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On the number of critical equilibria separating two equilibria

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It is shown that two arbitrary equilibria in the general equilibrium model without sign restrictions on endowments can be joined by a continuous equilibrium path that contains at most two critical equilibria. This property is strengthened by showing that regular equilibria having an index equal to 1, a necessary condition for stability, can be joined by a path containing no critical equilibrium. These properties follow from the real-algebraic nature of the set of critical equilibria in any fiber of the equilibrium manifold.

Keywords. Equilibrium prices, equilibrium manifold, equilibrium path, critical equilibrium, catastrophe.

JEL classification. D41, D51.

1. Introduction

An equilibrium is by definition regular if the sufficient condition stated by the implicit function theorem for equilibrium prices to be locally a smooth function of the fundamentals defining the economy is satisfied Debreu (1970, 1976). A contrario, a critical equilibrium is an equilibrium that is not regular. At a critical equilibrium, the equilibrium price selection function may even fail to depend continuously on the fundamentals. A singular economy is an economy that possesses at least one critical equilibrium. A regular economy is an economy that is not singular. In other words, a regular economy has no critical equilibria. Debreu proved that the set of singular economies is closed with measure zero in the space of economies, which is equivalent to the set of regular economies being open with full measure. An economy picked up at random is therefore singular with probability 0, in which case one does not have to worry about singular economies. The situation is different when economies evolve by following continuous paths instead of remaining stuck in one point. The set of regular economies is path-connected if and only if all economies have a unique equilibrium, a property that is not generic on preferences (Ghiglino and Tvede 1997). Therefore, for an open and dense set of preferences, the set of regular economies has several path-connected components. It then suffices to pick up two regular economies in two different path-connected components to see that all the continuous paths linking these two economies intersect the set of singular economies. The price selection mechanism is generally discontinuous

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when the economy crosses the set of singular economies along these paths (Thom 1975, Arnol'd 1992).

The path-connectedness of the equilibrium manifold implies, however, that there always exist continuous paths linking two arbitrary equilibria, making it possible for the fundamentals and their associated equilibrium prices to move continuously between two different sets of values (Balasko 1975a, 1975b). But how do we implement such equilibrium paths? There is a real problem only at critical equilibria because of the failure of the implicit function theorem at these equilibria. As with singular economies, the set of critical equilibria is a closed subset of measure zero of the equilibrium manifold (Balasko 1992). But here also, this is not sufficient to prevent continuous paths from intersecting the set of critical equilibria. In the absence of any other criterion, equilibrium paths should therefore be selected so that they contain the smallest number of critical equilibria. In a recent paper, Loi and Matta (2008) show that in the exchange model, two regular equilibria can always be joined by a continuous path that contains only a finite number of critical equilibria. This interesting result, however, falls short of giving us an idea on the number of these critical equilibria.

The minimal number of critical equilibria over all the equilibrium paths linking two equilibria defines a "distance" or, better, a pseudodistance on the equilibrium manifold. (It is not a distance because two different regular equilibria that belong to the same path-connected component (of the set of regular equilibria) can be joined by a path with no critical equilibrium, which makes their pseudodistance equal to zero.) The main goal of this paper is therefore to give an upper bound on this pseudodistance. The model used for this purpose is the pure exchange general equilibrium model with constant total resources and no sign restrictions on endowments, a model that is standard in this kind of questions. The fact that total resources are fixed is justified by the lack of control for this parameter. The possibility of negative coordinates for individual endowments is a way to accommodate in this relatively frugal model some aspects of financial markets, even if the latter are generally handled with the help of far more complex models.

The pseudodistance is first shown to be finite-valued on the full equilibrium manifold (i.e., not only on the subset of regular equilibria). This extends to arbitrary equilibria the property proved for regular equilibria by Loi and Matta (2008). This finiteness property is then improved by showing that the pseudodistance is in fact less than or equal to 2. Another result deals with regular equilibria that have an index equal to 1, a necessary condition for (tatonnement) stability. It is shown that the pseudodistance of two equilibria with index equal to 1 is equal to 0. The paper ends by extending the definition of the pseudodistance to equilibria restricted to have only strictly positive endowments. There is still a finite upper bound on that pseudodistance, but the question remains open as to whether two equilibria with index equal to 1 have a pseudodistance equal to 0. All these results are derived from the real-algebraic nature of the set of critical equilibria in every fiber of the equilibrium manifold.

The paper is organized as follows. Section 2 contains the main definitions and sets the notation. Section 3 is devoted to defining the pseudodistance of two equilibria and to showing that this pseudodistance is not only finite but bounded from above. Section 4 shows that the pseudodistance of two equilibria is less than or equal to 2. Section 5

proves that the pseudodistance of two regular equilibria with index 1 is equal to 0, i.e., the two equilibria belong to the same path-connected component of the set of regular equilibria. Section 6 deals with the extension of the pseudodistance to the case where endowments are restricted to be strictly positive. Section 7 is devoted to concluding comments, while Appendixes end this paper. The technical aspects of the proofs of the main results of this paper are placed in Appendix A. Properties of regular, critical, and no-trade equilibria are recalled in Appendix B.

The mathematical prerequisites for reading this paper are basic knowledge of linear algebra, the fundamental theorem of algebra, i.e., the property that a real polynomial of degree n has at most n real roots (see, for example, Courant and Robbins 1996) and the partition of the set $\mathrm{GL}(n,\mathbb{R})$ of invertible real matrices of order n into two path-connected components consisting of matrices with positive (resp. negative) determinant (see Chevalley 1946). All the properties of the equilibrium manifold and of regular and critical equilibria used in this paper are recalled in Appendix B. No knowledge of differential topology is necessary, because all smooth manifolds considered in the current paper are diffeomorphic to Euclidean spaces.

2. Exchange economies: Definitions and a few properties

2.1 *Goods and prices*

Let ℓ denote the number of goods. All goods are divisible. The price vector $p = (p_1, \dots, p_\ell)$ is normalized by the numeraire convention $p_\ell = 1$. The set of numeraire normalized price vectors is denoted by $S = \{p \in \mathbb{R}^\ell_{++} \mid p_1 > 0, \dots, p_{\ell-1} > 0, \text{ and } p_\ell = 1\}$.

2.2 *Consumption sets and preferences*

Consumer i's consumption set is the strictly positive orthant $X = \mathbb{R}^\ell_{++}$. Consumer i's preferences are defined by a smooth utility function $u_i : X \to \mathbb{R}$ that satisfies the following assumptions that are standard in the literature on smooth economies. 1. Smooth monotonicity: $Du_i(x_i) \in X$. 2. Smooth strict quasi-concavity: the restriction of the quadratic form $Z \in \mathbb{R}^\ell \to Z^T D^2 u_i(x_i) Z$ to the hyperplane $Z^T Du_i(x_i) = 0$ of $\mathbb{R}^\ell$ is negative definite. 3. The indifference surface $\{y_i \in X \mid u_i(y_i) = u_i(x_i)\}$ is closed in $\mathbb{R}^\ell$ for any $x_i \in X$.

2.3 *Demand functions*

Maximization of utility $u_i(x_i)$ subject to the budget constraint $p \cdot x_i \le w_i$ for $w_i > 0$ has a unique solution denoted by $f_i(p, w_i)$. The demand function $f_i : S \times \mathbb{R}_{++} \to X$ is then a smooth map that satisfies Walras' law $p \cdot f_i(p, w_i) = w_i$. For properties of demand functions used in the current paper, see Appendix B.

2.4 *The exchange model*

The exchange model is defined by m consumers characterized by their preferences and endowments. Preferences and total resources are fixed. An exchange economy is identified by its endowment vector $\boldsymbol{\omega} = (\omega_1, \dots, \omega_m) \in (\mathbb{R}^\ell)^m$, where the total resource vector

$r = \sum_i \omega_i$ is fixed. Let $F = \{\boldsymbol{\omega} \in (\mathbb{R}^\ell)^m \mid \sum_i \omega_i = r\}$ denote the set of these endowments compatible with the total resource vector $r \in \mathbb{R}^\ell_{++}$. The endowment vector $\boldsymbol{\omega}$ is assumed to vary in some subset of F that is known as the endowment set. This endowment set is denoted by Ω and is defined shortly.

2.4.1 *Equilibrium* The pair $(p, \boldsymbol{\omega}) \in S \times F$ is an *equilibrium* if the consumer's wealth $w_i = p \cdot \omega_i$ is strictly positive for $i = 1, \ldots, m$ and the equilibrium equation

$$\sum_i f_i(p, p \cdot \omega_i) = \sum_i \omega_i = r$$

is satisfied.

Note that consumers can be endowed with negative quantities of some goods. These negative quantities can be interpreted as debts contracted toward the market. These negative endowments contribute negatively to consumers' wealth. The endowment vector $\boldsymbol{\omega}$ and the price vector $p \in S$ are nevertheless such that each consumer has a strictly positive net wealth at equilibrium and the vector of total resources $\sum_i \omega_i = r$ is strictly positive.

2.4.2 *Equilibrium manifold E* The *equilibrium manifold E* is the subset of $S \times F$ consisting of all equilibria. Recall that the equilibrium manifold E is defined for the fixed vector of total resources $r \in \mathbb{R}^\ell_{++}$.

2.4.3 *Endowment set* Ω By definition, the *endowment set* Ω is the image of the equilibrium manifold E by the projection map $\pi : E \to F$, i.e., $\Omega = \pi(E) \subset F$. The *endowment set* Ω therefore coincides with the set of endowment vectors (with possibly negative coordinates) for which there exists at least one equilibrium. Note that the projection map $\pi : E \to F$ is not a surjection, since no equilibrium exists for negative individual endowment vectors (the consumer's wealth then cannot be strictly positive). The equivalent of an existence theorem when working with the endowment set Ω (where equilibrium exists by definition) becomes the search for characterizations of that set or, in the absence of complete characterizations, the identification of sufficient conditions for the endowment vector $\boldsymbol{\omega}$ to belong to the endowment set Ω. Global topological properties can shed some light on this characterization question. In that regard, path-connectedness is passed on from the equilibrium manifold E to the endowment set Ω, the natural projection $\pi : E \to F$ being a continuous map.

2.4.4 *The set of strictly positive endowments* Ω_{++} The set of strictly positive endowments Ω_{++} is the subset of F consisting of individual endowments $\omega_i \in \mathbb{R}^\ell_{++}$ that are strictly positive for all consumers and that sum up to the vector of total resources $r \in \mathbb{R}^\ell_{++}$.

The inclusion $\Omega_{++} \subset \Omega$ follows from the existence of equilibrium for exchange economies with strictly positive individual endowment vectors.

2.4.5 *An illustration: The* $(\ell, m) = (2, 2)$ *case* The $(\ell, m) = (2, 2)$ case with fixed total resources lends itself to a nice geometrical representation thanks to the Edgeworth box. The contract curve represents the set of Pareto optima and is the projection of the set

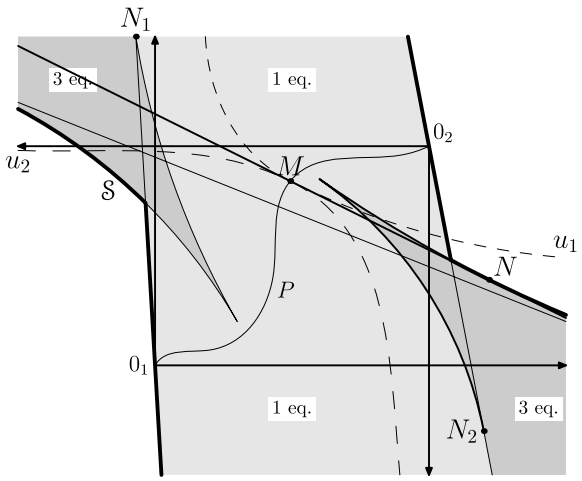

FIGURE 1. The endowment set Ω and the Edgeworth box.

of no-trade equilibria T. (See Section 3.1.1 for their definition.) With every Pareto optimum M is associated the tangent line $D(M)$ at the point M to the two agents' indifference curves. The collection of these lines $D(M)$ generates the endowment set Ω when the point M is varied in the contract curve $P = \pi(T)$. The set Ω is represented by the area in grey in Figure 1. These lines $D(M)$ are also the collection of tangent lines to the set of singular values of the natural projection $\pi : E \to F$, a set that is denoted by $\mathcal{S}$. The set $\mathcal{S}$ is therefore the envelope of the budget lines $D(M)$. On the figure, the contact point of $D(M)$ with $\mathcal{S}$ is denoted by N. The lighter grey area represents the set of endowments with a unique equilibrium and the darker grey areas represent those with multiple equilibria. The figure illustrates a case where the set of endowments with multiple equilibria is not path-connected.

The set Ω_{++} is just the interior of the rectangle known as the Edgeworth box.

3. Pseudodistance on the equilibrium manifold

3.1 *Definitions*

Let x and x' be two equilibria. Let $\gamma_{xx'}$ be a continuous path in the equilibrium manifold E linking these two points. Let $N(\gamma_{xx'})$ denote the number of critical equilibria along the path $\gamma_{xx'}$.

DEFINITION 1. Let $d(x, x') = \inf N(\gamma_{xx'})$ be the *minimal number of critical equilibria* over all continuous paths $\gamma_{xx'}$ linking x and x' in the equilibrium manifold E.

Our goal in this section is to show that $d(x, x')$ defines a pseudodistance on the equilibrium manifold E. The proof of this property is based on the (b, Y) coordinate system for the equilibrium manifold E. Before defining this coordinate system, we briefly recall the definitions of a no-trade equilibrium and of the linear fiber of the equilibrium manifold through a given no-trade equilibrium.

3.1.1 *No-trade equilibrium* A no-trade equilibrium $(p, \boldsymbol{\omega})$ is such that $\omega_i = f_i(p, p \cdot \omega_i)$ for $i = 1, \ldots, m$. Let $w_i = p \cdot \omega_i$ with $i = 1, \ldots, m$ and $b = (p, w_1, \ldots, w_m) \in B = S \times \mathbb{R}^m_{++}$. We then have $\sum_i f_i(p, w_i) = \sum_i \omega_i = r$. The subset of B consisting of the price–income vectors $b = (p, w_1, \ldots, w_m)$ such that $\sum_i f_i(p, w_i) = r$ is denoted by $B(r)$.

Let $f(b) = (p, f_1(p, w_1), \ldots, f_m(p, w_m))$ denote the no-trade equilibrium associated with the price–income vector $b \in B(r)$. Let $T = \{f(b) \in E \mid b \in B(r)\}$ denote the set of no-trade equilibria compatible with the total resources $r \in \mathbb{R}^\ell_{++}$. The map $b \to f(b)$ is a diffeomorphism between $B(r)$ and the set of no-trade equilibria T, and both sets are diffeomorphic to $\mathbb{R}^{m-1}$, hence are path-connected (Balasko 2009, Proposition A.6.2 combined with Proposition 3.3.1).

3.1.2 *The fibers of the equilibrium manifold* Let $b = (p, w_1, \ldots, w_m) \in B$ be some price–income vector. The fiber $V(b)$ is the set of pairs $(p, \boldsymbol{\omega}) \in S \times (\mathbb{R}^\ell)^m$ such that

$$V(b) = \left\{ (p, \boldsymbol{\omega}) \in S \times \mathbb{R}^{\ell m} \;\middle|\; p \cdot \omega_i = w_i,\ i = 1, \ldots, m; \sum_i \omega_i = \sum_i f_i(p, w_i) \right\}.$$

Obviously, the elements of the fiber $V(b)$ are all equilibria. The fiber $V(b)$ is an affine subspace of $S \times (\mathbb{R}^\ell)^m$ of dimension $(\ell - 1)(m - 1)$. The fiber $V(b)$ is a subset of the equilibrium manifold E associated with the total resources $r \in \mathbb{R}^\ell_{++}$ if and only if $b \in B(r)$.

3.1.3 *The (b, Y) coordinate system for the equilibrium manifold* Coordinates for the points of the fiber $V(b)$ are provided by the coefficients of the $(m - 1) \times (\ell - 1)$ (real) matrix Y, where

$$y_i^j = \omega_i^j - f_i^j(p, w_i)$$

for $1 \leq i \leq m - 1$ and $1 \leq j \leq \ell - 1$. Every equilibrium $x = (p, \boldsymbol{\omega}) \in E$ can then be represented by its coordinates (b, Y), which we write as $x = (p, \boldsymbol{\omega}) = (b, Y)$ (Balasko 2009, Section 4.4.4).

The unique no-trade equilibrium of the fiber $V(b)$ is the equilibrium $f(b) = (p, f_1(p, w_1), \ldots, f_m(p, w_m))$ represented by its coordinates $(b, 0)$ in the (b, Y) coordinate system. The (b, Y) coordinate system reflects the structure of the equilibrium manifold consisting of linear fibers parameterized by the no-trade equilibria.

3.2 *Finiteness of $d(x, x')$*

LEMMA 1. *The inequality $d(x, x') \leq 2 \times \inf(\ell, m) - 2$ is satisfied for any $(x, x') \in E^2$.*

PROOF. Let $x = (p, \boldsymbol{\omega}) = (b, Y)$ and $x' = (p', \boldsymbol{\omega}') = (b', Y')$ be two equilibria. Let $V(b)$ (resp. $V(b')$) be the fiber associated with the price–income vector b (resp. b') in $B(r)$. Let $f(b)$ (resp. $f(b')$) denote the unique no-trade equilibrium of the fiber $V(b)$ (resp. $V(b')$). The line segment $[(b, Y), (b, 0)]$ (resp. $[(b', Y'), (b', 0)]$) contains at most $\inf(\ell, m) - 1$ critical equilibria by Corollary 1 in Appendix A. The set of no-trade equilibria $T = f(B(r))$ being path-connected (Section 3.1.1 above), the two no-trade equilibria $f(b)$ and $f(b')$

can be joined by a continuous path $\gamma_{f(b)f(b')} \subset T$. That path contains no critical equilibrium since all no-trade equilibria are regular (Appendix B.6). The continuous path defined by first following the line segment $[(b, Y), (b, 0)]$, continuing with the path $\gamma_{f(b)f(b')}$, and ending with the line segment $[(b', 0), (b', Y')]$ links x to x' and contains at most $2 \times \inf(\ell, m) - 2$ critical equilibria. This implies the inequality $d(x, x') \leq 2\inf(\ell, m) - 2$. □

PROPOSITION 1. *The function $(x, x') \to d(x, x')$ is a finite-valued pseudodistance on the equilibrium manifold E.*

PROOF. That $d(x, x')$ is finite-valued is an obvious consequence of Lemma 1. The properties satisfied by a pseudodistance are $d(x, x') \geq 0$, $d(x, x') = d(x', x)$, and $d(x, x') + d(x', x'') \leq d(x, x'')$ (triangle inequality). At variance with a distance, $d(x, x') = 0$ does not necessarily imply the equality $x = x'$.

Only the triangle inequality requires a proof; the other two properties are obvious. Let x, x', and x'' be three equilibria. Then $d(x, x')$ and $d(x', x'')$ are the numbers of critical equilibria on some equilibrium paths $\gamma_{xx'}$ and $\gamma_{x'x''}$. The path obtained by combining the paths $\gamma_{xx'}$ and $\gamma_{x'x''}$ joins x to x'' and contains at most $d(x, x') + d(x', x'')$ critical equilibria (some critical equilibria may be common to the two paths), from which follows the inequality $d(x, x'') \leq d(x, x') + d(x', x'')$. □

4. UPPER BOUND ON THE PSEUDODISTANCE $d(x, x')$

It follows from Lemma 1 that $2 \times \inf(\ell, m) - 2$ is an upper bound for the pseudodistance $d(x, x')$ on the equilibrium manifold E. By introducing the ranks $\kappa(x)$ and $\kappa(x')$ of the equilibria x and x' (see Lemma 12 in Appendix B), it is possible to improve on Lemma 1 with the following statement.

LEMMA 2. *The inequality $d(x, x') \leq \kappa(x) + \kappa(x') - 2$ is satisfied for any $(x, x') \in E^2$*

PROOF. It suffices to observe that, in the proof of Lemma 1, the line segment $[(b, Y), (b, 0)]$ (resp. $[(b', 0), (b', Y')]$) contains at most $\kappa(x) = \kappa(b, Y)$ (resp. $\kappa(x') = \kappa(b', Y')$) critical equilibria by Lemma 13 in Appendix B. □

This lower upper bound is significantly improved in the following proposition.

PROPOSITION 2. *The inequality $d(x, x') \leq 2$ is satisfied for all $(x, x') \in E^2$.*

PROOF. It follows from Lemmas 8 and 9 in Appendix A that there exists a continuous path linking $x = (b, Y)$ to the no-trade equilibrium $f(b) = (b, 0)$ by a continuous path $\gamma_{(b,Y),(b,0)}$ that contains no more than one critical equilibrium.

Similarly, there exists a continuous path $\gamma_{(b',Y'),(b',0)}$ in the fiber $V(b')$ linking $x' = (p', \boldsymbol{\omega}') = (b', Y')$ to $f(b') = (b', 0)$ and containing no more than one critical equilibrium.

The path obtained by combining the continuous path $\gamma_{(b,Y),(b,0)}$, the path $\gamma_{(b,0)(b',0)}$ in the set of no-trade equilibria T used in the proof of Proposition 1, and the reversed path to the path $\gamma_{(b',Y'),(b',0)}$ then links the two equilibria $x = (b, Y)$ and $x' = (b', Y')$, and contains no more than two critical equilibria. □

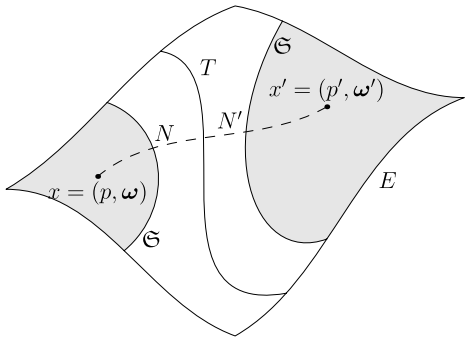

FIGURE 2. The path-connected components of the set of regular equilibria.

PROPOSITION 3. *The inequality $d(x, x') \leq 2$ cannot be improved for arbitrary x and x' in E.*

PROOF. The proof proceeds by way of an example set for $(\ell, m) = (2, 2)$ and constant total resources $r \in \mathbb{R}^2_{++}$, the setup of the standard Edgeworth box. The set of regular equilibria $\mathfrak{R}$ is partitioned into several path-connected components. See Figure 2. The equilibria with index equal to $+1$ make up one of the path-connected components of the set $\mathfrak{R}$ as follows from the forthcoming Proposition 4.

The example is designed in such a way that the set made of the regular equilibria $x \in \mathfrak{R}$ with index $\iota(x) = -1$ is not path-connected. For details, see Appendix A.6, in particular, Lemmas 10 and 11.

Pick two equilibria x and x' belonging to two different path-connected components of $\mathfrak{R}$ having index $\iota(x) = \iota(x') = -1$. Then any path linking these two equilibria must go through the path-connected component of $\mathfrak{R}$ made of the equilibria with index 1. Such a path must then intersect the set of critical equilibria $\mathfrak{S}$ in at least two points. □

5. PATH-CONNECTEDNESS OF THE SET OF REGULAR EQUILIBRIA WITH INDEX 1

PROPOSITION 4. *The pseudodistance of two regular equilibria x and x' with index $\iota(x) = \iota(x') = 1$ is equal to 0: $d(x, x') = 0$.*

PROOF. The proposition is equivalent to the path-connectedness of the set of regular equilibria $\mathfrak{R}$ with index $\iota(x) = 1$. It follows from Lemma 7 that the intersection of that set with the fiber $V(b)$ is path-connected for every $b \in B(r)$. Therefore, the regular equilibrium x (with index $\iota(x) = 1$) can be linked by a continuous path to the no-trade equilibrium $f(b)$ of the fiber $V(b)$ containing x. A similar construction is made with the regular equilibrium x' that is linked to the no-trade equilibrium $f(b')$. It then suffices to link the two no-trade equilibria $f(b)$ and $f(b')$ by a continuous path in the set of no-trade equilibria T as in the proof of Lemma 1. □

6. The case of strictly positive endowments

In this section, all the coordinates of the endowment vector $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_m)$ are strictly positive, i.e., $\boldsymbol{\omega} \in \Omega_{++}$. Let $E_{++} = E \cap (S \times \Omega_{++})$ denote the equilibrium manifold with strictly positive endowments and let $V_{++}(b) = V(b) \cap (S \times \Omega_{++})$ denote the fiber of E_{++} associated with $b \in B(r)$.

Let $x = (p, \boldsymbol{\omega})$ and $x' = (p', \boldsymbol{\omega}')$ be two equilibria in E_{++}. The pseudodistance $d_{++}(x, x')$ is defined as the minimal number of critical equilibria over all continuous paths $\gamma_{xx'}$ linking x and x' in E_{++} (cf. Definition 1). Note the obvious inequality $d(x, x') \leq d_{++}(x, x')$.

The analog of Proposition 1 is true for $d_{++}(x, x')$, since the main argument in its proof consists of counting the number of critical equilibria along the segments $[x, f(b)]$ and $[x', f(b')]$, where $f(b)$ and $f(b')$ are the no-trade equilibria contained in the fibers through the equilibria x and x', respectively. For x and x' in E_{++}, it follows from the convexity of Ω_{++} that these segments are contained in E_{++}.

At variance with the case of endowments with no sign constraints, the integer 2 is not necessarily an upper bound on the pseudodistance $d_{++}(x, x')$. Note that $2 \times \inf(\ell, m) - 2$ is still an upper bound in that case. It is possible to improve on that bound by using the rank $\kappa(x)$ of the equilibrium $x = (p, \omega) = (b, Y)$ defined in Appendix B.10. The best bound available at the moment is given by the following proposition.

Proposition 5. *The inequality* $d_{++}(x, x') \leq \kappa(x) + \kappa(x') - 2$ *is satisfied for any* $(x, x') \in E_{++}^2$.

To prove the proposition, it suffices to observe that the proof of Lemma 2 works for the pseudodistance $d_{++}(x, x)$.

It would be interesting to get a better bound on the distance $d_{++}(x, x')$. The problem is that little is known about the number of path-connected components of the set of regular equilibria in a given fiber when endowments are constrained to be strictly positive. It is also an open problem whether the analog of Proposition 4 holds true with strictly positive endowments.

7. Conclusion

The impact of criticality is negligible for equilibria picked up randomly because the set of critical equilibria is a closed subset with measure 0 of the equilibrium manifold. When moving from one equilibrium to another along some continuous equilibrium path, the impact of criticality can be measured by the number of critical equilibria along the path. This paper shows that this impact can be limited by the appropriate choice of the continuous path between the two equilibria. For two regular equilibria, there are paths with at most two critical equilibria if there are no sign restrictions on endowments. If the two equilibria have index $+1$—a necessary condition for tatonnement stability—there is a path that avoids all critical equilibria. In other words, the set of regular equilibria with index $+1$ is path-connected. If the two regular equilibria have opposite indices (i.e., one with index $+1$ and the other -1), the two equilibria can be linked by a continuous path that contains only one critical equilibrium.

Appendixes

Appendix A deals with specific properties of the sets of regular and critical equilibria that belong to a given fiber of the equilibrium manifold, properties that are crucial to the proofs of the main results of this paper. Appendix B is devoted to recalling a few general definitions and properties of regular and critical equilibria, and, more generally, of the equilibrium manifold.

Appendix A: Some useful lemmas

A.1 *Number of critical equilibria along a segment with endpoint the corresponding no-trade equilibrium*

Let $b = (p, w_1, \ldots, w_m) \in B = S \times \mathbb{R}^m_{++}$ be an arbitrary price–income vector. The associated fiber $V(b)$ contains the no-trade equilibrium $f(b) = (b, 0)$. Let $x = (p, \boldsymbol{\omega}) = (b, Y)$ be an arbitrary equilibrium in the fiber $V(b)$. Let $\kappa(b) = \kappa(b, Y)$ denote the rank of the equilibrium (b, Y) defined in Lemma 2.

Lemma 3. *The number of critical equilibria in the line segment* $[x, f(b)] = [(b, Y), (b, 0)]$, *where* $x = (b, Y) \in V(b)$, *is less than or equal to* $\kappa(b)$, *the rank of the equilibrium* (b, Y).

Proof. The critical equilibria (b, tY) of the line segment $[(b, Y), (b, 0)]$ correspond to the roots $t \in [0, 1]$ of the polynomial function $v(t) = (-1)^{\ell-1} \det J_{\ell\ell}(b, tY)$. This polynomial function is not identically zero and its degree is less than or equal to the rank $\kappa(b)$ by Lemma 13. Therefore, the number of zeros or roots of the function $v(t)$ is less than or equal to $\kappa(b)$. □

Corollary 1. *The number of critical equilibria in the line segment* $[x, f(b)] = [(b, Y), (b, 0)]$, *where* $x = (b, Y) \in V(b)$, *is less than or equal to* $\inf(\ell, m) - 1$.

The proof follows from the inequality $\kappa(b) \leq \inf(\ell, m) - 1$.

A.2 *Alternative expression for* $\det J_{\ell\ell}(b, Y)$ *for* $b \in B$ *fixed*

The function $Y \to \det J_{\ell\ell}(b, Y)$ for $b \in B$ fixed is now expressed in a form that makes it more suitable for the study of the path-connected components of the regular equilibria in the fiber $V(b)$.

Lemma 4. *There is a* $(\ell - 1) \times (\ell - 1)$ *invertible matrix* $U(b)$ *such that* $U(b)J_{\ell\ell}(b, Y)$ *is a* $(\ell - 1) \times (\ell - 1)$ *matrix* M *whose last* $\ell - \kappa(b) - 1$ *rows define a submatrix* M_0 *that is independent of* Y. *Conversely, given any matrix* M *whose submatrix, consisting of the last* $\ell - \kappa(b) - 1$ *rows, is equal to* M_0, *there exists a* $(m - 1) \times (\ell - 1)$ *matrix* Y *such that* $J_{\ell\ell}(b, Y) = U(b)^{-1}M$.

Proof. In the expression $J_{\ell\ell}(b, Y) = J_{\ell\ell}(b, 0) + F(b)Y$ ((1) in Appendix B.9), there is no loss in generality in assuming that the first $k = \kappa(b)$ column vectors of matrix $F(b)$ are

linearly independent. These vectors are denoted by $\mathbf{f}_1, \ldots, \mathbf{f}_k$ and are completed in a base $\mathbf{f}_{k+1}, \mathbf{f}_{k+2}, \ldots, \mathbf{f}_{\ell-1}$ of $\mathbb{R}^{\ell-1}$. Let $U^{-1}(b)$ be the matrix whose column vectors in the canonical base of $\mathbb{R}^{\ell-1}$ are the coordinates of those $\ell-1$ vectors $\vec{\mathbf{f}}_1, \vec{\mathbf{f}}_2, \ldots, \vec{\mathbf{f}}_{\ell-1}$. Then the matrix product $U(b)F(b)$ takes the block form

$$U(b)F(b) = \begin{bmatrix} I_k & A \\ 0 & B \end{bmatrix},$$

where I_k is the $k \times k$ identity matrix and 0 is the $(\ell-k-1) \times k$ matrix with all coefficients equal to zero. The rank of $F(b)$ is equal to k and also to the rank of $U(b)F(b)$. Therefore, the column vectors of the submatrix $\left[\begin{smallmatrix} A \\ B \end{smallmatrix}\right]$ have to be linear combinations of the k linearly independent column vectors of the $(\ell-1) \times k$ matrix $\left[\begin{smallmatrix} I_k \\ 0 \end{smallmatrix}\right]$. This implies $B = 0$, from which follows the equality

$$U(b)F(b) = \begin{bmatrix} I_k & A \\ 0 & 0 \end{bmatrix}.$$

Then

$$U(b)J_{\ell\ell}(b, Y) = U(b)J_{\ell\ell}(b, 0) + U(b)F(b)Y.$$

This implies that the $\ell-k-1$ last rows of $U(b)J_{\ell\ell}(b, Y)$ do not depend on matrix Y that represents the equilibrium (b, Y) of the fiber $V(b)$.

Conversely, let the matrix be

$$\begin{bmatrix} N \\ M_0 \end{bmatrix},$$

where M_0 is the submatrix of $U(b)J_{\ell\ell}(b, 0)$ consisting of those fixed last $\ell-k-1$ rows. The issue is to find a $(m-1) \times (\ell-1)$ matrix Y such that

$$\begin{bmatrix} N \\ M_0 \end{bmatrix} = U(b)J_{\ell\ell}(b, 0) + \begin{bmatrix} I_k & A \\ 0 & 0 \end{bmatrix} Y.$$

It then suffices to take $Y = \left[\begin{smallmatrix} Y_0 \\ 0 \end{smallmatrix}\right]$ with

$$Y_0 = N - N_0,$$

where N_0 is the $k \times k$ principal submatrix of $U(b)J_{\ell\ell}(b, 0)$. □

REMARK 1. Lemma 4 tells us that after left multiplication by a fixed invertible matrix, the set of matrices $J_{\ell\ell}(b, Y)$ for $b \in B$ fixed can be identified with the set of $(\ell-1) \times (\ell-1)$ matrices that have $\ell-1-\kappa(b)$ fixed rows or—and this is the same thing—only $\kappa(b)$ rows that can vary. Taking the transpose of a matrix defines a homeomorphism between this set and the set of matrices with $\ell-1-\kappa(b)$ fixed columns. It is under this form that this set is studied in this appendix.

REMARK 2. Note that it also follows from (1) in Appendix B.9 that the set of matrices Y that give the same matrix $J = J_{\ell\ell}(b, Y)$ is an affine space.

A.3 *Set of matrices with a fixed number of columns*

Let n be some integer and let M be some $n \times n$ matrix. For $0 \leq k \leq n-1$, let N_1 be the $n \times (n-k)$ matrix made of the last $n-k$ columns of M. Let $\mathcal{M}(N_1)$ denote the set of $n \times n$ matrices $M = [N \;\; N_1]$, where the block matrix N_1 is fixed.

For $k = 0$, the set $\mathcal{M}(N_1)$ consists of just one element: the matrix N_1. For $k = n$, the set $\mathcal{M}(0)$ is the set of all square $n \times n$ matrices.

LEMMA 5. *The set* $\mathrm{GL}(n, R)$ *of invertible real matrices of order* n *consists of two path-connected components,* $\mathrm{GL}_+ = \{M \in \mathrm{GL}(n, \mathbb{R}) \mid \det M > 0\}$ *and* $\mathrm{GL}_- = \{M \in \mathrm{GL}(n, \mathbb{R}) \mid \det M < 0\}$.

For a proof, see, for example, Chevalley (1946).

LEMMA 6. *Let* $1 \leq k \leq n$ *and let the* $n \times (n-k)$ *matrix* N_1 *with* $\mathrm{rank}(N_1) = n-k$ *be given. The intersection* $\mathrm{GL}(n, \mathbb{R}) \cap \mathcal{M}(N_1)$ *consists of two path-connected components, namely* $\mathrm{GL}_+ \cap \mathcal{M}(N_1)$ *and* $\mathrm{GL}_- \cap \mathcal{M}(N_1)$*, the sets consisting of matrices with strictly positive and negative determinants, respectively.*

PROOF. For $k = n$, the set $\mathrm{GL}(n, \mathbb{R}) \cap \mathcal{M}(N_1)$ coincides with the linear group $\mathrm{GL}(n, \mathbb{R})$ and we can apply Lemma 5.

Let us now assume $1 \leq k \leq n-1$: the last $n-k$ columns of the matrices in $Gl(n, \mathbb{R}) \cap \mathcal{M}(N_1)$ are fixed and equal to N_1. Let $e_1, \ldots, e_k, e_{k+1}, \ldots, e_n$ be an arbitrary base of $\mathbb{R}^n$. Let f be the linear map from $\mathbb{R}^n$ into itself defined by some matrix $M \in \mathrm{GL}(n, \mathbb{R}) \cap \mathcal{M}(N_1)$. The linear map $g = f^{-1} \circ f = id_{\mathbb{R}^n}$ is the identity map of $\mathbb{R}^n$ and its matrix is the $n \times n$ identity matrix I_n. Let now $M' \in \mathrm{GL}(n, \mathbb{R}) \cap \mathcal{M}(N_1)$. It is the matrix of some linear map f' from $\mathbb{R}^n$ into itself.

The matrix of the map $g' = f^{-1} \circ f'$ takes the form

$$Z' = \begin{bmatrix} Z'_k & 0 \\ Z'_{n-k} & I_{n-k} \end{bmatrix},$$

where I_{n-k} is the $(n-k) \times (n-k)$ identity matrix and 0 is the $k \times (n-k)$ matrix with coefficients all equal to zero. Conversely, for any linear map g' from $\mathbb{R}^n$ into itself defined by a matrix Z' of the above form, the map $f' = f \circ g'$ is represented in the base $e_1, \ldots, e_k, \ldots, e_n$ by a matrix with its last $n-k$ columns fixed and defining the block matrix N_1.

Assume now $\det M^{-1} \det M' > 0$. We want to build a continuous path from the identity matrix I_n associated with the map g to the matrix Z' of g' that belongs to the set of matrices whose last $n-k$ columns make up the block matrix $\left[\begin{smallmatrix} 0 \\ I_{n-k} \end{smallmatrix}\right]$.

From $\det Z'_k = \det Z' = \det g' > 0$, Lemma 5 implies the existence of a continuous path $t \in [0,1] \to Z_k(t)$ with $Z_k(0) = I_k$, the $k \times k$ identity matrix, and $Z_k(1) = Z'_k$. Let $g(t)$ be the linear map defined by the matrix

$$Z(t) = \begin{bmatrix} Z_k(t) & 0 \\ tZ'_{n-k} & I_{n-k} \end{bmatrix}.$$

The matrix $M(t)$ of the map $f \circ g(t)$ has its last $n-k$ columns defining the block matrix N_1 that is fixed for $t \in [0,1]$. In addition, the sign of $\det M(t)$ is constant and $M(0) = M$ and $M(1) = M'$. □

A.4 *Path-connected components in the fiber $V(b)$ of the set of regular equilibria*

LEMMA 7. *For $b \in B$ such that $\kappa(b) \geq 1$, the set of regular equilibria $\mathfrak{R} \cap V(b)$ is made of two path-connected components, one component consisting of the equilibria of index $+1$ and the other consisting of equilibria with index -1. For $b \in B$ such that $\kappa(b) = 0$, all the equilibria in the fiber $V(b)$ are regular and of index $+1$.*

PROOF. If $\kappa(b) = 0$, then the Jacobian matrix $J_{\ell\ell}(b, Y)$ does not depend on Y and all the equilibria (b, Y) of the fiber $V(b)$ are regular and of index $+1$.

For $\kappa(b) \geq 1$, it follows from Lemma 4 that the set of matrices $J_{\ell\ell}(b, Y)$ becomes the set of matrices with fixed last $\ell - 1 - \kappa(b)$ columns after suitable matrix multiplication followed by transposition. In addition, given that the matrix $J_{\ell\ell}(b, 0)$ is invertible (see Appendix B.6), it then suffices to apply Lemma 6. □

A.5 *Application to paths with endpoints the no-trade equilibrium $f(b)$ of the fiber $V(b)$*

LEMMA 8. *Let $x = (p, \boldsymbol{\omega}) = (b, Y)$ be a regular equilibrium with index -1. There exists a continuous path in the fiber $V(b)$ linking x and the no-trade equilibrium $f(b) = (b, 0)$ with the property that the path intersects the set of critical equilibria $\mathfrak{S} \cap V(b)$ in just one point.*

PROOF. The polynomial function $t \to v(t) = (-1)^{\ell-1} \det J_{\ell\ell}(b, tY)$ is not identically equal to zero because $v(0) > 0$ and $v(1) < 0$. This polynomial function has a finite number of roots $t_1, \ldots, t_h$, with $1 \leq h \leq \kappa(b)$, where $\kappa(b)$ is the rank of the fiber $V(b)$. Therefore, there exists at least one root t_j such that the function $v(t)$ changes sign at that root. By this, it is meant that for $\varepsilon > 0$ small enough, $v(t')$ and $v(t'')$ have opposite signs for $t_j - \varepsilon < t' < t_j$ and $t_j < t'' < t_j + \varepsilon$. See Figure 3. This implies that the segment $[(b, t'Y), (b, t''Y)]$ intersects the set $\mathfrak{S}$ at only one point, the equilibrium (b, t_jY), and that its extremities are in the two different path-connected components of the set $\mathfrak{R} \cap V(b)$. Assuming, for example, that the index $\iota(b, t'Y)$ is equal to $+1$ and the index $\iota(b, t''Y)$ is equal to -1, there exist continuous paths linking $x = (b, Y)$ to $(b, t''Y)$ and $f(b) = (b, 0)$ to $(b, t'Y)$. The combination of these three paths gives us the required path. In case the indices of $(b, t'Y)$ and $(b, t''Y)$ are -1 and $+1$, respectively, it then suffices to link $x = (b, Y)$ to $(b, t''Y)$ and $f(b) = (b, 0)$ to $(b, t'Y)$. □

The previous result can be extended to the case where the equilibrium $x = (p, \boldsymbol{\omega}) = (b, Y)$ is critical equilibrium as follows:

LEMMA 9. *Let $x = (b, Y)$ be a critical equilibrium. There exists a continuous path in the fiber $V(b)$ linking $x = (b, Y)$ to the no-trade equilibrium $f(b) = (b, 0)$ of the fiber with the property that all its points except $x = (b, Y)$ are regular and have index $+1$.*

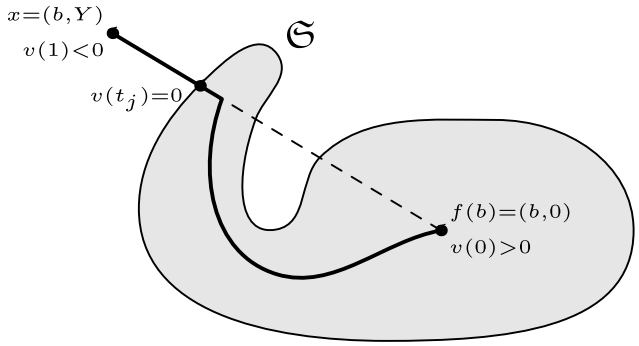

FIGURE 3. The path-connected components of $V(b)\cap\mathfrak{R}$.

PROOF. It suffices to find a line through $x=(b,Y)$ in the fiber such that $\det J_{\ell\ell}(b,Y')$ changes sign at $Y'=Y$ when matrix Y' is varied along this line.

Since the equilibrium $x=(b,Y)$ is critical, this means that the rank $\kappa(b)$ of the fiber $V(b)$ is different from zero; otherwise, $\det J_{\ell\ell}(b,Y)$ would be constant and different from zero. The set of matrices $J_{\ell\ell}(b,Y')$ is isomorphic by Lemma 4 to the set of matrices having at least one variable column with the determinant being a non-constant function of these variable columns. This implies that the determinant is not constant with respect to at least one coefficient $a(Y)$ of the matrix Y. Let us fix all the other coefficients of Y. Then $\det J_{\ell\ell}(b,Y')=A+Ba(Y')$, where $B\neq 0$. It follows from $A+Ba(Y)=\det J_{\ell\ell}(b,Y)=0$ that $\det J_{\ell\ell}(b,Y)$ changes sign at (b,Y). □

A.6 *Example of a non-path-connected set of regular equilibria with index* -1

We assume $(\ell,m)=(2,2)$ and fixed total resources $r\in\mathbb{R}^2_{++}$. Let $p=(p_1,1)$ denote the numeraire normalized price vector. The price–income vector (p,w_1,w_2) is compatible with the total resources r if $f_1(p,w_1)+f_2(p,w_2)=r$. The set $B(r)$ of price–income vectors compatible with r is diffeomorphic to the open interval $(0,1)$ and can be ordered by the (indirect) utility $u_1(f_1(p,w_1))$ of the price–income vector $b=(p,w_1,w_2)$. The set $B(r)$ is also diffeomorphic to the set of no-trade equilibria T for the fixed total resources r.

We denote by $b(t)=(p_1(t),w_1(t),w_2(t))$ the price–income vector associated with $t\in(0,1)$ and denote by $f(b(t))$ the corresponding no-trade equilibrium.

The fiber $V(b(t))$ is diffeomorphic to the set of real numbers $\mathbb{R}$. The equilibrium manifold E is therefore diffeomorphic to the Cartesian product $(0,1)\times\mathbb{R}$. The coordinate system (b,Y) of the equilibrium manifold can then be identified to the coordinates $(t,y)\in(0,1)\times\mathbb{R}$.

For $t\in(0,1)$ given, the fiber $V(b(t))$ contains at most one critical equilibrium. This critical equilibrium is defined by the coordinate $y_1^1(t)=\omega_1^1-f_1^1(p(t),w_1(t))$ satisfying the equation

$$K(b(t))+F(b(t))\bigl(\omega_1^1-f_1^1(p(t),w_1(t))\bigr)=0,$$

where $K(b(t)) < 0$ for $t \in (0,1)$ and

$$F(b(t)) = \frac{\partial f_1^1(p(t), w_1(t))}{\partial w_1} - \frac{\partial f_2^1(p(t), w_2(t))}{\partial w_2}.$$

This formula defines a function that associates with $t \in (0,1)$ the coordinate $y_1^1(t)$ (possibly equal to ∞) of the critical equilibrium of the fiber $V(b(t))$.

The sign of the Jacobian determinant $v(t,y) = -\det J_{\ell\ell}(b(t), Y)$ is positive if the point (t,y) is above the graph of the function $t \to y_1^1(t)$ and $F(b(t))$ is negative. This sign is negative if $F(b(t))$ is positive.

To get an example of a disconnected set of regular equilibria with index -1, it then suffices to have a function $y_1^1(t)$ with a graph such that the set $\{(t,y) \mid v(t,y) < 0\}$ is disconnected.

The idea is to have a graph with at least one vertical asymptote and with points above the t axis left of the vertical asymptote and below the t axis right of the vertical asymptote or, vice versa, points below the t axis left of the vertical asymptote and above the t axis and right of the vertical asymptote. The existence of such a function $y_1^1(t)$ results from the following lemma.

LEMMA 10. *There exist preferences represented by utility functions u_1 and u_2 and three price–income vector $b(t_0)$, $b(t_1)$, and $b(t_2)$ compatible with the total resources r such that*

$$u_1\big(f_1(p(t_0), w_1(t_0))\big) < u_1\big(f_1(p(t_1), w_1(t_1))\big) < u_1\big(f_1(p(t_2), w_1(t_2))\big)$$

and

$$F(b(t_0)) = -F(b(t_2)) \quad \textit{and} \quad F(b(t_1)) = 0.$$

PROOF. A simple example does the trick. Let $u_1 = u_2$. Then $f_1(p,w) = f_2(p,w)$. Pick for $p(t_1)$, the price vector supporting the allocation $r/2$ and let $w_1(t_1) = w_2(t_1) = p(t_1) \cdot r/2$. Then $\frac{\partial f_1^1}{\partial w_1}(p(t_1), w_1(t_1)) = \frac{\partial f_2^1}{\partial w_2}(p(t_1), w_2(t_1))$, which implies $F(b(t_1)) = 0$.

Now pick for $b(t_0) = (p(t_0), w_1(t_0), w_2(t_0))$ any price–income equilibrium compatible with r and such that $u_1(f_1(p(t_0), w_1(t_0))) < u_1(f_1(p(t_1), w_1(t_1)))$. Then let $b(t_2) = (p(t_2), w_1(t_2), w_2(t_2))$, where $p(t_2) = p(t_0)$, $w_1(t_2) = w_2(t_0)$, and $w_2(t_2) = w_1(t_0)$. Then

$$\begin{aligned} f_1(p(t_2), w_1(t_2)) + f_2(p(t_2), w_2(t_2)) &= f_1(p(t_0), w_2(t_0)) + f_2(p(t_0), w_1(t_0)) \\ &= f_2(p(t_0), w_2(t_0)) + f_1(p(t_0), w_1(t_0)) = r, \end{aligned}$$

which proves that $b(t_2)$ is indeed compatible with r. In addition, $F(b(t_2))$ is equal to $-F(b(t_0))$. □

LEMMA 11. *The set of regular equilibria with index -1 of an exchange economy satisfying the conditions of Lemma 10 is disconnected.*

To prove the lemma, draw the graph of the function $t \to y_1^1(t)$. The conclusion is then obvious on Figure 4.

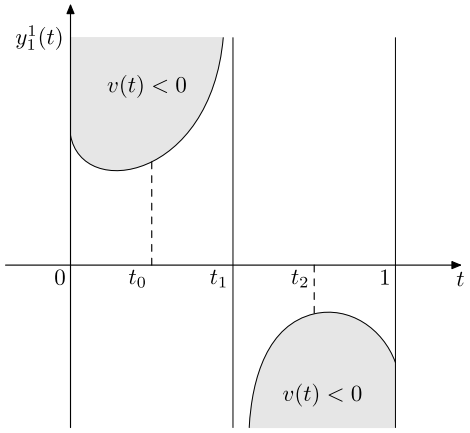

FIGURE 4. Graph of the function $t \to y_1^1(t)$.

APPENDIX B: REGULAR AND CRITICAL EQUILIBRIA

B.1 *Slutsky matrix of individual demand*

Let $\bar{f}_i(p, w_i) \in \mathbb{R}^{\ell-1}$ denote the vector defined by the first $\ell - 1$ coordinates of $f_i(p, w_i)$. The $(\ell - 1) \times (\ell - 1)$ Slutsky matrix $S_{\ell\ell} f_i(p, w_i)$ associated with the numeraire normalized price vector $p \in S$ is the Jacobian matrix of the map $p \in S \to \bar{f}_i(p, p \cdot \omega_i)$, where $\omega_i = f_i(p, w_i)$. (Recall that we then have $w_i = p \cdot \omega_i$ by Walras's law.) It is a standard property of consumer theory that this matrix is negative definite. (See, e.g., Balasko 1988, Theorem 2.5.9.)

B.2 *The Jacobian matrix of aggregate excess demand*

We denote by $\bar{z}(p, \boldsymbol{\omega}) \in \mathbb{R}^{\ell-1}$ the aggregate excess demand $\sum_i \bar{f}_i(p, p \cdot \omega_i) - \sum_i \bar{\omega}_i$ in the first $\ell - 1$ goods.

Let $(p, \boldsymbol{\omega}) = (b, Y) \in E$ be an equilibrium. The map $p' \to \bar{z}(p', \boldsymbol{\omega})$ is obviously defined in a neighborhood of the price vector $p \in S$. The Jacobian matrix of this map at $p' = p$ is denoted by $J_{\ell\ell}(p, \boldsymbol{\omega}) = J_{\ell\ell}(b, Y)$.

B.3 *Regular equilibria*

By definition, the equilibrium $x = (p, \boldsymbol{\omega}) = (b, Y) \in E$ is regular if $\det J_{\ell\ell}(b, Y)$ is different from zero. The implicit function theorem can then be applied to the equilibrium equation $\bar{z}(p', \boldsymbol{\omega}) = 0$. This implies that the equilibrium price vector $p \in S$ is a smooth function of the fundamentals represented by the endowment vector.

The set of regular equilibria $\mathfrak{R}$ is an open subset with full measure of the equilibrium manifold E. (See Balasko 2009, Proposition 4.7.1.)

It follows from the negative definiteness of the individual Slutsky matrices that the Jacobian matrix of aggregate excess demand $J_{\ell\ell}(b, 0)$ at the no-trade equilibrium $f(b) = (b, 0)$, is negative definite. This implies that every no-trade equilibrium is regular. (Under the assumptions of this paper that preferences are defined by smooth utility

functions, the individual Slutsky matrices are also symmetric, which implies the symmetry of the matrix $J_{\ell\ell}(b,0)$. Note, however, that the results of this paper do not need that property.)

B.4 *Index of a regular equilibrium*

By definition, the index $\iota(x) = \iota(b,Y)$ of the regular equilibrium $x = (b,Y) \in E$ takes the value $+1$ or -1, the sign being determined by the condition that the product $(-1)^{\ell-1}\iota(b,Y)\det J_{\ell\ell}(b,Y)$ is strictly positive. For properties of this index number, see Dierker (1972) or Balasko (2009, Corollary 2.5.8). Only the definition of the index is used in this paper.

B.5 *Index of a stable equilibrium*

Let (b,Y) be a tatonnement stable regular equilibrium. (See, for example, Balasko 2009, Chapter 7.) The Jacobian determinant $\det J_{\ell\ell}(b,Y)$ is the product of its $\ell-1$ eigenvalues. The eigenvalues either are strictly negative when real or are complex conjugate with nonpositive real parts. The product of the complex conjugate eigenvalues is strictly positive. The product of all $\ell-1$ eigenvalues has therefore the same sign as $(-1)^{\ell-1}$. The index $\iota(b,Y)$ of the stable equilibrium (b,Y) is therefore equal to $+1$.

B.6 *Regularity of every no-trade equilibrium*

The matrix $J_{\ell\ell}(b,0)$ being negative definite (see Appendix B.9), the inequality $(-1)^{\ell-1} \times \det J_{\ell\ell}(b,0) > 0$ is satisfied at every no-trade equilibrium $(b,0) = f(b) \in T$, from which follows that every no-trade equilibrium is regular with an index equal to $+1$. (Recall that every no-trade equilibrium is tatonnement stable; see, for example, Balasko (2009, Chapter 7.)

B.7 *Critical equilibrium*

The equilibrium $(p,\boldsymbol{\omega}) = (b,Y)$ is critical if $\det J_{\ell\ell}(p,\boldsymbol{\omega}) = \det J_{\ell\ell}(b,Y) = 0$. An equilibrium is critical if it is not regular and conversely. The set of critical equilibria is denoted by $\mathfrak{S}$. It is the complement of the set of regular equilibria: $\mathfrak{S} = E \setminus \mathfrak{R}$.

B.8 *Sets of regular and critical equilibria*

The set $\mathfrak{S}$ of critical equilibria is a closed subset of measure zero of the equilibrium manifold E (Balasko 1992 or 2009, Proposition 4.7.1). Note that the set of critical equilibria $\mathfrak{S} \cap V(b)$ within the fiber $V(b)$ is actually defined by a polynomial equation in the coordinates Y and, as such, is an algebraic set, a crucial property in the proof that $\mathfrak{S}$ is closed with measure zero in the equilibrium manifold.

B.9 *Expression of the Jacobian matrix of aggregate excess demand in the (b, Y) coordinate system*

Many issues about regular equilibria deal with properties of the Jacobian matrix of aggregate excess demand. For a definition, see Appendix B.2. Here, we recall the remarkable expression of this matrix in the (b, Y) coordinate system,

$$J_{\ell\ell}(b, Y) = J_{\ell\ell}(b, 0) + F(b)Y, \tag{1}$$

where $J_{\ell\ell}(b, 0)$ is the sum of the individual Slutsky matrices,

$$J_{\ell\ell}(b, 0) = \sum_i S_{\ell\ell} f_i(p, w_i),$$

and the $(\ell - 1) \times (m - 1)$ matrix $F(b)$ is equal to

$$F(b) = \left[\frac{\partial \bar{f}_1}{\partial w_1}(p, w_1) - \frac{\partial \bar{f}_m}{\partial w_m}(p, w_m), \ldots, \frac{\partial \bar{f}_{m-1}}{\partial w_{m-1}}(p, w_{m-1}) - \frac{\partial \bar{f}_m}{\partial w_m}(p, w_m) \right],$$

with $\bar{f}_i(p, w_i)$ representing consumer i's demand of the first $\ell - 1$ goods. (See, Balasko 2009, Proposition 4.5.6.)

An immediate application of the above expression is the following.

LEMMA 12. *The function $Y \to \det J_{\ell\ell}(b, Y)$ is polynomial in the coefficients of Y.*

The proof is obvious.

B.10 *Rank of an equilibrium*

DEFINITION 2. The rank of the fiber $V(b)$ associated with the price–income vector $b = (p, w_1, \ldots, w_m) \in B$ is the rank of matrix $F(b)$. By extension, this number is also the rank of any equilibrium $x = (b, Y)$ in the fiber $V(b)$ and is denoted by $\kappa(x) = \kappa(b, Y)$.

This definition of the rank of the equilibrium $x \in E$ is new. This rank $\kappa(x)$ is always less than or equal to $\inf(\ell, m) - 1$.

LEMMA 13. *The function $t \in [0, 1] \to v(t) = \det J_{\ell\ell}(b, tY)$ is polynomial and not identically equal to zero. Its degree is less than or equal to $\kappa(b, Y) = \operatorname{rank} F(b)$.*

PROOF. That the function is polynomial and its degree is less than or equal to $\kappa(b, Y)$ are obvious. That this polynomial is not identically equal to zero follows from $v(0) = \det J_{\ell\ell}(b, 0) \neq 0$. □

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