildly weaker than the corresponding condition of Theorem 1, as I establish in Theorem 2. Furthermore, when this necessary and sufficient condition is met, it is possible to assign strict preferences to all pairwise comparisons in the voting record except those that are entangled in a directly revealed (i.e., via a sequence of votes) individual preference cycle, in accordance with the following intermediate criterion for rationalizability.

Definition 4. A utility function $u: X \to \mathbb{R}$ *almost strictly rationalizes* the voting record V if it rationalizes that record and, in addition, it strictly rationalizes the restricted voting record[4]

$$V_a = \{(y, z) \in V \mid \nexists V' \subseteq V \text{ such that } (y, z) \in V' \text{ and } Y(V') = N(V')\}. \tag{3}$$

When a voting record is almost strictly rationalized, indifference between any pair of alternatives is imputed by the rationalizing function in a minimal way. As shown in the next theorem, a voting record can be almost strictly rationalized by a strictly (quasi)concave utility function whenever it can be rationalized by such a function.

Theorem 2. *Assume $X = \mathbb{R}^d$ and let $V = \{(y_j, z_j)\}_{j=1}^m$ be a voting record. Then the following conditions are equivalent.*

(W) For all nonempty $V' \subseteq V$, either there exists $x \in \mathcal{E}(X(V'))$ such that $x \notin Y(V')$ or there exists a nonempty $V'' \subseteq V'$ such that $N(V'') = Y(V'') \subseteq \mathcal{E}(X(V'))$ and $Y(V'') \cap Y(V' \setminus V'') = \varnothing$.

(W′) There exists a sequence of restricted voting records $\{V_t\}_{t=1}^{k+1}$ and sets of alternatives $\{X_t\}_{t=1}^{k}$ such that $V_1 = V$ and $V_{k+1} = \varnothing$, and for all $t = 1, \dots, k$, $X_t \subseteq \mathcal{E}(X(V_t))$, $V_{t+1} = \{(y, z) \in V_t \mid z \notin X_t\}$, and either $X_t = \{x_t\}$ with $x_t \notin Y(V_t)$ or $X_t = N(V_t \setminus V_{t+1}) = N(V_t') = Y(V_t')$ for some $V_t' \subseteq V_t \setminus V_{t+1}$ and $X_t \cap Y(V_{t+1}) = \varnothing$.

[4]Alternatively, V_a is the intersection of V and the antisymmetric part of the transitive closure of V.

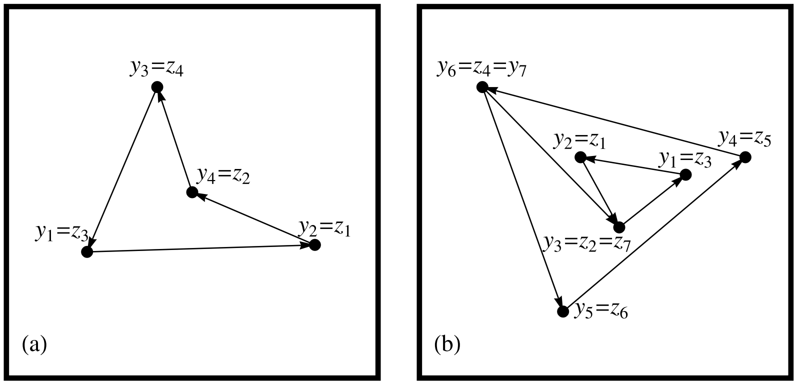

FIGURE 2. Rationalizability and admissible cycles. An arrow emanating from y and pointing to z indicates $(y, z) \in V$. (a) $V = \{(y_j, z_j)\}_{j=1}^4$ violates (W) because the indifference contour crossing $y_1, \dots, y_4$ cannot delineate a convex set. (b) $V = \{(y_j, z_j)\}_{j=1}^6$ satisfies (W) but $V = \{(y_j, z_j)\}_{j=1}^7$ does not since the nested indifference contour crossing through y_1, y_2, y_3 cannot be ranked above the contour crossing through y_4, y_5, y_6.

(W_c') *There exists a strictly concave utility function that almost strictly rationalizes V.*

(W_q') *There exists a strictly quasiconcave utility function that almost strictly rationalizes V.*

(W_c) *There exists a strictly concave utility function that rationalizes V.*

(W_q) *There exists a strictly quasiconcave utility function that rationalizes V.*

Furthermore, if $d = 1$, then (W) is equivalent to the following condition.

(W_1) *For all $V' \subseteq V$ with $|V'| = 2$, either there exists $x \in \mathcal{E}(X(V'))$ such that $x \notin Y(V')$ or $N(V') = Y(V')$.*

Condition (W) requires that, on the basis of any restricted voting record $V' \subseteq V$, if we can refute the possibility that one of the extreme points of $X(V')$ is least preferred among all alternatives in $X(V')$, then we cannot refute the possibility that a subset of extreme points of $X(V')$ belonging in an indifference class are least preferred among alternatives in $X(V')$. An inspection of condition (W) reveals that voting records that cannot be strictly rationalized but can be rationalized by strictly (quasi)concave functions exhibit a particular type of violation of acyclicity, (A). To rationalize voting records that violate (A), all alternatives that are entangled in the revealed voting cycle must be assigned to the same indifference contour. While this is possible in the case of Figure 1(a) without violating the convexity requirement on the voter's preferences, not all revealed preference cycles can be so rationalized without violating convexity of the voter's preferences as illustrated in Figure 2. In the case of Figure 2(a), the required indifference contour cannot delineate a convex set; in the case of Figure 2(b), two nested indifference contours that rationalize cycles cannot be ranked in ascending order, due to the fact that $(y_7, z_7) \in V$. Clearly, if violations of acyclicity are ruled out, such as is the case when (N) holds, then conditions (S) and (W) are equivalent.

Remark 1. If $X = \mathbb{R}^d$ and the voting record V satisfies (A), then (S)$\Leftrightarrow$(W).

Condition (W) is equivalent to condition (W′), which (like condition (S′)) is much easier to verify. Indeed, a modification of Algorithm 1 can be employed to ascertain the validity of condition (W′): the details are contained in the Appendix.

It is useful to compare conditions (S) and (W) with the following (slightly restated) necessary and sufficient conditions of Richter and Wong for the existence of a strictly concave function (Richter and Wong 2004, Theorem 2) that rationalizes a reflexive, transitive, and complete preference relation $\succeq$ over a finite set $K \subset \mathbb{R}^d$.

(G′) For all $Y \subseteq K$ such that $|Y| \leq d+1$ and $\mathcal{E}(Y) = Y$, and for all $x \in K \setminus Y$ such that $x \in \mathcal{C}(Y)$, there exists $x' \in Y$ such that $x \succeq x'$ and $x' \not\succeq x$.

Note that, since (G′) is necessary and sufficient, if the revealed preference relation, V, defined by the voting record can be extended to a reflexive, transitive, and complete preference relation $V^* \supset V$ on $X(V)$ that can be rationalized by a strictly concave utility function, then condition (G′) must hold for that extension, V^*. But condition (G′) applied to the incomplete revealed preference relation V defined by the voting record is neither necessary nor sufficient for the existence of a strictly concave rationalizing utility function. The fact that (G′) is not sufficient is illustrated with the voting record V depicted in Figure 1(a) in the case of strict rationalizability and with the voting record V depicted in Figure 2(a) in the case of mere rationalizability. In particular, when (W) fails, as is the case in Figure 2(a), there does not exist a reflexive, transitive, and complete preference relation $V^* \supset V$ that satisfies (G′), even though V satisfies (G′). But note that if the voting record V satisfies both (A) and (G′), then it is strictly rationalizable by a strictly concave utility function, as is the case for the voting record illustrated in Figure 1(d). In that example, the fact that the voter reveals that $(y_2, z_2) \in V$ ensures that (G′) holds for any extension of V to a reflexive, transitive, and complete $V^* \supset V$.[5] Condition (G′) is also not necessary for (strict) rationalizability. Indeed, in typical situations, condition (G′) does not hold on the basis of the information directly or indirectly[6] revealed by the voting record, as is the case in Figure 1(b) and (c). Nevertheless, a rationalizing strictly concave utility function does exist in the case of Figure 1(b), but not in the case of Figure 1(c).

In this section I have derived necessary and sufficient conditions that must be satisfied by a voting record for it to be strictly rationalized by a (strictly) (quasi)concave function. If the rationalizing utility function is required to be strictly quasiconcave, then mildly weaker conditions are necessary and sufficient to (merely) rationalize a voting record. These conclusions are summarized in Table 1. In the next section, I use these

[5] Richter and Wong (2004) discuss the additional acyclicity condition required to render condition (G′) (or its counterpart, condition (G), for mere concavity) sufficient for concave rationalizability of a pair of incomplete strict preference and indifference relations in their Remark 4, page 344, and footnote 8.

[6] That is, even if we consider the transitive closure of the directly revealed preferences, V.

	u need not be quasiconcave	u is (quasi)concave	u is strictly (quasi)concave
There exists u that rationalizes V	no restrictions	no restrictions	(W)
There exists u that strictly rationalizes V	(A)	(S)	(S)

Table 1. Necessary and sufficient conditions for rationalizability of voting record V.

conditions to test the hypothesis that the voter has preferences that are satiated at some voting alternative.

4. Ideal points

4.1 *One voter, known set of alternatives*

Assuming a voting record V is (strictly) rationalizable, the voter may have an *ideal point*, i.e., there may exist an alternative $\hat{x} \in X$ such that the voter strictly prefers $\hat{x}$ over all other alternatives. In particular, the evidence from the voting record V cannot refute the existence of such an ideal point $\hat{x}$ whenever V can be rationalized by a utility function that is uniquely maximized at $\hat{x}$.

Definition 5. A utility function $u : X \to \mathbb{R}$ *(strictly) rationalizes the voting record V with ideal point* $\hat{x} \in X$ if it (strictly) rationalizes V and

$$u(\hat{x}) > u(x) \text{ for all } x \in X, x \neq \hat{x}.$$

Armed with the above criterion, we may then inquire whether a voting record places any testable restrictions on the location of the voter's ideal point? Obviously, this question has a trivial answer without any restrictions on voter preferences: If a voting record can be rationalized, then it can be rationalized with any ideal point $\hat{x} \notin N(V)$. On the other hand, under convexity restrictions on preferences, the results contained in the previous section provide a more promising approach to the problem. In fact, as I discuss shortly, the following lemma reduces the question on the nature of testable restrictions on a voter's ideal point from voting data to a question of rationalizability of an augmented voting record.

Lemma 1. *Assume $X = \mathbb{R}^d$, let V be a finite voting record, and assume that there exists a strictly concave utility function $u : \mathbb{R}^d \to \mathbb{R}$ that (strictly) rationalizes V. If there exists alternative $\hat{x} \in \mathbb{R}^d \setminus N(V)$ such that $u(\hat{x}) \geq u(x)$ for all $x \in X(V)$, then there exists another strictly concave function $\tilde{u} : \mathbb{R}^d \to \mathbb{R}$ that (strictly) rationalizes V with ideal point $\hat{x}$.*

Thus, if the finite voting record V can be rationalized by a strictly concave function and this rationalizing utility function assigns at least as high a utility level to an alternative $\hat{x} \notin N(V)$ compared to the utility level assigned to every other alternative in the agenda of the voting record, then we cannot reject the hypothesis that the voter has a strictly concave utility function with ideal point $\hat{x}$. Lemma 1 suggests a straightforward test for the hypothesis that the voting record V can be strictly rationalized by a strictly concave utility function with ideal point $\hat{x}$ by testing the rationalizability of an augmented voting record that includes $|X(V)\setminus\{\hat{x}\}|$ additional voting items of the form $(\hat{x}, z)$, one for each of the alternatives $z \in X(V)\setminus\{\hat{x}\}$. Specifically, this augmented voting record is defined as follows.

Definition 6. Given a voting record V and an alternative $\hat{x} \in X$, the $\hat{x}$-*augmented voting record* is

$$\hat{V} = V \cup \{(\hat{x}, z) \mid z \in X(V)\setminus\{\hat{x}\}\}.$$

By Lemma 1 and Theorem 1, there exists a strictly concave utility function that strictly rationalizes the voting record V with ideal point $\hat{x}$ if and only if the $\hat{x}$-augmented voting record $\hat{V}$ satisfies (S). In Theorem 3, I state this necessary and sufficient condition as $(\widehat{\mathrm{S}}')$ and show that, in fact, it is equivalent to the apparently weaker condition $(\widehat{\mathrm{S}})$ of that theorem.

Theorem 3. *Assume $X = \mathbb{R}^d$ and $\hat{x} \in \mathbb{R}^d$, and let $V = \{(y_j, z_j)\}_{j\in M}$ be a voting record. Then the following conditions are equivalent.*

- $(\widehat{S})$ *For all nonempty $V' \subseteq V$, there exists $x \in \mathcal{E}(X(V') \cup \{\hat{x}\})$ such that $x \notin Y(V') \cup \{\hat{x}\}$.*
- $(\widehat{S'})$ *The $\hat{x}$-augmented voting record $\hat{V}$ satisfies (S) or (S′).*
- $(\widehat{S}_c)$ *There exists a strictly concave utility function that strictly rationalizes V with ideal point $\hat{x}$.*
- $(\widehat{S}_q)$ *There exists a strictly quasiconcave utility function that strictly rationalizes V with ideal point $\hat{x}$.*
- $(\widehat{S}'_c)$ *There exists a concave utility function that strictly rationalizes V with ideal point $\hat{x}$.*
- $(\widehat{S}'_q)$ *There exists a quasiconcave utility function that strictly rationalizes V with ideal point $\hat{x}$.*

If $d = 1$, then $(\widehat{S})$ is equivalent to the following condition.

- $(\widehat{S}_1)$ *For all $V' \subseteq V$, $1 \le |V'| \le 2$, there exists $x \in \mathcal{E}(X(V') \cup \{\hat{x}\})$ such that $x \notin Y(V') \cup \{\hat{x}\}$.*

Condition $(\widehat{\mathrm{S}})$ provides a precise set of testable restrictions on the location of the voter's ideal point that arises from her voting record, assuming that the voter has a (strictly quasi)concave utility function. Of course, condition $(\widehat{\mathrm{S}})$ implies condition (S). Furthermore, as is true for Theorem 1, the one-dimensional case admits a further simplification of condition $(\widehat{\mathrm{S}})$. I provide a graphical illustration of the implications of Theorem 3 in Figure 3, where I depict five voting alternatives associated with four voting items ($m = 4$) in a two-dimensional space. Application of condition $(\widehat{\mathrm{S}})$ restricts the voter's ideal point, $\hat{x}$, to lie outside the gray areas in Figure 3(b). Because Lemma 1 concerns both strict and mere rationalizability, virtually identical arguments deal with the

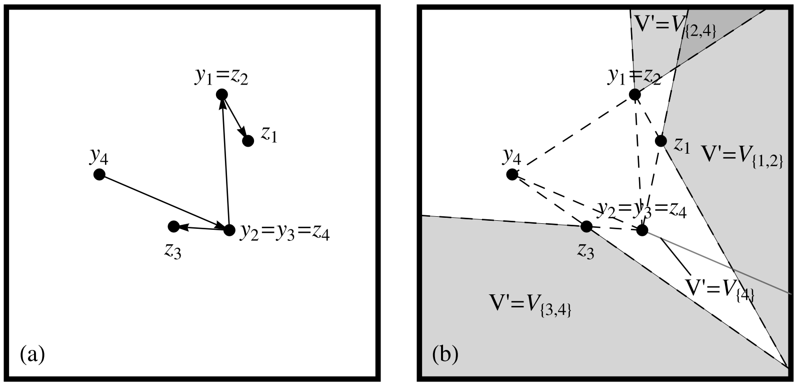

FIGURE 3. Ideal point restrictions. (a) Voting record $V = \{(y_j, z_j)\}_{j=1}^4$; an arrow emanating from y and pointing to z indicates $(y, z) \in V$. (b) The voter with voting record V of (a) cannot have a (strictly) (quasi)concave utility function with ideal point that lies in the gray areas. For each $\hat{x}$ in these areas, condition $(\widehat{\text{S}})$ is violated for the indicated restricted voting record V'.

case of mere rationalizability with an ideal point, and I state the corresponding result as Theorem 6 in the Appendix.

4.2 *Many voters, unknown set of alternatives*

Theorem 3 (and Theorem 6 in the Appendix) establishes that finite voting records impose nontrivial testable restrictions on the location of voter ideal points. Compared to existing parametric methods for the estimation of voters' ideal points, though, the nonparametric tests suggested by Theorems 3 and 6 impose a significant burden on the analyst, because they require knowledge of the location of the voting alternatives across all voting items. On the contrary, most existing techniques for the estimation of voter ideal points rely only on partial information that typically reduces to mere knowledge of the vector of voting decisions, yes or no, of a set of voters on a collection of voting items. In particular, these techniques simultaneously estimate both the voting agenda, $\{(p_j, q_j)\}_{j\in M}$ and the voters' ideal points.[7] Given that in many voting contexts the voting agenda involves intangible issues that are hard to represent on numerical scales,[8] it is important to ask whether the testable restrictions on ideal points derived so far have any bearing if we relax the assumption that the location of the voting alternatives in the voting agenda, (p_j, q_j), is known. I devote the rest of this section to this question.

[7]Much of this literature bears a close relationship with item response models used in the psychometrics literature on educational testing (e.g., see discussion in Clinton et al. 2004, p. 356) in which the data are given by, for example, true or false responses of a number of test subjects (the analogue of the voters) on a series of test questions, and the estimator simultaneously recovers the *ability* (which corresponds to the ideal points of the voters in the roll-call analysis) of the test subjects and the *discriminating* power of the items/questions on the test (which in the case of roll-call data corresponds to statistics of the location of the two voting alternatives in the voting agenda).

[8]Of course there are exceptions, such as when voting takes place over financial legislation that disburses funds in different policy areas.

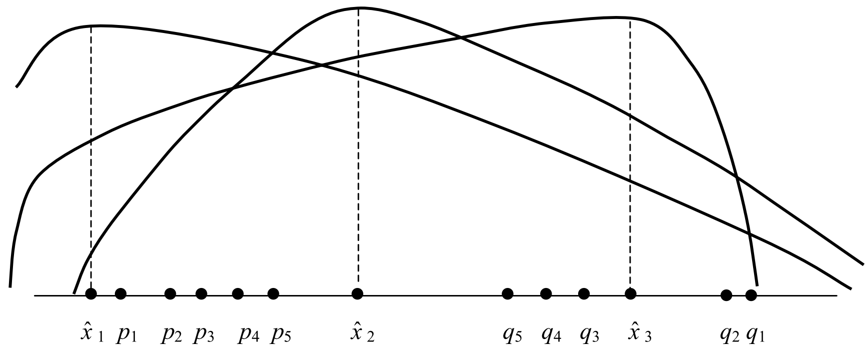

FIGURE 4. Illustration of Theorem 4. An example of the construction in the proof of Theorem 4 for $n = 3$ voters, $m = 5$ voting items, and ideal points that satisfy $\hat{x}_1 < \hat{x}_2 < \hat{x}_3$. The voting alternatives are located so that condition (S_1) is satisfied for any voting decisions by these voters. In this example, the voting decisions are given by $v_j^1 =$ yes for all $j = 1, \ldots, 5$ for voter 1, $v_1^2 = v_3^2 = v_4^2 =$ no and $v_2^2 = v_5^2 =$ yes for voter 2, and $v_3^3 = v_4^3 = v_5^3 =$ no and $v_1^3 = v_2^3 =$ yes, for voter 3.

Recall that the voting record $V = \{(y_j, z_j)\}_{j \in M}$ is generated from the voting agenda and voting decisions $\{((p_j, q_j), v_j)\}_{j \in M}$ if $(y_j, z_j) = (p_j, q_j)$ when $v_j =$ yes and $(y_j, z_j) = (q_j, p_j)$ when $v_j =$ no for all j. First, I show that the conditions of Theorem 3 (or those of Theorem 6) are vacuously met for all voters and for every number of issue dimensions $d \geq 1$ if the location of the voting alternatives that appear on the voting agenda is unrestricted. Specifically, the following theorem holds.

THEOREM 4. *Let $v^i \in \{\text{yes}, \text{no}\}^m$, $i = 1, \ldots, n$, represent the voting decisions of each of n voters $1, \ldots, n$ on m voting items indexed by $M = \{1, \ldots, m\}$. For every $d \geq 1$ and every n-tuple of points $\hat{x}_1, \ldots, \hat{x}_n \in \mathbb{R}^d$, there exist alternatives $p_j, q_j \in \mathbb{R}^d$, $p_j \neq q_j$, $j \in M$, such that for every voter i there exists a (strictly) (quasi)concave utility function u_i with ideal point $\hat{x}_i$ that strictly rationalizes the voting record generated from the voting agenda and voting decisions $\{((p_j, q_j), v_j^i)\}_{j \in M}$.*

Note that Theorem 4 states that all possible voting decisions and all possible ideal points for the n voters can be jointly rationalized by appropriately choosing the location of the voting alternatives across the m voting items. That is, one choice of the location of the voting alternatives works for all voters at the same time. In the one-dimensional case, Theorem 4 is shown by constructing an agenda that satisfies condition $(\widehat{S})$ independent of the voting decisions on this agenda. An illustration of this construction is provided in Figure 4. In the example of Figure 4, there are five voting items and the voting alternatives p_j, q_j, $j = 1, \ldots, 5$, are ordered according to $p_1 < p_2 < \cdots < p_5 < q_5 < q_4 < \cdots < q_1$. This arrangement ensures that all possible voting records generated from this agenda and any voting decisions on that agenda necessarily satisfy condition (S_1) of Theorem 1. In particular, for any pair of voting items j, h with $j < h$, it is the case that $\mathcal{E}(X(V_{j,h})) = \{p_j, q_j\}$, and it cannot be that any voter votes both for p_j and for q_j. Clearly, the construction in this example can be easily generalized to arbitrary numbers of voting items. It is then a simple additional step to translate the arrangement of the voting alternatives around the predetermined ideal points to ensure that the added restrictions of condition ($\widehat{\mathrm{S}}_1$) of Theorem 3 are not violated. In the generic case when voters' ideal points are distinct, this construction can be achieved while at the same time ensuring that at least one of the voting alternatives p_j, q_j lies in the voters' Pareto set for each voting item $j \in M$, as is the case in Figure 4.

Theorem 4 forecloses nonparametric estimation of agnostic (Londregan 1999) models of legislator ideal points, i.e., models in which the voting alternatives are unknowns to be estimated along with the legislators' ideal points. Barring knowledge of the voting alternatives, nonparametric estimation of voter preferences requires at least some restrictions on their location for identification purposes. One such extra identification restriction in the context of a parametric probabilistic voting model is used by Clinton and Meirowitz (2001). They assume that the agenda of each voting item j, (p_j, q_j), comprises a proposal p_j and a status quo q_j, and the status quo q_{j+1} is equal to one of the two alternatives p_j, q_j that prevailed in the vote for the jth item. With this extra restriction, the conclusion of Theorem 4 no longer obtains. In particular, consider an example with $m = 2$ voting items, an agenda that satisfies $p_1 = a$, $q_1 = b$, $p_2 = c$, and $q_2 = a$, and $n = 3$ voters with voting records $V^1 = \{(a, b), (c, a)\}$, $V^2 = \{(b, a), (a, c)\}$, and $V^3 = \{(b, a), (c, a)\}$. Note that there is no location for the three alternatives a, b, c on the real line, so that all three voters' records satisfy condition (S_1) of Theorem 1, since $Y(V^1) = \{a, c\}$, $Y(V^2) = \{a, b\}$, and $Y(V^3) = \{b, c\}$.[9] Thus, in contrast to the conclusion of Theorem 4, due to the restriction that $p_1 = q_2$, it is not possible to strictly rationalize the voting record of all three voters in $d = 1$ dimension, a fact that leaves open the possibility for nonparametric estimation of the one-dimensional probabilistic voting model. Yet, as the following theorem shows, the identifying role of this additional restriction, while possibly strong in one dimension, has no bite in higher dimensions.

Theorem 5 (Embedding). *Let $V^1, \dots, V^n$ be the finite voting records of n voters $1, \dots, n$. For every $d \geq 2$ and for every n-tuple of points $\hat{x}_1, \dots, \hat{x}_n \in \mathbb{R}^d$, there exists a one-to-one function $f : \bigcup_{i=1}^{n} X(V^i) \to \mathbb{R}^d$ such that for every voter $i \in \{1, \dots, n\}$*

(i) there exists a (strictly) (quasi)concave utility function $u_i : \mathbb{R}^d \to \mathbb{R}$ with ideal point $\hat{x}_i$ that almost strictly rationalizes the voting record $\{(f(y), f(z)) \mid (y, z) \in V^i\}$

(ii) if V^i satisfies (A), then there exists a (strictly) (quasi)concave utility function $u_i : \mathbb{R}^d \to \mathbb{R}$ with ideal point $\hat{x}_i$ that strictly rationalizes the voting record $\{(f(y), f(z)) \mid (y, z) \in V^i\}$.

According to Theorem 5, it is always possible to embed *any* collection of finite voting records for *any* set of voters in a two-dimensional space while at the same time endowing each voter with a strictly concave rationalizing utility function that has an arbitrarily prespecified ideal point in that space. The domain of the function f and the fact the f is

[9]This example replicates an argument by Bogomolnaia and Laslier (2007, Proposition 17). Recently, Schwartz (2007, Theorem 4), working in a discrete space of alternatives, gave a sufficient condition on the voting record that guarantees violation of single-peakedness of the preferences of at least one voter over the voting alternatives, X_M.

injective ensure that the new embedded voting record $\{(f(y), f(z)) \mid (y, z) \in V^i\}$ respects all the equality restrictions on the voting alternatives across different voting items in the original voting record V^i. Thus, unlike the construction in Theorem 4, the embedded voting records $\{(f(y), f(z)) \mid (y, z) \in V^i\}$, like the original voting records V^i, $i = 1, \ldots, n$, may feature voting agendas such that the victorious alternative in early voting items becomes the status quo in subsequent voting items or feature any possible recurrence of voting alternatives across voting items. In fact, the theorem places no other restrictions on the location of the original voting alternatives, so that the original voting record may involve alternatives drawn from some space X other than Euclidean space. Also, as long as the original voting record does not reveal any individual preference cycles, this representation can be achieved while at the same time ensuring that every voting record is *strictly* rationalized by part (ii) of the theorem. Last, note that the theorem does not require that all n voters vote on the same set of voting items, a generality that accommodates the possibility that, for example, some voters may have abstained in some voting items or may have voted on different agendas because they have nonoverlapping careers in the legislature, etc. Over all, Theorem 5 provides a new twist on the common finding of many parametric ideal point estimation techniques that two-dimensional representations are sufficient to fit voting patterns in existing roll-call data as is the case, for example, in the analysis of the history of U.S. Congressional roll-call votes by Poole and Rosenthal (1997).[10]

The proof of Theorem 5 proceeds by positioning all the voting alternatives in two-dimensional space so that they are all extreme points of the union of the voting alternatives and the given ideal points. The positioning of all the voting alternatives so that they all have the property of being extreme points is possible in two (or more) dimensions, but not in one dimension, which accounts for the fact that Theorem 5 does not cover the one-dimensional case. To the degree that it relies on the positioning of voting alternatives in the set of extreme points, the proof of Theorem 5 bears a connection with a result developed independently by Bogomolnaia and Laslier (2007, Theorem 16) that all individual transitive and complete preferences over a finite set of alternatives, $\tilde{X} \subset \mathbb{R}^2$, can be rationalized by convex preferences in $\mathbb{R}^2$ if and only if $\tilde{X} = \mathcal{E}(\tilde{X})$. Despite this similarity, both the content and the proof of Theorem 5 differ significantly from that of Bogomolnaia and Laslier (2007). Bogomolnaia and Laslier (2007) prove their theorem by constructing the required convex preferences. These constructed preferences are not continuous and cannot be represented by a strictly concave utility function. Theorem 5 is proved by invoking conditions ($\widehat{\mathrm{S}}$) and ($\widehat{\mathrm{W}}$) of Theorems 3 and 6 instead of constructing the required utility function, and yields preferences over $\mathbb{R}^2$ that can be represented by strictly concave utility functions. Finally, Theorem 5 also ensures that the revealed preference relations of all n voters are jointly rationalized with arbitrarily prespecified ideal points, which need not be extreme points of the union of the set of voting alternatives and ideal points.

[10]See Heckman and Snyder (1997) for different conclusions on the dimensionality of the policy space in U.S. Congressional voting.

5. Vote prediction

In this section, I turn to the question of predicting the future voting behavior of an individual voter on the basis of past observations of that individual's voting choices. Theorems 1 and 2 suggest a straightforward strategy for the task. Suppose the following: $X = \mathbb{R}^d$, the voter's preferences are represented by an unobserved (strictly) (quasi)concave utility function $u : \mathbb{R}^d \to \mathbb{R}$, the voting record $V = \{(y_j, z_j)\}_{j\in M}$ is available, each past voting decision indicates strict preference, and the voter is faced with a decision between an alternative $x \in \mathbb{R}^d$, and some alternative $x' \in \mathbb{R}^d$. Then, by Theorem 1, it follows that this voter must weakly prefer x' over x ($u_i(x') \geq u_i(x)$) if x' belongs in the set[11]

$$R(x) = \bigl\{x' \in \mathbb{R}^d \setminus \{x\} \mid \text{the record } V \cup \{(x, x')\} \text{ violates (S)}\bigr\}.$$

In particular, if $u(x) > u(x')$, instead, then the voting record $V \cup \{(x, x')\}$ is strictly rationalized by the voter's utility function u, which is impossible since that voting record violates (S). An identical argument ensures that we must have $u(x) \geq u(x')$ if x' belongs in the set

$$R^{-1}(x) = \bigl\{x' \in \mathbb{R}^d \setminus \{x\} \mid \text{the record } V \cup \{(x', x)\} \text{ violates (S)}\bigr\}.$$

In fact, stronger conclusions obtain by relaxing the assumption that the voter's choices indicate strict preference, while strengthening the assumption on the voter's unobserved utility function. In particular, I now assume that the voter has a strictly (quasi)concave utility function $u : \mathbb{R}^d \to \mathbb{R}$ and that the record $V = \{(y_j, z_j)\}_{j\in M}$ of past votes reveals weak preference with each voting decision. Then, if the voter is faced with a decision between an alternative $x \in \mathbb{R}^d$ and some alternative $x' \in \mathbb{R}^d$, it must be that $u(x') > u(x)$ if x' belongs in the set

$$P(x) = \bigl\{x' \in \mathbb{R}^d \setminus \{x\} \mid \text{the record } V \cup \{(x, x')\} \text{ violates (W)}\bigr\}.$$

The stronger conclusion obtains, because now it suffices to have $u(x) \geq u(x')$ so that u rationalizes the voting record $V \cup \{(x, x')\}$, in contradiction to Theorem 2. Analogously, it must be that $u(x) > u(x')$ if x' belongs in

$$P^{-1}(x) = \bigl\{x' \in \mathbb{R}^d \setminus \{x\} \mid \text{the record } V \cup \{(x', x)\} \text{ violates (W)}\bigr\}.$$

In summary, we have the following corollary of Theorems 1 and 2.

Corollary 1. *Assume $X = \mathbb{R}^d$, let $V = \{(y_j, z_j)\}_{j\in M}$ be a voting record, and consider any $x \in \mathbb{R}^d$.*

(i) If a (strictly) (quasi)concave function $u : \mathbb{R}^d \to \mathbb{R}$ strictly rationalizes V, then $u(x') \geq u(x)$ for all $x' \in R(x)$ and $u(x') \leq u(x)$ for all $x' \in R^{-1}(x)$.

[11] Alternatively, $R(x)$ can be more explicitly defined as $R(x) = \bigcup_{V'\subseteq V}\{x' \in \mathbb{R}^d \setminus \{x\} \mid \mathcal{E}(X(V') \cup \{x, x'\}) \subseteq Y(V') \cup \{x\}\}$.

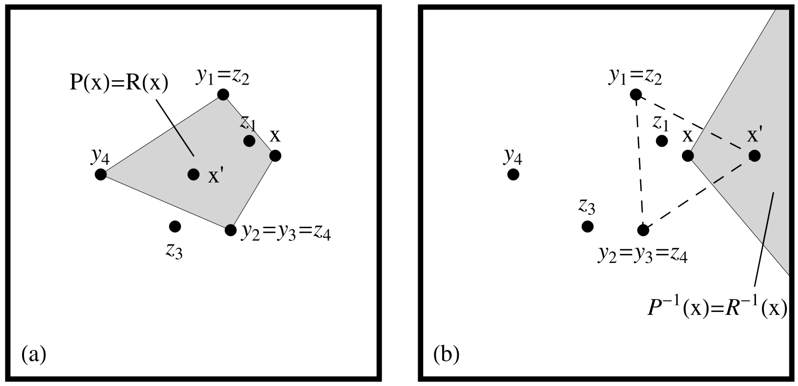

FIGURE 5. Vote prediction. (a) Any quasiconcave (strictly quasiconcave) utility function u that strictly rationalizes (rationalizes) the voting record V of (a) must satisfy $u(x') \geq u(x)$ ($u(x') > u(x)$) for all x' in the gray area; otherwise condition (S) (condition (W)) is violated for $V' = \{(x, x'), (y_1, z_1), (y_2, z_2), (y_4, z_4)\}$. (b) Any quasiconcave (strictly quasiconcave) utility function u that strictly rationalizes (rationalizes) the voting record V of (a) must satisfy $u(x) \geq u(x')$ ($u(x) > u(x')$) for all x' in the gray area; otherwise condition (S) (condition (W)) is violated for $V' = \{(x', x), (y_1, z_1), (y_2, z_2)\}$.

(ii) *If a strictly (quasi)concave function* $u: \mathbb{R}^d \to \mathbb{R}$ *rationalizes* V*, then* $u(x') > u(x)$ *for all* $x' \in P(x)$ *and* $u(x') < u(x)$ *for all* $x' \in P^{-1}(x)$*.*

(iii) *If* V *satisfies (A), then* $R(x) \setminus Y(V) \subseteq P(x) \subseteq R(x)$ *and* $R^{-1}(x) \setminus N(V) \subseteq P^{-1}(x) \subseteq R^{-1}(x)$*.*

Figure 5 depicts the same voting record as the one depicted in Figure 3(a) and displays the set of alternatives $P(x)$ that must be strictly preferred over alternative x by the voter, given the observed voting behavior and assuming that the voter has a strictly (quasi)concave utility function that dictates his/her voting decisions. Similarly, the voter must strictly prefer x over all alternatives in the set $P^{-1}(x)$ of Figure 5(b). Part (iii) of Corollary 1 states that if the voting record satisfies the acyclicity condition (A), then the sets $P(x)$ and $R(x)$ ($P^{-1}(x)$ and $R^{-1}(x)$) differ from each other by at most a finite number of alternatives. Thus, if (A) holds, the set of future voting agendas (x, x') for which we can predict the voter's decisions is almost identical whether we assume that each past vote indicates weak preference (as in part (ii) of Corollary 1) or strict preference (as in part (i) of Corollary 1).

6. General revealed preference data

I have focused the analysis on preference revelation from binary voting choices, but the analysis readily generalizes to arbitrary choice situations over a finite set of alternatives. To be concrete, suppose an individual makes a choice $x_j \in B_j$ in each of m choice situations $j = 1, \ldots, m$, where $B_j \subset \mathbb{R}^d$ is a finite budget set with $|B_j| \geq 2$. Cast in that language, the voting record generated from voting agenda and voting decisions

$\{((p_j, q_j), v_j)\}_{j=1}^m$ can be equivalently represented by $\{(B_j, x_j)\}_{j=1}^m$, where $B_j = \{p_j, q_j\}$ and $x_j = p_j$ if $v_j =$ yes and $x_j = q_j$ if $v_j =$ no. Conversely, the data $\{(B_j, x_j)\}_{j=1}^m$ with budget sets of arbitrary finite cardinality can be summarized in the form of a voting record with $\sum_{j=1}^m (|B_j| - 1)$ voting items, that is, by creating $|B_j| - 1$ voting items (x_j, z), one for each $z \in B_j \setminus \{x_j\}$. In that form, these multiple-choice data yield the relation

$$V = \bigcup_{j=1}^{m} \{(x_j, z) \mid z \in B_j \setminus \{x_j\}\}.$$

Clearly, the rationalizability conditions established in Theorems 1 and 2 are also necessary and sufficient when applied to the record V obtained from multiple-choice data $\{(B_j, x_j)\}_{j=1}^m$ and finite budget sets.

More generally, independent of their origin, the available data may consist of any irreflexive finite revealed preference relation V. It is straightforward to extend the definitions of the various rationalizability criteria: The relation V is strictly rationalized if there exists a utility function $u:\mathbb{R}^d \to \mathbb{R}$ such that $u(y) > u(z)$ for all $(y, z) \in V$, merely rationalized if $u(y) \geq u(z)$ instead, and almost strictly rationalized if it is rationalized and the subrelation V_a defined in (3) is strictly rationalized.

Then the following result generalizes Theorems 1 and 2.

Corollary 2. *Let $K \subset \mathbb{R}^d$ be a finite set, let $P \subset K \times K$ be irreflexive, and let $I \subset K \times K$ be irreflexive and symmetric.*

(i) Let $V = P$. Then there exists a (strictly) (quasi)concave utility function $u:\mathbb{R}^d \to \mathbb{R}$ that strictly rationalizes V if and only if V satisfies (S) (or (S′)).

(ii) Let $V = P$. Then there exists a strictly (quasi)concave utility function $u:\mathbb{R}^d \to \mathbb{R}$ that (almost strictly) rationalizes V if and only if V satisfies (W) (or (W′)).

(iii) Let $V = P \cup I$ and further assume that there does not exist $V' \subseteq V$ such that $N(V') = Y(V')$ and $V' \cap P \neq \varnothing$. Then there exists a strictly (quasi)concave utility function $u:\mathbb{R}^d \to \mathbb{R}$ such that $u(x) > u(x')$ for all $(x, x') \in P$ and $u(x) = u(x')$ for all $(x, x') \in I$ if and only if V satisfies (W) (or (W′)).

Parts (i) and (ii) of Corollary 2 are mere restatements of Theorems 1 and 2, respectively. The only difference is in the interpretation of the source of the data V. Instead of a voting record, V in Corollary 2 is an arbitrary revealed preference relation that may arise from, for example, multiple-choice data $\{(B_j, x_j)\}_{j=1}^m$. Part (iii) of the theorem demonstrates the reach of the analysis outside a revealed preference context. The data for part (iii) consist of an "indifference" relation I and a "strict preference" relation P. These relations may comprise only a fraction of the strict preference and indifference pairs of the individual's actual preferences within the set K. Part (iii) states that if P is acyclic and $V = P \cup I$ reveals no cycle of the form $x\, I\, x' \cdots y\, P\, z\, I\, x$, then condition (W) is necessary and sufficient for the existence of a strictly (quasi)concave utility function that represents all the indifferences in I and all the strict preferences in P. This essentially follows from part (ii) of the corollary. In particular, the symmetry of I ensures that $u(x) = u(x')$ for all $(x, x') \in I$ when u rationalizes V. Furthermore, $V_a = P \subseteq V$ (where V_a is defined in (3)) since there does not exist $V' \subseteq V$ such that $N(V') = Y(V')$ and $V' \cap P \neq \varnothing$, that is, P is the component of V that is strictly rationalized by the utility function u that almost strictly rationalizes $V = P \cup I$.

7. Conclusions

I have derived necessary and sufficient conditions for observed binary voting choices to be consistent with the hypothesis that the voter has preferences that admit (quasi)concave utility representations. The derived conditions are computationally tractable and can be verified by solving a finite sequence of linear programming problems. The analysis demonstrates that the hypothesis that the voter has preferences represented by a (strictly) concave utility function and the hypothesis that the voter has preferences represented by a (strictly) quasiconcave utility function are observationally equivalent on the basis of finite data. Furthermore, the conditions that ensure the existence of a rationalizing (quasi)concave utility function consistent with observed voting behavior imply simple testable restrictions on the location of the voter's ideal point, and can be used to predict future voting decisions. If the location of voting alternatives is unknown and unrestricted (as is assumed in prevalent political methodology techniques for the estimation of legislators' ideal points), then the derived conditions are vacuously satisfied for arbitrary ideal points and arbitrary voting decisions by a group of voters, even if the voting alternatives are restricted to lie in one dimension. The same is true in two or more dimensions if the location of the voting alternatives is unknown but these alternatives satisfy known equality restrictions across voting items. The analysis is readily applicable to the nonparametric study of general deterministic choice situations over general finite budget sets with only convexity restrictions on individual preferences.

Appendix A: Verification of conditions (W) and (W′)

The following algorithm can be used to verify the validity of condition (W) (or (W′)) of Theorem 2.

Algorithm 2. *Assume $X = \mathbb{R}^d$ and let $V = \{(y_j, z_j)\}_{j=1}^m$ be a voting record.*

Step 1. *Set $V_1 = V$ and proceed to Step 2 with $t = 1$.*

Step 2. *For each $x \in N(V_t) \setminus Y(V_t)$, test whether $x \in \mathcal{E}(X(V_t))$. If $x \in \mathcal{E}(X(V_t)) \cap (N(V_t) \setminus Y(V_t))$ is found, set $X_t = \{x\}$ and proceed to Step 4. If $\mathcal{E}(X(V_t)) \cap (N(V_t) \setminus Y(V_t)) = \varnothing$, proceed to Step 3.*

Step 3. *Enumerate all $x \in \mathcal{E}(Y(V_t))$. Set $V' = \{(y, z) \mid y \in \mathcal{E}(Y(V_t))\}$ and, using the function $T : 2^{V'} \to 2^{V'}$ defined as $T(A) = \{(y, z) \in A : y \in N(A)\}$, construct the sequence $V_0 = V'$, $V_1 = T(V_0)$, $V_{t'+1} = T(V_{t'})$ until $T(V_{t'}) = V_{t'} = V''$ for some t'.*[12]

[12] See footnote 13 for a justification of this step.

If $V'' \neq \varnothing$, set $X_t = N(V'')$ and proceed to Step 4. If $V'' = \varnothing$, V does not satisfy (W).

Step 4. Set $V_{t+1} = \{(y,z) \in V_t \mid z \notin X_t\}$ and proceed to Step 5.

Step 5. If $V_{t+1} = \varnothing$, V satisfies (W); else, proceed to Step 2 with $t = t+1$.

The most challenging computation in Algorithm 2 is the enumeration of all extreme points of $\mathcal{E}(Y(V_t))$ in Step 3—a task that can be executed by solving at most $|V_t|$ linear programs.

Appendix B: Ideal points and rationalizability

In the following theorem, I derive testable restrictions on the location of voter ideal points when votes may indicate weak preference.

Theorem 6. *Assume $X = \mathbb{R}^d$, $\hat{x} \in \mathbb{R}^d$ and let $V = \{(y_j, z_j)\}_{j=1}^m$ be a voting record. Then the following conditions are equivalent.*

($\widehat{W}$) *For all nonempty $V' \subseteq V$, either there exists $x \in \mathcal{E}(X(V') \cup \{\hat{x}\})$ such that $x \notin Y(V') \cup \{\hat{x}\}$ or there exists nonempty $V'' \subseteq V'$ such that $N(V'') = Y(V'') \subseteq \mathcal{E}(X(V') \cup \{\hat{x}\})$ and $Y(V'') \cap Y(V' \setminus V'') = \varnothing$.*

($\widehat{W}'$) *The $\hat{x}$-augmented voting record $\hat{V}$ satisfies (W) or (W′), and $\hat{x} \notin N(V)$.*

($\widehat{W}_c$) *There exists a strictly concave utility function that (almost strictly) rationalizes V with ideal point $\hat{x}$.*

($\widehat{W}_q$) *There exists a strictly quasiconcave utility function that (almost strictly) rationalizes V with ideal point $\hat{x}$.*

If $d = 1$, then ($\widehat{W}$) is equivalent to the following condition.

($\widehat{W}_1$) *For all $V' \subseteq V$, $1 \leq |V'| \leq 2$, there exists $x \in \mathcal{E}(X(V') \cup \{\hat{x}\})$ such that $x \notin Y(V') \cup \{\hat{x}\}$ or $N(V') = Y(V') \subseteq \mathcal{E}(Y(V') \cup \{\hat{x}\})$.*

Proof. First note that ($\widehat{\mathrm{W}}_c$)⇒($\widehat{\mathrm{W}}_q$)⇒[$\hat{V}$ satisfies (W_q) and $\hat{x} \notin N(V)$]⇒($\widehat{\mathrm{W}}'$), the latter implication by Theorem 2. Next, I show that ($\widehat{\mathrm{W}}'$)⇒($\widehat{\mathrm{W}}_c$). Note that by Theorem 2, if the $\hat{x}$-augmented voting record $\hat{V}$ satisfies (W), then there exists a strictly concave utility function that (almost strictly) rationalizes $\hat{V}$. Since $\hat{x} \notin N(V)$, $\hat{x} \notin N(\hat{V})$ and, by Lemma 1, there exists a strictly concave utility function that (almost strictly) rationalizes $\hat{V}$ with ideal point $\hat{x}$. As a result, Theorem 2 and Lemma 1 jointly imply the equivalence ($\widehat{\mathrm{W}}'$)⇔($\widehat{\mathrm{W}}_c$)⇔($\widehat{\mathrm{W}}_q$). To establish that ($\widehat{\mathrm{W}}'$)⇔($\widehat{\mathrm{W}}$), I first show ($\widehat{\mathrm{W}}'$)⇒($\widehat{\mathrm{W}}$). Consider any nonempty $V' \subseteq V$ and define $\hat{V}' = V' \cup \{(\hat{x}, z) \mid z \in X(V')\} \subseteq \hat{V}$. By ($\widehat{\mathrm{W}}'$), $\hat{V}$ satisfies (W), so either there exists $x \in \mathcal{E}(X(\hat{V}')) = \mathcal{E}(X(V') \cup \{\hat{x}\})$ such that $x \notin Y(\hat{V}') = Y(V') \cup \{\hat{x}\}$, or there exists a nonempty $\hat{V}'' \subseteq \hat{V}'$ such that $N(\hat{V}'') = Y(\hat{V}'') \subseteq \mathcal{E}(X(\hat{V}')) = \mathcal{E}(X(V') \cup \{\hat{x}\})$ and $Y(\hat{V}'') \cap Y(\hat{V}' \setminus \hat{V}'') = \varnothing = Y(\hat{V}'') \cap Y(V' \setminus \hat{V}'')$. By ($\widehat{\mathrm{W}}'$), $\hat{x} \notin N(V)$ so that $\hat{x} \notin N(\hat{V}'') = Y(\hat{V}'')$. It follows that $\hat{V}'' \subseteq V'$, thus establishing that ($\widehat{\mathrm{W}}'$)⇒($\widehat{\mathrm{W}}$). It remains to show

($\widehat{\mathrm{W}}$)$\Rightarrow$($\widehat{\mathrm{W}}'$), i.e., to show that the $\hat{x}$-augmented voting record $\hat{V}$ satisfies condition (W) and that $\hat{x} \notin N(V)$ when ($\widehat{\mathrm{W}}$) holds. To show that $\hat{x} \notin N(V)$, assume $\hat{x} \in N(V)$ to get a contradiction. Then there exists $(y, \hat{x}) \in V$ which contradicts the assumption that ($\widehat{\mathrm{W}}$) holds, since $Y(V') \cup \{\hat{x}\} = X(V')$ and $N(V') \neq Y(V')$ for $V' = \{(y, \hat{x})\}$. Thus, $\hat{x} \notin N(V)$ when ($\widehat{\mathrm{W}}$) holds and I need to show that $\hat{V}$ satisfies condition (W). Consider any $V' \subseteq \hat{V}$ and, assuming ($\widehat{\mathrm{W}}$) holds, distinguish three possibilities.

Case 1: $V' \subseteq \hat{V} \setminus V$. Then $Y(V') = \{\hat{x}\}$ and, since $\hat{x} \notin N(V')$, there exists $x \in \mathcal{E}(X(V'))$ such that $x \notin Y(V')$.

Case 2: $V' \cap V \neq \varnothing$ and there exists $x \in \mathcal{E}(X(V' \cap V) \cup \{\hat{x}\})$ such that $x \notin Y(V' \cap V) \cup \{\hat{x}\}$. It follows that $x \in X(V' \cap V) \setminus \mathcal{C}(Y(V' \cap V) \cup \{\hat{x}\})$. Furthermore, $X(V') \supseteq X(V' \cap V)$ and $Y(V') \subseteq Y(V' \cap V) \cup \{\hat{x}\}$, since $Y(\{(y,z)\}) = \{\hat{x}\}$ for all $(y,z) \in V' \setminus V$. Thus, $x \in X(V') \setminus \mathcal{C}(Y(V'))$, hence there exists $x' \in \mathcal{E}(X(V'))$ such that $x' \notin Y(V')$.

Case 3: $V' \cap V \neq \varnothing$, and there exists nonempty $V'' \subseteq V' \cap V$ such that $Y(V'') = N(V'') \subseteq \mathcal{E}(X(V' \cap V) \cup \{\hat{x}\})$ and $Y(V'') \cap Y((V' \cap V) \setminus V'') = \varnothing$. I distinguish two subcases.

Subcase 3.1: $N(V' \setminus V) \setminus \mathcal{C}(X(V' \cap V) \cup \{\hat{x}\}) \neq \varnothing$. Since $N(V' \setminus V) \setminus \mathcal{C}(X(V' \cap V) \cup \{\hat{x}\}) \neq \varnothing$, then $V' \setminus V \neq \varnothing$ so that $Y(V' \setminus V) = \{\hat{x}\}$, and there exists $x \in \mathcal{E}(X(V')) \cap N(V' \setminus V)$ such that $x \notin \mathcal{C}(X(V' \cap V) \cup \{\hat{x}\})$. Hence, since $Y(V') = Y(V' \cap V) \cup Y(V' \setminus V) = Y(V' \cap V) \cup \{\hat{x}\} \subseteq \mathcal{C}(X(V' \cap V) \cup \{\hat{x}\})$, it follows that there exists $x \in \mathcal{E}(X(V'))$ such that $x \notin Y(V')$.

Subcase 3.2: $N(V' \setminus V) \subset \mathcal{C}(X(V' \cap V) \cup \{\hat{x}\})$. Consider nonempty $V'' \subseteq V' \cap V$ such that $Y(V'') = N(V'') \subseteq \mathcal{E}(X(V' \cap V) \cup \{\hat{x}\})$, and $Y(V'') \cap Y((V' \cap V) \setminus V'') = \varnothing$ assumed to exist in Case 3. Recall that condition ($\widehat{\mathrm{W}}$) implies $\hat{x} \notin N(V'')$. Thus, since $X(V') \cup \{\hat{x}\} = X(V' \cap V) \cup (N(V' \setminus V) \cup \{\hat{x}\})$ and $N(V' \setminus V) \subset \mathcal{C}(X(V' \cap V) \cup \{\hat{x}\})$, it follows that $Y(V'') = N(V'') \subseteq \mathcal{E}(X(V' \cap V) \cup \{\hat{x}\}) \setminus \{\hat{x}\} = \mathcal{E}(X(V') \cup \{\hat{x}\}) \setminus \{\hat{x}\} \subseteq \mathcal{E}(X(V'))$. Furthermore, $Y(V' \setminus V'') \subseteq Y((V' \cap V) \setminus V'') \cup \{\hat{x}\}$ since $Y(V' \setminus V) \subseteq \{\hat{x}\}$. Because $Y(V'') \cap Y((V' \cap V) \setminus V'') = \varnothing$ and $\hat{x} \notin Y(V'')$, it follows that $Y(V'') \cap Y(V' \setminus V'') = \varnothing$.

In sum, in all three cases, either there exists $x \in \mathcal{E}(X(V'))$ such that $x \notin Y(V')$ or there exists nonempty $V'' \subseteq V'$ such that $Y(V'') = N(V'') \subseteq \mathcal{E}(X(V'))$ and $Y(V'') \cap Y(V' \setminus V'') = \varnothing$, completing the proof of ($\widehat{\mathrm{W}}$)$\Rightarrow$($\widehat{\mathrm{W}}'$), so that ($\widehat{\mathrm{W}}$)$\Leftrightarrow$($\widehat{\mathrm{W}}'$).

Also, from the same arguments, it follows that ($\widehat{\mathrm{W}}_1$)$\Rightarrow$[$\hat{V}$ satisfies (W_1) and $\hat{x} \notin N(V)$]. If $d = 1$, (W_1)$\Leftrightarrow$(W) by Theorem 2; hence ($\widehat{\mathrm{W}}_1$)$\Rightarrow$($\widehat{\mathrm{W}}'$)$\Leftrightarrow$($\widehat{\mathrm{W}}$), thus establishing ($\widehat{\mathrm{W}}_1$)$\Leftrightarrow$($\widehat{\mathrm{W}}$) when $d = 1$. □

Appendix C: Proofs

In this appendix, I prove the results stated in the main body of the paper. I start with four lemmas (including Lemma 1) and then prove the remaining results in the order in which they appear in the text.

Lemma 1 (Restated). *Assume $X = \mathbb{R}^d$, let V be a finite voting record, and assume that there exists a strictly concave utility function $u : \mathbb{R}^d \to \mathbb{R}$ that (strictly) rationalizes V. If there exists alternative $\hat{x} \in \mathbb{R}^d \setminus N(V)$ such that $u(\hat{x}) \geq u(x)$ for all $x \in X(V)$, then there exists another strictly concave function $\tilde{u} : \mathbb{R}^d \to \mathbb{R}$ that (strictly) rationalizes V with ideal point $\hat{x}$.*

Proof. I start by constructing a system of equalities and inequalities with unknowns $u^x \in \mathbb{R}$ and $d^x \in \mathbb{R}^d$, one for each $x \in X(V) \cup \{\hat{x}\}$:

$$u^y - u^z > 0 \text{ for all } y, z \in X(V) \cup \{\hat{x}\} \text{ such that } u(y) > u(z)$$
$$u^y - u^z = 0 \text{ for all } y, z \in X(V) \cup \{\hat{x}\} \text{ such that } u(y) = u(z)$$
$$u^y - u^z - (d^y)^T(y - z) > 0 \text{ for all distinct } y, z \in X(V) \cup \{\hat{x}\}.$$

Observe that since u is strictly concave, the above system has a solution (by setting $u^x = u(x)$ and d^x equal to a supergradient of u at x for all $x \in X(V) \cup \{\hat{x}\}$). By setting $d^{\hat{x}} = 0$, $u^{\hat{x}} = u(\hat{x}) + \eta$ for small enough $\eta > 0$, and maintaining the remaining values of the original solution, we obtain a solution to the modified system:

$$u^{\hat{x}} - u^x > 0 \text{ for all } x \in X(V) \setminus \{\hat{x}\}$$
$$u^y - u^z > 0 \text{ for all } y, z \in X(V) \setminus \{\hat{x}\} \text{ such that } u(y) > u(z)$$
$$u^y - u^z = 0 \text{ for all } y, z \in X(V) \setminus \{\hat{x}\} \text{ such that } u(y) = u(z)$$
$$u^y - u^z - (d^y)^T(y - z) > 0 \text{ for all distinct } y, z \in X(V) \cup \{\hat{x}\}.$$

This solution to the latter system produces a strictly concave utility function $\tilde{u}: \mathbb{R}^d \to \mathbb{R}$ (as in Matzkin and Richter 1991 or Richter and Wong 2004) defined as

$$\tilde{u}(x) = \min_{z \in X(V) \cup \{\hat{x}\}} \{u^z + (d^z)^T(x - z) - \varepsilon(x - z)^T(x - z)\}.$$

For small enough $\varepsilon > 0$, this new utility function $\tilde{u}$ (strictly) rationalizes the voting record V with ideal point $\hat{x}$. □

The first of the next three lemmas concerns well known properties of the minimizers of quasiconcave functions on a polytope and is stated without proof.

Lemma 2. *Consider a finite set $K \subset \mathbb{R}^d$ and a quasiconcave function $u: \mathbb{R}^d \to \mathbb{R}$.*

(i) There exists $x \in \mathcal{E}(K)$ that minimizes u over $\mathcal{C}(K)$.

(ii) If u is strictly quasiconcave, then $\arg\min\{u(x) \mid x \in \mathcal{C}(K)\} \subseteq \mathcal{E}(K)$.

Lemma 3. *Consider disjoint finite sets $K, K' \subset \mathbb{R}^d$ (K possibly empty) such that $K' \subseteq \mathcal{E}(K \cup K')$ and consider a strictly concave function $u: \mathbb{R}^d \to \mathbb{R}$. Then there exists another strictly concave $u': \mathbb{R}^d \to \mathbb{R}$ such that $u'(x) \geq u'(x')$ if and only if $u(x) \geq u(x')$ for all $x, x' \in K$, $u'(x) = u'(x')$ for all $x, x' \in K'$ and such that $u'(x) > u'(x')$ for all $x \in K$, $x' \in K'$.*

Proof. Given u, define a reflexive, complete, and transitive preference relation $\succeq$ on $K \cup K' \times K \cup K'$ as follows: For all $x, x' \in K$, let $x \succeq x'$ if and only if $u(x) \geq u(x')$, let $x \succeq x'$ and $x' \succeq x$ for all $x, x' \in K'$, and let $x \succeq x'$ for all $x \in K$, $x' \in K'$. By definition, the function u represents the restriction of $\succeq$ on K. Thus, since u is strictly concave, the restriction of $\succeq$ on K satisfies condition (G′) of Richter and Wong (2004). Consider any $X \subseteq K \cup K'$ such that $|X| \leq d+1$ and $X \cap K' \neq \varnothing$. For every $x \in \mathcal{C}(X) \setminus X$, we have $x \notin K'$ since $K' \subseteq \mathcal{E}(K \cup K')$. Furthermore, there exists $x' \in X \cap K'$, since $X \cap K' \neq \varnothing$, and x' satisfies $x \succeq x'$, $x' \not\succeq x$. Thus, $\succeq$ also satisfies (G′) on $K \cup K'$, ensuring the existence of the required function u' by Theorem 2 of Richter and Wong (2004). □

Lemma 4. *Assume $X = \mathbb{R}^d$ and let V' be a nonempty finite voting record such that $N(V') \subseteq Y(V')$. Then there exists nonempty $V'' \subseteq V'$ such that $N(V'') = Y(V'')$ and $Y(V' \setminus V'') \cap Y(V'') = \varnothing$.*

Proof. Define the function $T : 2^{V'} \to 2^{V'}$ as $T(A) = \{(y,z) \in A \mid y \in N(A)\}$. Consider the sequence $V_0 = V'$, $V_1 = T(V_0)$, $V_{t+1} = T(V_t)$. By the definition of T, $Y(V_t) \supseteq N(V_t) = Y(V_{t+1}) \supseteq N(V_{t+1}) \neq \varnothing$. Since $2^{V'}$ is finite, the sequence converges to a fixed point[13] $V'' = T(V'') \neq \varnothing$ that also satisfies $N(V'') = Y(V'')$. To show $Y(V'') \cap Y(V' \setminus V'') = \varnothing$, note that otherwise there exists $(y,z) \in V' \setminus V''$ such that $y \in Y(V'') = N(V'')$ and, as a consequence, $(y,z) \in T(V_t)$ for all t; hence, $(y,z) \in V''$, which is absurd. □

To avoid duplication of arguments, the proof of Theorem 1 relies on Theorem 2.

Theorem 1 (Restated). *Assume $X = \mathbb{R}^d$ and let $V = \{(y_j, z_j)\}_{j=1}^m$ be a voting record. Then (S)⇔(S′)⇔(S$_c$)⇔(S$_q$)⇔(S$'_c$)⇔(S$'_q$). Furthermore, if $d = 1$, then (S)⇔(S$_1$).*

Proof. Note that (S$_c$)⇒(S$'_c$)⇒(S$'_q$) and (S$_c$)⇒(S$_q$)⇒(S$'_q$). Thus, it suffices to show (S$'_q$)⇒(S), (S)⇒(S′), and (S′)⇒(S$_c$).

(S$'_q$)⇒(S): Let quasiconcave $u : \mathbb{R}^d \to \mathbb{R}$ strictly rationalize V. Fix any nonempty $V' \subseteq V$. There exists an alternative $x \in \mathcal{E}(X(V'))$ such that $u(x) \leq u(y)$ for all $y \in \mathcal{C}(X(V'))$ by Lemma 2, part (i). If $x \in Y(V')$, then, since u strictly rationalizes V, there exists $x' \in X(V')$ such that $u(x) > u(x')$, a contradiction. Thus, $x \notin Y(V')$. Proving (S$'_q$)⇒(S).

(S)⇒(S′): Since (S)⇒(W), then (S)⇒(W′) by Theorem 2. Furthermore (S)⇒(A), so that (S)⇒[(W′) and (A)]⇒(S′).

(S′)⇒(S$_c$): I will first show that (S′)⇒(A). To get a contradiction, suppose not, i.e., suppose (S′) holds and there exists nonempty $V' \subseteq V$ such that $N(V') = Y(V')$. Let $\{x_t\}_{t=1}^k$ be the sequence identified by condition (S′) and set $t' = \min\{t \mid x_t \in N(V')\}$ so that $V' \subseteq V_{t'}$. Thus, $x_{t'} \in N(V')$ but $x_{t'} \notin Y(V_{t'}) \supseteq Y(V')$, contradicting $N(V') = Y(V')$. Hence, (S′)⇒[(A) and (W′)]⇒[(A) and (W$'_c$)] by Theorem 2. But [(A) and (W$'_c$)]⇒(S$_c$) since we have $V = V_a$ defined in (3), when (A) is true.

For the last part of the theorem, since (S)⇒(S$_1$), it remains to show one last relationship.

[$d = 1$ and (S$_1$)]⇒(S): Assume $d = 1$ and (S$_1$) holds, and suppose (S) fails, so as to get a contradiction. Then there exists $V' \subseteq V$ with $|V'| > 2$ for which $\mathcal{E}(X(V')) \subseteq Y(V')$. Since

[13]Indeed, V'' is the greatest fixed point of T, which exists by the Knaster–Tarski Fixed Point Theorem since the mapping T is monotone ($T(A) \subseteq T(A')$ for any A, A' such that $A \subseteq A'$) and $2^{V'}$ is a complete lattice ordered by set inclusion.

$d=1$ and $|X(V')| \geq 2$, $\mathcal{E}(X(V')) = \{x, x'\}$ for some distinct $x, x' \in Y(V')$. Let $j, h \in M$ be such that $x \in Y(V_{\{j\}})$ and $x' \in Y(V_{\{h\}})$. Then $\mathcal{E}(X(V_{\{j,h\}})) = Y(V_{\{j,h\}})$, which contradicts the assumption that condition (S_1) holds. □

Theorem 2 (Restated). *Assume $X = \mathbb{R}^d$ and let $V = \{(y_j, z_j)\}_{j=1}^m$ be a voting record. Then (W)⇔(W′)⇔(W'_c)⇔(W'_q)⇔(W_c)⇔(W_q). Furthermore, if $d = 1$, then (W)⇔(W_1).*

Proof. Since (W'_c)⇒(W_c)⇒(W_q) and (W'_c)⇒(W'_q)⇒(W_q), it suffices to show (W_q)⇒(W), (W)⇒(W′), and (W′)⇒(W'_c).

(W_q)⇒(W): Let u be a strictly quasiconcave function that rationalizes V. Consider any $V' \subseteq V$. If there does not exist $x \in \mathcal{E}(X(V'))$ such that $x \notin Y(V')$, then $\mathcal{E}(X(V')) \subseteq Y(V')$. Furthermore, by part (ii) of Lemma 2 it follows that $\arg\min\{u(x) \mid x \in \mathcal{C}(X(V'))\} \subseteq \mathcal{E}(X(V')) \subseteq Y(V')$. I now show that there exists nonempty $V'' \subseteq V'$ such that $Y(V'') = N(V'') \subseteq \mathcal{E}(X(V'))$ and $Y(V'') \cap Y(V' \setminus V'') = \varnothing$. Set $K = \arg\min\{u(x) \mid x \in \mathcal{C}(X(V'))\}$ and define

$$V_m = \{(y, z) \in V' \mid y \in K\}.$$

Since $K \subseteq Y(V')$, it must be that $Y(V_m) = K$ and V_m is nonempty. I also claim that $N(V_m) \subseteq Y(V_m)$. If not, then there exists $x \in N(V_m)$ such that $x \in X(V') \setminus K$. But then $u(x) > u(y)$ for all $y \in Y(V_m) = K$, contradicting the assumption that u rationalizes V. Hence it must indeed be that $N(V_m) \subseteq Y(V_m) = K \subseteq \mathcal{E}(X(V'))$. It follows by Lemma 4 that there exists nonempty $V'' \subseteq V_m$ such that $N(V'') = Y(V'') \subseteq \mathcal{E}(X(V))$ and $Y(V'') \cap Y(V_m \setminus V'') = \varnothing$. It remains to show that $Y(V'') \cap Y(V' \setminus V'') = \varnothing$. If not, there exists $(y, z) \in V' \setminus V_m$ such that $y \in Y(V'') = N(V'') \subseteq K$. But then $y \notin K$ by the definition of V_m, a contradiction proving that $Y(V'') \cap Y(V' \setminus V'') = \varnothing$.

(W)⇒(W′): To construct the required sequence, I proceed inductively starting with $V_1 = V$. Consider the tth step with V_t already specified. Note that by (W) either there exists $x_t \in \mathcal{E}(X(V_t))$ such that $x_t \notin Y(V_t)$ or there exists $V'' \subseteq V_t$ such that $N(V'') = Y(V'') \subseteq \mathcal{E}(X(V_t))$ and $Y(V'') \cap Y(V_t \setminus V'') = \varnothing$. Accordingly, I distinguish two cases:

Case 1: *There exists $x_t \in \mathcal{E}(X(V_t))$ such that $x_t \notin Y(V_t)$.* Let $X_t = \{x_t\}$ and $V_{t+1} = \{(y, z) \in V_t \mid z \notin X_t\}$. Clearly $X_t \subseteq \mathcal{E}(X(V_t))$ and $x_t \notin Y(V_t)$ as required by (W′).

Case 2: *There exists $V'' \subseteq V_t$ such that $N(V'') = Y(V'') \subseteq \mathcal{E}(X(V_t))$ and $Y(V'') \cap Y(V_t \setminus V'') = \varnothing$.* Let $X_t = N(V'')$, $V'_t = V''$, and $V_{t+1} = \{(y, z) \in V_t \mid z \notin X_t\}$. Note that $X_t \subseteq \mathcal{E}(X(V_t))$. Since $V'_t = V''$, $X_t = N(V'_t) = Y(V'_t)$. Furthermore, by the definition of V_{t+1}, $V_t \setminus V_{t+1} = \{(y, z) \in V_t \mid z \in X_t\} \supseteq V'' = V'_t$ and it follows that $X_t = N(V_t \setminus V_{t+1})$. Finally, $X_t = Y(V'')$ and $V_{t+1} \subseteq V_t \setminus V''$, so that $Y(V_{t+1}) \subseteq Y(V_t \setminus V'')$ and it follows that $X_t \cap Y(V_{t+1}) = \varnothing$ because $Y(V'') \cap Y(V_t \setminus V'') = \varnothing$.

Since V is finite, we obtain the sequences required by (W′) at the $k+1$th step, $k \leq |N(V)|$, with $V_{t+1} = \varnothing$.

(W′)⇒(W'_c): The proof proceeds by induction, first establishing the existence of the required function for the record V_k. Consider any strictly concave function $u : \mathbb{R}^d \to \mathbb{R}$. By (W′), $N(V_k) \subseteq \mathcal{E}(X(V_k))$, so now Lemma 3 (applied on $K = X(V_k) \setminus N(V_k)$, $K' = N(V_k)$) ensures the existence of a strictly concave u^k that strictly rationalizes the voting record V_k if $|N(V_k)| = 1$ and almost strictly rationalizes this record if $|N(V_k)| > 1$, since in that case $K = Y(V_k) \setminus Y(V'_k)$, and $K' = N(V_k) = N(V'_k) = Y(V_k)$ for some $V'_k \subseteq V_k$ by condition (W′). But now, suppose there exists a strictly concave function $u^t : \mathbb{R}^d \to \mathbb{R}$ that almost strictly rationalizes the record V_t, $t > 1$, $t \leq k$. I wish to show that there also exists such a function u^{t-1} that almost strictly rationalizes the record V_{t-1}. By (W′) there exists $X_{t-1} = N(V_{t-1} \setminus V_t) \subseteq \mathcal{E}(X(V_{t-1}))$ and $N(V_{t-1} \setminus V_t) \cap X(V_t) = \varnothing$. Set $K = X(V_{t-1}) \setminus X_{t-1}$ and $K' = X_{t-1}$. By Lemma 3 there exists a strictly concave $u^{t-1} : \mathbb{R}^d \to \mathbb{R}$ that satisfies

$$u^{t-1}(x) > u^{t-1}(x') \Leftrightarrow u^t(x) > u^t(x') \text{ for all } x, x' \in K$$

$$u^{t-1}(x) > u^{t-1}(x') \text{ for all } x \in K, x' \in K'$$

$$u^{t-1}(x) = u^{t-1}(x') \text{ for all } x, x' \in K'.$$

Thus, u^{t-1} almost strictly rationalizes the voting record V_{t-1}; in particular, concerning the additional voting items in $V_{t-1} \setminus V_t$, $u^{t-1}(x) > u^{t-1}(x')$ for all $x \in Y(V_{t-1} \setminus V_t) \setminus X_{t-1} \subseteq K$ and all $x' \in N(V_{t-1} \setminus V_t) = X_{t-1} = K'$, and $u^{t-1}(x) = u^{t-1}(x')$ for all $x, x' \in X_{t-1} = K'$, where $K' = N(V_{t-1} \setminus V_t) = N(V'_{t-1}) = Y(V'_{t-1}) \subseteq Y(V_{t-1} \setminus V_t)$ for some $V'_{t-1} \subseteq V_{t-1} \setminus V_t$ when $|X_{t-1}| > 1$.

To complete the proof, I must show [$d = 1$ and (W$_1$)]⇒(W), as it is clearly the case that (W)⇒(W$_1$): If (W) holds and $\mathcal{E}(X(V')) \subseteq Y(V')$ for some $|V'| = 2$, then $N(V') = Y(V')$, since $y \neq z$ for all $(y, z) \in V$. Thus, to show that (W$_1$)⇒(W) when $d = 1$, consider an arbitrary $V' \subseteq V$ with $|V'| > 2$. If there does not exist $x \in \mathcal{E}(X(V'))$ such that $x \notin Y(V')$, then $\mathcal{E}(X(V')) \subseteq Y(V')$ and it must be shown that there exists nonempty $V'' \subseteq V'$ such that $N(V'') = Y(V'') \subseteq \mathcal{E}(X(V''))$ and $Y(V'') \cap Y(V' \setminus V'') = \varnothing$. Since $\mathcal{E}(X(V')) \subseteq Y(V')$, $\mathcal{E}(X(V')) = \{x_L, x_R\}$ for some $x_L, x_R \in Y(V')$. Let $V'' \subseteq V'$ be the largest subset of V' such that $Y(V'') = \{x_L, x_R\}$, so that $Y(V'') \cap Y(V' \setminus V'') = \varnothing$. It suffices to show that $N(V'') = Y(V'') = \{x_L, x_R\}$. First, it cannot be that $N(V'') = \{x\} \subseteq \{x_L, x_R\}$ because in that case, $y = z$ for some $(y, z) \in V''$, which is absurd. Thus, the proof is complete if $N(V'') \subseteq \{x_L, x_R\}$ because then it must be that $N(V'') = Y(V'') = \{x_L, x_R\}$. Suppose instead that there exists $x \in N(V'')$ such that $x \notin \{x_L, x_R\}$ so as to get a contradiction. Without loss of generality, let $(x_L, x) \in V''$ and select any $(x_R, z) \in V''$. Then we have both $\mathcal{E}(X(\{(x_L, x), (x_R, z)\})) \subseteq Y(\{(x_L, x), (x_R, z)\})$ and $Y(\{(x_L, x), (x_R, z)\}) \neq N(\{(x_L, x), (x_R, z)\})$, which is absurd because (W$_1$) holds. Thus, $N(V'') = Y(V'') = \{x_L, x_R\}$ and the proof is complete. □

Theorem 3 is proved using Theorem 6.

Theorem 3 (Restated). *Assume $X = \mathbb{R}^d$ and $\hat{x} \in \mathbb{R}^d$, and let $V = \{(y_j, z_j)\}_{j \in M}$ be a voting record. Then $(\widehat{\text{S}}) \Leftrightarrow (\widehat{\text{S}'}) \Leftrightarrow (\widehat{\text{S}}_c) \Leftrightarrow (\widehat{\text{S}}_q) \Leftrightarrow (\widehat{\text{S}'_c}) \Leftrightarrow (\widehat{\text{S}'_q})$. Furthermore, if $d = 1$, then $(\widehat{\text{S}}) \Leftrightarrow (\text{S}_1)$.*

Proof. First $(\widehat{\text{S}'_q}) \Rightarrow [\hat{V}$ satisfies $(\text{S}'_q)] \Leftrightarrow (\widehat{\text{S}'})$, the latter equivalence by Theorem 1. Furthermore, $(\widehat{\text{S}'})$ implies that $\hat{x} \notin N(V)$. Indeed, if $\hat{x} \in N(V)$, then $(y, \hat{x}) \in V \subset \hat{V}$ and $(\hat{x}, y) \in \hat{V}$, and $\hat{V}$ violates (S) since it violates (A) which is absurd. Thus, $\hat{x} \notin N(V)$ and, by Lemma 1, $(\widehat{\text{S}'}) \Rightarrow (\widehat{\text{S}}_c)$. Since $(\widehat{\text{S}}_c) \Rightarrow (\widehat{\text{S}}_q) \Rightarrow (\widehat{\text{S}'_q})$ and $(\widehat{\text{S}}_c) \Rightarrow (\widehat{\text{S}'_c}) \Rightarrow (\widehat{\text{S}'_q})$, it follows that $(\widehat{\text{S}'}) \Leftrightarrow (\widehat{\text{S}}_c) \Leftrightarrow (\widehat{\text{S}}_q) \Leftrightarrow (\widehat{\text{S}'_c}) \Leftrightarrow (\widehat{\text{S}'_q})$. Further note that $(\widehat{\text{S}}) \Rightarrow (\widehat{\text{S}'})$ follows from Cases 1 and 2 in the proof of Theorem 6. To also show that $(\widehat{\text{S}}')\Rightarrow(\widehat{\text{S}})$, consider any $V' \subseteq V$ and define $\hat{V}' = V' \cup \{(\hat{x}, z) \mid z \in X(V')\}$. By $(\widehat{\text{S}}')$ there exists $x \in \mathcal{E}(X(\hat{V}')) = \mathcal{E}(X(V') \cup \{\hat{x}\})$ such that $x \notin Y(\hat{V}') = Y(V') \cup \{\hat{x}\}$. Thus, $(\widehat{\text{S}}')\Leftrightarrow(\widehat{\text{S}})$, proving the first part of the theorem.

From the arguments in Cases 1 and 2 in the proof of Theorem 6 it follows that $(\widehat{\text{S}}_1)\Rightarrow[\hat{V}$ satisfies $(\text{S}_1)]$. If $d = 1$, $(\text{S}_1)\Leftrightarrow(\text{S})$ by Theorem 1; hence $(\widehat{\text{S}}_1)\Rightarrow[\hat{V}$ satisfies $(\text{S})]\Rightarrow(\widehat{\text{S}})$ when $d = 1$. Thus, if $d = 1$, $(\widehat{\text{S}}_1)\Leftrightarrow(\widehat{\text{S}})$. □

Theorem 4 (Restated). *Let $v^i \in \{\text{yes}, \text{no}\}^m$, $i = 1, \ldots, n$, represent the voting decisions of each of n voters $1, \ldots, n$ on m voting items indexed by $M = \{1, \ldots, m\}$. For every $d \geq 1$ and every n-tuple of points $\hat{x}_1, \ldots, \hat{x}_n \in \mathbb{R}^d$, there exist alternatives $p_j, q_j \in \mathbb{R}^d$, $p_j \neq q_j$, $j \in M$, such that for every voter i there exists a strictly concave utility function u_i with ideal point $\hat{x}_i$ that strictly rationalizes the voting record generated from the voting agenda and voting decisions $\{((p_j, q_j), v^i_j)\}_{j\in M}$.*

Proof. Assume $d = 1$. Without loss of generality, let $\hat{x}_1 \leq \hat{x}_2 \leq \cdots \leq \hat{x}_n$. The proof proceeds by constructing the agenda and then verifying condition $(\widehat{\text{S}}_1)$. Position alternatives $p_1, \ldots, p_m$ in the interval $(-\infty, \hat{x}_1)$ so that $p_1 < \cdots < p_m < \hat{x}_1$. Position alternatives $q_1, \ldots, q_m$ in $(\hat{x}_n, +\infty)$ so that $\hat{x}_n < q_m < \cdots < q_1$. Now, for every voter i, consider the voting record V^i generated from voting agenda and voting decisions $\{((p_j, q_j), v^i_j)\}_{j\in M}$. For every pair of voting items $h, j \in M$ and for every voter i, $N(V^i_{\{j\}}) \cap (Y(V^i_{\{j,h\}}) \cup \{\hat{x}_i\}) = \varnothing$. Furthermore, if $h > j$, we have $p_j < p_h < q_h < q_j$. Thus, for every voter i the alternative $x \in N(V^i_{\{j\}})$ is such that $x \in \mathcal{E}(X(V^i_{\{j,h\}}) \cup \{\hat{x}_i\})$ and $x \notin Y(V^i_{\{j,h\}}) \cup \{\hat{x}_i\}$. Thus, the voting record V^i generated from the voting agenda and voting decisions $\{((p_j, q_j), v^i_j)\}_{j\in M}$ satisfies condition $(\widehat{\text{S}}_1)$, so, by Theorem 3, for every i the voting record V^i is rationalizable by a strictly concave utility function u_i with ideal point $\hat{x}_i$.

For the case of $d' > 1$ dimensions, note that the above constructed voting records, $V^1, \ldots, V^n$ in $d = 1$, dimension are strictly rationalizable and satisfy (N). As a result, each V^i also satisfies (A). Then, by part (ii) of Theorem 5, for every ideal point $\hat{x}_1, \ldots, \hat{x}_n \in \mathbb{R}^{d'}$, there exists a function $f : \bigcup_{j\in M}\{p_j, q_j\} \to \mathbb{R}^d$ such that for every voter i, the voting record $V^{i\prime} = \{(f(y), f(z)) \mid (y, z) \in V^i\}$ is strictly rationalizable by a strictly concave utility function u_i with ideal point $\hat{x}_i$. □

Theorem 5 (Embedding) (Restated). *Let $V^1, \ldots, V^n$ be the voting records of n voters $1, \ldots, n$. For every $d \geq 2$ and for every n-tuple of points $\hat{x}_1, \ldots, \hat{x}_n \in \mathbb{R}^d$, there exists a one-to-one function $f : \bigcup_{i=1}^n X(V^i) \to \mathbb{R}^d$ such that for every voter $i \in \{1, \ldots, n\}$*

(i) there exists a strictly concave utility function $u_i : \mathbb{R}^d \to \mathbb{R}$ with ideal point $\hat{x}_i$ that almost strictly rationalizes the voting record $\{(f(y), f(z)) \mid (y, z) \in V^i\}$

(ii) if V^i satisfies (A), then there exists a strictly concave utility function $u_i : \mathbb{R}^d \to \mathbb{R}$ with ideal point $\hat{x}_i$ that strictly rationalizes the voting record $\{(f(y), f(z)) \mid (y, z) \in V^i\}$.

Proof. Let $\hat{X} = \{\hat{x}_1, \ldots, \hat{x}_n\}$ and construct a finite set $\tilde{X} \subset \mathbb{R}^d$ such that $\tilde{X} \cap \hat{X} = \varnothing$, $|\tilde{X}| = |\bigcup_{i=1}^n X(V^i)|$, and $\tilde{X} \subseteq \mathcal{E}(\tilde{X} \cup \hat{X})$. Since $d > 1$, such a set $\tilde{X}$ trivially exists. Consider any onto function $f:\bigcup_{i=1}^{n} X(V^i) \to \tilde{X}$ and, for every i and every $V' \subseteq V^i$, let $f[V'] = \{(f(y), f(z)) \mid (y,z) \in V'\}$ denote the restricted voting record V' embedded in $\mathbb{R}^d$ via the function f. I show that for every voter i, the embedded voting record $f[V^i]$ satisfies ($\widehat{\text{W}}$). In particular, I show that if $\mathcal{E}(X(f[V']) \cup \{\hat{x}_i\}) \subseteq Y(f[V']) \cup \{\hat{x}_i\}$ for some nonempty $V' \subseteq f[V^i]$, then there exists nonempty $V'' \subseteq V'$ such that $N(V'') = Y(V'') \subseteq \mathcal{E}(X(V') \cup \{\hat{x}_i\})$ and $Y(V'') \cap Y(V' \setminus V'') = \varnothing$. Since $X(f[V^i]) \subseteq \mathcal{E}(X(f[V^i]) \cup \hat{X})$, it must be that $N(V') \subseteq X(V') \subseteq \mathcal{E}(X(V') \cup \{\hat{x}_i\}) \subseteq Y(V') \cup \{\hat{x}_i\}$. Since $\hat{x}_i \notin X(V')$, $N(V') \subseteq Y(V')$. By Lemma 4, there exists nonempty $V'' \subseteq V'$ such that $N(V'') = Y(V'') \subseteq \mathcal{E}(X(V') \cup \{\hat{x}_i\})$ and $Y(V'') \cap Y(V' \setminus V'') = \varnothing$. We conclude that the voting record $f[V^i]$ satisfies ($\widehat{\text{W}}$) for every i, so part (i) follows by Theorem 6. Under the additional assumption of part (ii), the voting record $f[V^i]$ satisfies (A), because the original record V^i does and because f is one-to-one. Thus, since $f[V^i]$ satisfies ($\widehat{\text{W}}$) and (A), it also satisfies ($\widehat{\text{S}}$), and part (ii) now follows from Theorem 3. □

Corollary 1 (Restated). *Assume $X = \mathbb{R}^d$, let $V = \{(y_j, z_j)\}_{j \in M}$ be a voting record, and consider any $x \in \mathbb{R}^d$.*

(i) If a (strictly) (quasi)concave function $u:\mathbb{R}^d \to \mathbb{R}$ strictly rationalizes V, then $u(x') \geq u(x)$ for all $x' \in R(x)$ and $u(x') \leq u(x)$ for all $x' \in R^{-1}(x)$.

(ii) If a strictly (quasi)concave function $u:\mathbb{R}^d \to \mathbb{R}$ rationalizes V, then $u(x') > u(x)$ for all $x' \in P(x)$ and $u(x') < u(x)$ for all $x' \in P^{-1}(x)$.

(iii) If V satisfies (A), then $R(x) \setminus Y(V) \subseteq P(x) \subseteq R(x)$ and $R^{-1}(x) \setminus N(V) \subseteq P^{-1}(x) \subseteq R^{-1}(x)$.

Proof. Parts (i) and (ii) follow immediately from Theorems 1 and 2, so it remains to show part (iii). By Remark 1, a voting record that satisfies (A) violates (W) if and only if it violates (S). Since V satisfies (A), $V \cup \{(x, x')\}$ must satisfy (A) if $x' \notin Y(V)$ and $V \cup \{(x', x)\}$ must satisfy (A) if $x' \notin N(V)$. □

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