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On the consistency of data with bargaining theories

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We develop observable restrictions of well known theories of bargaining over money. We suppose that we observe a finite data set of bargaining outcomes, including data on allocations and disagreement points, but no information on utility functions. We ask when a given theory could generate the data. We show that if the disagreement point is fixed and symmetric, the Nash, utilitarian, and egalitarian max-min bargaining solutions are all observationally equivalent. Data compatible with these theories are, in turn, characterized by the property of comonotonicity of bargaining outcomes.

We establish different tests for each of the theories under consideration in the case in which the disagreement point can be variable. Our results are readily applicable, outside of the bargaining framework, to testing the tax code for compliance with the principle of equal loss.

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1. Introduction

This paper is an inquiry into the testable implications of bargaining theory. We have in mind the allocation of a single-dimensional resource: we can essentially focus on the allocation of money among a set of agents. We suppose that we have available certain data on how money was divided among a fixed number of agents. The data describe final allocations of money and the agents' disagreement points, but include no information on agents' preferences or on the method (or protocol or extensive-form game) that led to the division. We want to know when observed allocations are consistent with standard bargaining theory.

Several well known theories could explain a given collection of data. We focus on the *utilitarian*, *Nash*, and *egalitarian* models. The utilitarian model divides the resource so as to maximize the sum of agents' utilities. The Nash bargaining model (after Nash 1950)

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seeks to maximize the product of the gains from splitting the resource. Finally, the egalitarian solution (commonly identified with Rawls 1971) chooses a division to maximize the utility of the worst off agent. Our goal here is to investigate the implications of these theories for certain kinds of data sets.

Data sets consist of observations of how a resource is divided among a set of agents and of the relevant "disagreement point." That is, we observe how much money was at stake, how it was divided among the agents, and which outcome would have prevailed if the agents had failed to agree on a division. We do not, however, have any data on agents' preferences or on the protocol (or underlying strategic interactions) that produced the observed allocations.

Such data are used by Hamermesh (1973), for example. He uses data on union wage bargaining. More generally, the data could be the outcome of bankruptcy liquidation proceedings, or government subsidies. We investigate the restrictions that each of the three models, utilitarian, Nash, and egalitarian, place on the allocations of money. Essentially, we want to test the hypotheses that resources are allocated according to these bargaining theories when utility functions are assumed to be increasing and concave, but otherwise can differ across individuals.

We present two sets of results.

First, in Section 4, we assume that the disagreement points are fixed across observations and the same for all agents. A case in point is the wage bargaining data mentioned above. We provide a joint test of the hypotheses that a particular bargaining protocol is used and that the disagreement outcome is zero for all agents. *We discover that the empirical content of the three models is identical: No data set of the kind we assume allows us to distinguish between them.* A data set either refutes all three or is consistent with all three.

Furthermore, the theories have very weak predictive power. The only empirical prediction of any of these theories is that data should be perfectly strictly ordinally correlated, or co-monotonic, and that each agent should get at least a positive amount of consumption if any other agent does. This means that when the total amount of resource increases, all agents must receive a higher amount of resource.

In fact, we show more. We show that with our assumption on data sets, comonotonicity characterizes the empirical content of theories based a large class of social welfare functions, namely, any that takes a *generalized utilitarian* form: that is, $\sum_{i\in N} g(u_i(x_i))$. The utilitarian and Nash models are special cases. Our result implies a rather strong form of observational equivalence. Whenever the data are consistent with one of these theories (say Nash bargaining), then *the same* rationalizing utility functions serve to rationalize the observed data as generated by any of the other theories. As far as we know, our result is the first to document this strong form of observational equivalence in the revealed preference literature. We are considering three "nonnested" models and, given a rationalizable data set, we can find utility functions that serve to rationalize the data using any of the three models.

Second, in Section 5, we turn to data in which the disagreement point can vary. We observe here that the result from Section 4 readily extends to the utilitarian model and we present a simple result on the testable implications of the Nash bargaining solution. The most interesting case, though, deals with the egalitarian model of bargaining.

Under the additional assumption that utility functions are the same, the egalitarian model embodies the principle of "equal gain." The increase in utility between the final outcome and the disagreement point should be equal across agents.

An interesting empirical application is to taxation. Our formal model is characterized by the similar primitives as in axiomatic models of taxation (for example, Young 1988, 1990). There is a natural relation between the equal gains bargaining solution and a classical egalitarian principle of taxation. We can interpret data on taxation (specifically, the tax code) as bargaining data: the disagreement point represents agents' post-tax incomes and the division of money represents the amount they earn before taxes. Data of this kind can be readily inferred from the tax code.

The egalitarian theory we discuss is the "equal loss" principle (Young 1988). A tax code is consistent with this principle if there is a utility function for which the "loss" to all agents (as measured by the difference in utility between pre- and post-tax income) is equalized. The principle of equal loss has a long history in the economics of taxation.[1] We are concerned with the empirical problem of testing a tax code for compliance with the principle of equal loss. The tax code is not (necessarily) a bargaining outcome, but our model can be reinterpreted to fit tax data instead of bargaining. Formally, the test coincides with our test for the equal gains bargaining solution by reinterpreting the primitives appropriately.

We present a test of this theory in the case when utility is unknown. Young (1990) studies the same problem, but using a parametric estimation approach to find the best-fitting utility index to the tax code in the United States. We present instead a nonparametric test, which can be applied to the data used by Young.

Section 5 discusses the case when utilities may differ among agents and presents an extension to when utility is required to be concave, as it is in the rest of the paper. The results of Section 5 have an interesting by-product: the testable implications of Hotelling's model of spatial competition (Hotelling 1929). Section 5.8 demonstrates how this problem is a special case of the environment studied in Section 5.

1.1 *Related literature*

The closest papers to ours may be Cherchye et al. (2013) and Carvajal and González (forthcoming). These are independently conducted investigations into the testable implications of Nash bargaining.

Cherchye, Demuynck, and De Rock consider a model where a pair of agents bargain over consumption decisions, so the framework is different from our focus on bargaining over money. They assume that the disagreement points vary endogenously because individual agents have the option of making consumption purchases on their own, and

[1] As Young (1990) notes, it was championed by John Stuart Mill and spawned a large literature on the normative virtues of equal sacrifice.

they characterize the rationalizable data as those that satisfy a system of quadratic inequalities. Recognizing that such a system is hard to solve, they provide a sufficient condition and a necessary condition that can be operationalized computationally. Finally, they carry out a laboratory experiment and show how one can use their tests.

Carvajal and González suppose that consumption is over monetary units (as we do) and develop polynomial tests of the Nash bargaining model under various hypotheses about the behavior of the disagreement point. Most of their tests characterize rationalizable data as those that satisfy a system of quadratic inequalities. They use the Tarski–Seidenberg algorithm to construct direct tests of rationalizability in terms of data alone. A version of our Proposition 10 appears already in Carvajal and González.

The setup and methodology in both Cherchye et al. (2013) and Carvajal and González (forthcoming) are distinct from ours and perhaps closest to our discussion in Section 5.2. In fact, probably the method we suggest there can be applied in their frameworks and vice versa. The boundary of the problems we can solve in revealed preference analysis is given by polynomial problems, such as the ones they analyze. It is interesting to see complementary approaches emerge.

The recent contribution of Chiappori et al. (2012) investigates the empirical content of Nash bargaining. There are several important differences between that work and ours. The main difference is that their framework assumes disagreement points are unobserved. Instead, they suppose that some vector of underlying, observable Euclidean characteristics uniquely determines both the utility functions of agents and the disagreement point. Without assuming any kind of structure on the joint dependence of disagreement point and utility on these underlying characteristics, their model obviously has no testable implications (this is their Proposition 2). To have any empirical content, they must assume some structure on the dependence of the utility function and disagreement point on these characteristics. They assume that this dependence is known to satisfy certain properties (differentiability and "exclusion restrictions") both within and across characteristics. By contrast, in our model, disagreement point observations are part of the observed data, and this leads to the falsifiability of the model.

The other main distinction between their work and ours is that they are concerned with understanding the testable implications of the model in a continuous sense—the implications of the model if we could observe the division across *all* possible problems. Our work, alternatively, assumes only that a finite number of possible division problems are observed (with their solutions). The distinction in the two approaches can be best understood by considering the classical demand model: their approach is analogous to characterizing rationalizability by conditions on the Slutsky matrix, while our approach is analogous to Afriat's (1967) discussion of finite data sets that are rationalizable. Their work also notes that the testable implications of the Nash, utilitarian, and max-min model are identical in certain frameworks.

de Clippel and Eliaz (2012) also provide an interesting study of the empirical content of a particular bargaining solution, which they call the *fallback* solution (which shares some ideas with the max-min solution). Their framework is a general ("Arrovian") choice environment, where two agents decide from a finite choice set. The paper by Kıbrıs (2012) also studies bargaining solutions (albeit in a claims framework) and derives a revealed preference relation from the choices made by the solutions. Kıbrıs finds conditions on the solutions so that the implied revealed preference relation is rational.

Earlier works that discuss the empirical content of Nash bargaining, usually assuming all individuals are identical and risk neutral, include Hamermesh (1973) and Bowlby and Schriver (1978). Svejnar (1980) provides a critique of these ideas.

As earlier noted, Young (1990) constructs a test of the max-min hypothesis using empirical data on U.S. income taxes from 1957 to 1987. His approach is estimation based and he finds that tax data are reasonably close to predicted data from the max-min model in most years (there are exceptions). He assumes specific parametric forms for the utility function. By contrast, we provide an exact test of the max-min model, assuming no parametric functional form. Young (1988) provides a kind of exact empirical test of the max-min model, assuming the solutions to all possible problems are observed and further assuming observations *across* different populations.

2. The theories

We consider the most commonly used (cooperative) theories of bargaining. We assume an environment where some quantity $m \in \Re$ of a single-dimensional resource (e.g., money) is available and a group of n agents needs to decide on an allocation of m. Each problem possesses a disagreement point, which is the vector of outcomes received by agents if negotiations break down. For each agent, d_i is the monetary value of this outcome and the vector $d = (d_1, \dots, d_n)$ is the *disagreement point*. The set

$$B(m, d) = \left\{(x_1, \dots, x_n) \in \Re^n_+ : \sum_{i=1}^{n} x_i \leq m \text{ and, for all } i, x_i \geq d_i\right\}$$

represents all the allocations of m in which each agent gets at least their disagreement points. The disagreement point is observed and known.

A bargaining theory uses information on agents' preferences to predict an outcome in $B(m, d)$. Suppose that each agent i is described by a strictly increasing and concave utility function $u_i : \Re_+ \to \Re$.

The *utilitarian* theory says that m is divided so as to maximize the sum $\sum_{i=1}^{n} u_i(x_i)$ over $B(m, d)$. We consider a generalization of the utilitarian theory, where for some function $g : A \subset \Re \to \Re$, the sum $\sum_{i=1}^{n} g(u_i(x_i) - u_i(d_i))$ is maximized over $B(m, d)$.[2]

The *Nash bargaining* theory predicts a choice in $B(m, d)$ that maximizes the so-called *Nash product*,

$$\prod_{i=1}^{n} [u_i(x_i) - u_i(d_i)].$$

Note that the Nash bargaining theory is a special case of our generalization of the utilitarian theory, letting $g = \log$.

[2]Note that this is indeed a generalization, as the problems of optimizing $\sum_{i=1}^{n} u_i(x_i)$ and of optimizing $\sum_{i=1}^{n} [u_i(x_i) - u_i(d_i)]$ have the same solutions over $B(m, d)$.

Finally, the *egalitarian* (or max-min) theory says that $x \in B(m, d)$ should be chosen to maximize

$$\min_{i \in N}[u_i(x_i) - u_i(d_i)].$$

These three theories have both positive and normative interpretations. Under the positive interpretation, it is evident that we may want to understand the empirical content of the theory. The theories are commonly assumed to predict an outcome in applied economic models. Probably the most commonly used is the utilitarian model, but the Nash solution also finds applications. From macroeconomics to contract theory and applied mechanism design, authors often use the Nash solution as a "reduced form" model to capture the outcome of some bargaining stage.[3]

From the normative viewpoint, our three theories have know axiomatizations that relate them to basic principles of justice or social welfare (see, for example, Thomson 1981, 2010, Lensberg 1987, Thomson and Lensberg 1989, Kalai 1977).

3. The data

We assume a finite collection of observations of bargaining outcomes. A *data set* is a set $D = \{(d^k, x^k)\}_{k=1}^K$. Each observation k specifies a pair $(d^k, x^k) \in \Re^{2n}$, where $x_i^k \geq d_i^k$ for all i and k. For each k, d^k represents a disagreement point and x^k represents an outcome of bargaining. Let the set $N = \{1, \ldots, n\}$.

Any study of the empirical content of a theory depends crucially on what one assumes is observable. If we observe m, d, and agents' utility functions, then the theories we described in Section 2 are all testable and each one of them makes predictions that are not made by the others.

In contrast, we assume that utility functions are not observable. Our assumption follows the mainstream revealed preference view of preference and utility: utilities are purely theoretical constructs and do not have any meaning beyond what they predict about agents' behavior—in this case joint behavior. Utilities are not observable or even meant to be observable.

The revealed preference view is rooted in the use of field data in economics. Most data sets in economics consist of field data and they do not normally contain information on utility functions. In the particular case of bargaining theory, the econometric studies of bargaining use data with no information on utility functions (see, for example, Hamermesh 1973).

Experimental researchers can, and often do, collect partial information on utility functions by the design of the experiment or by using supplementary surveys. Many experimentalists are, however, skeptical about the idea that one can control utility effectively in the laboratory. In experiments specifically designed to test bargaining theory, attempts have been made to assume as little as possible about agents' preferences. Roth, in a survey of the experimental literature on bargaining (see pp. 41–43 in Roth 1995),

[3] See, for example, Hart (1995) for the use of Nash bargaining in the literature on incomplete contracts; see Rogerson et al. (2005) for macroeconomic search models.

argues that experiments that assume a specific utility function are problematic, and he describes designs that attempt to assume as little as possible about subjects' preferences (typically only that the subjects are expected utility maximizers, but nothing about the form of the utility function). Our assumptions about data are obviously in line with such experimental designs.

4. Fixed and symmetric disagreement point

In this section, we suppose that the disagreement point is fixed and symmetric. Our result is that the only aspect of bargaining theory that can be tested is a basic solidarity principle. A discussion of this solidarity principle in different economic environments is presented in Sprumont (2008).

The punchline is twofold. First, the three models of Section 2 have identical testable implications. Thus, for the kinds of data described in Section 3, if the disagreement point is fixed and symmetric, then the three most popular models of cooperative bargaining are observationally equivalent.

Second, the empirical predictions of these models are relatively weak: if one agent's consumption increases, then so does the consumption of all remaining agents.

With a fixed disagreement point, we assume that the disagreement point is normalized to 0. A data set (Section 3) then takes the special form $\{x^k\}_{k=1}^K$, where $x^k \in \Re^n_+$. Importantly, the disagreement point must be the same for all observations.

Our assumption of a common disagreement can be justified when disagreement points are observed to be fixed across observations, of course, but also when they are unobserved. In the latter case, we may choose a disagreement point as part of the exercise of constructing a rationalizing instance of the model: we are free to choose disagreements just as we are free to construct rationalizing utility functions. Now, the assumption that disagreement points can vary in arbitrary ways leads to an untestable theory; one can choose disagreement points (and utilities) to rationalize any data using, for example, Nash bargaining (Chiappori et al. 2012). So a researcher may want to assume that the variability of the disagreement point is limited: the most natural such assumption is that it is fixed (and can then be normalized to be zero; Hamermesh 1973 is a case in point). Our test then becomes a joint test for the particular bargaining solution together with the assumption of a fixed disagreement point at zero.[4]

We consider the general model described in Section 2. Let $g : [0,\infty) \to \Re \cup \{-\infty\}$ be a strictly increasing, smooth, and concave function. We say that data $\{x^k\}_{k=1}^K$ are *g-rationalizable* if there exist strictly increasing, smooth, and strictly concave functions u_i for which $u_i(0) = 0$ and $u_i'(0) = \infty$ (Inada conditions), and for which $\sum_{i\in N} g(u_i(x_i^k)) \geq \sum_{i\in N} g(u_i(y_i))$ for all allocations $(y_1, \dots, y_n) \in B(\sum_i x_i^k, 0)$ and $k = 1, \dots, K$. As we remarked in Section 2, the utilitarian and Nash models are special cases of g-rationalizability.

Finally, data $\{x^k\}_{k=1}^K$ are *max-min rationalizable* if there exist strictly increasing and strictly concave u_i, normalized so that $u_i(0) = 0$, for which $\min_{i\in N} u_i(x_i^k) \geq \min_{i\in N} u_i(y_i)$ for all $(y_1, \dots, y_n) \in B(\sum_i x_i^k, 0)$ and $k = 1, \dots, K$.

[4]Of course, other assumptions on the disagreement point are possible, but they fall outside the scope of our paper.

Our result establishes a property of the data that is equivalent to rationalizability by these theories: this property yields a nonparametric test for bargaining theory. We say that data $\{x^k\}_{k=1}^K$ are *co-monotonic* if for all $i, j \in N$ and all k, l, $x_i^k < x_i^l$ implies $x_j^k < x_j^l$, and for all $i, j \in N$, $x_i^k = 0$ if and only if $x_j^k = 0$. Co-monotonicity requires that outcomes are perfectly strictly ordinally correlated (when 0 is also considered an outcome).

Theorem 1. *Given data $\{x^k\}_{k=1}^K$ and a strictly increasing concave g, the following statements are equivalent.*

(i) The data are co-monotonic.

(ii) The data are g-rationalizable.

(iii) The data are max-min rationalizable.

Remark 2. Our original proof of this theorem establishes only the equivalence of the utilitarian model, the Nash model, and the max-min model. An anonymous referee (at the *American Economic Review*) showed us how to generalize the result to the one stated above.

Remark 3. We fix g, thereby fixing a theory, and ask if there are utility functions that rationalize the observations. In principle, the rationalizing utility functions *could* depend on the particular g under consideration. For example, the utilities that rationalize some data set as coming from Nash bargaining may differ from the utilities that give a utilitarian rationalization. *Surprisingly, it turns out that the utilities we construct in the proof serve to rationalize the data for any g.* (This was shown to us by the anonymous referee mentioned in Remark 2.)

In fact, it is evident from our proof that the constructed utilities also allow rationalization by any symmetric, increasing, and quasiconcave (even quasiconcavity can be weakened) function $\varphi : \Re^n \to \Re$, independently of whether φ is additively separable.[5]

Thus the models we consider are observationally equivalent in a particularly strong sense. Rationalizable data allow us to fix the unobservable utility functions in a way that is consistent with any of the models under consideration. We cannot differentiate one model from another in terms of their implied behavior about rationalizing utility functions.

Remark 4. Dropping the hypothesis of smoothness of the u_i functions and of g results in a weaker notion of co-monotonicity. Dropping the hypothesis that u is strictly concave can result in models with no testable implications on our class of data. For example the utilitarian model with linear utility functions is not testable, as the theory allows for any individually rational division of surplus.

[5]The converse statement, that any symmetric, increasing, and quasiconcave social welfare function generates co-monotonic data, is not true. It seems that the additive welfare functions we consider are the most general ones with this property.

Remark 5. Note that our notion of rationalizability requires utility functions to satisfy an Inada condition. Without the Inada condition, the theories in Section 2 may possess noninterior solutions, and the resulting data may not be co-monotonic as we have defined co-monotonicity (we would have a notion of co-monotonicity with weak inequalities instead of strict). The equivalence between g-rationalizability and max-min rationalizability also fails without Inada conditions.

Remark 6. Suppose that our data $\{x^k\}_{k=1}^K$ are such that $x^k \in \Re^n_{++}$ for all k. Then the condition in the theorem is equivalent to the statement that for all $i, j \in N$ and all k, l, $x_i^k < x_i^l$ implies $x_j^k < x_j^l$. Section 5.1 below exploits this idea further.

Remark 7. We could suppose that for each $i \in N$, there is d_i, potentially different from zero, that serves as a fixed disagreement point and then test our theories with the corresponding d_i. Our rationalizations would be required to satisfy $u_i'(d_i) = +\infty$, and co-monotonicity would be redefined as $x_i^k = d_i$ for some $i \in N$ implies that $x_j^k = d_j$ for all $j \in N$. In particular, if we are given data that satisfy $x_i^k > x_i^l$ if and only if $x_j^k > x_j^l$ for all $i, j \in N$ and k, l, then we can *always* rationalize such data by fixing, for all i, some $d_i < \min_{k=1}^K x_i^k$ and using any of the models previously discussed.

Remark 8. We could also ask about rationalization by "weighted" versions of the rules. For example, with weights $\alpha \in \Re^n_{++}$, a *weighted g rule* might be one that is chosen to maximize $\sum_{i\in N} g(\alpha_i u_i(x_i))$ over $B(m, 0)$, subject to the constraint that $\sum_{i\in N} x_i = m$. By Theorem 1, the only implication of the maximization of such rules is co-monotonicity. This can be seen by finding u_i functions that g-rationalize the data and then rescaling. A similar statement holds for max-min rationalizability.

Proof. It follows from the first order conditions that if the data are either g-rationalizable or max-min rationalizable, then they are co-monotonic.[6]

For the other direction, we show something slightly stronger: If the data are co-monotonic, then there exist strictly concave, continuous, and increasing functions u_i such that if $\varphi : [0, \infty) \to \Re \cup \{-\infty\}$ is a increasing, symmetric and quasiconcave function, then $\varphi(u_1(x_1^k), \dots, u_n(x_n^k)) \geq \varphi(u_1(y_1), \dots, u_n(y_n))$ for all allocations $(y_1, \dots, y_n)$ that satisfy $\sum_{i\in N} x_i^k = \sum_{i\in N} y_i$.[7] As a special case, we have $\varphi(z_1, \dots, z_n) = \sum_{i=1}^n g(z_i)$. Note the order of the quantifiers used above: the same profile of utility functions $u_1, \dots, u_n$ works across all φ.

[6] The argument for the utilitarian model is as follows: Suppose that all agents' marginal utilities are equalized at $x \in B(m, 0)$ and at $x' \in B(m', 0)$ with $m = \sum_i x_i$ and $m' = \sum_i x_i'$. (Note that this uses the Inada conditions assumed on utilities.) Suppose that $m < m'$. Some agent i must have $x_i' > x_i$; then all agents' marginal utilities must be smaller at the allocation x' than at x and the concavity of utility implies that all $x_i' > x_i$ for all i. The argument is almost identical for generalized g-utilitarianism. For the max-min model, it is clear that all agents must have equal utility at a given allocation. Thus, if one agent's utility goes up, so must all others.

[7] Symmetry means that if σ is a permutation on $\{1, \dots, n\}$, then $\varphi(x_{\sigma(1)}, \dots, x_{\sigma(n)}) = \varphi(x_1, \dots, x_n)$. Increasing here means that if $x_i > y_i$ for all i, then $\varphi(x_1, \dots, x_n) > \varphi(y_1, \dots, y_n)$.

To this end, we suppose the data are co-monotonic and we ignore replications as well as points where every agent consumes 0. Without loss of generality, let us suppose that $x_i^1 < x_i^2 < \cdots < x_i^K$ for all $i \in N$ (that this is possible follows from co-monotonicity). Below we construct a profile of utility functions $u_1, \ldots, u_n$ with the property that for all $k = 1, \ldots, K$, $\sum_{i \in N} u_i(x_i^k)$ is maximal across all allocations $y_1, \ldots, y_n$ for which $\sum_{i \in N} x_i^k = \sum_{i \in N} y_i$, and that $\min_{i \in N} u_i(x_i^k)$ is also maximal across all such allocations; it follows that since each u_i is strictly increasing, $u_i(x_i^k) = u_j(x_j^k)$ for all $i, j \in N$.

We first argue that such a construction suffices to establish the result: Let φ be as above, and suppose, by way of contradiction, that there is a k and a feasible allocation $(y_1, \ldots, y_n)$ for which $\varphi(u_1(y_1), \ldots, u_n(y_n)) > \varphi(u_1(x_1^k), \ldots, u_n(x_n^k))$. Note then, by symmetry of φ, that for any permutation of the agents $\sigma : N \to N$, $\varphi(u_{\sigma(1)}(y_{\sigma(1)}), \ldots, u_{\sigma(n)}(y_{\sigma(n)})) = \varphi(u_1(y_1), \ldots, u_n(y_n))$. Quasiconcavity of φ then implies that

$$\varphi\left(\sum_{i \in N} \frac{u_i(y_i)}{n}, \ldots, \sum_{i \in N} \frac{u_i(y_i)}{n}\right) > \varphi(u_1(x_1^k), \ldots, u_n(x_n^k)).$$

By strict increasingness of φ and using the fact that $u_i(x_i^k) = u_j(x_j^k)$ for all $i, j \in N$, this implies that

$$\sum_{i \in N} \frac{u_i(y_i)}{n} > \sum_{i \in N} \frac{u_i(x_i^k)}{n},$$

contradicting

$$\sum_{i \in N} u_i(x_i^k) \geq \sum_{i \in N} u_i(y_i)$$

for all feasible allocations $y_1, \ldots, y_n$.

We finish the proof by constructing, for each i, a strictly decreasing, continuous, and positive function f_i, with the property that if we set u_i to be the integral of f_i, then the profile of utility functions $(u_1, \ldots, u_n)$ works as required by the first part of the proof.

We proceed by induction. We ensure that, for each $i \in N$ and each k, the following equalities are true:

(i) $\int_0^{x_i^k} f_i(x)\, dx = \int_0^{x_j^k} f_j(x)\, dx$

(ii) $f_i(x_i^k) = f_j(x_j^k)$.

In the first place, for $k = 1$, we define for each agent j, $f_j(0) = +\infty$. The construction is done in a series of steps, labeled Step 1 to Step 6.

Step 1. For K, define $f_i(x_i^K) = 1$ for all $i \in N$.

Step 2. For $x > x_i^K$, define $f_i(x)$ to be any strictly decreasing function, taking value everywhere less than 1 and rendering f_i continuous.

STEP 3. Proceed by induction. Let $k > 1$ be arbitrary, and suppose that $f_i(x)$ has been defined for all $x \geq x_i^k$. We assume that for all $k' \geq k$, $f_i(x_i^{k'}) = f_j(x_j^{k'})$ and

$$\int_{x_i^k}^{x_i^K} f_i(x)\,dx = \int_{x_j^k}^{x_j^K} f_j(x)dx \quad \text{for all } i, j \in N.$$

Recall that we have $x_i^1 < x_i^2 < \cdots < x_i^K$. We choose a finite $f_j(x_j^{k-1})$, but we must choose it to be sufficiently large. Specifically, let z be large enough so that there is $\varepsilon > 0$ for which $z(x_j^k - x_j^{k-1}) - \varepsilon > \max_{i\in N} f_i(x_i^k)(x_i^k - x_i^{k-1}) + \varepsilon$ for all j. We can then set $f_j(x_j^{k-1}) = z$ for all j.

STEP 4. Observe that, given $f_j(x_j^{k-1})$ and $f_j(x_j^k)$, for any $\varepsilon > 0$ and any

$$y \in (f_j(x_j^k)(x_j^k - x_j^{k-1}) + \varepsilon, f_j(x_j^{k-1})(x_j^k - x_j^{k-1}) - \varepsilon),$$

we can define f_j continuous and decreasing on $x \in (x_j^{k-1}, x_j^k)$ so that

$$\int_{x_j^{k-1}}^{x_j^k} f_j(x)\,dx = y.$$

This follows as we can approximate the constant functions $h(x) = f_j(x_j^{k-1})(x_j^k - x_j^{k-1})$ and $h^*(x) = f_j(x_j^k)(x_j^k - x_j^{k-1})$ arbitrarily closely pointwise by strictly decreasing and continuous functions.

STEP 5. Complete $f_j(x)$ on $x \in (x_j^{k-1}, x_j^k)$ so that

$$\int_{x_j^{k-1}}^{x_j^k} f_j(x)\,dx$$

is equalized across all agents, by picking

$$y \in \bigcap_{i\in N} (f_i(x_i^k)(x_i^k - x_i^{k-1}) + \varepsilon, f_i(x_i^{k-1})(x_i^k - x_i^{k-1}) - \varepsilon)$$

and choosing $f_j(x)$ on $x \in (x_j^{k-1}, x_j^k)$ so that

$$\int_{x_j^{k-1}}^{x_j^k} f_j(x)\,dx = y.$$

STEP 6. In the case of $k = 1$, also maintain that

$$\int_0^{x_j^1} f_j(x)\,dx < +\infty.$$

The functions f_j so constructed satisfy the conditions we ask for: that for all k, $f_j(x_j^k)$ is equalized across j and

$$\int_0^{x_j^k} f_j(x)\,dx$$

is equalized across j. By setting

$$u_j(x) = \int_0^x f_j(x)\,dx,$$

we have the required u_j. □

Remark 9. An anonymous referee suggested to us the following explicit construction, which works if we want to find rationalizations that do not necessarily satisfy the Inada conditions:

Step 1. Set $f_i(x_i^K) = 1$ for all $i \in N$.

Step 2. Complete f_i above x_i^K with any continuous, strictly decreasing function.

Step 3. Pick any

$$z_{K-1} > \left(2\frac{\max_{j\in N}(x_j^K - x_j^{K-1})}{\min_{j\in N}(x_j^K - x_j^{K-1})} - 1\right),$$

and set $f_i(x_i^{K-1}) = z_{K-1}$ for every i.

Step 4. Complete f_i on (x_{K-1}^i, x_K^i) such that each f_i is continuous and strictly decreasing, and $\int_{x_i^{K-1}}^{x_i^K} f_i(x)\,dx$ is the same for every i. One way to do this is to pick $i^* \in \arg\min_{j\in N}(x_j^K - x_j^{K-1})$ and set

$$f_i(x) = z_{K-1} - \left(\frac{x - x_i^{K-1}}{x_i^K - x_i^{K-1}}\right)^{\alpha_i}(z_{K-1} - z_K)$$

with $\alpha_{i^*} = 1$ and the other $\alpha_i \leq 1$ chosen to equalize integrals (z_{K-1} was chosen high enough so that such an α_i exists for every i).

Step 5. Repeat Steps 3 and 4 for $k = K-1, K-3$, and so on. Once each f_i has been defined down to x_i^1, let $x_i^0 = 0$, define z_0 as above, and complete f on $[0, x_i^1]$ in the same way.

This results in a set of functions $\{f_i\}$ with $f_i(x_i^k) = f_j(x_j^k)$ and $\int_0^{x_i^k} f_i(x)\,dx = \int_0^{x_j^k} f_j(x)\,dx$ for every (i, j, k); setting $u_i(x) = \int_0^x f_i(x)\,dx$ completes the construction.

5. Variable disagreement point

We now turn to data as defined in Section 3, where the disagreement point is allowed to vary from one observation to another.

5.1 *The utilitarian model*

Suppose that our data set satisfies $x_i^k > d_i^k$ for all k and i. Then co-monotonicity of the x variables is all that is required for utilitarian rationalizability. This is because there exist utility functions that rationalize the data in the fictitious environment in which each $d_i^k = 0$; it is easily checked that these same utility functions therefore rationalize the given data. Therefore, the equivalence between co-monotonicity and rationalizability by the utilitarian theory extends to the case of a variable disagreement point.[8,9]

5.2 *The Nash model*

A data set $D = \{(d^k, x^k)\}_{k=1}^K$, with variable disagreement point, is *Nash rationalizable* if there are strictly increasing and concave u_i for which

$$\prod_{i\in N}[u_i(x_i^k) - u_i(d_i^k)] \geq \prod_{i\in N}[u_i(y_i^k) - u_i(d_i^k)]$$

for all $(y_1, \dots, y_n) \in B(\sum_{i\in N} x_i^k, d^k)$. A version of the following result appears in Carvajal and González (forthcoming).

Proposition 10. *A data set D is Nash rationalizable if and only if for all $i \in N$, there are numbers $U_i(d_i^k)$, $U_i(x_i^k)$, $M_i(d_i^k)$, $M_i(x_i^k)$ for $k = 1, \dots, K$ that solve the following equations: for all i, j, and k,*

$$\frac{M_i(x_i^k)}{U_i(x_i^k) - U_i(d_i^k)} = \frac{M_j(x_j^k)}{U_j(x_j^k) - U_j(d_j^k)},$$

and for all $z, z' \in \bigcup_{i=1}^N \{d_i^k, x_i^k\}$,

$$\begin{cases} U_i(z) - U_i(z') > 0 & \text{if } z < z' \\ M_i(z')(z - z') \geq U_i(z) - U_i(z'). \end{cases}$$

Proposition 10 is straightforward. It says simply that we need numbers $U_i(z)$ to signify levels of utility and $M_i(z)$ for supergradients or marginal utilities. The first system of equalities ensures that the first-order conditions for the maximization of the Nash product hold. The second set of inequalities makes sure that utility is increasing and that marginal utilities are supergradients of the utilities.

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[8]An alternative model with variable disagreement would be one that seeks to maximize the sum of utilities $u_i(x_i - d_i)$, but this is the same problem studied in Section 4.

[9]One may also be interested in data for which we may have $x_i^k = d_i^k$, basically allowing for corner solutions in the maximization of utilitarian welfare. One can set up a result like that in Proposition 10 for this case, based on writing the corresponding Kuhn–Tucker conditions.

Proof of Proposition 10. We first show that if we are given increasing and concave utility functions u_i, then $(x_1^k, \dots, x_n^k)$ is a solution to $\max_{\sum_{i\in N} x_i = M} \prod_{i\in N}[u_i(x_i) - u_i(d_i^k)]$ if and only if for each i, there is a supergradient μ_i of u_i at x_i^k for which

$$\frac{\mu_i}{U_i(x_i^k) - U_i(d_i^k)} = \frac{\mu_j}{U_j(x_j^k) - U_j(d_j^k)}.$$

To this end, define $\mathcal{U} = \{(u_1(x_1) - u_1(d_1^k), \dots, u_n(x_n) - u_n(d_n^k)) : \sum_{i\in N} x_i = x_i^k, x_i \geq d_i^k \text{ for all } i\}$. Consider maximizing the function $f(y) = \prod_{i\in N} y_i$ subject to $y \in \mathcal{U}$. A point $u \in \mathcal{U}$ maximizes f if and only if $(\prod_{i\neq 1} u_i, \dots, \prod_{i\neq n} u_i)$ supports $\mathcal{U}$ at u (by definition). Because f is strictly convex, and since $\mathcal{U}$ is convex and compact, there is a unique such maximizer u^*. It is clear that $u_i^* > 0$ for all $i \in N$.

This states that there is a unique solution $(x_1^k, \dots, x_n^k)$ to the Nash problem for which $u_i(x_i^k) - u_i(d_i^k) = u_i^*$ for all $i \in N$. We define $\lambda_j = \prod_{i\neq j}[u_i(x_i^k) - u_i(d_i^k)]$. We know that $\sum_{i\in N} \lambda_i u_i(x_i)$ is maximized at $x_1^k, \dots, x_n^k$ across all x_i for which $\sum_{i\in N} x_i = m$. Our next step is to show that this can happen if and only if the vector $(1/\lambda_1, \dots, 1/\lambda_n)$ is proportional to a vector of supergradients.

Since the constraints $x_i \geq d_i^k$ are not binding, we can set up the Lagrangian for the problem, say $L(x, \mu) = \sum_{i\in N} \lambda_i u_i(x_i) + \mu(m - \sum_{i\in N} x_i)$, and note that it is equal to $L(x, \mu) = \sum_{i\in N}[\lambda_i u_i(x_i) - \mu x_i] + \mu m$. We know the constraint $\sum_{i\in N} x_i = m$ is binding so that the solution to the max-min problem features $\mu^* > 0$. For μ^*, we know that $\max_x L(x, \mu^*)$ is equal to the maximum Nash product subject to the constraint and has the same solution. This is equivalent to saying that $(\lambda_i/\mu^*)u_i(x_i^k) - x_i^k \geq (\lambda_i/\mu^*)u_i(x_i) - x_i$ for all x_i or, rewriting,

$$u_i(x_i) + (\mu^*/\lambda_i)(x_i^k - x_i) \leq u_i(x_i^k).$$

This is equivalent to saying that $(\mu^*/\lambda_1, \dots, \mu^*/\lambda_n)$ is a supergradient, or that the vector $(1/\lambda_1, \dots, 1/\lambda_n)$ is proportional to a supergradient.

Another way to say that $(1/\lambda_1, \dots, 1/\lambda_n)$ is proportional to a supergradient is to say that for all $i \in N$, there is a supergradient $M_i(x_i^k)$ of u_i at x_i^k for which, for all i, j,

$$\frac{\lambda_i}{\lambda_j} = \frac{M_j(x_j^k)}{M_i(x_i^k)}.$$

Writing out the explicit form of λ and eliminating terms, this is equivalent to saying that

$$\frac{M_i(x_i^k)}{u_i(x_i^k) - u_i(d_i^k)} = \frac{M_j(x_j^k)}{u_j(x_j^k) - u_j(d_j^k)},$$

which is precisely the condition in the theorem. The other conditions simply say that M_i is a supergradient, and that u_i is strictly increasing.

Conversely, the details of how to construct a utility function from these numbers essentially follow from Afriat, defining $u_i(x) = \inf_{z\in\bigcup_{k=1}^K \{x_i^k, d_i^k\}} U_i(z) + M_i(z)(x - z)$, where the infimum is taken over all data points. It is then simple to verify by construction that for all $z \in \bigcup_{k=1}^{K}\{x_i^k, d_i^k\}$, $M_i(z)$ is a supergradient of u_i at z. From this, the fact that the equality in the statement of the theorem is solved implies that the Nash product is maximized for this collection of utility functions (by the previous argument). □

For Proposition 10 to be useful, it must be accompanied by a procedure that one can perform on a data set and decide whether the data are Nash rationalizable. Such a procedure is discussed in Chambers and Echenique (2011); suffice it to say here that it is based on a computational version of results in real algebraic geometry. There is a sort of "theorem of the alternative" that applies to systems of polynomial inequalities. In Chambers and Echenique (2011), we explain how these results can be used to operationalize the test in Proposition 10.

5.3 *The egalitarian model*

Finally, we turn to the egalitarian, or max-min, model. A data set D is egalitarian rationalizable if there are continuous and strictly increasing utility functions $u_i : \Re_+ \to \Re$ such that, for all k, i, and j,

$$u_i(x_i^k) - u_i(d_i^k) = u_j(x_j^k) - u_j(d_j^k).$$

We first discuss a strong version of the theory, called the *equal gains* theory, where we require that all agents share the same utility function u.

We have emphasized that observations (x^k, d^k) should be interpreted as bargaining outcomes, where d^k is a disagreement point, but there are other interpretations. We can, instead, think of x_i^k as agent i's pre-tax income and of d_i^k as his income after taxes. This interpretation is completely unrelated to bargaining, but leads to the same mathematical formalism. Then, to require that there be some increasing function u for which, for all i, j,

$$u(x_i^k) - u(d_i^k) = u(x_j^k) - u(d_j^k),$$

says that the tax code is compatible with all agents sharing equally in the loss of utility derived from taxation. Young (1990) studies the equal gains (or equal loss) model, under the taxation interpretation. Young's empirical results are of a parametric nature. In contrast, we present a nonparametric test for the compliance of the tax code with the principle of equal gains.

Our first result assumes only that u is increasing. Our second result requires u also to be concave (as in the results of Section 4). Below we also discuss the (easy) extension to when utility is allowed to differ across agents.

To begin to understand the empirical content of the equal gains model, let us suppose we have two agents, so that $N = \{1, 2\}$, and that we observe the two data points

$$d^1 = (0, 7), \qquad x^1 = (5, 8) \quad \text{and} \quad d^2 = (1, 3), \qquad x^2 = (2, 8).$$

We claim that this data set cannot be rationalized. To see why, suppose that u is a utility function that rationalizes these data using the equal gains model. Then we would have

$u(5) - u(0) = u(8) - u(7)$ and $u(2) - u(1) = u(8) - u(3)$. Therefore, since $u(5) - u(0) + u(7) - u(8) = 0$ and $u(8) - u(3) + u(1) - u(2) = 0$, we must have

$$[u(7) - u(8)] + [u(8) - u(3)] + [u(5) - u(0)] + [u(1) - u(2)] = 0. \tag{1}$$

But we can regroup terms in this expression, obtaining

$$[-u(2) + u(7)] + [-u(8) + u(8)] + [-u(3) + u(5)] + [-u(0) + u(1)] = 0. \tag{2}$$

The contradiction arises because the increasingness of u implies that each term in brackets in (2) is nonnegative, and at least one of them is strictly positive (in fact, each of the terms $[-u(2) + u(7)]$, $[-u(3) + u(5)]$, and $[-u(0) + u(1)]$ is strictly positive). Therefore, the terms cannot add up to zero.

To develop a feeling for the kinds of data that are rationalizable, consider a somewhat more involved example. The example helps to motivate the condition that we arrive at in the next result. Consider the data

$$\begin{aligned} d^1 &= (1, 8), \qquad & x^1 &= (3, 9) \\ d^2 &= (2, 8), \qquad & x^2 &= (5, 9) \\ d^3 &= (2, 9), \qquad & x^3 &= (4, 10) \\ d^4 &= (0, 9), \qquad & x^4 &= (4, 10). \end{aligned}$$

A rationalizing utility u must satisfy $u(3) - u(1) = u(9) - u(8)$, $u(5) - u(2) = u(9) - u(8)$, $u(4) - u(2) = u(10) - u(9)$, and $u(4) - u(0) = u(10) - u(9)$. By adding and subtracting, we obtain

$$\begin{aligned} &\big([u(1) - u(3)] + [u(5) - u(2)] + [u(2) - u(4)] + [u(4) - u(0)]\big) \\ &\qquad + \big([u(8) - u(9)] + [u(9) - u(10)] + [u(10) - u(9)] + [u(9) - u(8)]\big) = 0. \end{aligned}$$

But note again, by regrouping, we obtain

$$\begin{aligned} &\big([-u(0) + u(1)] + [-u(3) + u(5)] + [-u(2) + u(2)] + [-u(4) + u(4)]\big) \\ &\qquad + \big([-u(8) + u(8)] + [-u(9) + u(9)] + [-u(10) + u(10)] + [-u(9) + u(9)]\big) = 0. \end{aligned}$$

And again, using the increasingness of u, each of the terms inside the brackets is nonnegative and some are strictly positive. This results in a contradiction.

In each of the two examples, we have taken data points that, if rationalizable, should force a certain expression to add to zero. By regrouping the terms, the increasingness of u forces a contradiction; the expression could not possibly add to zero if utility is increasing. It turns out that the inability to regroup data in this sense is necessary and sufficient for the data to be rationalizable. The inability to regroup data in the appropriate way is a condition (or a nonparametric test) that is equivalent to rationalization by the equal gains model.

To formalize our condition, we have to be more specific in what we mean by "regrouping data." It is easiest to think of this in graph theoretic terms. In (1), we can think

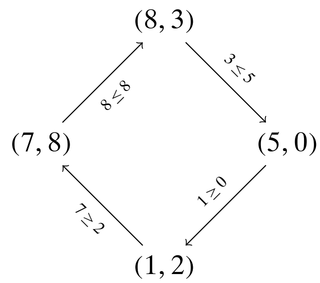

Figure 1. A cycle.

of edges pointing from 7 to 8, from 8 to 3, from 5 to 0, and from 1 to 2. Note that these edges come in "pairs," for example, the edge pointing up from 7 to 8 comes from the data point $(d^1, x^1) = ((0, 7), (5, 8))$ and is naturally paired with the edge pointing down from 5 to 0. Likewise, the edge pointing up from 1 to 2 is naturally paired with the edge pointing down from 8 to 3.

The interesting point is that when we put these edges together in the appropriate sequence, they form a cycle: see Figure 1. Consider the "edges" $(7, 8)$, $(8, 3)$, $(5, 0)$, $(1, 2)$. The endpoints of adjacent edges here are ordered, where we treat $(1, 2)$ and $(7, 8)$ as adjacent. That is, the second number (number 8) of the node $(7, 8)$ is less than or equal (in fact, equal) to the first number in $(8, 3)$ and so forth for each pair of adjacent edges. In fact, the second number (3) of the node $(8, 3)$ is strictly less than the first number (5) in $(5, 0)$. And returning to (2), we see that when we regroup the data, the term $-u(3) + u(5)$ appears.

We proceed to define our condition formally. First, we define a *cycle* to be a finite sequence of ordered pairs of real numbers, $\{(z_l^1, z_l^2)\}_{l=1}^L$, for which for all $l = 1, \dots, L-1$, $z_l^2 \le z_{l+1}^1$ and $z_L^2 \le z_1^1$. A *strict cycle* is a cycle $\{(z_l^1, z_l^2)\}_{l=1}^L$ for which for some l, $z_l^2 < z_{l+1}^1$ or $z_L^2 < z_1^1$. A finite sequence $\{(z_l^1, z_l^2)\}_{l=1}^L$ *defines a (strict) cycle* if there exists a bijection $\sigma : L \to L$ for which $\{(z_{\sigma(l)}^1, z_{\sigma(l)}^2)\}_{l=1}^L$ is a (strict) cycle. Then the ordered pairs $\{(7, 8), (8, 3), (5, 0), (1, 2)\}$ from our first example form a strict cycle.

We might conjecture that for data not to be rationalizable, we should be able to pair "up" edges with "down" edges in a way that forms a strict cycle. But this is not quite enough. If we look at the regrouping in the second example, we again pair up edges with down edges, but we do not end up with a single cycle. In fact, we end up with *two* cycles, only one of which is strict. Namely, the edges

$$\{(1, 3), (5, 2), (2, 4), (4, 0), (8, 9), (9, 10), (10, 9), (9, 8)\}$$

do not themselves form a cycle, but the two sets of edges $\{(1, 3), (5, 2), (2, 4), (4, 0)\}$ and $\{(8, 9), (9, 10), (10, 9), (9, 8)\}$ each form a cycle. Only the first cycle here is strict, but that is all we need.

In general, we can see that there is no reason that a sequence of paired edges needs to correspond to one, two, or even k cycles. All that we need to obtain a contradiction is that data can be grouped into paired edges that can be partitioned into cycles, at least one of which is strict. These observations motivate the following definitions.

Let L be a natural number, and let $\{(a_l, b_l)\}_{l=1}^L$ and $\{(a'_l, b'_l)\}_{l=1}^L$ be two sequences of L ordered pairs. Say that $\{(a_l, b_l)\}_{l=1}^L$ and $\{(a'_l, b'_l)\}_{l=1}^L$ can be *partitioned into cycles* if there exists a natural number T and for each $t \leq T$, there exists a collection of finite sequences $\{(z^1_{tl}, z^2_{tl})\}_{l=1}^{L_t}$ that define cycles (at least one cycle of which is strict) for which there exists a bijection

$$f : \{(t, l) : t \leq T, l \leq L_t\} \to \{(l, i) : l \leq L, i = \{1, 2\}\}$$

such that

$$(z^1_{tl}, z^2_{tl}) = \begin{cases} (a_{f_1(t,l)}, b_{f_1(t,l)}) & \text{if } f_2(t, l) = 1 \\ (a'_{f_1(t,l)}, b'_{f_1(t,l)}) & \text{if } f_2(t, l) = 2. \end{cases}$$

The inability to partition paired data points into cycles is exactly the necessary and sufficient condition needed to guarantee that data are rationalizable.

Proposition 11. *The data set $D = \{(d^k, x^k) : k = 1, \dots, K\}$ is rationalizable if and only if there are no sequences $(d^l, x^l)_{l=1}^L$ in D, and agents $i_l \neq j_l$ for all l, such that $(d^l_{i_l}, x^l_{i_l})_{l=1}^L$ and $(x^l_{j_l}, d^l_{j_l})_{l=1}^L$ can be partitioned into cycles, at least one of which is strict.*

Two points are worth mentioning. The definition of cycle does not preclude repetition of elements; neither does the notion of "sequence of data points" referred to in the statement of the proposition. As a result, it may not be obvious how to operationalize the test.

An alternative, computationally viable, test to the condition in Proposition 11 is the following. There is a rationalizing utility if and only if the following linear program has a solution (u, e) with $e > 0$:

$$\max_{(e,u)\in\Re\times\Re^{|X|}} e$$

$$\text{s.t.} \begin{cases} (\forall i, j, k) & ((1_{x^k_i} - 1_{d^k_i}) + (1_{d^k_j} - 1_{x^k_j})) \cdot u \geq 0 \\ (\forall z, z' \in X) & z < z' \Rightarrow (1_{z'} - 1_z) \cdot u \geq e. \end{cases}$$

Here, $X \subseteq \Re^n$ is a finite set such that $d^k, x^k \in X$ for all k. For any $z \in X$, 1_z denotes the vector in $\Re^{|X|}$ with a 0 in all its entries, except that corresponding to z. A vector $u \in \Re^{|X|}$ is simply a utility function: it assigns a real number to each $z \in X$. With this interpretation in mind, the constraints

$$((1_{x^k_i} - 1_{d^k_i}) + (1_{d^k_j} - 1_{x^k_j})) \cdot u \geq 0$$

in the program above simply recast the conditions of the egalitarian model (we have expressed the equality in the egalitarian model as two weak inequalities). The constraints that

$$(1_{z'} - 1_z) \cdot u > 0$$

when $z < z'$ express the fact that u is monotonically increasing. The scalar e is simply an artifact to capture the existence of a solution: it is a (standard) way to write the problem of the existence of a solution to a system of inequalities as an optimization problem.[10]

The relation between Proposition 11 and the linear program above is obvious from the proof. In fact, the proof of Proposition 11 follows from using the following version of a result in linear programming (Farkas's lemma).

LEMMA 12 (Integer-real Farkas). *Let $\{A_i\}_{i=1}^K$ be a finite collection of vectors in $\mathbf{Q}^n$. Let L be an integer with $1 \leq L \leq K$. Then one and only one of the following statements is true:*

(i) There exists $y \in \Re^n$ such that for all $i = 1, \dots, L$, $A_i \cdot y \geq 0$, and for all $i = L+1, \dots, K$, $A_i \cdot y > 0$.

(ii) There exists $z \in \mathbf{Z}_+^K$ with $\sum_{i=L+1}^K z_i > 0$ such that $\sum_{i=1}^K z_i A_i = 0$.

PROOF. Statements (i) and (ii) cannot simultaneously hold. To see why, suppose that there exist y and z as stated. Then $A_i \cdot y \geq 0$ for all $i = 1, \dots, L$ and $A_i \cdot y > 0$ for all $i = L+1, \dots, K$. Consider $\sum_{i=1}^K z_i A_i \cdot y$. Since $\sum_{i=1}^K z_i A_i = 0$, we know that $\sum_{i=1}^K z_i A_i \cdot y = 0$. Furthermore, since there is some $j \in \{L+1, \dots, K\}$ for which $z_j A_j \cdot y > 0$, and for all i, $z_i A_i \cdot y \geq 0$, we conclude that $\sum_{i=1}^K z_i A_i \cdot y > 0$, a contradiction.

We now establish that if (ii) does not hold, then (i) holds. By Theorem 3.2 of Fishburn (1973), if (ii) does not hold, there exists $y \in \mathbf{Q}^n$ such that for all $i = 1, \dots, L$, $A_i \cdot y \geq 0$, and for all $i = L+1, \dots, K$, $A_i \cdot q > 0$. Since $\mathbf{Q}^n \subset \Re^n$, then $y \in \Re^n$. □

5.4 *Proof of Proposition 11*

Let $X \subseteq \Re^n$ be a finite set such that $d^k, x^k \in X$ for all k.

The notation 1_x refers to a vector of 0's and 1's, with a 1 in the x coordinate and 0 elsewhere (an indicator function).

There is a rationalizing u if and only if there is a solution to the system of linear inequalities

$$(\forall i, j, k) \qquad ((1_{x_i^k} - 1_{d_i^k}) + (1_{d_j^k} - 1_{x_j^k})) \cdot u \geq 0 \tag{3}$$

$$(\forall z, z' \in X) \quad z < z' \quad \Rightarrow \quad (1_{z'} - 1_z) \cdot u > 0. \tag{4}$$

Statement (3) defines a collection of inequalities, one for each i, j, and k. Statement (4) defines another set of inequalities, one for each $z', z \in X$ with $z < z'$.

Once a solution to the linear inequalities is obtained, the function u can be completed by linear interpolation.

By Lemma 12, there is no solution to system (3)–(4) if and only if there are vectors $\lambda \in \mathbf{Z}_+^{KN^2}$ and $\theta \in \mathbf{Z}_+^{|X|^2}$ with

$$\sum_{k,i,j} \lambda_{k,i,j}((1_{x_i^k} - 1_{d_i^k}) + (1_{d_j^k} - 1_{x_j^k})) + \sum_{(z,z'):z'>z} \theta_{z,z'}(1_{z'} - 1_z) = 0$$

[10] We thank an anonymous referee for suggesting this formulation of the argument.

and

$$\sum_{(z,z'):z'>z} \theta_{z,z'} > 0.$$

Without loss of generality, we can assume that $d_i^k \neq d_j^k$ and $x_i^k \neq x_j^k$ for all k and all $i \neq j$. To see this note that if $d_i^k = d_j^k$, then $x_i^k < x_j^k$ implies that there is no rationalizing increasing u; but then the intervals $[d_i^k, x_i^k]$ and $[d_j^k, x_j^k]$ define a strict cycle $\{(d_i^k, x_i^k)\}$ $\{(x_j^k, d_j^k)\}$. A similar result obtains if $x_j^k < x_i^k$. Alternatively, $x_i^k = x_j^k$ implies that the inequalities that correspond to k, i, and j in (3) are always satisfied. So these inequalities are irrelevant to the existence of a rationalizing u. The argument is analogous when $d_i^k \neq d_j^k$ and $x_i^k = x_j^k$.

Let the vectors $(\lambda_{k,i,j})$ and $(\theta_{z,z'})$ be as above. Consider the following collections of vectors in $\{-1, 0, 1\}^X$: Let A_D be the collection of vectors with $\lambda_{k,i,j}$ copies of $(1_{d_j^k} - 1_{x_j^k})$ and let A_U be the collection with $\lambda_{k,i,j}$ copies of $(1_{x_i^k} - 1_{d_i^k})$. Let $f: A_D \to A_U$ be the bijection that associates each $(1_{d_j^k} - 1_{x_j^k})$ with a different copy of $(1_{x_i^k} - 1_{d_i^k})$. Such a bijection exists given the way that A_D and A_U are constructed.

Let A_M be the collection with $\theta_{z,z'}$ copies of $1_{z'} - 1_z$ for each $z, z' \in X$ with $z' > z$. By definition of λ and θ, we know that the sum of the elements of A_D, A_U, and A_M equals the null vector. We also have $A_M \neq \emptyset$.

Let $G = (X, E)$ be the graph obtained by letting there be an edge pointing from x to x' if and only if there is a vector $1_{x'} - 1_x$ in one of the collections A_D, A_U, or A_M. By the Poincaré–Veblen–Alexander theorem (see Berge 2001, p. 148, Theorem 5), since the sum of the elements of the vectors in A_D, A_U, and A_M equals the null vector, then G can be partitioned into circuits $C_1, \ldots, C_T$. Note that if $e = (v, v') \in A_U \cup A_M$, then $v \leq v'$. If $e = (v, v') \in A_D$, then $v \geq v'$.

Consider the edges in circuit C_t: Let $[d_{i_l}^l, x_{i_l}^l]$, $l = 1, \ldots, L_t^U$ be the set of intervals defined by edges $(d_{i_l}^l, x_{i_l}^l) \in A_U$ and let $[d_{j_l}^l, x_{j_l}^l]$, $l = 1, \ldots, L_t^D$ be the set of intervals defined by edges $(x_{j_l}^l, d_{j_l}^l) \in A_D$. For any edge $e = (v, v') \in A_U \cup A_D$ in C_t, let (v'', v''') be the first edge in C_t after e that is in $A_U \cup A_D$. Then either $v' = v''$ or there are edges in A_M between e and (v'', v''') in C_t; so $v' \leq v''$. Hence, for any $e = (v, v') \in A_U \cup A_D$ in C_t, the successor edge $(v'', v''') \in A_U \cup A_D$ satisfies that $v' \leq v''$. Hence the intervals $(d_{i_l}^l, x_{i_l}^l)$ $l = 1, \ldots, L_t^U$ and $(x_{j_l}^l, d_{j_l}^l)$, $l = 1, \ldots, L_t^D$ define a cycle.

In addition, since $A_M \neq \emptyset$, at least one of the sets of intervals defined by a circuit C_t defines a strict cycle.

Finally, since there is a bijection between the edges in A_U and in A_D, we have $\sum_t L_t^U = \sum_t L_t^D = L$. So if we let $([d_{i_l}^l, x_{i_l}^l])_{l=1}^L$ collect the sequences $[d_{i_l}^l, x_{i_l}^l]$, $l = 1, \ldots, L_t^U$, and let $([x_{j_l}^l, d_{j_l}^l])_{l=1}^L$ collect the sequences $[d_{j_l}^l, x_{j_l}^l]$, $l = 1, \ldots, L_t^D$, then we have a sequence of intervals in the condition in the statement of the proposition.

5.5 *Strengthening*

The condition in Proposition 11 is necessary and sufficient for rationalizability by *some* increasing utility function. In axiomatic bargaining, however, we often assume that utility functions are concave; this was our motivation in Theorem 1 of Section 4. We now turn to ask for the additional testable implications of requiring the rationalizing u to be concave in the egalitarian model.

The answer is not so difficult and is closely related to work of Afriat (1967), Richter and Wong (2004), and Kalandrakis (2010). To simplify matters, we suppose in this section that all observed data points are rational. Thus, say that a data set D is *rational-valued* if all d_i^k and all x_i^k are elements of $\mathbf{Q}$.

We say that a data set D is *concave rationalizable* if there is a strictly increasing, continuous, concave $u : \Re_+ \to \Re$ for which for all k, i, and j,

$$u(x_i^k) - u(d_i^k) = u(x_j^k) - u(d_j^k).$$

Our first task is to describe an example whereby concave rationalizability fails. It is easy to construct such an example: consider the two agent case and the one point data set

$$d^1 = (0, 2), \qquad x^1 = (2, 3).$$

How can we see that there is no concave u that concave-rationalizes this data set? Clearly, if a concave u exists, it must be that $u(2) \geq (2/3)u(3) + (1/3)u(0)$ or $3u(2) \geq 2u(3) + u(0)$. This expression is obviously equivalent to $2[u(2) - u(3)] + [u(2) - u(0)] \geq 0$. Finally, we know that $[u(3) - u(2)] + [u(0) - u(2)] = 0$, so by adding the two terms, we obtain $[u(2) - u(3)] \geq 0$, which we know to be a contradiction to increasingness.

If we think in the context of the previous section, what we are doing is adding new types of "edges" to our graph. In the previous section, we could add an edge from d_i^k to x_i^k as long as we added a corresponding edge from some x_j^k to d_j^k. Now, we are also allowed to add certain "collections" of edges, namely, any collection $\{(a_1, b), \dots, (a_n, b)\}$ for which $nb = \sum_{l=1}^n a_l$; that is, b is a rational convex combination of the a_l terms. This is precisely what we did in the previous example. We have the collection of edges $(3, 2)$, $(3, 2)$, $(0, 2)$, which comes from the fact that $2 = (2/3)3 + (1/3)0$. To this, we add the edges $(2, 3)$ and $(2, 0)$, which comes from the fact that we have equal gains. Combining these together results in the cycles

$$((3, 2), (2, 3))$$

$$((0, 2), (2, 0))$$

and

$$(3, 2).$$

Note that the singleton element $(3, 2)$ is by itself a strict cycle.

To this end, we discuss the following generalization of the concept from the previous section. We say a collection of ordered pairs $\{(a_1, b), \dots, (a_m, b)\}$ is a *convex collection* if $\sum_{o=1}^m a_o = mb$. That is, b is a convex combination of the a_o terms. Note that we do not preclude the possibility of several a_o terms being equal.

Let P be a natural number. For each $p \leq P$, let L_p be a natural number and let $\{(a_l^p, b_l^p)\}_{l=1}^{L_p}$ be a sequence of ordered pairs. Say the sequences $\{(a_l^p, b_l^p)\}_{l=1}^{L_p}$ can be *partitioned into cycles* if there exists a natural number T and for each $t \leq T$, a collection of finite sequences $\{(z^1_{tl}, z^2_{tl})\}_{l=1}^{L_t}$ that define cycles (at least one cycle of which is strict) for which there exists a bijection $f:\{(t,l): t \le T, l \le L_t\} \to \{(l,i): l \le L_i, i \in \{1,\dots,P\}\}$ for which $(z^1_{tl}, z^2_{tl}) = (a^p_{f_1(t,l)}, b^p_{f_1(t,l)})$ if $f_2(t,l) = p$.

Proposition 13. *The data $D = \{(d^k, x^k): k = 1,\dots,K\}$ are rationalizable if and only if there are no sequences of data points $(d^l, x^l)_{l=1}^L$ in D, agents $i_l \neq j_l$ for all l, and Q convex collections $\{(a^q_o, b^q_o)\}_{o=1}^{m_q}$, $q = 1,\dots,Q$, with*

$$a^q_o, b^q_o \in \bigcup_i \bigcup_k \{x^k_i, d^k_i\},$$

such that $\{(d^l_{i_l}, x^l_{i_l})\}_{l=1}^L$, $\{(x^l_{j_l}, d^l_{j_l})\}_{l=1}^L$, and $\{(a^q_o, b^q_o)\}_{o=1}^{m_q}$ can be partitioned into cycles, at least one of which is strict.

Remark 14. The assumption in this section that the data are rational is not without loss of generality. The reason has to do with our notion of "convex collection," which allows only for b to be a rational convex combination of the a_o terms. However, a counterpart of Proposition 13 that allows for irrational data could be presented in terms of graphs with weighted edges (which would allow edges to have irrational weights), requiring conservation of flow and precluding strict cycles. The case in Proposition 13 corresponds to the case in which the weights of all edges are rational-valued.

5.6 *Proof of Proposition 13*

Let $X \subseteq \Re^n$ be a finite set such that $d^k, x^k \in X$ for all k. We introduce two copies of X, one whose indicator functions are written in the standard way (1_x). The indicator functions for the second copy are written $1'_x$.

There is a rationalizing u if and only if there is a solution to the system of linear inequalities in the X-dimensional variables u and α,

$$((1_{x^k_i} - 1_{d^k_i}) + (1_{d^k_j} - 1_{x^k_j}))\cdot(u,\alpha) \ge 0 \tag{5}$$

$$(1_{z'} - 1_z)\cdot(u,\alpha) > 0 \tag{6}$$

$$(1_{z'} - 1_z) + (z - z')1'_{z'}\cdot(u,\alpha) \ge 0, \tag{7}$$

where (6) is required for z and z' with $z' > z$.

To see why, suppose there is a rationalizing u. Without loss, we may suppose that u is piecewise linear (if u concave-rationalizes the data, then so does the piecewise linear function that takes the same values as u for every x^k_i and d^k_i). Then u has a supergradient α at every x^k_i and d^k_i. This is the content of (7). The other two inequalities are obviously satisfied.

Conversely, suppose the three equations are satisfied. First, we claim that without loss, each α term is greater than 0. For example, if we consider α_x, where $x < y$ for some $y \in X$, then by (7), it follows that $\alpha_x(y - x) + u_x - u_y > 0$ or $\alpha_x > (u_y - u_x)/(y - x)$ (since $u_y > u_x$ by (6)). Furthermore, if there is no $y \in X$ for which $y > x$, then we can always redefine $\alpha_x = \min_{y \in X, x \neq y}(u_y - u_x)/(y - x) > 0$, which results in another system of consistent weights.

Then it is a standard trick, due to Afriat (1967), to define

$$u(y) = \min_{x \in X} u_x + \alpha_x(y - x),$$

and note that this function is concave, strictly increasing, and rationalizes the data.

Now, we can sketch the argument as to why the satisfaction of the system of inequalities is equivalent to the absence of cycles as stated in Proposition 13. It is again an application of Lemma 12. Equations (5)–(7) have no solution if and only if there are there are vectors $\lambda \in \mathbf{Z}_+^{KN^2}$, $\theta \in \mathbf{Z}_+^{|X|^2}$, and $\eta \in \mathbf{Z}_+^{|X|^2}$, with

$$\sum_{k,i,j} \lambda_{k,i,j}((1_{x_i^k} - 1_{d_i^k}) + (1_{d_j^k} - 1_{x_j^k})) + \sum_{(z,z'):z'>z} \theta_{z,z'}(1_{z'} - 1_z) + \sum_{(z,z')} \eta_{z,z'}(1_{z'} - 1_z) + (z - z')1'_{z'} = 0$$

and $\sum_{(z,z'):z'>z} \theta_{z,z'} > 0$. Importantly, from this equation, we can infer that the two equations

$$\sum_{k,i,j} \lambda_{k,i,j}((1_{x_i^k} - 1_{d_i^k}) + (1_{d_j^k} - 1_{x_j^k})) + \sum_{(z,z'):z'>z} \theta_{z,z'}(1_{z'} - 1_z) + \sum_{(z,z')} \eta_{z,z'}(1_{z'} - 1_z) = 0$$

and

$$\sum_{(z,z')} \eta_{z,z'}(z - z')1'_{z'} = 0 \tag{8}$$

are jointly satisfied.

The proof now proceeds in the same way as the proof of Proposition 11. There is only one change. Now, we also consider a collection of X-dimensional vectors $\{-1, 0, 1\}$, which we call A_C, that consists of $\eta_{z,z'}$ copies of each $1_z - 1_{z'}$. The graph $G = (X, E)$ is now constructed in the same way as in the proof of Proposition 11, letting there be one edge from x to x' for each copy of $1_{x'} - 1_x$ in A_D, A_U, A_M, or A_C. Now, consider (8). This equation implies, in particular, that for all z', $\sum_z \eta_{z,z'}(z - z') = 0$; in other words, the collection of edges pointing to z' in A_C form a concave collection.

5.7 *Generalization*

We have asked for data to be rationalized by a single utility function, common to all $i \in N$. If, instead, we ask that for each i, there exists $u_i : \Re \to \Re$ for which, for all $i, j \in N$,

$$u_i(x_i^k) - u_i(d_i^k) = u_j(x_j^k) - u_j(d_j^k),$$

we obviously get a weaker condition. The weakening required here is simply that when partitioning data into cycles, each cycle can contain only edges that correspond to a *single* agent. The proof is similar to the preceding proof and is hence omitted.

5.8 *An application to spatial competition*

The result in Proposition 11 has a simple application to the testable implications of Hotelling's model of spatial competition (Hotelling 1929). Hotelling's model concerns the location of two vendors on a unidimensional space and a distribution of consumers. Consumers always buy from the vendor closest to them (in the case of equidistant vendors, half of the consumers go to one vendor and the other half to the other vendor) and each vendor's profit consists of how many consumers buy from him. The unique Nash equilibrium of this game has both vendors locating at the median of the distribution of consumers. Hotelling's model is not about bargaining, but it seems potentially useful to point out the application of our results.

In our version of Hotelling's model, we observe a finite collection of closed intervals $\{[a^k, b^k]\}_{k=1}^K$ and for each observed interval, a location $m^k \in (a^k, b^k)$. We want to know, when does there exist a full-support distribution μ of agents on $[0, 1]$ such that for each k, m^k is the median of μ conditional on $[a^k, b^k]$? This provides us with the testable implications of the Hotelling model when the distribution of agents is unobserved, but when the boundaries of spatial competition can vary.

The relation to Section 5 is as follows. A distribution μ that satisfies the properties exists if and only if there is a strictly increasing $F : [0, 1] \to \Re$ (a cumulative distribution function (c.d.f.)) for which for all k, $F(b^k) - F(m^k) = F(m^k) - F(a^k)$. Now, imagine that in the previous section we had only two agents ($|N| = 2$), and $d^k = (m^k, a^k)$ and $x^k = (b^k, m^k)$.

This leads us directly to the following corollary.

Corollary 15. *A finite list of intervals $[a^k, b^k]$ and locations m^k is consistent with the Hotelling model if and only if there are no sequences of intervals $[a^l, b^l]_{l=1}^L$, $[a^l, b^l]_{l=L+1}^{L'}$ for which $\{(a^l, m^l)\}_{l=1}^L$, $\{(b^l, m^l)\}_{l=1}^L$, $\{(m^l, a^l)\}_{l=L+1}^{L'}$, and $\{(m^l, b^l)\}_{l=L+1}^{L'}$ can be partitioned into cycles.*

6. Conclusion

We consider finite sets of observations of bargaining outcomes. Assuming that utility function are unobservable, we develop testable implications of some of the best known models in bargaining theory.

We consider two basic frameworks. Our results are sharpest for the case where we assume that disagreement points are fixed across observations. We show that the utilitarian, Nash bargaining, and egalitarian max-min models are all observationally equivalent. Furthermore, we show that a simple test for these models consists of checking that the observed allocations are co-monotonic.

When disagreement points can vary, we present a characterization of the data that are consistent with a form of egalitarianism, namely the model of equal gains/losses. By appropriately interpreting the model, we can apply our results to data on the tax code: we can check for consistency of the tax code with the principle of equal loss when the utility function is unknown.

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