Theoretical Economics 5 (2010), 241–274 1555-7561/20100241

Competing auctions: Finite markets and convergence

Gábor Virág
Department of Economics, University of Rochester

The literature on competing auctions offers a model where sellers compete for buyers by setting reserve prices. An outstanding conjecture (e.g., Peters and Severinov 1997) is that the sellers post prices close to their marginal costs when the market becomes large. This conjecture is confirmed in this paper: we show that if all sellers have zero costs, then the equilibrium reserve price converges to 0 in distribution. Under further conditions there is a symmetric pure strategy equilibrium. In this equilibrium, if the ratio of buyers to sellers increases, then the equilibrium reserve price increases, and the reserve price is decreasing in the size of the market. Convergence of reserve prices occurs at the fast rate of $1/n$ if the ratio of buyers to sellers is held constant.

Keywords. Competing auctions, finite markets, convergence.

JEL classification. D44, D82, D83.

1. Introduction

In many markets sellers compete by not simply posting prices, but by posting more complicated market mechanisms. A prime example is online auction sites where many different sellers are selling similar objects, and thus the sellers are in direct competition. When sellers post auctions the choice variable is the reserve price: a seller can try to attract more buyers by lowering his reserve price or can try his luck by posting a high reserve price and hoping that buyers will not be deterred. A natural question is how the equilibrium reserve prices depend on the market size, the buyer to seller ratio, and other market characteristics. McAfee (1993), Peters (1997), Peters and Severinov (1997), Burguet and Sákovics (1999), and others study sellers who compete by posting second-price auctions setting the reserve price as they wish, and the buyers decide which seller to visit given the reserve prices posted.[1] Peters and Severinov (1997) consider the case where the market is infinitely large and show that in equilibrium the sellers post reserve prices that are equal to their production costs. They also show that *if* there is an equilibrium for each finite market size where the sellers post identical (and deterministic)

Gábor Virág: gvirag@mail.rochester.edu
I would like to thank Angel Hernando-Veciana, Stephan Lauermann, Michael Peters, József Sákovics, and audiences at the University of Rochester, the 2007 Fall Midwest Theory Conference, and the 2008 Econometric Society Summer Meetings for their useful comments. All remaining errors are mine.

[1] The environment in Peters (1991, 2000) is also similar, but he focuses on the case where sellers post prices and not auctions. For this case, Galenianos and Kircher (2009) provides a convergence result similar to ours.

DOI: 10.3982/TE538

reserve prices, then the equilibrium reserve price converges to the cost of production, which is normalized to zero. However, as Burguet and Sákovics (1999) argue, such an equilibrium does not exist in the case when there are two sellers, and we extend their argument to the case of any (finite) number of sellers. Therefore, the equilibrium reserve price in large but finite markets is not settled by those articles. Hernando-Veciana (2005) shows that if only a finite number of reserve prices are allowed, then the equilibrium reserve price in large, but finite markets converge to the cost of the sellers. While this result is interesting, it is dependent on the restriction on the set of admissible reserve prices.

Our paper revisits the question of convergence by providing two results. First, we show that in all equilibria the reserve price each seller posts converges to zero in distribution (and in support) as the market becomes large. The logic is that as the market becomes large each seller loses his effect on the utility levels of the buyers and thus has limited incentives to increase his reserve price. Alternatively, by decreasing the reserve price a seller is able to attract extra visitors. Since the utility effect is small in large markets, the seller has just to provide a market utility to the buyers visiting him and can capture the surplus generated beyond that utility level. This extra surplus stays positive, since each seller has finitely many expected visitors regardless of the market size. Thus an extra visitor increases the probability of sale, increasing the surplus generated by the seller. As a consequence, decreasing the reserve price offers benefits at little cost to the seller as the market becomes large.

To confirm this conjecture, we need to show that as the market becomes large, each seller loses his effect on the utility levels of the buyers. We consider a game with finitely many players and use properties of the binomial distribution to derive the desired result.[2] In our proof, we concentrate on the highest reserve price in the support of the equilibrium and show that the seller who posted that price has an incentive to reduce his reserve price regardless of what reserve price the other sellers posted if the market is large enough. By focusing on the upper end of the support we can characterize revenues in a simple formula, utilizing the fact that the seller posting the highest price is visited only by the buyers with the highest valuations. To do this, we first characterize the utility of a buyer type who does *not* visit the seller who posted the highest reserve price, and thus his utility changes only because the change in the highest reserve price changes the visiting probabilities of the other buyers. The change of utility resulting from such a change in visiting probabilities is relatively easy to characterize by appealing to the envelope theorem. We then calculate the utility of buyer types who *do* visit the seller with the highest reservation price by using the envelope theorem and the utility of a type who does *not* visit this seller. This indirect method establishes that the utility effect vanishes in large markets, leading to the convergence result.

[2]Our approach is different from that of Hernando-Veciana (2005), who approximates large finite markets with the limiting case of infinitely many agents. As a consequence, he works with the more tractable limiting distribution instead of the binomial distribution that arises in games with finitely many players. Since his game has only finitely many strategies (reserve prices), this approximation works well, but in our game with a continuous action space the same approximation does not directly deliver the desired convergence result. Our different approach is also useful to obtain closed form solutions for *exact* equilibria. This is essential when, in Section 3, we analyze pure strategy equilibria to understand under what conditions convergence occurs and at what rate.

Our second result concerns cases in which a pure strategy equilibrium exist, i.e., where the sellers post nonrandom reserve prices. This is a deviation from the previous literature that makes assumptions that preclude the existence of a pure strategy equilibrium. More precisely, the above articles assume that the lowest possible valuation of the buyers is equal to the production cost (which is normalized to zero in this paper). Then the first order condition for seller optimality suggests that the sellers choose a zero reserve price. Then it is costless to increase the reserve price, since only buyers with the lowest possible valuations (which are zero) are lost. So the second order condition fails and a pure strategy equilibrium does not exist as was recognized by Burguet and Sákovics (1999) for the case of two sellers. Therefore, we consider the "gap case," where the lowest possible valuation is positive $a > 0$, and we provide a sufficient condition for a pure strategy equilibrium to exist. We provide intuitive comparative statics results and also characterize the rate of convergence of the reserve price to zero. If the number of buyers (sellers) increases, then the equilibrium reserve price increases (decreases), and the reserve price decreases in the size of the market. The equilibrium reserve price converges to 0 at the quick rate of $1/n$ if the ratio of sellers to buyers is constant. Moreover, we show that as long as the number of sellers *squared* over the number of buyers tends to infinity as the market becomes large, the equilibrium reserve price must converge to zero. This finding highlights that one needs much weaker conditions for convergence to occur than were previously stipulated.

2. Model and analysis

There are k sellers, each with one unit of an indivisible good, and n risk neutral buyers, each with a unit demand for the good. The valuation of each buyer is his private information, and the valuations are independent and identically distributed (i.i.d.) according to cumulative distribution function (c.d.f.) F and density f with support $[a, a+1]$, where $a \geq 0$. For simplicity, assume that f is continuous, bounded, and strictly positive on the support. The timing of the game is simple. First, the k sellers each post a nonnegative reserve price r_k and then the buyers simultaneously decide which seller to visit after observing all the reserve prices posted. At seller j the buyers present engage in a second-price auction with reserve price r_j without observing the number of other buyers present.[3] The winner of the auction is the buyer with the highest bid; in case of a tie, the seller flips a fair coin. The payment of the winner is equal to the reserve price if no other buyer visited seller j and is the highest other bid if there was a competing bid, while losing bidders do not pay. Each seller maximizes his expected revenue. Each buyer i obtains a von Neumann–Morgenstern utility (surplus) equal to his valuation v_i minus his payment m_i if he wins the auction, and zero otherwise. Finally, buyers maximize their expected utility.

We now turn to the analysis starting with the buyer's stage game, taking the reserve prices set by the sellers $(r_1, r_2, \ldots, r_k)$ as given. Given that the sellers post second-price auctions, it is a dominant strategy for each buyer to submit a bid equal to his

[3] Similarly to Burguet and Sákovics (1999), any mechanism that is efficient when the buyers are symmetric, like first-price or all-pay auctions, yields similar results. More precisely, there would still be an equilibrium in which the sellers post reserve prices as characterized below.

valuation at the auction where he participates. Assuming that each buyer follows his dominant strategy when bidding, the only decision a buyer needs to make is which seller to visit. We concentrate on equilibria where the buyers employ symmetric visiting strategies. Formally, the probability that buyer i with type v visits seller j when the reserve prices posted by the k sellers are $(r_1, r_2, \dots, r_k)$ is such that $\pi_i^j(v, r_1, r_2, \dots, r_k) = \pi^j(v, r_1, r_2, \dots, r_k)$. This requirement means that the visiting decision of a buyer depends only on his valuation for the object, but not on the name of the buyer. This requirement is standard in the competing mechanisms literature, and captures the notion that in large markets buyers are unable to coordinate their actions and behave in an anonymous manner.

Let us start the analysis by characterizing the equilibrium visiting strategies of the buyers given the reserve prices posted by the sellers. Lemma 1 exploits a simple single crossing property: a buyer with lower valuation is less eager to obtain the good and thus he is willing to pay less. Therefore, buyers with lower valuations are more likely to visit sellers who posted lower reserve prices as in the two-seller analysis of Burguet and Sákovics (1999).

Lemma 1. *Assume that the sellers post reserve prices* $(r_1, r_2, \dots, r_k)$ *such that* $r_1 \le r_2 \le \cdots \le r_{k-1} \le r_k < a+1$. *Then there exists a unique symmetric equilibrium in the buyers' stage game. This equilibrium is characterized by cutoff strategies: there exist (unique) cutoff types* $(t_1, \dots, t_{k-1})$ *such that* $\max\{a, r_1\} = t_0 \le t_1 \le t_2 \le \cdots \le t_{k-1} \le t_k = a+1$, *and buyer types between* t_l *and* t_{l+1} *visit the first* $l+1$ *sellers each with probability* $1/(l+1)$, *while types less than* t_0 *do not visit any seller.*

The proofs of Lemmas 1, 3, and 4 are given in the Appendix.

While very high types (above t_{k-1}) randomize between all k sellers, lower types restrict their visits to sellers who posted lower reserve prices. Very low types (types lower than t_1) visit only the seller with the lowest reserve price. The existence of such cutoff types can be established following the two-seller analysis provided by Burguet and Sákovics (1999). We establish the uniqueness of the cutoff types by making use of the incentive conditions of the buyers in an iterative procedure.

Let us characterize the equilibrium utility of the buyers in their stage game when the reserve prices posted are already fixed at $(r_1, r_2, \dots, r_k)$. Since all buyers follow symmetric strategies by assumption, the equilibrium utility of a buyer depends only on his valuation, but not on his name. Also, as the lemma above claims, there is a unique equilibrium in the buyers' stage game and thus the equilibrium utility of the buyers is uniquely determined. Therefore, let $u(x, r_1, r_2, \dots, r_k)$ denote the equilibrium (expected) utility of a buyer with type $x \in [a, a+1]$ in the buyers' stage game given the posted reserve prices. For brevity, we simply refer to this utility as $u(x)$ when this does not create any confusion.

The result that the (symmetric) equilibrium of the buyers' stage game is unique has important implications also for continuity properties. As we show in the proof of the above lemma, the cutoff types $(t_0, t_1, \dots, t_{k-1}, t_k)$ are solutions to a system of equations that are polynomial in the reserve prices. Therefore, the correspondence that describes the equilibrium cutoff values for any reserve price vector is upper-hemicontinuous. The above lemma implies that the correspondence is also single-valued and thus the cutoff types are continuous in the reserve prices.

We now show that the equilibrium (expected) revenue functions are continuous in the cutoff types and reserve prices. For simplicity let us consider a small deviation of seller j such that for all $l \neq j$ we have $r_l \neq r_j$.[4] To calculate the revenue of seller j it is necessary and sufficient to know the second highest type visiting if there are at least two visitors; otherwise one needs to know whether a visit occurred at all. Using Lemma 1, the probability that any given buyer visits seller j, the seller with the jth lowest reserve price, can be written as

$$P_j = \frac{F(t_j) - F(t_{j-1})}{j} + \frac{F(t_{j+1}) - F(t_j)}{j+1} + \cdots + \frac{1 - F(t_{k-1})}{k}. \tag{1}$$

Let M_j denote the distribution function of the type of a buyer *conditional* on visiting seller j. Using Bayes' rule, for all $x \geq t_{k-1}$, we have

$$M_j(x) = \frac{P_j - \frac{1-F(x)}{k}}{P_j}.$$

Similarly, for all $x \in [t_{k-2}, t_{k-1})$, we have

$$M_j(x) = \frac{P_j - \frac{1-F(t_{k-1})}{k} - \frac{F(t_{k-1})-F(x)}{k-1}}{P_j},$$

and then using this procedure one can calculate M_j for all $x \geq t_{j-1}$. By construction, $M_j(t_{j-1}) = 0$, since types lower than t_{j-1} do not visit seller j. Let $\widetilde{M}_j$ denote the distribution function of the second order statistics of buyer types that visit, setting it to zero if fewer than two buyers visited. Then for all $x \geq t_{j-1}$,

$$\begin{aligned}\widetilde{M}_j(x) = {} & (1-P_j)^n + nP_j(1-P_j)^{n-1} \\ & + \sum_{l=2}^{n} \binom{n}{l} P_j^l (1-P_j)^{n-l} \big[(M_j(x))^l + l(M_j(x))^{l-1}(1-M_j(x))\big].\end{aligned}$$

To explain this formula, note that the probability that no buyer visits is $(1-P_j)^n$, and the probability that one buyer visits is $nP_j(1-P_j)^{n-1}$. In both cases the second order statistic is set to zero. If there are $l \geq 2$ buyers that visit, then the probability that the second highest type is less than x is given in the brackets above. Let $\widetilde{m}_j(x)$ denote the derivative of $\widetilde{M}_j$ at x. Using the above calculations, one can write seller j's expected revenue for any small deviation that still leaves him with the jth lowest reserve:

$$R_j = (1-P_j)^n * 0 + nP_j(1-P_j)^{n-1} r_j + \int_{t_{j-1}}^{a+1} x\widetilde{m}_j(x)\,dx. \tag{2}$$

[4]This assumption is only for expositional purposes. If it does not hold, then the whole exercise can be done the same way except that one needs to handle the case of a small increase and a small decrease separately to show that continuity is sustained in both directions.

This revenue formula is a simple consequence of the second-price auction format. If no buyer visits, the revenue is zero; if one visits, then it is r_j; if several visit, then the revenue is equal to the second highest bid (which is equal to the second highest type).

Inspecting formula (2), it is clear that the expected revenue is continuous in the reserve prices and cutoff types, because functions P_j and $\widetilde{m}_j(x)$ do not depend directly on the reserve prices and depend continuously on the cutpoints. We have already argued that cutoff types are unique and continuous in reserves. Let us then take the game among the sellers, taking as given that the buyers play the unique symmetric equilibrium in their stage game. This defines revenue functions for the sellers only in terms of reserve prices, once one substitutes in the cutoff types in the revenue formula. Given our continuity results, one can conclude that the revenues in this game are continuous in the decision variables, the reserve prices. The following lemma summarizes this discussion.

Lemma 2. *The expected revenue of seller j is continuous in the reserve prices posted.*

Also, each seller posts a reserve price r_j in the compact, convex set $[0, a+1]$. Therefore, standard existence results (appealing to Glicksberg's fixed-point theorem) imply that there exists an equilibrium in mixed strategies in the sellers' game,[5] and thus our entire game has an equilibrium where the buyers follow symmetric strategies. The following corollary recaps our discussion.

Corollary 1. *There exists a perfect Bayesian equilibrium where the buyers follow symmetric strategies.*

Before proceeding to our main result, it is useful to further characterize how the cutoff types respond to changes in the reserve prices. The result below establishes that the cutoffs are continuously differentiable in the reserve prices as long as all reserve prices are different.

Lemma 3. *If no two reserve prices are equal, then at point $(r_1, r_2, \ldots, r_k)$ the cutoff types and the expected revenues are continuously differentiable functions of the reserve prices. Moreover, if $r_j \neq r_l$ for all $l \neq j$, then at that point the cutoff types and expected revenues are continuously differentiable functions of r_j and the partial derivative with respect to r_j is also continuous in r_l for all $l \neq j$. If $r_j = r_l$ for some $l \neq j$, then right hand partial derivatives and left hand partial derivatives of the cutoff types and expected revenues with respect to r_j still exist.*

To prove this result we first note that any cutoff type t_j is indifferent between visiting seller $j-1$ and j. This gives us $k-1$ indifference conditions in $k-1$ unknowns, the cutoff values. We are able to exploit the special structure of these conditions to show that one can use the implicit function theorem to obtain the desired conclusion. The logic of the

[5] See Lemma 7 of Dasgupta and Maskin (1986).

exercise does not allow us to handle the case where the reserve prices are not all different, since in this case slightly increasing or slightly decreasing the reserve price would change a seller's ranking, and then the whole set of equations would need to be rewritten to reflect this change in the ranking. Indeed, one can show that differentiability fails at a point $(r_1, r_2, \ldots, r_k)$ where $r_l = r_m$ for $m \neq l$, but left hand and right hand derivatives with respect to all reserve prices exist. It is also useful to point out that given the nature of the problem, the utility function of the buyers u also satisfies similar differentiability properties.

With these results in hand we are ready to consider our main question, that is, whether the equilibria of games with a finite number of players converge (as the market size becomes large) to the equilibrium of the infinite game where all sellers post a zero reserve price. We prove that such a convergence result indeed holds. The proof follows our intuitive argument in the Introduction. We first show that as the market grows large, each seller loses his effect on the equilibrium utilities of the buyers. Then we conclude the proof by showing that serving an extra type has a nonvanishing positive effect on total welfare, which is thus captured by the seller as revenue in the limit.

Theorem 1. *Take a sequence of games where* $\lim_{l\to\infty} k^l, n^l = \infty$ *and for all* l *we have* $n^l/k^l \leq \rho$ *for some* $\rho > 0$*. Then for any sequence of equilibria where the buyers use symmetric strategies, the reserve prices posted by the sellers converge to zero in distribution.*

Proof. Take a sequence of games characterized by the number of sellers k^l and the number of buyers n^l, and assume that $k^l, n^l \to \infty$ as $l \to \infty$. Denote the upper end of the support of the equilibrium strategies of seller i by d_i^l and denote the distribution function of the equilibrium reserve prices[6] by H_i^l. Note that it is without loss of generality to assume that it is a best response to post reserve price d_i^l, because the game between the sellers is continuous in the reserve prices. In the main text we provide the proof only for a simpler case (Case 1), relegating the more complicated case (Case 2) to the Appendix.

Case 1. First, assume that the equilibrium reserve price distribution does not put positive probability on d_i^l, i.e., for all i and l we have

$$\lim_{x \nearrow d_i^l} H_i^l(x) = 1.$$

The proof is presented in steps. In Step 1 we show that it is sufficient to establish that the seller with the highest strictly positive reserve price always wants to decrease his reserve if the market is large enough. In Step 2 we introduce a useful lemma and reformulate our problem. In Step 3 we provide an upper bound to the sensitivity of buyer utilities to a slight change in the action of the seller with the highest reserve price. In Step 4 we show that if the market is large enough, then it is profitable for the seller with the highest reserve to decrease his reserve price, which concludes the proof.

Step 1. Let $m(l) \in \{1, 2, \ldots, k^l\}$ be such that $d_{m(l)}^l \geq d_j^l$ for all j. For our purposes it is sufficient to show that $\lim_{l\to\infty} d_{m(l)}^l = 0$ for any sequence of equilibria, because this

[6]The language of the proof implicitly assumes that we have a mixed strategy equilibrium, but the case of pure strategies is covered in Case 2 in the Appendix.

would prove that all equilibria get close to all sellers posting a zero reserve price as the market becomes large. Our starting point is that (under the assumption for Case 1) by posting a reserve price $d^l_{m(l)}$, seller $m(l)$ posts a higher reserve price than all other sellers with probability 1. To establish the conclusion of the theorem, Steps 2–4 below show that *for any strictly positive reserve price r_i, there exists a threshold $\widetilde{l}(r_i)$ such that if $r_i > r_j$ for all $j \neq i$ and $l > \widetilde{l}(r_i)$, then seller i can increase his profit by slightly decreasing his reserve price*. Intuitively, this implies that seller $m(l)$ who posts $d^l_{m(l)}$, has a profitable downward deviation if l is large enough, which contradicts with the assumption that $d^l_{m(l)}$ is in the support of the strategy of seller $m(l)$.

The rest of Step 1 formalizes the italicized statement above and shows that this statement is sufficient to conclude that $\lim_{l\to\infty} d^l_{m(l)} = 0$ indeed holds. Let $r_{-i} = (r_1, r_2, \ldots, r_{i-1}, r_{i+1}, \ldots, r_{k^l})$, let $G^l_{-i}(r_{-i}) = \prod_{j\neq i} H^l_j(r_j)$ denote the distribution function of the equilibrium reserve prices posted by the other sellers, and let $R^l_i(r_1, \ldots, r_{k^l})$ be the (expected) revenue of seller i in market l if the vector of reserve prices posted is $(r_1, \ldots, r_{k^l})$. Note that formula $R^l_i(r_1, \ldots, r_{k^l})$ is well defined for any market size l by the uniqueness claim of Lemma 1. Let

$$ER^l_i(r_i) = \int_{[0,a+1]^{k^l-1}} R^l_i(r_1, \ldots, r_{k^l})\, dG^l_{-i}$$

denote the expected revenue of seller i in market l if he posts reserve price r_i. By Lemma 3, R^l_i is continuously differentiable with respect to r_i if $r_i > r_j$ for all $j \neq i$. Fixing the market size l and the strategies of the other sellers, the derivative of the expected revenue of seller i can be written as $\int_{[0,a+1]^{k^l-1}} \partial R^l_i(r_1, \ldots, r_{k^l})/\partial r_i\, dG^l_{-i}$.

To formalize the italicized statement, suppose that for all $r_i > 0$, there exists $\widetilde{l}(r_i)$ such that if $l > \widetilde{l}(r_i)$ and $r_j < r_i$ for all $j \neq i$, then

$$\frac{\partial}{\partial r_i} R^l_i(r_1, \ldots, r_{k^l}) < 0. \tag{3}$$

Then it follows that if $i = m(l)$ and $r_i = d^l_{m(l)} > 0$, and $l > \widetilde{l}(r_i)$ then $\int_{[0,a+1]^{k^l-1}} \partial R^l_i(r_1, \ldots, r_{k^l})/\partial r_i\, dG^l_{-i} < 0$ and thus $\partial ER^l_i(r^l_i)/\partial r^l_i < 0$. Therefore, if $l > \widetilde{l}(r_i)$ holds, then it is profitable for seller $m(l)$ to decrease his reserve price from r_i. Since this statement is true for any $r_i > 0$, therefore $\lim_{l\to\infty} d^l_{m(l)} = 0$ must hold; otherwise seller $m(l)$ has no incentive to post reserve price $d^l_{m(l)}$, which contradicts with our starting assumption that $d^l_{m(l)}$ is in the support of the strategy of seller $m(l)$.

Step 2. The rest of the proof (Steps 2–4) finds an appropriate threshold value $\widetilde{l}$ for any value of r_i such that (3) holds under the conditions stated before (3). For the rest of the proof, assume without loss of generality that $r_1 \leq r_2 \leq \cdots \leq r_{k-2} \leq r_{k-1} < r_k < a+1$, i.e., we assume that $m(l) = k^l$ for all l. Moreover, we will not indicate superscript l if its omission does not create confusion. By Lemma 3, R_k is continuously differentiable with respect to r_k in this case. The following useful lemma helps our analysis.

Lemma 4. *If $r_1 \leq r_2 \leq \cdots \leq r_{k-2} \leq r_{k-1} < r_k < a+1$, then we have $\partial t_{k-1}/\partial r_k > 0$, $\partial t_{k-2}/\partial r_k > 0$, and for any $x \geq t_{k-2}$, we have $\partial u(x)/\partial r_k \leq 0$.*

The above lemma implies that one can take t_{k-1} as the decision variable of seller k, since t_{k-1} is one-to-one with r_k if one fixes r_{-i}. Letting

$$\widetilde{R}_i(t_{k-1}) = \frac{k}{n} R_i(r_1, \ldots, r_k(t_{k-1})),$$

given Lemma 4, inequality (3) is equivalent to inequality $\partial \widetilde{R}_i(t_{k-1})/\partial t_{k-1} < 0$. However, it is more useful to establish the stronger condition that for some negative α, we have

$$\frac{\partial}{\partial t_{k-1}} \widetilde{R}_i(t_{k-1}) \leq \alpha < 0. \tag{4}$$

The main advantage of establishing the stronger condition (4) is that under (4) it is sufficient to cover the simplest case where $r_1 \leq r_2 \leq \cdots \leq r_{k-2} < r_{k-1} < r_k$ holds, i.e., where the *two* largest reserve prices are posted by a single seller. To see this, note that by Lemma 3 the expected revenue and the cutoffs are continuously differentiable in r_{k-2} and thus

$$\frac{\partial}{\partial t_{k-1}} \widetilde{R}_i(t_{k-1}) = \frac{k}{n} \frac{\partial R_i}{\partial r_k} \frac{\partial r_k}{\partial t_{k-1}} = \frac{k}{n} \frac{\partial R_i}{\partial r_k} \bigg/ \frac{\partial t_{k-1}}{\partial r_k}$$

is continuous in r_{k-2}. Therefore, if one increases r_{k-2} until $r_{k-2} = r_{k-1}$, it must be still true that $\partial \widetilde{R}_i(t_{k-1})/\partial t_{k-1} \leq \alpha < 0$.

The rest of the proof (Steps 3 and 4) establishes that for all $r_k > 0$, there exists $\widetilde{l}(r_k)$ such that if $l > \widetilde{l}(r_k)$ and $r_1 \leq r_2 \leq \cdots \leq r_{k-2} < r_{k-1} < r_k$, then (4) holds, which is sufficient to establish the desired conclusion by the above discussion.

Step 3. In this step we characterize the change in utilities of the buyers who visit seller $i = m(l)$ if seller i reduces t_{k-1} slightly by reducing his reserve price slightly from r_k. Using the incentive conditions of the buyers, one can establish that for all $x > t_{k-1}$ we have

$$0 \geq \frac{\partial u(x)}{\partial t_{k-1}} \geq -\frac{n-1}{k(k-1)} f(t_{k-1})(a+1). \tag{5}$$

The formula (5) is the key step in the argument, as it implies that the utility effect of a single seller vanishes in large markets. Formula (5) is formally proven as Lemma 5 in the Appendix, but given its significance, the rest of Step 3 goes through the main steps of the argument in an informal manner. The formal proof is somewhat involved as it involves the function $u(x)$, for which there is no simple explicit formula. To tackle the difficulties, we follow an indirect method. Instead of directly characterizing the utility changes of types who visit seller k (i.e., types above t_{k-1}), we characterize the changes in the utility of those types who visit seller $k-1$ but not seller k, i.e., types between t_{k-2} and t_{k-1}. This indirect method has the advantage that one does not need to calculate how sensitive the cutoff types are to a change in r_k, a calculation that involves a complicated system of equations in $k-1$ variables. For types on interval $[t_{k-2}, t_{k-1}]$ who visit seller $k-1$, the relevant reserve price r_{k-1} is unchanged, and thus their utility changes only because of changes in t_{k-1} and t_{k-2}. Moreover, for types close to (and above) t_{k-2}, the change in utility occurs only in the case where there is no other buyer visiting seller

$k-1$. (Otherwise, the utility is close to zero anyway, since the second highest type visiting seller $k-1$ is greater than t_{k-2}, and thus a type close to t_{k-2} who pays an amount of at least t_{k-2} obtains a utility level very close to zero.) The utility of a buyer with valuation x for the case when no other buyer visits seller $k-1$ can be written as the probability that no other buyer visits seller $k-1$ times his surplus $x-r_{k-1}$ in this case. Formally, the utility in this case is

$$(1-P_{k-1})^{n-1}(x-r_{k-1}) = \left(\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)}\right)^{n-1}(x-r_{k-1}).$$

The derivative of this expression with respect to t_{k-1} depends on how t_{k-2} responds to a change in t_{k-1}. Fortunately, our analysis only uses the fact that $\partial t_{k-2}/\partial t_{k-1} \geq 0$, which holds by Lemma 4.[7] Substituting $\partial t_{k-2}/\partial t_{k-1} \geq 0$ into expression

$$\frac{d}{dt_{k-1}}\left(\left(\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)}\right)^{n-1}(x-r_{k-1})\right)$$

yields an upper bound to how big the utility effect can be for a type x, which is close to t_{k-2}.[8] Then one can use the envelope theorem to estimate the utility changes for all the types who visit seller k, i.e., types greater than t_{k-1}. The details are in the Appendix.

Step 4. The rest of the proof formalizes the key intuition of the proof, i.e., we formally show that the benefit from increasing participation remains positive, while the utility costs tend to zero and therefore it is beneficial for the seller with the highest reserve price to decrease his reserve. We first calculate the expected revenue of seller k as the total surplus generated at seller k minus the total utilities of the types visiting seller k. For all $x \geq t_{k-1}$, let $G(x)$ denote the probability that seller k sells to a buyer with type less than x or does not sell at all. This event happens if and only if no buyer with type greater than x visits seller k, and thus $G(x) = (1-(1-F(x))/k)^n$. Letting $g(x) = \partial G/\partial x$, the total surplus generated at k is

$$W_k = \int_{t_{k-1}}^{a+1} x g(x)\,dx.$$

Since such types visit seller k with probability $1/k$, it follows that the sum of utilities generated at k can be written as

$$C_k = n\int_{t_{k-1}}^{a+1} \tfrac{1}{k} u(x) f(x)\,dx.$$

[7]Note, that by the chain rule and Lemma 4,

$$\frac{\partial t_{k-2}}{\partial t_{k-1}} = \frac{\partial t_{k-2}}{\partial r_k} \Big/ \frac{\partial t_{k-1}}{\partial r_k} > 0.$$

[8]Note, that the partial derivative with respect to t_{k-1} (ignoring the effect through the change in t_{k-2}) converges to zero if $(n-1)/(k(k-1))$ converges to zero. This suggests that the utility effect vanishes in the limit.

Then the expected revenue is $R_k = W_k - C_k$ and one can assume that seller k maximizes $\widetilde{R}_k$ with respect to t_{k-1}. We have

$$\frac{\partial \widetilde{R}_k}{\partial t_{k-1}} = -f(t_{k-1})\left[\left(1-\frac{1-F(t_{k-1})}{k}\right)^{n-1} t_{k-1} - u(t_{k-1})\right] - \int_{t_{k-1}}^{a+1} f(x)\frac{\partial u(x)}{\partial t_{k-1}}\,dx. \tag{6}$$

Since type t_{k-1} wins with probability $(1-(1-F(t_{k-1}))/k)^{n-1}$ and pays r_k at seller k, we have

$$u(t_{k-1}) = \left(1-\frac{1-F(t_{k-1})}{k}\right)^{n-1}(t_{k-1}-r_k)$$

and thus

$$\left(1-\frac{1-F(t_{k-1})}{k}\right)^{n-1}(t_{k-1}-u(t_{k-1})) = \left(1-\frac{1-F(t_{k-1})}{k}\right)^{n-1} r_k. \tag{7}$$

Using (5) and (7), one obtains from (6) that

$$\frac{\partial \widetilde{R}_k}{\partial t_{k-1}} \leq -f(t_{k-1})\left(1-\frac{1-F(t_{k-1})}{k}\right)^{n-1} r_k + \frac{n-1}{k(k-1)} f(t_{k-1})(a+1).$$

Inequality $n/k \leq \rho$ implies that for all $k \geq 2$,

$$\left(1-\frac{1-F(t_{k-1})}{k}\right)^{n-1} \geq \left(1-\frac{1}{k}\right)^{n-1} \geq \left(1-\frac{1}{k}\right)^{k\rho} \geq \left(1-\frac{1}{2}\right)^{2\rho} = \left(\frac{1}{4}\right)^{\rho}.$$

Therefore, for all l such that $n/k \leq \rho$ (and $k \geq 2$), we have

$$\frac{\partial \widetilde{R}_k}{\partial t_{k-1}} \leq f(t_{k-1})\left\{-r_k\left(\frac{1}{4}\right)^{\rho} + \frac{\rho}{(k-1)}(a+1)\right\} \leq \alpha < 0$$

if

$$k \geq \frac{\rho(a+1)}{r_k\left(\frac{1}{4}\right)^{\rho} + \frac{\alpha}{\max_x f(x)}}. \tag{8}$$

Therefore, for all $r_k > 0$, if l is large enough such that (8) holds and $r_1 \leq r_2 \leq \cdots \leq r_{k-2} < r_{k-1} < r_k$, then (4) holds, which concludes the proof for Case 1 as we indicated in Step 2.

Case 2. Now, assume that the equilibrium bid distribution may put positive probability on d_i^l, i.e., it *does not* hold that

$$\lim_{l\to\infty} \lim_{x \nearrow d^l} H^l(x) = 1.$$

The formal proof for this case is given in the Appendix. The idea behind the proof is the same as in Case 1: the utility effect of any seller vanishes in the limit. Then each seller maximizes the total surplus created at his store, which is achieved by selling the object if any buyer visits at all. The main technical complications compared to Case 1 are twofold. First, when a seller with the highest reserve considers a deviation downward, he will no longer be the seller with the highest reserve price if other sellers posted the exact same reserve price that he did before the deviation. Therefore, the deviating seller's profit can be described in a more complicated manner that depends on the last *two* cutpoints, and not only the last one. One then needs to calculate how both of those cutpoints respond to a change in the reserve price, which complicates the analysis substantially. The second, and more major, complication is that now the number of sellers q who posted the very highest reserve price (in a "tie") can take any numbers between 1 and k, where k becomes large as the market becomes large. We need to distinguish different cases, depending on whether q is large or small compared to k as the market grows large, and we need to show that a uniform bound (that is now independent of not only the reserve prices posted by the other seller, but also of q) exists. □

Using the above argument one can also show that the rate of convergence is fairly quick under our assumptions. Inspecting formula (8), which is valid in Case 1, suggests that the upper end of the equilibrium reserve price distribution $\bar{r}$ satisfies if $\bar{r} < D/k$ for some constant D. The Proof for Case 2 in the Appendix shows that this result holds there as well. Therefore, the posted reserve prices must converge to zero in distribution at the fast rate of $1/k$ under our assumption that $n/k \leq \rho$ for all markets l.[9]

It is appropriate here to compare our results with those of Hernando-Veciana (2005), who considers a convergence result as well. The main difference is that he assumes that the sellers are restricted to a finite grid that includes production costs. This assumption leads to a pure strategy equilibrium for a large enough (but finite!) market size where all sellers post a reserve price equal to their production costs. The discrete grid also allows him to consider the case where sellers have heterogeneous production costs, a case that is not formally studied here.[10] A further major difference between Hernando-Veciana (2005) and our paper is that when the market becomes large, he assumes that the ratios of buyers to sellers are fixed, while we allow it to change as long as in the limit the ratio of buyers to sellers remains bounded. Finally, we are able to study a pure strategy equilibrium for finite markets (see Section 3), which is not possible using the large market approximations of Hernando-Veciana (2005). In particular, we give sufficient conditions for when such an equilibrium exists and, if it does, what the speed of convergence is, and we also show interesting comparative statics results.

[9]If one lifts this assumption, then it is our conjecture that convergence also holds (at a rate possibly slower than $1/k$) under the weaker condition that $\lim_{l\to\infty}(n^l-1)/(k^l(k^l-1)) = 0$, but a proof is unavailable. We see in the next section that this condition is sufficient in the case when a pure strategy equilibrium exists.

[10]While a formal analysis of the heterogeneous cost case is beyond the scope of this work, the logic behind Theorem 1 extends to this case to some extent. More precisely, our proof can be directly adapted to show that the seller with the highest cost (also the one with the highest reserve price) must post a reserve price that converges to his cost of production as the market becomes large. However, a convergence result for sellers with lower cost levels does not follow directly.

3. Symmetric pure strategy equilibrium

In this section we study whether an equilibrium exist for a given f, n, k, a, where sellers use pure strategies, and characterize such an equilibrium if it exists, providing comparative statics and convergence results. Burguet and Sákovics (1999) establishes that when $a = 0$ and there are $k = 2$ sellers, such a pure strategy equilibrium does not exist. For reasons of tractability, we concentrate on *symmetric* pure strategy equilibria, where the sellers play symmetric and pure strategies all posting reserve price $r \geq 0$. (The buyers are assumed to use symmetric strategies as before.) In what follows we assume that $r \leq a$ in equilibrium and thus all types are served.[11] The following result characterizes such an equilibrium when it exists and provides a necessary second order condition for existence.

Theorem 2. *There is at most one symmetric pure strategy equilibrium of the game. In the symmetric pure strategy equilibrium, each seller posts a reserve price*

$$r = \frac{(n-1)a}{(n-1)+(k-1)^2}.$$

A necessary condition for such an equilibrium to exist is that

$$af(a) \geq a^* = \frac{k-1+\frac{n-k}{k}}{k^2-2+n-k}. \tag{9}$$

Proof. Suppose that all other sellers post reserve price r and seller 1 deviates to r_1.[12] First, take the case where $r_1 > r$. By Lemma 1, the unique symmetric Bayesian equilibrium in the buyers' stage game is such that types less than y visit sellers 2 through k with equal probability and types above y visit all k sellers with equal probability. By Lemma 4, y and r_1 are one-to-one, so the deviating seller's problem can be written using decision variable y instead of r_1.

Next, note that type y has the same probability of winning at any seller; this probability is $(1-(1-F(y))/k)$. At seller 1, type y wins if and only if all the other buyers visited other sellers and his payment is r_1 conditional on winning. Therefore, the expected payment of type y at seller 1 is $(1-(1-F(y))/k)r_1$. If type y visits seller $j \neq 1$, then he wins if and only if all the other buyers have visited a seller other than j or those visiting j have types less than y. If no one else visited j, then buyer i pays r, and if l other buyers visited j, then the expected payment is the expected value of the highest type among the l buyers conditional on them all having types less than y (otherwise buyer i does not win). Let $Em^{(l)}$ denote this expected value when l other buyers visited.

[11] One can show that there is no symmetric pure strategy equilibrium where $r > a$ holds. Moreover, our conjecture is that a pure strategy equilibrium with asymmetric reserve prices does not exist, but a proof is unavailable.

[12] The calculations below can be obtained as a special case of our first two numbered formulas. However, we find it more instructive to simplify the analysis by using that all other sellers post the same reserve price, and we derive the revenue formula from "scratch."

Since his probability of winning is the same whether he visited seller 1 or $j \neq 1$, the payment of type y must be equal at seller 1 and at the other sellers. Therefore,

$$\begin{aligned}\left(1-\frac{1-F(y)}{k}\right)^{n-1} r_1 &= \left(1-\frac{1-F(y)}{k}-\frac{F(y)}{k-1}\right)^{n-1} r \\ &\quad + \sum_{l=2}^{n} \binom{n-1}{l-1}\left(1-\frac{1-F(y)}{k}-\frac{F(y)}{k-1}\right)^{n-l}\left(\frac{F(y)}{k-1}\right)^{l-1} Em^{(l-1)}.\end{aligned} \tag{10}$$

Let us rewrite expression $\varphi = \sum_{l=2}^{n}\binom{n-1}{l-1}(1-(1-F(y))/k - F(y)/(k-1))^{n-l} \times (F(y)/(k-1))^{l-1} Em^{(l-1)}$ as an expected value of a random variable. Let τ take value zero when there is no other buyer at seller $j \neq 1$; otherwise let τ take the highest type among the other buyers who visited j. Let function $\rho(z) = z$ if $z \leq y$ and let $\rho(z) = 0$ if $z > y$. Then φ is the expected value of random variable $\rho(\tau)$. Letting t be the density function of τ, we have

$$\varphi = \int_a^{a+1} \rho(z) t(z)\, dz = \int_a^{y} z t(z)\, dz.$$

Also, by construction, for all $z \leq y$, we have $t(z) = (n-1)/(k-1)(1 - 1/k - F(y)/(k(k-1)) + F(z)/(k-1))^{n-2} f(z)$ and thus

$$\varphi = \int_a^{y} \frac{n-1}{k-1} z \left(1 - \frac{1}{k} - \frac{F(y)}{k(k-1)} + \frac{F(z)}{k-1}\right)^{n-2} f(z)\, dz. \tag{11}$$

Now, we describe the expected revenue of seller 1. The revenue of seller 1 is equal to r_1 if exactly one buyer visited him and equal to zero if no buyer did. If more than two buyers visited him, then the revenue is equal to the value of the second highest type. Therefore, the expected revenue from having $l \geq 2$ buyers visit is equal to $Es^{(l)}$, the expected value of the second highest type conditional on all buyers having types above y (otherwise they would have visited other sellers). With this shorthand notation, the expected revenue of seller 1 is

$$\begin{aligned}R_1 = n\frac{1-F(y)}{k}\left(1-\frac{1-F(y)}{k}\right)^{n-1} r_1 \\ + \sum_{h=2}^{n}\binom{n}{h}\left(\frac{1-F(y)}{k}\right)^{h}\left(1-\frac{1-F(y)}{k}\right)^{n-h} Es^{(h)}.\end{aligned} \tag{12}$$

A similar reasoning as before formula (11) yields that

$$\begin{aligned}\sum_{h=2}^{n}\binom{n}{h}\left(\frac{1-F(y)}{k}\right)^{h}\left(1-\frac{1-F(y)}{k}\right)^{n-h} Es^{(h)} \\ = \frac{n}{k}\int_y^{a+1} z(n-1)\left(1-\frac{1-F(z)}{k}\right)^{n-2}\frac{1-F(z)}{k} f(z)\, dz.\end{aligned} \tag{13}$$

Equations (10), (11), (12), and (13) yield that

$$\begin{aligned}
R_1 &= \overline{R}_1(y)\\
&= \frac{n}{k}\Bigg[\int_y^{a+1} z(n-1)\left(1-\frac{1-F(z)}{k}\right)^{n-2}\frac{1-F(z)}{k}f(z)\,dz\\
&\qquad + (1-F(y))\Bigg\{\left(1-\frac{1}{k}-\frac{F(y)}{k(k-1)}\right)^{n-1} r\\
&\qquad + \int_a^y \frac{n-1}{k-1}z\left(1-\frac{1}{k}-\frac{F(y)}{k(k-1)}+\frac{F(z)}{k-1}\right)^{n-2} f(z)\,dz\Bigg\}\Bigg].
\end{aligned}$$

Now, let $r_1 < r$. In this case the unique symmetric Bayesian equilibrium in the buyers' stage game is such that types less than x visit seller 1 and types above x visit all k sellers with equal probability. For all $z \leq x$, let

$$G(z) = \frac{(1-F(x))(k-1)}{k} + F(z),$$

and for all $z > x$, let

$$G(z) = \frac{(k-1)+F(z)}{k}.$$

A similar analysis as above yields that inducing $x \in [a, a+1]$ yields an expected revenue of

$$\begin{aligned}
R_1 &= \widehat{R}_1(x)\\
&= \int_a^x zn(n-1)G(z)^{n-2}(1-G(z))f(z)\,dz\\
&\quad + \int_x^{a+1} \frac{zn(n-1)G(z)^{n-2}(1-G(z))f(z)}{k}\,dz\\
&\quad + n\frac{(1+(k-1)F(x))}{k}\left[\left(1-\frac{1-F(x)}{k}\right)^{n-1} r - \int_a^x z(n-1)f(z)G(z)^{n-2}\,dz\right].
\end{aligned}$$

The optimality condition for the seller has two parts. First, it requires that he should not have an incentive to increase his reservation price above r, which is equivalent to requiring that $a \in \arg\max_{y\geq a} \overline{R}_1(y)$, since choosing $y > a$ in terms of cutoffs is equivalent to choosing a reserve price greater than r. The corresponding first order condition is $\partial\overline{R}_1/\partial y|_{y=a} \leq 0$, which is equivalent to $r \geq (n-1)a/((n-1)+(k-1)^2)$, after making the necessary calculations. Second, it requires that he should not have an incentive to choose a reserve price less than r, which is equivalent to requiring that $a \in \arg\max_{x\geq a} \widehat{R}_1(x)$, since choosing $x > a$ in terms of cutoffs is equivalent to choosing a reserve price less than r. The corresponding first order condition is $\partial\widehat{R}_1/\partial x|_{x=a} \leq 0$, which is equivalent to $r \leq (n-1)a/((n-1)+(k-1)^2)$, after making the necessary calculations. Hence, the first order condition(s) uniquely pins down the reserve price at $r = (n-1)a/((n-1)+(k-1)^2)$. For the local second order conditions, we need that when we let $r = (n-1)a/((n-1)+(k-1)^2)$, the second derivatives of $\overline{R}_1$ and $\widehat{R}_1$ are nonpositive at $y = a$ and $x = a$, respectively. The second derivative of $\overline{R}_1$ at $y = a$ is nonpositive if $af(a) \geq a^*$, while that of $\widehat{R}_1$ at $x = a$ is nonpositive for all values of a and $f(a)$. (The Appendix contains the details of the calculus.) □

The above result adds a few important insights that are not possible to draw when a pure strategy equilibrium does not exist. First, Theorem 2 implies that as the number of sellers (buyers) increases, the sellers post lower (higher) equilibrium reserve prices to compete effectively. This result shows that market forces work in the intuitive directions. Second, the theorem also shows that the equilibrium reserve price decreases in the size of the market due to the fact that each seller has less effect on the equilibrium utility levels of the buyers and thus increasing his reserve price becomes less appealing for each seller. To see how the market size effect works, let $(n-1)/(k-1) = \beta$ be fixed and note that the candidate equilibrium has a reserve price

$$r = \frac{\beta a}{\beta + k - 1},$$

which is decreasing in market size (represented by k in this formula as β is held fixed). Third, one is able to obtain a rate of convergence that turns out to be quite high. To see this, assume again that $(n-1)/(k-1) = \beta$ is held fixed and note that in this case the posted reserve price converges to zero at rate $1/k$. Fourth, one is able to relax the assumption of Hernando-Veciana (2005) that the ratio of buyers to sellers is constant. More precisely, Theorem 2 shows that if $\lim(n-1)/(k-1)^2 = 0$, then the equilibrium reserve price converges to zero, even if the ratio of buyers to sellers (n/k) converges to infinity. However, if the ratio of buyers to sellers converges to infinity fast enough, then convergence does not occur, since the sellers try to take advantage of the large number of buyers by posting high reserve prices. Fifth, it may be interesting to see how the results change when the sellers are known to have lower valuations than the buyers, a case that was mostly neglected by the previous literature. Sixth, our result also shows that if a symmetric pure strategy equilibrium exists, then the equilibrium reserve price depends only on the number of buyers and sellers, and the lowest possible valuation of the buyers, but not on the distribution of valuations.[13] Finally, it is interesting to point out that as the market becomes large, we have $a^* \to 0$ if the ratio of buyers to sellers is fixed, and thus for any (fixed) positive values of a and $f(a)$, the local second order condition $af(a) \geq a^*$ is satisfied in the limit. This suggests that as the market becomes large, it is more likely that there is a pure strategy equilibrium. According to our conjecture (see below), for any convex distribution function F and $a > 0$ (the gap case), there exists a pure strategy equilibrium if the market is large enough. It is also interesting to point out that if a pure strategy equilibrium exists, then the equilibrium is ex post efficient. The reason is that efficiency dictates trade to occur, and the equilibrium allocation is such

[13]The reason is that as a seller changes his reserve price from r, the change in the cutoff type t_1 depends only on n, k, r, a, and $f(a)$. Therefore, the revenue of the deviator may depend only on those variables, but variable $f(a)$ cancels out (at least for the first order condition) when the calculations are actually made.

that trade always occurs as long as at least one buyer visited a certain seller, since $r < a$ holds.[14]

The above results are, of course, conditional on the existence of a pure strategy equilibrium. It is important to understand when such an equilibrium exists and when it does not. In what follows we provide an explicit sufficient condition that is not, however, on the primitives of the model except for the case of two buyers. After the formal result below, we discuss our conjectures about when those sufficient conditions may hold in the general case of more than two buyers.

Before the formal result, we provide a discussion that shows why the second order condition of seller optimization may hold in this model, although it always fails in the setup of Burguet and Sákovics (1999). Our starting point is that as Burguet and Sákovics (1999) have already pointed out the first order conditions are not sufficient for seller optimization, and this is the cause for nonexistence of pure strategy equilibrium when $a = 0$. Indeed, when $a = 0$ the case studied by Burguet and Sákovics (1999), the local second order condition fails and a symmetric pure strategy equilibrium does not exist. In our model with a large enough, this problem does not arise, because with a high enough level of a we have $\partial^2\overline{R}_1/\partial y^2|_{y=a} \leq 0$. In less technical terms, the key is to study the incentives to increase the reserve price slightly from r when all the other sellers posted r. As a seller changes his reserve price from r, the change in the cutoff type t_1 depends only on $f(a)$, but not the entire distribution of types F. Therefore, the revenue of the deviator may depend only on $f(a)$, but not on function F. It turns out that $f(a)$ cancels out in the first order condition as a balance of two effects. On one hand, the higher $f(a)$ is, the more costly it is to lose visits from types less than t_1 for any fixed value of t_1. On the other hand, the higher $f(a)$ is, the less the cutoff t_1 is influenced by the reserve price. These two effects are equally important, and thus $f(a)$ does not have a first order effect on how the revenue changes when r_1 changes. But the density $f(a)$ does not cancel out in the second order condition. Moreover, the more buyers have types close to a (i.e., the higher $f(a)$ is) and the higher a is, the more costly it is to lose visits from types close to a, and therefore the more likely that increasing ones reserve price is not profitable. This intuition is confirmed formally in the above proof leading to formula (9). Therefore, if a and $f(a)$ are large enough, then the second order conditions hold at $r_1 = r$, and one only needs to make sure that the second order condition (9) (together with the first order condition) is sufficient to rule out that any *large* deviation is profitable. This is done in the next corollary. More precisely, the following corollary provides a sufficient condition for existence of a symmetric pure strategy equilibrium and states a specific case in which those conditions hold.

Corollary 2. *A symmetric pure strategy equilibrium exists if functions $\overline{R}_1$ and $\widehat{R}_1$ are quasiconcave for all $y \in [a, a+1]$ and $x \in [a, a+1]$ when $r = (n-1)a/((n-1)+(k-1)^2)$. If the necessary conditions stated in Theorem 1 hold, $n = 2$, and F is convex, then there exists a symmetric pure strategy equilibrium.*

[14]This result is different from Burguet and Sákovics (1999), who study the case where $a = 0$ (with two sellers) and show that a mixed strategy equilibrium exists where the reserve prices are positive with probability 1. Their result underlines that if a pure strategy equilibrium does not exist, then ex post efficiency may not hold.

Proof. The first part of Corollary 2 is a simple consequence of the above analysis. For the $n = 2$ case, a simple substitution yields that $\partial^2\overline{R}_1/\partial y^2$ and $\partial^2\widehat{R}_1/\partial x^2$ are decreasing functions for all $y \in [a, a+1]$ and $x \in [a, a+1]$ when $r = (n-1)a/((n-1)+(k-1)^2)$ and F is convex. Therefore, the local second order conditions at $y = a$ and $x = a$ are sufficient globally. □

Unfortunately, a satisfactory sufficient condition stated on the primitives is only available in the not very interesting special case where there are two buyers ($n = 2$). For this reason, we conducted numerical calculations, and obtained the conjecture that convexity of F seems to be sufficient for a pure strategy equilibrium to exists (for any n, k) as long as the local second order condition in Theorem 2 is satisfied.[15] More precisely, using Mathematica we were not able to find any cases with F convex where the global sufficient conditions ($\overline{R}_1$ and $\widehat{R}_1$ being quasiconcave) fail. Since functions $\overline{R}_1$ and $\widehat{R}_1$ are highly complicated polynomial functions involving high order powers of F and r, a proof to support our conjecture remains unavailable.

4. Conclusion

Our contribution to the literature on competing auctions is twofold. First, we analyze the case of large markets and show that as the market becomes large, the utility effect of each single seller approaches zero. Therefore, as long as his reserve price is larger than his cost, it is strictly profitable for each seller to attract extra buyers by decreasing his reserve price. Hence in large markets, we prove that the sellers post reserve prices close to their production costs. Second, we provide conditions under which a pure strategy equilibrium exists and provide explicit solutions for the case of finite markets. This allows us to obtain intuitive comparative statics results, showing that if the number sellers (buyers) increases, then the equilibrium reserve price goes down (up), and as the market size increases, the equilibrium reserve price decreases.

Appendix

Proofs of Lemmas 1, 3, and 4. Proving that any equilibria can be characterized by such cutoff values as stated in Lemma 1 is a straightforward extension of the proof for the case of two sellers as covered in Burguet and Sákovics (1999), and is thus omitted. The other results are proved in the following order. We first prove the content of Lemma 3. The uniqueness part of Lemma 1 then follows from that proof almost immediately. At the end of the proof, we cover Lemma 4.

Note, that to prove Lemma 3 we need only to prove continuous differentiability of the cutoff types with respect to the reserves, since the revenue functions are continuously differentiable in the reserve prices and the cutoff types as formula (2) in the

[15] It is easy to provide examples with a decreasing density function where there is a profitable global deviation and thus a regular equilibrium does not exist. This is not very surprising, since when f is decreasing, the virtual utilities may be nonmonotone, in which case losing visits from higher types may be less costly for a seller than losing visits from types close to a.

main text implies. Let us calculate how the cutoff types are calculated given a reserve price vector $(r_1, r_2, \ldots, r_k)$, again assuming that $0 \le r_1 \le r_2 \le \cdots \le r_k \le a+1$. First note that type t_{k-1} is indifferent between visiting sellers 1 and 2. Also, by construction, he wins with the same probability at the two sellers. To see this, note that since t_{k-1} is the lowest type visiting seller k, he wins there if and only if there is no other buyer visiting, i.e., with probability $(1-P_k)^{n-1}$. At seller $k-1$, he wins if and only if either no other buyer visits or if the other buyer with the highest type that visits is less than t_{k-1}. This occurs with probability $(1-P_{k-1}+(F(t_{k-1})-F(t_{k-2}))/(k-1))^{n-1}$, since by Lemma 1 a type below t_{k-1} visits seller $k-1$ if his type is above t_{k-2} and the probability of a visit is $1/(k-1)$ in this case. As formula (1) implies, $P_{k-1} = P_k + (F(t_{k-1})-F(t_{k-2}))/(k-1)$, and thus the two winning probabilities are equal: $(1-P_k)^{n-1} = (1-P_{k-1}+(F(t_{k-1})-F(t_{k-2}))/(k-1))^{n-1}$.

Therefore, the indifference condition of type t_1 implies that such a buyer type has to make the same expected payment at the two sellers. At seller k, the expected payment conditional on winning is simply r_k, since that type pays only if no other buyer is present and the payment is equal to the reserve price. At seller $k-1$, the formula is more complicated. The payment is r_{k-1} if no other buyer is present, but it is higher if competing buyers are present. If h other buyers are present, then the expected payment is equal to the first order statistics of h types that are i.i.d. on $[t_{k-2}, t_{k-1}]$ with distribution function F. Let v_{k-1}^1 denote the highest other type that visits seller $k-1$, and set v_{k-1}^1 equal to zero if no other buyer visits. Then the expected payment conditional on winning at seller $k-1$ is equal to $E[\max\{r_{k-1}, v_{k-1}^1\} \mid v_{k-1}^1 \le t_{k-1}]$ and the indifference condition becomes

$$r_k = E[\max\{r_{k-1}, v_{k-1}^1\} \mid v_{k-1}^1 \le t_{k-1}]. \tag{14}$$

This equation involves only two cutoff type values, t_{k-1} and t_{k-2}. To see this, note that the above equation involves only the probability distribution of v_{k-1}^1, which we denote by Γ_{k-1}. One can then rewrite (14) as

$$r_k = \int_a^{t_{k-1}} \max\{r_{k-1}, x\}\, d\Gamma_{k-1}(x). \tag{15}$$

To describe the distribution function Γ_{k-1}, note that its support is $r_{k-1} \cup [t_{k-2}, t_{k-1}]$ by construction, since all the competing buyers at seller $k-1$ have types on $[t_{k-2}, t_{k-1}]$. Then

$$\Gamma_{k-1}(x) = \begin{cases} 0 & \text{if } x < r_{k-1} \\ \left(\dfrac{1-P_{k-1}}{1-P_k}\right)^{n-1} & \text{if } x \in [r_{k-1}, t_{k-2}) \\ \left(\dfrac{1-P_{k-1}+\frac{F(x)-F(t_{k-2})}{k-1}}{1-P_k}\right)^{n-1} & \text{if } x \in [t_{k-2}, t_{k-1}) \\ 1 & \text{if } x \ge t_{k-1}. \end{cases}$$

This distribution function depends only on t_{k-1}, t_{k-2}, and the reserve prices, since P_k and P_{k-1} are only functions of t_{k-1} and t_{k-2}. Therefore, equation (15) involves only those two variables. Denoting the right hand side of (15) by β_{k-1} this equation can be rewritten as

$$\beta_{k-1}(t_{k-1}, t_{k-2}, r_{k-1}) = r_k.$$

We now proceed to show that $\partial\beta_{k-1}/\partial t_{k-2} < 0$, $\partial\beta_{k-1}/\partial t_{k-1} > 0$, and thus the implicit function theorem can be used to describe t_{k-2} as a continuously differentiable function of t_{k-1} and also to conclude that $\partial t_{k-2}/\partial t_{k-1} > 0$. First, we show that $\partial\beta_{k-1}/\partial t_{k-2} < 0$. To establish this, we show that a change in t_{k-2} causes a first order stochastic dominant change in distribution function Γ_{k-1}. To see this, note that $\Gamma_{k-1}(r_{k-1}) = ((1-P_k-(F(t_{k-1})-F(t_{k-2}))/(k-1))/(1-P_k))^{n-1}$ is a decreasing function of t_{k-2}. Moreover, for all $x \in [t_{k-2}, t_{k-1})$ we have

$$\Gamma_{k-1}(x) = \left(\frac{1-P_{k-1}+\frac{F(x)-F(t_{k-2})}{k-1}}{1-P_k}\right)^{n-1} = \left(\frac{1-P_k-\frac{F(t_{k-1})-F(x)}{k-1}}{1-P_k}\right)^{n-1}. \tag{16}$$

Let $\tilde{t}_{k-2} > t_{k-2}$ denote two possible values of the $(k-2)$nd cutpoint. Then since the formula (16) does not depend on t_{k-2}, we must have for all $x \in [\tilde{t}_{k-2}, t_{k-1})$ the two distribution functions are equal:

$$\tilde{\Gamma}_{k-1}(x) = \Gamma_{k-1}(x).$$

Moreover, for all $x \geq t_{k-1}$ we have $\tilde{\Gamma}_{k-1}(x) = \Gamma_{k-1}(x) = 1$. As we have already note above,

$$\begin{aligned}\Gamma_{k-1}(r_{k-1}) &= \left(\frac{1-P_k-\frac{F(t_{k-1})-F(t_{k-2})}{k-1}}{1-P_k}\right)^{n-1} \\ &< \left(\frac{1-P_k-\frac{F(t_{k-1})-F(\tilde{t}_{k-2})}{k-1}}{1-P_k}\right)^{n-1} = \tilde{\Gamma}_{k-1}(r_{k-1}).\end{aligned}$$

Also, by strict monotonicity of Γ_{k-1} on $[t_{k-2}, \tilde{t}_{k-2}]$ it follows that for all x on that interval,

$$\Gamma_{k-1}(x) < \Gamma_{k-1}(\tilde{t}_{k-2}) = \tilde{\Gamma}_{k-1}(\tilde{t}_{k-2}) = \tilde{\Gamma}_{k-1}(x),$$

where the first equality follows because the right hand side of (16) does not depend on the $(k-2)$nd cutpoint, while the second inequality follows from the fact that when the $(k-2)$nd cutoff takes value $\tilde{t}_{k-2}$, then no buyer with type lower than $\tilde{t}_{k-2}$ visits. Putting all these observations together implies that Γ_{k-1} first order stochastically dominates $\tilde{\Gamma}_{k-1}$ and thus $\beta_{k-1}(t_{k-1}, t_{k-2}, r_{k-1}) = \int_a^{t_{k-1}} \max\{r_{k-1}, x\}\, d\Gamma_{k-1}(x) > \beta_{k-1}(t_{k-1}, \tilde{t}_{k-2}, r_{k-1}) = \int_a^{t_{k-1}} \max\{r_{k-1}, x\}\, d\tilde{\Gamma}_{k-1}(x)$. Moreover, it is routine to establish that β_{k-1} is differentiable in its second component and that its derivative is always strictly negative.

Now, we show that $\partial\beta_{k-1}/\partial t_{k-1} > 0$. For this, it is sufficient to show that Γ_{k-1} is decreasing in t_{k-1}. First,

$$\frac{1-P_{k-1}}{1-P_k} = 1 - \frac{k}{k-1}\frac{F(t_{k-1})-F(t_{k-2})}{F(t_{k-1})+k-1}$$

is clearly decreasing in t_{k-1}. A similar argument shows that $(1 - P_{k-1} + (F(x) - F(t_{k-2}))/(k-1))/(1-P_k)$ is decreasing as well, which concludes this step and shows that given the reserve prices and t_{k-1}, there is a unique cutoff t_{k-2} that is a candidate for an equilibrium in the buyers' stage game. Moreover, t_{k-2} is a differentiable function of t_{k-1} and $\partial t_{k-2}/\partial t_{k-1} > 0$.

Now, one can write the indifference conditions corresponding to cutoff type t_{k-2}. This condition involves only three endogenous variables: t_{k-3}, t_{k-2}, and t_{k-1}. To complete the inductive steps, one needs to conclude that

$$\frac{\partial \beta_{k-2}}{\partial t_{k-3}} < 0, \qquad \frac{\partial \beta_{k-2}}{\partial t_{k-1}}, \ \frac{\partial \beta_{k-2}}{\partial t_{k-2}} > 0. \tag{17}$$

These inequalities together with the property of the implicit function from the previous step $\partial t_{k-2}/\partial t_{k-1} > 0$ imply that one can again use the implicit function theorem to describe t_{k-3} as a function of t_{k-1} and also that $\partial t_{k-3}/\partial t_{k-1} > 0$. The proof of the inequalities in (17) again follows after establishing stochastic dominance of the distribution of the first order statistics. The distribution of the first order statistics can be written as

$$\Gamma_{k-2}(x) = \begin{cases} 0 & \text{if } x < r_{k-2} \\ \left(\dfrac{1 - P_{k-2}}{1 - P_{k-1}}\right)^{n-1} & \text{if } x \in [r_{k-2}, t_{k-3}) \\ \left(\dfrac{1 - P_{k-2} + \frac{F(x) - F(t_{k-3})}{k-2}}{1 - P_k}\right)^{n-1} & \text{if } x \in [t_{k-3}, t_{k-2}) \\ 1 & \text{if } x \geq t_{k-2}. \end{cases}$$

It takes almost an identical argument as above to show that Γ_{k-2} is increasing in t_{k-3} and then first order stochastic dominance implies that, indeed, $\partial \beta_{k-2}/\partial t_{k-3} < 0$. Establishing that $\partial \beta_{k-2}/\partial t_{k-1}, \partial \beta_{k-2}/\partial t_{k-2} > 0$ is done by a slight modification of the proof in the first step (for function Γ_{k-1}).

Iterating the above arguments, one can use $k-2$ indifference conditions corresponding to cutoffs $t_{k-1}, t_{k-2}, \ldots, t_3, t_2$. This procedure yields that all the cutoff values $t_{k-2}, t_{k-3}, \ldots, t_2, t_1$ can be expressed as a continuously differentiable function of t_{k-1} with $\partial t_j/\partial t_{k-1} > 0$ for all $j = 1, 2, \ldots, k-2$. Then one can use the indifference condition corresponding to type t_1 to obtain that

$$\beta_1(t_1, t_2, \ldots, t_{k-1}, r_1, r_2, \ldots, r_{k-1}, r_k) = r_1.$$

As before, the stochastic dominance argument implies that for all $j = 1, 2, \ldots, k-1$,

$$\frac{\partial \beta_1}{\partial t_j} > 0.$$

Substituting the implicit functions for lower cutoff types (as a function of t_{k-1}), one can rewrite the above equation as

$$\widehat{\beta}_1(t_{k-1}, r_1, r_2, \ldots, r_{k-1}, r_k) = \beta_1(t_1(t_{k-1}), t_2(t_{k-1}), \ldots, t_{k-1}, r_1, r_2, \ldots, r_{k-1}, r_k) = r_1.$$

Then by the above results, $\partial\widehat{\beta}_1/\partial t_{k-1} > 0$ holds and the implicit function theorem can be used to obtain a continuously differentiable function $t_{k-1}(r_1, r_2, \ldots, r_k)$. Then using the previous function, one can obtain continuously differentiable functions $t_j(r_1, r_2, \ldots, r_k)$ for all $j = 1, 2, \ldots, k-2$ as well. This concludes the proof of continuous differentiability for the case where all reserves are different. Now, if two reserves are equal, $r_j = r_{j+1}$, the above procedure does not guarantee that when we increase r_j, the same derivatives apply as when we decrease it. (Indeed this is not true in general!) Therefore, at those points only right hand and left hand derivatives exist.

This proof also implies that since $\partial\widehat{\beta}_1/\partial t_{k-1} > 0$, there exists a unique solution, and thus a unique equilibrium cutoff vector exists for any vector of reserve prices. Moreover, if $r_k > r_{k-1}$ and then if r_k is slightly changed, then the ranking of the sellers remains unchanged and thus continuous differentiability in r_k in the claim of Lemma 3 follows from the above argument. Continuity of the derivatives $\partial t_j/\partial r_k$ in r_i, $i \neq k$, $j = 1, 2, \ldots, k-1$, follows also from the above argument, since all the indifference conditions (the β functions) are continuously differentiable in r_i.

Finally, we prove the comparative statics results stated in Lemma 4 for the case where $r_k > r_{k-1}$. Take two situations where the first $k-1$ sellers have their reserve prices fixed and seller k considers choosing between r_k and $\widetilde{r}_k$. Variables without a tilde refer to variables in the case where seller k posts r_k, while variables with a tilde refer to the situations where seller k posts $\widetilde{r}_k$. Supposing that $\widetilde{r}_k > r_k$ and $t_{k-1} > \widetilde{t}_{k-1}$ hold at the same time, we establish a contradiction below. If seller k posts r_k, then there are less buyers visiting seller k and the reserve price is also lower compared to the situation where seller k posts $\widetilde{r}_k$. Therefore, we must have

$$u(a+1) > \widetilde{u}(a+1).$$

Let $\Omega(x)$ and $\widetilde{\Omega}(x)$ be the probability that a buyer with type x obtains the object when the reserve price is r_k or $\widetilde{r}_k$.[16] The envelope theorem implies that

$$u'(x) = \Omega(x) \tag{18}$$

and

$$\widetilde{u}'(x) = \widetilde{\Omega}(x). \tag{19}$$

Since by assumption $t_{k-1} > \widetilde{t}_{k-1}$, therefore for all the types $x > t_{k-1}$, we have

$$\Omega(x) = \widetilde{\Omega}(x) = \left(F(x) + \frac{k-1}{k}(1 - F(x))\right)^{n-1},$$

since type x loses if and only if there is a higher type visiting seller k, which occurs with the same probability in the two cases. Therefore, the last four formulas imply that

$$u(t_{k-1}) > \widetilde{u}(t_{k-1}). \tag{20}$$

[16]This probability is the same regardless of which seller a buyer with type x visits among his optimal choices.

Now, for the same reason as before, for all $x \in (\widetilde{t}_{k-1}, t_{k-1})$ we have

$$\widetilde{\Omega}(x) = \left(F(x) + \frac{k-1}{k}(1 - F(x))\right)^{n-1}.$$

However, for all $x \in (\widetilde{t}_{k-1}, t_{k-1})$,

$$\Omega(x) = \left(F(x) + \frac{k-2}{k-1}(F(t_{k-1}) - F(x)) + \frac{k-1}{k}(1 - F(t_{k-1}))\right)^{n-1},$$

because the probability that another buyer with type between x and t_{k-1} visits the same seller as type x (seller $k-1$) is $1/(k-1)$. A simple comparison yields that $\widetilde{\Omega}(x) > \Omega(x)$, and then (18), (19), and (20) imply

$$u(\widetilde{t}_{k-1}) > \widetilde{u}(\widetilde{t}_{k-1}). \tag{21}$$

Now, suppose that $\widetilde{t}_{k-2} > t_{k-2}$. Then a similar argument as above implies that for all $x \in (\widetilde{t}_{k-2}, \widetilde{t}_{k-1})$, we have $\widetilde{\Omega}(x) > \Omega(x)$ and thus $\widetilde{u}'(x) > u'(x)$. Then (21) implies that

$$u(\widetilde{t}_{k-2}) > \widetilde{u}(\widetilde{t}_{k-2}). \tag{22}$$

By construction, it is optimal for type $\widetilde{t}_{k-2}$ to visit seller $k-1$ when seller k posts a reserve price of $\widetilde{r}_k$. Also, since $t_{k-1} > \widetilde{t}_{k-1} > \widetilde{t}_{k-2} > t_{k-2}$, it follows that it is optimal for type $\widetilde{t}_{k-2}$ to visit seller $k-1$ when seller k posts a reserve price of r_k. Let us compare $u(\widetilde{t}_{k-2})$ and $\widetilde{u}(\widetilde{t}_{k-2})$ directly. Since $\widetilde{t}_{k-2} > t_{k-2}$ and $t_{k-1} > \widetilde{t}_{k-1}$, it follows that seller $k-1$ is visited with a strictly lower probability when k posts $\widetilde{r}_k$ than when he posts r_k, and simple calculations show that we must have

$$u(\widetilde{t}_{k-2}) < \widetilde{u}(\widetilde{t}_{k-2}),$$

since the reserve price posted by seller $k-1$ is unchanged.

But the last inequality contradicts (22) and thus $t_{k-2} > \widetilde{t}_{k-2}$ follows. Similar arguments as before formula (21) then imply that

$$u(\widetilde{t}_{k-2}) > \widetilde{u}(\widetilde{t}_{k-2}).$$

Proceeding iteratively one can show that for all j, we have for all $j = 1, 2, \ldots, k-1$,

$$t_j > \widetilde{t}_j \tag{23}$$

and also that for all x,

$$u(x) > \widetilde{u}(x). \tag{24}$$

However, take a low type $y \in (r_1, \widetilde{t}_1)$ who visits seller 1 both when seller k posts r_k and when he posts $\widetilde{r}_k$. It is easy to see that if the cutoff values decrease, then he is better off, because seller 1 is visited with a lower probability. Therefore, (23) implies that $\widetilde{u}(y) > u(y)$, which contradicts (24), establishing that $\partial t_{k-1}/\partial r_k > 0$. The result that $\partial t_{k-2}/\partial r_k > 0$ can be established by a similar iterative argument, which we omit. The claim that $u(x)$

decreases in r_k for all $x \geq t_{k-2}$ follows also from the envelope theorem and the iterative construction employed. Given our other differentiability results, it follows that $u(x)$ is (continuously) differentiable in r_k when r_k is not equal to any other reserves. $\square$

Lemma 5. *In Case 1, for all $x > t_{k-1}$ we have*

$$0 \geq \frac{\partial u(x)}{\partial t_{k-1}} \geq -\frac{n-1}{k(k-1)} f(t_{k-1})(a+1).$$

Proof. Let $\Omega(x)$ denote the probability that type x is receiving an object. Since a type $x \in (t_{k-2}, t_{k-1})$ visits seller $k-1$, *any* of the other $n-1$ buyers takes the object away from a buyer with type x if this other buyer has a type between x and t_{k-1} and visits seller $k-1$ or has a type higher than t_{k-1} and visits seller $k-1$. (In what follows, we do not explicitly use the superscript l, but all elements of strategic interaction k, n, t and the reserve prices depend on l.) Using Lemma 1, the first possibility occurs with probability $(F(t_{k-1}) - F(x))/(k-1)$, while the second occurs with probability $(1 - F(t_{k-1}))/k$. Therefore, any given other buyer does *not* take the object away from a buyer with type x with probability

$$1 - \frac{F(t_{k-1}) - F(x)}{k-1} - \frac{1 - F(t_{k-1})}{k} = \frac{k-1}{k} + \frac{F(x)}{k-1} - \frac{F(t_{k-1})}{k(k-1)}.$$

Therefore, considering the case where *none* of the other $n-1$ buyers takes the object away from a buyer with type x, one obtains

$$\Omega(x) = \left(\frac{k-1}{k} + \frac{F(x)}{k-1} - \frac{F(t_{k-1})}{k(k-1)}\right)^{n-1}.$$

After differentiation, we obtain that for all $x \in (t_{k-2}, t_{k-1})$,

$$\frac{\partial \Omega(x)}{\partial t_{k-1}} = -\frac{n-1}{k(k-1)} \left(\frac{k-1}{k} + \frac{F(x)}{k-1} - \frac{F(t_{k-1})}{k(k-1)}\right)^{n-2} f(t_{k-1}). \tag{25}$$

A similar argument implies that for all $x > t_{k-1}$,

$$\Omega(x) = \left(1 - \tfrac{1}{k}(1 - F(x))\right)^{n-1}, \tag{26}$$

which does not depend on t_{k-1}.

Let $x \in (t_{k-2}, t_{k-1})$ and calculate the utility such a type achieves in equilibrium. The probability that no other buyer visits seller $k-1$ is $((k-1)/k + F(t_{k-2})/(k-1) - F(t_{k-1})/(k(k-1)))^{n-1}$, in which case the buyer with type x pays r_{k-1} for the object, which he wins for sure. The probability that $g \in \{1, 2, \ldots, n-1\}$ other buyers visit seller $k-1$ and all visitors have types less than x is $\binom{n-1}{g}((k-1)/k + F(t_{k-2})/(k-1) - F(t_{k-1})/(k(k-1)))^{n-1-g}((F(x) - F(t_{k-2}))/(k-1))^g$. In this case, the payment of a buyer with type x is equal to the largest valuation among all the $g-1$ other buyers who visit seller $k-1$. Therefore, his (expected) utility can be written as

$$\begin{aligned} u(x) = {} & \left(\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)}\right)^{n-1}(x - r_{k-1}) \\ & + (n-1)\left(\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)}\right)^{n-2} \\ & \times \frac{F(x) - F(t_{k-2})}{k-1}\big(x - E[y \mid y \in [t_{k-2}, x]]\big) \\ & + \binom{n-1}{2}\left(\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)}\right)^{n-3} \\ & \times \left(\frac{F(x) - F(t_{k-2})}{k-1}\right)^2\big(x - E[y^1 \mid y^1, y^2 \in [t_{k-2}, x], y^1 > y^2]\big) + \cdots. \end{aligned} \tag{27}$$

Now take the decision problem of seller k in terms of choosing t_{k-1} (which we can do since t_{k-1} and r_k are in a one-to-one relationship by Lemma 4) and let us calculate the utility change of a type when seller k decreases his decision variable t_{k-1} slightly. One needs to allow all the other cutpoints to change to accommodate the change in t_{k-1}, and thus when derivatives are taken with respect to t_{k-1}, these indirect effects are also taken into account in what follows. Fixing x at the *initial* value of t_{k-2}, we obtain that

$$\left.\frac{\partial\Big(\big(\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)}\big)^{n-2}(F(x) - F(t_{k-2}))(x - E[y \mid y \in [t_{k-2}, x]])\Big)}{\partial t_{k-1}}\right|_{x=t_{k-2}} = 0,$$

because $F(x) - F(t_{k-2}) = x - E[y \mid y \in [t_{k-2}, x]] = 0$ when $x = t_{k-2}$. The derivatives of the other terms of $u(x)$ at $x = t_{k-2}$, except for the first one, are zero for the same reason and thus[17]

$$\begin{aligned} \left.\frac{\partial u(x)}{\partial t_{k-1}}\right|_{x=t_{k-2}} = {} & \frac{n-1}{(k-1)}\left(\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)}\right)^{n-2} \\ & \times (t_{k-2} - r_{k-1})\left(\frac{\partial t_{k-2}}{\partial t_{k-1}} f(t_{k-2}) - \frac{f(t_{k-1})}{k}\right). \end{aligned} \tag{28}$$

Now, by Lemma 4, it follows that $\partial u(x)/\partial t_{k-1}|_{x=t_{k-2}} \leq 0$ and $\partial t_{k-2}/\partial t_{k-1} \geq 0$. Therefore,

$$\begin{aligned} 0 & \leq \frac{n-1}{(k-1)}\left(\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)}\right)^{n-2}(t_{k-2} - r_{k-1})\frac{\partial t_{k-2}}{\partial t_{k-1}} f(t_{k-2}) \\ & \leq \frac{n-1}{k(k-1)}\left(\frac{k-1}{k} + \frac{F(t_{k-2})}{k-1} - \frac{F(t_{k-1})}{k(k-1)}\right)^{n-2}(t_{k-2} - r_{k-1}) f(t_{k-1}). \end{aligned}$$

[17]When calculating the utility of type x (fixed at the initial level of t_{k-2}), we are using the fact that when t_{k-1} decreases, then t_{k-2} decreases as well, so a type x that is equal to the initial value of t_{k-2} still visits seller $k-1$.

Then revisiting (28) yields that

$$
\begin{aligned}
0 &\geq \left.\frac{\partial u(x)}{\partial t_{k-1}}\right|_{x=t_{k-2}} \\
&\geq -\frac{n-1}{k(k-1)}\left(\frac{k-1}{k}+\frac{F(t_{k-2})}{k-1}-\frac{F(t_{k-1})}{k(k-1)}\right)^{n-2}(t_{k-2}-r_{k-1})f(t_{k-1}).
\end{aligned}
\tag{29}
$$

Using the envelope theorem, we obtain that for all $x \geq z$,

$$
u(x)=u(z)+\int_z^x \Omega(y)\,dy
$$

and thus

$$
\frac{\partial u(x)}{\partial t_{k-1}}=\left.\frac{\partial u(x)}{\partial t_{k-1}}\right|_{x=t_{k-2}}+\int_{t_{k-2}}^{x}\frac{\partial\Omega(y)}{\partial t_{k-1}}\,dy.
$$

Formulas (25) and (26) imply that for all $x > t_{k-1}$, we have

$$
\frac{\partial u(x)}{\partial t_{k-1}}=\left.\frac{\partial u(x)}{\partial t_{k-1}}\right|_{x=t_{k-2}}-\frac{n-1}{k(k-1)}\int_{t_{k-2}}^{x}\left(\frac{k-1}{k}+\frac{F(y)}{k-1}-\frac{F(t_{k-1})}{k(k-1)}\right)^{n-2}f(t_{k-1})\,dy.
$$

Then (29) implies that for all $x > t_{k-1}$, we have

$$
\begin{aligned}
0 &\geq \frac{\partial u(x)}{\partial t_{k-1}} \\
&\geq -\frac{n-1}{k(k-1)}\left(\frac{k-1}{k}+\frac{F(t_{k-2})}{k-1}-\frac{F(t_{k-1})}{k(k-1)}\right)^{n-2}(t_{k-2}-r_{k-1})f(t_{k-1}) \\
&\qquad -\frac{n-1}{k(k-1)}\int_{t_{k-2}}^{x}\left(\frac{k-1}{k}+\frac{F(y)}{k-1}-\frac{F(t_{k-1})}{k(k-1)}\right)^{n-2}f(t_{k-1})\,dy.
\end{aligned}
$$

Using that $((k-1)/k+F(t_{k-2})/(k-1)-F(t_{k-1})/(k(k-1)))^{n-2}<1$, $\int_{t_{k-2}}^{x}((k-1)/k+F(y)/(k-1)-F(t_{k-1})/(k(k-1)))^{n-2}<1$, and $x \leq a+1$ implies together with the above formula that indeed

$$
0 \geq \frac{\partial u(x)}{\partial t_{k-1}} \geq -\frac{n-1}{k(k-1)}f(t_{k-1})(a+1). \qquad \square
$$

Lemma 6. *If q sellers post the highest reserve price r_k, then if any single seller (seller k) of those q deviates in such a way that t_{k-q} decreases below the original value, then we have*

$$
0 \geq \int_{t^*_{k-q}}^{a+1} f(x)\left.\frac{\partial u(x)}{\partial t_{k-q}}\right|_{t_{k-q}=t^*_{k-q}}dx \geq -\frac{\rho f(t_{k-q})k}{(k-q)(q-1)}.
$$

Proof. Note that

$$
\frac{\partial u(a+1)}{\partial t_{k-q}}=\frac{\partial u(a+1)}{\partial t_{k-q+1}}\frac{\partial t_{k-q+1}}{\partial t_{k-q}},
\tag{30}
$$

and that since type $a+1$ visits the sellers who posted the highest reserve price and that reserve price does not change when seller k decreases r_k slightly, therefore $u(a+1)$ changes only because t_{k-q+1} changes. Let us show now that $\partial u(a+1)/\partial t_{k-q+1}$ is uniformly bounded. To see this, let $A(x)$ denote the probability that no other buyer with type above x visits seller $k-q+1$, the seller that the highest type, $a+1$, visits with positive probability. Then for all $x \geq t_{k-q+1}$,

$$A(x) = \left(\frac{k-1}{k} + \frac{F(x)}{k}\right)^{n-1}.$$

Then

$$u(a+1) = \left(\frac{k-1}{k} + \frac{F(t_{k-q+1})}{k}\right)^{n-1}(a+1-r_{k-q+1}) + \int_{t_{k-q+1}}^{a+1} A'(x)(a+1-x)\,dx$$

and

$$\begin{aligned}\frac{\partial u(a+1)}{\partial t_{k-q+1}} &= \frac{(n-1)}{k}\left(\frac{k-1}{k} + \frac{F(t_{k-q+1})}{k}\right)^{n-1} f(t_{k-q+1})(t_{k-q+1}-r_{k-q+1}) \\ &\leq \rho f(t_{k-q+1}).\end{aligned}$$

Therefore, the last equation and formula (30) imply that

$$-\frac{\rho f(t_{k-q})k}{(k-q)(q-1)} \leq \frac{\partial u(a+1)}{\partial t_{k-q}} \leq 0. \tag{31}$$

Let $\Omega(x)$ denote the probability of winning for a buyer with type x. For $x \geq t_{k-q+1}$, we have

$$\Omega(x) = \left(1 - \frac{1-F(x)}{k}\right)^{n-1}.$$

Using the envelope formula,

$$u(x) = u(a+1) - \int_x^{a+1} \Omega(y)\,dy.$$

Therefore, for all $x \geq t_{k-q+1}$,

$$\frac{\partial u(x)}{\partial t_{k-q}} = \frac{\partial u(a+1)}{\partial t_{k-q}}. \tag{32}$$

Using formulas (31) and (32) implies that indeed

$$\begin{aligned}0 &\geq \int_{t^*_{k-q}}^{a+1} f(x)\frac{\partial u(x)}{\partial t_{k-q}}\bigg|_{t_{k-q}=t^*_{k-q}} dx = \int_{t^*_{k-q}}^{a+1} f(x)\frac{\partial u(a+1)}{\partial t_{k-q}}\bigg|_{t_{k-q}=t^*_{k-q}} dx \\ &= (1-F(t^*_{k-q}))\frac{\partial u(a+1)}{\partial t_{k-q}} \geq \frac{\partial u(a+1)}{\partial t_{k-q}} \geq -\frac{\rho f(t_{k-q})k}{(k-q)(q-1)}.\end{aligned}$$

□

Proof for Case 2 of Theorem 1. In Case 2 the main difference is that, with positive probability, there may be $q > 1$ sellers who post the highest reserve prices. The proof below establishes exactly that if there are $q > 1$ sellers who post the highest reserve price, then any of those q sellers find it profitable to decrease their reserve price. First, take the case where $q = k$, i.e., all the other sellers post reserve price r_k as well. As we show subsequently in the Proof of Theorem 2, seller k has an incentive to decrease his reserve price if $r_k > a(n-1)/((n-1)+(k-1)^2)$. But this threshold is approaching zero and thus if the market is large enough, seller k has an incentive to decrease his price for any positive r_k.

Otherwise (if $q < k$), let $r_1 \le r_2 \le \cdots \le r_{k-q-1} < r_{k-q} < r_{k-q+1} = \cdots = r_k < a+1$ and suppose that seller k decreases r_k slightly. In this case, Lemma 1 implies that after this change, seller k is visited with probability $1/(k-q+1)$ by types between t_{k-q} and t_{k-q+1}, and visited with probability $1/k$ by types larger than t_{k-q+1}. When r_k is at the original level (and thus $r_{k-q+1} = \cdots = r_k$), then of course $t_{k-q} = t_{k-q+1}$, but when r_k is decreased, then $t_{k-q} < t_{k-q+1}$. The following useful result helps the analysis below.[18]

Lemma 7. *The cutoff values are differentiable in r_k from the left hand side, i.e.,*

$$\lim_{r \nearrow r_k} \frac{t_j(r_1, r_2, \ldots, r_k) - t_j(r_1, r_2, \ldots, r)}{r_k - r}$$

exists for $j = 1, 2, \ldots, k-1$. Moreover, for the left hand derivatives we have $\partial t_{k-q}/\partial r_k > 0$, $\partial t_{k-q-1}/\partial r_k > 0$, and $\partial t_{k-q+1}/\partial r_k < 0$. Moreover, for all $x \ge t_{k-q-1}$, we have $\partial u(x)/r_k < 0$.

Because of the above lemma, instead of r_k, one can take t_{k-q} as the choice variable of seller k. Take a buyer with type x that is equal to the original value of t_{k-q-1}. For that type it is optimal to visit seller $k-q$. Suppose that t_{k-q} goes down and thus t_{k-q-1} goes down as well by the Lemma 7. Then for type x it is still optimal to visit seller $k-q$. Lemma 6 implies that we can bound the utility effect of such a change as

$$0 \ge \int_{t^*_{k-q}}^{a+1} f(x) \frac{\partial u(x)}{\partial t_{k-q}}\bigg|_{t_{k-q}=t^*_{k-q}} dx \ge -\frac{\rho f(t_{k-q})k}{(k-q)(q-1)}. \tag{33}$$

We first provide a uniform convergence result for the case where q is not too small. For all $x \ge t_{k-q-1}$, let $G(x)$ denote the probability that seller k sells to a buyer with type less than x or does not sell at all, and let $g(x) = \partial G/\partial x$ denote the corresponding density function. If $x > t_{k-q+1}$, this event happens if and only if no buyer with type greater than x visits seller k, and thus $G(x) = (1-(1-F(x))/k)^n$. For $x \in [t_{k-q}, t_{k-q+1}]$, the following equality holds:

$$G(x) = \left(1 - \frac{1-F(t_{k-q+1})}{k} - \frac{F(t_{k-q+1}) - F(x)}{k-q+1}\right)^n.$$

[18] The first half of the lemma is a direct consequence of Lemma 3. The proof of the second half is very similar to that of Lemma 4 and is thus omitted.

The expected revenue of seller k is the total surplus generated at seller k minus the total utilities of the types visiting seller k or

$$R_k = W_k - C_k$$

with

$$W_k = \int_{t_{k-q}}^{a+1} x g(x)\,dx$$

and

$$C_k = n\int_{t_{k-q}}^{t_{k-q+1}} \frac{1}{k-q+1} u(x) f(x)\,dx + n\int_{t_{k-q+1}}^{a+1} \frac{1}{k} u(x) f(x)\,dx,$$

since types above t_{k-q+1} visit seller k with probability $1/k$ and types in $[t_{k-q}, t_{k-q+1}]$ visit with probability $1/(k-q+1)$. It is useful to describe the utility cost in an alternative way using the function u^* that describes the utility of a type conditional on obtaining the object from seller k:

$$C_k = \int_{t_{k-q}}^{a+1} u^*(x) g(x)\,dx.$$

Note that type t_{k-q} obtains the object from seller k if and only if no other buyer visited seller k and thus his utility conditional on obtaining the object is $u^*(t_{k-q}) = t_{k-q} - r_k$. With this formulation (and explicitly recognizing the cutpoints), one can rewrite the revenue as

$$\begin{aligned} W_k = \int_{t_{k-q}}^{t_{k-q+1}} (x - u^*(x)) \frac{n}{k-q+1}\left(1 - \frac{1-F(t_{k-q+1})}{k} - \frac{F(t_{k-q+1}) - F(x)}{k-q+1}\right)^{n-1} f(x)\,dx \\ + \int_{t_{k-q+1}}^{a+1} (x - u^*(x)) \frac{n}{k}\left(1 - \frac{1-F(x)}{k}\right)^{n-1} f(x)\,dx. \end{aligned}$$

Using the above definitions implies that

$$\begin{aligned} \frac{\partial R_k}{\partial t_{k-q}} = -\frac{n}{k-q+1}\left(1 - \frac{1-F(t_{k-q})}{k}\right)^{n-1} f(t_{k-q}) r_k \\ - \int_{t_{k-q}}^{t_{k-q+1}} (x - u^*(x)) \frac{\partial g(x)}{\partial t_{k-q+1}} \frac{\partial t_{k-q+1}}{\partial t_{k-q}}\,dx \\ + \frac{\partial t_{k-q+1}}{\partial t_{k-q}} f(t_{k-q+1})\left(\frac{1}{k-q+1} - \frac{1}{k}\right) - \frac{n}{k}\int_{t_{k-q+1}}^{a+1} f(x) \frac{\partial u(x)}{\partial t_{k-q}}\,dx \\ - \frac{n}{k-q+1}\int_{t_{k-q}}^{t_{k-q+1}} f(x) \frac{\partial u(x)}{\partial t_{k-q}}\,dx. \end{aligned}$$

We need to evaluate this derivative at the point where $r_k = r_{k-1} = \cdots = r_{k-q+1}$ and thus $t_{k-q} = t_{k-q+1} = t^*_{k-q}$, where t^*_{k-q} stands for the original cutpoint. Therefore,

$$\begin{aligned}\frac{\partial R_k}{\partial t_{k-q}}\bigg|_{t_{k-q}=t^*_{k-q}} = &-\frac{n}{k-q+1}\left(1-\frac{1-F(t^*_{k-q})}{k}\right)^{n-1} f(t^*_{k-q})r_k \\ &+ \frac{\partial t_{k-q+1}}{\partial t_{k-q}}\bigg|_{t_{k-q}=t^*_{k-q}} f(t^*_{k-q+1})\left(\frac{1}{k-q+1}-\frac{1}{k}\right) \\ &- \frac{n}{k}\int_{t^*_{k-q}}^{a+1} f(x)\frac{\partial u(x)}{\partial t_{k-q}}\bigg|_{t_{k-q}=t^*_{k-q}} dx.\end{aligned}$$

Using that $q > 1$ and $\partial t_{k-q+1}/\partial t_{k-q} \le 0$ by Lemma 7 and formula (33) yields that

$$\begin{aligned}\frac{\partial R_k}{\partial t_{k-q}}\bigg|_{t_{k-q}=t^*_{k-q}} &\le -\frac{n}{k-q+1}\left(1-\frac{1-F(t^*_{k-q})}{k}\right)^{n-1} f(t^*_{k-q})r_k \\ &\qquad - \frac{n}{k}\int_{t^*_{k-q}}^{a+1} f(x)\frac{\partial u(x)}{\partial t_{k-q}}\bigg|_{t_{k-q}=t^*_{k-q}} dx \\ &\le -\frac{n}{k-q+1}\left(1-\frac{1-F(t^*_{k-q})}{k}\right)^{n-1} f(t^*_{k-q})r_k + \frac{\rho f(t^*_{k-q})n}{(k-q)(q-1)} < 0\end{aligned}$$

if

$$\frac{k-q+1}{(k-q)(q-1)} < \left(1-\frac{1-F(t^*_{k-q})}{k}\right)^{n-1}\frac{r_k}{\rho}.$$

Alternatively, if $k \ge 2$, then

$$\left(1-\frac{1-F(t^*_{k-q})}{k}\right)^{n-1} \ge \left(1-\frac{1}{k}\right)^{n} \ge \left(1-\frac{1}{k}\right)^{k\rho} \ge \left(\frac{1}{4}\right)^{\rho}.$$

Therefore, it is sufficient to show that

$$\frac{k-q+1}{(k-q)(q-1)} < \left(\frac{1}{4}\right)^{\rho}\frac{r_k}{\rho} = T.$$

Let q^* solve $2/(q^*-1) = T$ and let $q \ge q^*+1$. Then since $k-q \ge 1$, it follows that

$$\frac{k-q+1}{(k-q)(q-1)} \le \frac{2}{q-1} < T$$

and thus for any value of $k \ge 2$ and any reserve prices posted by the other $k-q$ sellers who did not post r_k, if $q \ge q^*+1$, then seller k has an incentive to reduce his reserve price.

Next consider the case where $\lim_{l\to\infty} q \le q^* < \infty$. Note that in this case it is sufficient to show that the left hand derivative of R_k (when r_k is maximal) is negative for all $q \le q^*$ if $k > \underline{k}(q)$, because then letting $\underline{k} = \max\{\underline{k}(1), \underline{k}(2), \ldots, \underline{k}(q^*)\}$ may serve as a uniform bound, so that convergence is uniform in q.[19] Let η denote the probability that a given other buyer does not visit seller $k-q$. Then

$$\eta = 1 - \frac{F(t_{k-q}) - F(t_{k-q-1})}{k-q} - \frac{F(t_{k-q+1}) - F(t_{k-q})}{k-q+1} - \frac{1 - F(t_{k-q+1})}{k} \tag{34}$$

by using Lemma 1. A similar argument as before (27) implies that

$$u(x) = \eta^{n-1}(t_{k-q-1} - r_{k-q}) + (n-1)\frac{F(x) - F(t_{k-q-1})}{k-q}\eta^{n-2}E[y \mid y \in [t_{k-q-1}, x]] + \cdots.$$

Similar argument as after equation (27) implies that all the terms except for the first one are higher order in t_{k-q-1} when x is close to t_{k-q-1}. Therefore,

$$\left.\frac{\partial u(x)}{\partial t_{k-q}}\right|_{x=t_{k-q-1}} = (n-1)\eta^{n-2}(t_{k-q-1} - r_{k-q})\frac{d\eta}{dt_{k-q}}, \tag{35}$$

where $d\eta/dt_{k-q}$ stands for the derivative of η with respect to t_{k-q}, taking indirect effects through t_{k-q-1} and t_{k-q+1} into account.

Using (35) and Lemma 7 implies that $\partial\eta/\partial t_{k-q} \leq 0$. This implies that

$$\frac{f(t_{k-q})}{(k-q)(k-q+1)} \geq -\frac{f(t_{k-q+1})(q-1)}{k(k-q+1)}\frac{\partial t_{k-q+1}}{\partial t_{k-q}} + \frac{f(t_{k-q-1})}{(k-q)}\frac{\partial t_{k-q-1}}{\partial t_{k-q}}. \tag{36}$$

Lemma 7 also implies that $\partial t_{k-q-1}/\partial t_{k-q} \geq 0$ and thus it follows from the last inequality that

$$\frac{f(t_{k-q})}{(k-q)(k-q+1)} \geq -\frac{f(t_{k-q+1})(q-1)}{k(k-q+1)}\frac{\partial t_{k-q+1}}{\partial t_{k-q}}$$

or

$$\frac{f(t_{k-q})k}{f(t_{k-q+1})(k-q)(q-1)} \geq -\frac{\partial t_{k-q+1}}{\partial t_{k-q}}. \tag{37}$$

Also, by Lemma 7 we have $0 \geq \partial t_{k-q+1}/\partial t_{k-q}$ and thus

$$0 \geq \frac{\partial t_{k-q+1}}{\partial t_{k-q}} \geq -\frac{f(t_{k-q})k}{f(t_{k-q+1})(k-q)(q-1)}. \tag{38}$$

Equation (37) implies that $\partial t_{k-q+1}/\partial t_{k-q}$ is bounded in absolute value. Formula (36) implies together with Lemma 7 that

$$\frac{f(t_{k-q})}{k-q+1} \geq f(t_{k-q-1})\frac{\partial t_{k-q-1}}{\partial t_{k-q}}$$

and thus

$$\lim_{l\to\infty}\frac{\partial t_{k-q-1}}{\partial t_{k-q}} = 0. \tag{39}$$

[19] Convergence is uniform in the reserve prices posted by the other sellers just like in Case 1.

Now, we establish that $\lim \partial u(x)/\partial t_{k-q}|_{x=t_{k-q-1}} = 0$, which is equivalent to (see (35)) $\lim(n-1)\, d\eta/dt_{k-q} = 0$. By definition,

$$(n-1)\frac{d\eta}{dt_{k-q}} = (n-1)\frac{\partial\eta}{\partial t_{k-q}} + (n-1)\frac{\partial\eta}{\partial t_{k-q+1}}\frac{\partial t_{k-q+1}}{\partial t_{k-q}} + (n-1)\frac{\partial\eta}{\partial t_{k-q-1}}\frac{\partial t_{k-q-1}}{\partial t_{k-q}}.$$

Using formulas (34), (38), and (39) together with $q \leq q^*$ implies that, indeed, $\lim(n-1)\, d\eta/dt_{k-q} = 0$ and thus $\lim \partial u(x)/\partial t_{k-q}|_{x=t_{k-q-1}} = 0$. The envelope theorem implies that for all $x > t_{k-q-1}$,

$$u(x) = u(t_{k-q-1}) + \int_{t_{k-q-1}}^{x} \Omega(y)\, dy,$$

where $\Omega(y)$ denotes the winning probability of type y. It is straightforward to establish that $\lim \partial\Omega(y)/\partial t_{k-q} = 0$ for all $y > t_{k-q-1}$, and thus we have for all $x > t_{k-q}$ that

$$\lim_{l\to\infty} \frac{\partial u(x)}{\partial t_{k-q}} = 0.$$

To complete the proof it is sufficient to show that $\lim \partial W_k/\partial t_{k-q} > 0$, which can be done following similar arguments as in the other cases. □

Proof of Theorem 2. Let us duplicate the revenue formula from the main text for our convenience:

$$\begin{aligned}
R_1 &= \overline{R}_1(y) \\
&= \frac{n}{k}\Bigg[\int_y^{a+1} z(n-1)\left(1 - \frac{1-F(z)}{k}\right)^{n-2}\frac{1-F(z)}{k} f(z)\, dz \\
&\qquad\qquad + (1-F(y))\Bigg\{\left(1 - \frac{1}{k} - \frac{F(y)}{k(k-1)}\right)^{n-1} r \\
&\qquad\qquad + \int_a^y \frac{n-1}{k-1} z\left(1 - \frac{1}{k} - \frac{F(y)}{k(k-1)} + \frac{F(z)}{k-1}\right)^{n-2} f(z)\, dz\Bigg\}\Bigg].
\end{aligned}$$

After taking a derivative and evaluating it at $y = a$, one obtains

$$\overline{R}_1'(a) = \frac{n}{k^2(k-1)}\big[(n-1)a - r(n-1+(k-1)^2)\big].$$

Therefore, the first order condition $\overline{R}_1'(a) \leq 0$ becomes $r \geq r^* = (n-1)a/((n-1)+(k-1)^2)$ as was stated in the main text. The first order condition corresponding to function $\widehat{R}_1$ can be calculated in a similar fashion.

To characterize the second order condition, we take a derivative of the revenue with respect to y and substitute $r = r^* = (n-1)a/((n-1)+(k-1)^2)$ to obtain

$$\begin{aligned}
\overline{R}_1'(y) = \frac{nf(y)}{k}\Bigg[&-y(n-1)\left(1-\frac{1-F(y)}{k}\right)^{n-2}\frac{1-F(y)}{k} - \left(1-\frac{1}{k}-\frac{F(y)}{k(k-1)}\right)^{n-1} r^* \\
&- (1-F(y))\frac{n-1}{k(k-1)}\left(1-\frac{1}{k}-\frac{F(y)}{k(k-1)}\right)^{n-2} r^* \\
&+ (1-F(y))\frac{n-1}{k-1}y\left(1-\frac{1}{k}+\frac{F(y)}{k}\right)^{n-2} \\
&- \int_a^y \frac{n-1}{k-1}z\left(1-\frac{1}{k}-\frac{F(y)}{k(k-1)}+\frac{F(z)}{k-1}\right)^{n-2} f(z)\,dz \\
&- (1-F(y))\int_a^y \frac{n-1}{k-1}z\frac{n-2}{k(k-1)}\left(1-\frac{1}{k}-\frac{F(y)}{k(k-1)}+\frac{F(z)}{k-1}\right)^{n-3} f(z)\,dz\Bigg].
\end{aligned}$$

Let γ denote the bracketed term, i.e., $\gamma(y) = \overline{R}_1'(y)/(nf(y)/k)$. For the second order condition to hold, it is necessary and sufficient that $\gamma'(a) \leq 0$, since that is equivalent to showing that $\overline{R}_1'$ has the correct sign in a neighborhood of $y = a$. Then we need to inspect

$$\begin{aligned}
\gamma'(a) = -(n-1)\left(1-\frac{1}{k}\right)^{n-2}\frac{1}{k} &- \frac{(n-1)(n-2)}{k^2}af(a)\left(1-\frac{1}{k}\right)^{n-3} \\
&+ \frac{(n-1)}{k}af(a)\left(1-\frac{1}{k}\right)^{n-2} + \frac{n-1}{k(k-1)}r^* f(a)\left(1-\frac{1}{k}\right)^{n-2} \\
&+ \frac{n-1}{k(k-1)}\left(1-\frac{1}{k}\right)^{n-2} r^* f(a) + \frac{(n-1)(n-2)}{k^2(k-1)^2}\left(1-\frac{1}{k}\right)^{n-3} r^* \\
&+ \frac{n-1}{k-1}\left(1-\frac{1}{k}\right)^{n-2} - \frac{n-1}{k-1}af(a)\left(1-\frac{1}{k}\right)^{n-2} \\
&+ \frac{(n-1)(n-2)}{k(k-1)}af(a)\left(1-\frac{1}{k}\right)^{n-3} \\
&- \frac{n-1}{k-1}af(a)\left(1-\frac{1}{k}\right)^{n-2} - \frac{n-1}{k-1}af(a)\frac{n-2}{k(k-1)}\left(1-\frac{1}{k}\right)^{n-3}.
\end{aligned}$$

After substituting $r^* = a(n-1)/((n-1)+(k-1)^2)$, this expression can be rewritten as

$$\gamma'(a) = af(a)A(n,k) - B(n,k).$$

It turns out that $A(n,k) < 0$, $B(n,k) > 0$ and thus the condition that $\gamma'(a) \leq 0$ becomes $af(a) \geq a^* = -B(n,k)/A(n,k)$. Moreover, one can establish that the cutoff value is indeed $a^* = (k-1+(n-k)/k)/(k^2-2+n-k)$. The details of how to calculate these values are available in a supplementary file on the journal website, http://econtheory.org/supp/538/supplement.pdf. □

References

Burguet, Roberto and József Sákovics (1999), "Imperfect competition in auction designs." *International Economic Review*, 40, 231–247. [241, 242, 243, 244, 253, 257, 258]

Dasgupta, Partha and Eric S. Maskin (1986), "The existence of equilibrium in discontinuous economic games, I: Theory." *Review of Economic Studies*, 53, 1–26. [246]

Galenianos, Manolis and Philipp Kircher (2009), "On the game-theoretic foundations of competitive search equilibrium." Unpublished paper, Department of Economics, Pennsylvania State University. [241]

Hernando-Veciana, Angel (2005), "Competition among auctioneers in large markets." *Journal of Economic Theory*, 121, 107–127. [242, 252, 256]

McAfee, R. Preston (1993), "Mechanism design by competing sellers." *Econometrica*, 61, 1281–1312. [241]

Peters, Michael (1991), "Ex ante price offers in matching games: Non-steady states." *Econometrica*, 59, 1425–1454. [241]

Peters, Michael (1997), "A competitive distribution of auctions." *Review of Economic Studies*, 64, 97–123. [241]

Peters, Michael (2000), "Limits of exact equilibria for capacity constrained sellers with costly search." *Journal of Economic Theory*, 95, 139–168. [241]

Peters, Michael and Sergei Severinov (1997), "Competition among sellers who offer auctions instead of prices." *Journal of Economic Theory*, 75, 141–179. [241]

Submitted 2009-3-19. Final version accepted 2010-1-11. Available online 2010-1-11.