Theoretical Economics 9 (2014), 253–277 1555-7561/20140253

Two axiomatic approaches to the probabilistic serial mechanism

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This paper studies the problem of assigning a set of indivisible objects to a set of agents when monetary transfers are not allowed and agents reveal only ordinal preferences, but random assignments are possible. We offer two characterizations of the *probabilistic serial* mechanism, which assigns lotteries over objects. We show that it is the only mechanism that satisfies *non-wastefulness* and *ordinal fairness*, and the only mechanism that satisfies *sd-efficiency*, *sd-envy-freeness*, and *weak invariance* or *weak truncation robustness* (where "sd" stands for first-order stochastic dominance).

Keywords. Random assignment, probabilistic serial, ordinal fairness, sd-efficiency, sd-envy-freeness, weak invariance, weak truncation robustness.

JEL classification. C71, C78, D71, D78.

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We would like to thank Christian Basteck, Anqi Fu, Fuhito Kojima, Mike Ostrovsky, Al Roth, Michael Schwarz, Jay Sethuraman, and participants at the 2011 Asian Meeting of the Econometric Society, CORE, the Duke "Roth–Sotomayor: 20 Years After" Conference, Kyoto, Maastricht, Osaka, and Stanford, Tokyo, Tsukuba, 6th Pan Pacific Conference on Game Theory at Tokyo Tech, and Université Libre de Bruxelles for comments. Kurino acknowledges financial support from the Max Planck Institute of Economics and Maastricht University when he was affiliated there. Ünver acknowledges the research support of Microsoft Research Lab, New England. The current paper supersedes two previous working papers, Kesten et al. (2011) and Hashimoto and Hirata (2011), in characterizing the probabilistic serial mechanism. We thank the editor, an associate editor, and referees for their constructive comments.

DOI: 10.3982/TE1010

1. Introduction

A wide range of real-life resource allocation problems—student placement in public schools, organ transplantation through live or deceased donors, on-campus housing allocation, and course allocation at business schools—involves the assignment of indivisible objects without the use of monetary transfers.

Most of these markets rely on *ordinal* mechanisms, where participants reveal only their preference rankings over given choices to the central authority rather than their cardinal preferences. Ensuring fairness of a deterministic allocation can entail significant inefficiencies.[1] Therefore, it has become commonplace to use random mechanisms, which allow the allocation of divisible probabilities, to achieve fairness ex ante.[2]

In spite of this use of randomization, most such markets rely on ordinal mechanisms: participants reveal only their preferences over objects, rather than their preferences over random allocations of objects. However, from an agent's ordinal ranking $\succ_i$ over the set A of objects (assumed strict), one can define first-order stochastic dominance (f.o.s.d.), which is a partial order $\geq_i$ over the set of random allocations (probability measures on A). These partial orders can be used to evaluate random mechanisms. Using the prefix "sd-" to indicate f.o.s.d., we say that a random assignment P is *sd-efficient* if it is Pareto efficient with respect to the f.o.s.d. orderings. We say that it is *sd-envy-free* if $P_i \geq_i P_j$ for all i, j. Then we can make comparisons:

- Sd-efficiency is stronger than ex post efficiency, though not as strong as ex ante efficiency would be if one had access to the complete von-Neumann–Morgenstern (vNM) utilities.
- Sd-envy-freeness is weaker than ex post envy-freeness, though not as weak as ex ante envy-freeness would be with the vNM utilities.

A common mechanism used in practice is the *random serial dictatorship* (RSD). Agents are randomly ordered (with a uniform distribution over permutations) and then, in the realized order, agents successively pick their favorite objects from those available. However, in spite of the apparent equal treatment of agents, the resulting random assignment may not be sd-envy-free; neither need it be sd-efficient.

In a seminal paper, Bogomolnaia and Moulin (2001) (BM hereafter) proposed the probabilistic serial mechanism (PS), which is sd-efficient and sd-envy-free. The outcome of PS is defined by the simultaneous eating algorithm (SEA): Consider each object as a continuum of probability shares. Agents simultaneously "eat away" from their favorite objects at the same speed; once an agent's favorite object is gone, he turns to his next favorite object, and so on. The amount of an object eaten away by an agent

[1] See, for example, Kesten and Yazıcı (2012).

[2] For example, the assignment mechanisms used in the context of student placement operate through a collection of strict priority orders of schools over students. In practice, determining these orders often involves randomization (Abdulkadiroğlu and Sönmez 2003b, Erdil and Ergin 2008, Pathak and Sethuraman 2011, Kesten and Ünver 2013). Similarly, in the exchange of live-donor kidneys among kidney patients for transplantation, the egalitarian approach requires the design of a random mechanism (Roth et al. 2005).

throughout the process is interpreted as the probability with which he is assigned this object by PS.[3]

The purpose of this paper is to provide two axiomatizations of PS. Our first axiomatization is built around a new property, *ordinal fairness*. Fix a random assignment and for any agent i and each $a \in A$, let $F_i(a)$ be the probability that i obtains a or an object better than a; this is called i's surplus at a. The random assignment is ordinally fair if for all i, j and $a \in A$ such that j obtains a with a positive probability, i's surplus at a is as large as j's surplus at a.

Though related in spirit, ordinal fairness and sd-envy-freeness are quite different, as we illustrate with this example. Suppose there are two agents, $i = 1, 2$, and two objects, $A = \{a, b\}$. Agent 1 prefers a to b; agent 2 prefers b to a. Suppose we give each object to each agent with an equal probability. Agent 1 does not wish he had agent 2's random allocation, yet he might envy the fact that agent 2 always gets an object that she likes at least as much as a, whereas this happens to agent 1 only half of the time. The allocation is not ordinally fair.

In this example, the random assignment is not sd-efficient. The only sd-efficient allocation gives a to agent 1 for sure and b to agent 2 for sure, but then ordinal fairness obtains. This suggests a link between ordinal fairness and both sd-efficiency and sd-envy-freeness. In fact, we show that ordinal fairness implies both of these properties in the BM setting in which the total supply of objects exactly equals the number of agents. Furthermore, it provides a full characterization of PS in the same setting. (This is the first redefinition of an algorithmic matching mechanism, that we are aware of, through a single tight property.) In the more general setting when the total supply of objects exceeds the number of agents, it characterizes PS in combination with a mild assumption called *non-wastefulness* (Theorem 1).

We obtain a second characterization of PS using sd-efficiency and sd-envy-freeness. These are implied by PS, but do not fully characterize it. Our Theorem 2 and Corollary 2 show that a complete characterization is obtained by adding either *weak invariance* or *weak truncation robustness*; these axioms impose invariance of the assignment to certain perturbations of the ordinal preferences.

1.1 *Related literature*

There are very few papers that discuss the random assignment problem prior to the new millennium. The earliest account of the problem is due to Hylland and Zeckhauser (1979), who propose a pseudo-market mechanism that relies on cardinal preferences of agents. Much later, Zhou (1990) proves an important impossibility result for the cardinal domain: There exists no strategy-proof, Pareto-efficient, and symmetric mechanism. A similar negative result is obtained by Chambers (2004) in the ordinal domain:

[3]However, RSD is sd-strategy-proof, unlike PS, which is sd-strategy-proof only in a weak sense. Nevertheless, Kojima and Manea (2010) show that in large but finite problems where each object has a sufficiently large supply, PS regains sd-strategy-proofness. In related work, Che and Kojima (2010) show that in the limit of discrete economies with finite object types, PS converges to RSD.

all ex post consistent, symmetric, and strategy-proof mechanisms should coincide with uniformly random assignment of objects.

Following the seminal work of BM that introduced PS, the literature on the random assignment problem has grown rapidly. Contrary to the early literature, the new strand of literature often restricts attention to the case when agents' preferences are ordinal.[4]

The PS was initially proposed by Crès and Moulin (2001) for a simple model where agents have the same rankings over objects. A characterization for this special context is given by Bogomolnaia and Moulin (2002). Kojima and Manea (2010) show that PS recovers strategy-proofness when the market size becomes sufficiently large. Manea (2009) shows that ordinal inefficiency of RSD prevails even for large assignment problems. Katta and Sethuraman (2006) extend PS to the domain of weak preferences. Yılmaz (2009, 2010) adapts it to environments where there may be initial property rights over some of the objects. Athanassoglou and Sethuraman (2011) further extend this model and the mechanism to the case with probabilistic endowments. Kojima (2009) offers a generalization of PS to multiple assignment problems.

Abdulkadiroğlu and Sönmez (1998) show that RSD is equivalent to a core mechanism that uniformly randomly selects an initial assignment of objects and then utilizes Gale's celebrated *top trading cycles* (Shapley and Scarf 1974) procedure. Sönmez and Ünver (2005), Pathak and Sethuraman (2011), and Carroll (2013) extend this result to different random matching domains. Kesten (2009) shows a similar connection between PS and the top trading cycles procedure: PS is equivalent to a particular top trading cycles mechanism that initially endows each agent with an equal share of each object. He also provides a "replicated" RSD mechanism that becomes equivalent to PS in the limit. Budish et al. (2013) characterize the constraints on a random assignment that can also be satisfied by each of the deterministic assignments in the support of a lottery that induces it.

The compelling notion of sd-efficiency is also the focus of other related papers. Abdulkadiroğlu and Sönmez (2003a) offer a characterization of ordinally efficient random assignments. McLennan (2002) proves an interesting result on the relationship between sd-efficiency and ex ante efficiency. Manea (2008) provides a constructive proof of this result.

The axiomatic characterization of PS for unrestricted preference domains began with three independent studies: Hashimoto and Hirata (2011) (hereafter, HH), Heo (2013), and Kesten et al. (2011) (hereafter, KKÜ).[5] KKÜ is the paper that originally presents Theorem 1 of this paper, and is the first paper that characterizes PS in the general case using sd-efficiency and sd-envy-freeness. HH characterize the mechanism with these axioms in the environment where the null object always exists. They also provide an axiomatization based on the Rawlsian principle. Heo (2013) considers an environment where agents may demand multiple units and shows that the generalized

[4]Three common justifications for the ordinal approach are as follows: First, since agents are boundedly rational, cardinal preferences are difficult to elicit. Second, ordinal mechanisms are relatively simpler and more practical than cardinal mechanisms. Third, real-life matching markets function mostly through elicitation of ordinal preferences.

[5]The first versions of the papers by Heo and KKÜ were circulated in 2010.

PS mechanism is characterized by sd-efficiency, the proportional division lower bound, and several auxiliary axioms.

In a more recent work, Bogomolnaia and Heo (2012) (hereafter, BH) replace the invariance axioms in KKÜ and HH with a weaker condition called bounded invariance and offer new, shorter proofs in a unifying framework for the KKÜ and HH results. This proof technique was based on the observations of Heo (2013) regarding PS and probabilistic assignments in general. Our characterizations in this paper using sd-efficiency and sd-envy-freeness (Theorem 2 and Corollary 2), which build on KKÜ and HH, are stronger than all three previous results mentioned (HH Theorem 1, KKÜ Theorem 2, and BH Theorem 2), as weak invariance is implied by both upper invariance of KKÜ and bounded invariance of BH, in the general case when a null object does not necessarily exist; and weak truncation robustness, when a null object exists, is implied by both truncation robustness of HH and bounded invariance of BH.[6,7,8]

Most notably, whereas all the previously considered invariance conditions mentioned above require that whenever the preferences of an agent change with reference to a fixed object in a specific way, *all* agents' probability shares of the particular object remain the same, weak invariance makes a much less demanding requirement: only the particular agent's probability share of the particular object should remain the same.[9]

Alternatively, Liu and Pycia (2011) look at large markets in which all types of agents are represented. They show that in this case, there is a unique mechanism that is sd-efficient and sd-envy-free, and that in the limit of large markets, uniformly random versions of many known deterministic mechanisms such as serial dictatorships, hierarchical exchange rules (Pápai 2000), and trading cycles mechanisms (Pycia and Ünver 2011) coincide with this unique mechanism.

2. Model

Our object of study is a discrete resource allocation problem (cf. Hylland and Zeckhauser 1979, Shapley and Scarf 1974). Let N be the finite set $\{1, \dots, n\}$ of *agents* to whom objects are allocated. In BM, there are exactly n distinct objects to be allocated, one per agent. We generalize this slightly: each agent still receives one object, but the pool of objects to be distributed can include duplicates, that is, objects that are equivalent for all the agents. We let A denote the set of *types of objects* and, for $a \in A$, let q_a denote the *quota* or supply of object a. There may be a surplus of objects: $\sum_{a\in A} q_a \geq |N|$.

[6]We thank an anonymous referee for suggesting that we weaken HH's definition of truncation robustness to the current definition (Definition 3). Upon showing that this new definition is strong enough to characterize PS, we observed that the proof also extends to the general case where the null object may not exist. This motivated us to obtain our second characterization result using the current definition of weak invariance (Definition 2), which is the counterpart of Definition 3 in environments without the null object.

[7]Also, our proof immediately implies that we can weaken sd-efficiency in Theorem 2 and Corollary 2 as in HH and BH.

[8]A more recent paper by Heo and Yılmaz (2012) extends the results of BH to the case with weak preferences for Katta and Sethuraman's (2006) extended probabilistic serial correspondence.

[9]This axiom was previously introduced by Heo (2013) as one of her auxiliary axioms. She referred to it as "limited invariance."

An *assignment* specifies an object for each agent such that for each $a \in A$, the number of agents receiving object a does not exceed q_a. Let $\mathcal{A}$ refer to the set of possible assignments. We assume objects can be allocated randomly. A *lottery* is a probability distribution over assignments. Each agent $i \in N$ cares only about his own *random allocation*, that is, the resulting probability distribution $P_i = [p_{i,a}]_{a\in A}$ over A, where $p_{i,a}$ is the probability with which he receives object a. We refer to the matrix $P = [P_i]_{i\in N}$ of random allocations, where each row P_i is the random allocation of an agent and each column P_a allocates probability shares of an object a to the agents, as a *random assignment*; it has the property that $\sum_{i\in N} p_{i,a} \leq q_a$ for each $a \in A$ and $\sum_{a\in A} p_{i,a} = 1$ for $i \in N$. Let $\mathcal{R}$ refer to the set of possible random assignments. Each lottery induces such a random assignment and each such random assignment is induced by some lottery (cf. von Neumann 1953).[10] Therefore, we can focus our attention on random assignments as the outcome of a mechanism.

We require that a mechanism elicits only each agent i's *ordinal preference relation* $\succ_i$ over objects. This preference ordering is assumed to be strict. Although we implicitly allow for some indifference by letting there be duplicates of each object, any indifference must be shared by all agents. Let $\mathbf{P}$ be the set of such strict preferences. We sometimes represent $\succ_i$ by the ordered list of objects; e.g., $\succ_i = (b, c, a)$ or $\succ_i = (bca)$ means that $b \succ_i c \succ_i a$ (here assuming that $A = \{a, b, c\}$).

Although agents' preferences over random allocations are unspecified, we can construct a partial order that can be used to compare random allocations based on (first-order) stochastic dominance. Given $a \in A$ and $\succ_i \in \mathbf{P}$ for agent i, let $U(\succ_i, a) = \{b \in A \mid b \succeq_i a\}$ be the *upper contour set of object a at $\succ_i$*. Given a random allocation P_i, let $F(\succ_i, a, P_i) = \sum_{b\in U(\succ_i,a)} p_{i,b}$ be the probability that i is assigned an object at least as good as a under P_i; we simply refer to it as *i's surplus at a under P_i*. For agent i, given $\succ \in \mathbf{P}^N$ and $P, R \in \mathcal{R}$, P_i *stochastically dominates* R_i at $\succ_i$ if $F(\succ_i, a, P_i) \geq F(\succ_i, a, R_i)$ for all $a \in A$. In addition, P *stochastically dominates* R at $\succ$ if P_i stochastically dominates R_i at $\succ_i$ for all $i \in N$.

Throughout the paper, whenever it is not ambiguous, we suppress N, A, and q, and denote an *allocation problem* by a preference profile. Formally, a *mechanism* is a systematic way to find a random assignment for a given problem, that is, it is an allocation rule $\phi : \mathbf{P}^N \to \mathcal{R}$.

Our model is general enough to contain various interesting special cases:

(i) *Unacceptable objects*: There is a specific object referred to as the *null object* and assigned a quota of at least $|N|$. By interpretation, agents who are assigned the null object are viewed as taking their outside options or, using the matching jargon, they *remain unassigned*. The objects ranked below the null object are called *unacceptable*. This case models assignment under voluntary participation.[11]

[10] This classical result is also commonly credited to Garrett Birkhoff and referred to as the Birkhoff–von Neumann Theorem.

[11] In this setting, the standard *individual rationality* requirement, i.e., that no agent be assigned an unacceptable object with some positive probability, is implied by either efficiency property to be subsequently introduced; namely, by either non-wastefulness or sd-efficiency.

(ii) *Perfect supply with unit quotas*: Each object has a quota of 1 and there are exactly $|N|$ objects. This is the original setting of BM.[12]

Three properties of random assignments are essential in our characterizations. A random assignment is *sd-efficient* if it is not stochastically dominated by another random assignment.[13]

Next is a much weaker efficiency property. A random assignment is non-wasteful if the surplus of no agent at any object can be raised through the use of an unassigned probability share of some object. Formally, given $\succ \in \mathbf{P}^N$, $P \in \mathcal{R}$ is *non-wasteful* at $\succ$ if for all $i \in N$ and all $a \in A$ such that $p_{i,a} > 0$, we have $\sum_{j \in N} p_{j,b} = q_b$ for all $b \in A$ with $b \succ_i a$.

Our first fairness property is a fundamental principle in mechanism design theory originally proposed by Foley (1967). A random assignment is sd-envy-free if each agent, regardless of his vNM utilities, prefers his random allocation to that of any other agent. Formally, given $\succ \in \mathbf{P}^N$, $P \in \mathcal{R}$ is *sd-envy-free* at $\succ$ if for all $i \in N$, P_i stochastically dominates P_j for all $j \in N$ at $\succ_i$.

A mechanism is said to satisfy a property if its outcome, for any problem, satisfies that property.

3. Two new axioms

Our second fairness property, which is essential to our first characterization, is a natural and intuitive axiom for the random assignment setting. A random assignment is ordinally fair if whenever an agent is assigned some object with positive probability, his surplus at this object is no greater than that of any other agent at the same object. It follows that whenever an agent is assigned some object x with zero probability, he must be assigned a better object (for him) with a probability no less than any agent who is assigned object x with positive probability.

Definition 1. Given $\succ \in \mathbf{P}^N$, $P \in \mathcal{R}$ is *ordinally fair* at $\succ$ if for all $a \in A$ and all $i, j \in N$ with $p_{i,a} > 0$, we have $F(\succ_i, a, P_i) \leq F(\succ_j, a, P_j)$.

One interpretation of the problem we are studying here is to entitle each agent to an equal probability share of each object initially. Under such an interpretation, ordinal fairness makes it possible for agents to efficiently redistribute their initial shares among themselves so that every agent can enjoy a higher object-specific surplus, provided that this surplus does not exceed that of another agent. In this sense, ordinal fairness can be viewed as an analogue for the current setup of Varian's fairness notion, which encompasses Pareto efficiency and envy-freeness in exchange economies with perfectly divisible goods (cf. Varian 1974, 1975, 1976). Remarkably, ordinal fairness implies both sd-efficiency and sd-envy-freeness, and it is implied by these two properties in conjunction

[12]In this setting, one of our properties—non-wastefulness—to be subsequently introduced, is satisfied vacuously.

[13]Equivalently, under any alternative random assignment, the surplus of some agent at some object is less than that under the original assignment.

with a weak technical property when the total supply of objects is equal to the number of agents.

We next introduce an auxiliary robustness axiom—weak invariance—that is essential to our second characterization. Given $\succ_{-i}$, the axiom requires that the probability of agent i getting object a depends only on i's preference ranking down to a. When the null object is available, we can interpret weak invariance as robustness against truncations, which are practically important manipulations. We formalize this interpretation in Section 6. Let $\succ_i|_B$ be the *restriction of* $\succ_i \in \mathbf{P}$ *to* $B \subseteq A$; that is, $\succ_i|_B$ is a preference relation over B such that for all $a, b \in B$, $a \succ_i|_B b \Leftrightarrow a \succ_i b$.

Definition 2. A mechanism ϕ is *weakly invariant* if for all $\succ \in \mathbf{P}^N$, $i \in N$, $a \in A$, and $\succ'_i \in \mathbf{P}$, $\phi_{i,a}(\succ) = \phi_{i,a}(\succ'_i, \succ_{-i})$ whenever $U(\succ'_i, a) = U(\succ_i, a)$ and $\succ'_i|_{U(\succ'_i,a)} = \succ_i|_{U(\succ'_i,a)}$.[14]

Most mechanisms studied in the literature are weakly invariant. Examples include PS, the agent-proposing deferred acceptance mechanism, the Boston mechanism, and hierarchical exchange rules (Pápai 2000), which include serial dictatorship and the top trading cycles mechanism as special cases.[15] The RSD is also weakly invariant since it is a convex combination of weakly invariant mechanisms.

4. Probabilistic serial mechanism

BM introduced the *probabilistic serial mechanism* (PS), the outcome of which can be computed via the following *simultaneous eating algorithm* (SEA):

Given a problem $\succ$, think of each object a as an infinitely divisible good with supply q_a that agents eat in the time interval [0, 1].

Step 1. Each agent eats away from his favorite object at the same unit speed. Proceed to the next step when an object is completely exhausted.

⋮

Step s (for $s \in \{2, \dots, S\}$). Each agent eats away from his remaining favorite object at the same speed. Proceed to the next step when an object is completely exhausted.

The procedure terminates after $S \leq |N|$ steps when each agent has eaten exactly 1 total unit of objects (i.e., at time 1). The random allocation of an agent i by PS is then given by the amount of each object he has eaten until the algorithm terminates. Let $\text{PS}(\succ) \in \mathcal{R}$ denote the outcome of PS for problem $\succ$.

5. First characterization of probabilistic serial

In our first result, we establish that for each problem there is a unique ordinally fair and non-wasteful random assignment and that this random assignment is the outcome of

[14] This property is weaker than both the upper invariance condition of KKÜ and the bounded invariance condition of BH.

[15] The "object-proposing" deferred acceptance mechanism, however, violates weak invariance. This is because agents may benefit from truncation (see, e.g., Example 2 of Roth and Rothblum 1999), which is not possible under a weakly invariant mechanism.

SEA. In other words, PS fully characterizes ordinal fairness with non-wastefulness and vice versa.

THEOREM 1. *A mechanism is ordinally fair and non-wasteful if and only if it is* PS.

PROOF. Fix $\succ \in \mathbf{P}^N$. We drop $\succ$ from all arguments below. We reinterpret the SEA such that at each step, at most one object is fully exhausted: If two objects a, b are exhausted in a step according to the original definition, we order these objects arbitrarily and say that one is exhausted first and the other one is exhausted in the next step. We redefine step S as the first step when each agent has eaten exactly 1 total unit of objects. Let $h^1, \dots, h^{S-1}$ denote the objects exhausted in Steps 1 to $S-1$ and let the remaining objects be arbitrarily ordered as $h^S, \dots, h^{|A|}$.

($\Leftarrow$) PS is *non-wasteful* as it is *sd-efficient*. We show that PS is *ordinally fair*. First, consider $s < S$: Each agent has eaten away weakly better objects than h^s until s at the same speed. Thus, for any $i \in N$ who eats away h^s at s and any $j \in N$ who eats away some $b_j \succeq_j h^s$, since they continue eating at the same speed and b_j is not exhausted before h^s, we have $F(\succ_i, h^s, \mathrm{PS}_i) \leq F(\succ_j, b_j, \mathrm{PS}_j) \leq F(\succ_j, h^s, \mathrm{PS}_j)$. Next, consider $s \geq S$: At step S, each $j \in N$ eats away some $b_j \succeq_j h^s$. When SEA terminates after S, j's surplus at b_j is 1 and, hence, $F(\succ_j, h^s, \mathrm{PS}_j) = 1$. Thus, in either case *ordinal fairness* is satisfied for h^s.

($\Rightarrow$) Let $P \in \mathcal{R}$ be *ordinally fair* and *non-wasteful* at the fixed $\succ$. We show that $\mathrm{PS} = P$. Define $\pi(a) = \min_i F(\succ_i, a, P_i)$ for all $a \in A$. Relabel objects as $a^1, \dots, a^{|A|}$ so that $\pi(a^s) \leq \pi(a^{s+1})$ for all $s \leq |A| - 1$. Let $A^0 = \varnothing$, let $A^s = \{a^1, \dots, a^s\}$, and let $\overline{A^s} = A \setminus A^s$ be the set complement of A^s. For all $s \geq 1$ and all $a \in \overline{A^{s-1}}$, let $N^s(a) = \{k \in N \mid a \succeq_k b \text{ for all } b \in \overline{A^{s-1}}\}$.

We argue by induction. Fix some $s \geq 1$. Assume that for all $t < s$ and all $i \in N^t(a^t)$, $F(\succ_i, a^t, P_i) = F(\succ_i, a^t, \mathrm{PS}_i) = \pi(a^t)$; for all $k \notin N^t(a^t)$, $p_{k,a^t} = \mathrm{PS}_{k,a^t} = 0$; and a^t is the object exhausted at step t of SEA for $t < S$. Each statement in the inductive assumption holds vacuously for $s = 1$. We prove that they also hold for step s and thus $P = \mathrm{PS}$:

Step 1. We show that for all $k \notin N^s(a^s)$, $p_{k,a^s} = 0$. For a contradiction, suppose for some $b \in \overline{A^s}$ and $k \in N^s(b)$, we have $p_{k,a^s} > 0$. For an agent j with $\pi(a^s) = F(\succ_j, a^s, P_j)$, we have $F(\succ_k, a^s, P_k) > F(\succ_k, b, P_k) \geq F(\succ_j, a^s, P_j)$, where the last inequality follows from the ordering of a^s before b through π. However, this inequality violates *ordinal fairness* of P.

Step 2. We show that for all $i \in N^s(a^s)$, $F(\succ_i, a^s, P_i) = \pi(a^s)$. Let $i \in N^s(a^s)$. Either $p_{i,a^s} > 0$ or $p_{i,a^s} = 0$. If $p_{i,a^s} > 0$, then by *ordinal fairness*, for all $j \in N$, $F(\succ_i, a^s, P_i) \leq F(\succ_j, a^s, P_j)$ and, thus, $F(\succ_i, a^s, P_i) = \pi(a^s)$ by the definition of $\pi(a^s)$. Suppose $p_{i,a^s} = 0$. Let t^* be the earliest step t such that $i \in N^t(a^s)$. If $t^* = 1$, then by $p_{i,a^s} = 0$, $F(\succ_i, a^s, P_i) = 0$ and, thus, $F(\succ_i, a^s, P_i) = \pi(a^s)$ by the definition of $\pi(a^s)$. Next, suppose $t^* > 1$. Then $i \in N^{t^*-1}(a^{t^*-1})$. By the inductive assumption (as $t^* \leq s$), $\pi(a^{t^*-1}) = F(\succ_i, a^{t^*-1}, P_i)$. Thus, as a^s is ranked just below a^{t^*-1} in $\succ_i$ and $p_{i,a^s} = 0$, then $F(\succ_i, a^s, P_i) = \pi(a^{t^*-1})$. Moreover, since a^s is ordered after a^{t^*-1} according to π, then $\pi(a^s) \geq \pi(a^{t^*-1})$. Thus, as $F(\succ_i, a^s, P_i) \geq \pi(a^s)$, $F(\succ_i, a^s, P_i) = \pi(a^s)$.

Step 3. We show that at step s of SEA for $s < S$, for any agent $i \in N^s(a^s)$, $F(\succ_i, a^s, \mathrm{PS}_i) \geq \pi(a^s)$. By the inductive assumption, for each $b \in \overline{A^{s-1}}$, at the end of step

$s-1$ of SEA, the amount that each $i \in N^s(b)$ has eaten away from objects in $U(\succ_i, b)$ is $x = \pi(a^{s-1})$ if $s > 1$ and is $x = 0$ if $s = 1$; moreover, all objects in $U(\succ_i, b) \setminus \{b\}$ are also fully exhausted. For all $j \in N^s(a^s)$, all $b \in \overline{A^{s-1}}$, and all $k \in N^s(b)$, we have $\pi(a^s) \leq \pi(b)$ by the definition of a^s, and this, together with Step 2 and the definition of π, implies $F(\succ_j, a^s, P_j) - x = \pi(a^s) - x \leq \pi(b) - x \leq F(\succ_k, b, P_k) - x$. Thus, the remaining amount of object b is sufficiently large for each agent in $N^s(b)$ so that when each agent in $N^s(a^s)$ has eaten away $\pi(a^s) - x$ of a^s, no agent in $N^s(b)$ has yet started eating an object different from b. Therefore, each $j \in N^s(a^s)$ eats away by the end of step s at least $(\pi(a^s) - x) + x = \pi(a^s)$, the total amount from objects in $U(\succ_j, a^s)$, implying that $\pi(a^s) \leq F(\succ_j, a^s, \mathrm{PS}_j)$.

Step 4. We show that for all $j \in N^s(a^s)$, $\pi(a^s) = F(\succ_j, a^s, \mathrm{PS}_j)$ and that $\mathrm{PS}_{k,a^s} = 0$ for all $k \notin N^s(a^s)$ for $s < S$. Proving the first claim is sufficient (by Step 3). Suppose, to the contrary, for some $i \in N^s(a^s)$, $F(\succ_i, a^s, P_i) = \pi(a^s) < F(\succ_i, a^s, \mathrm{PS}_i) \leq 1$; but then

$$\sum_j p_{j,a^s} = \sum_{j \in N^s(a^s)} \left\{ F(\succ_j, a^s, P_j) - \sum_{b \succ_j a^s} p_{j,b} \right\}$$

$$< \sum_{j \in N^s(a^s)} \left\{ F(\succ_j, a^s, \mathrm{PS}_j) - \sum_{b \succ_j a^s} \mathrm{PS}_{j,b} \right\} = \sum_{j \in N^s(a^s)} \mathrm{PS}_{j,a^s} \leq q_{a^s},$$

where $\sum_{j \in N^s(a^s)} F(\succ_j, a^s, P_j) < \sum_{j \in N^s(a^s)} F(\succ_j, a^s, \mathrm{PS}_j)$ by Steps 2 and 3 and the supposition, and $p_{j,b} = \mathrm{PS}_{j,b}$ for all $j \in N^s(a^s)$ and all $b \succ_j a^s$ by the inductive assumption. This violates *non-wastefulness* of P. We have shown that for all $j \in N^s(a^s)$, $F(\succ_i, a^s, \mathrm{PS}_i) = \pi(a^s)$. Then step s of SEA ends when a^s is fully exhausted by Step 3 of the proof. Moreover, $\mathrm{PS}_{k,a^s} = 0$ for all $k \notin N^s(a^s)$ as none of these agents has started eating a^s before it gets fully exhausted under SEA.

Step 5. We show that the rest of the inductive claim holds for $s \geq S$. The SEA terminates at step S when each agent has eaten exactly 1 total unit of objects. Any agent $i \in N^S(a)$ eats away $a \in \overline{A^{S-1}}$ at step S of SEA. Thus, $F(\succ_i, a, \mathrm{PS}_i) = 1$ and for any $k \notin N^S(a)$, $\mathrm{PS}_{i,a} = 0$. By *non-wastefulness* of P (through the same argument in Step 4 applied to a instead of a^s), for any $i \in N^s(a)$, $F(\succ_i, a, \mathrm{PS}_i) = \pi(a) = F(\succ_i, a, P_i)$. □

Remarkably, ordinal fairness turns out to be a very powerful axiom as it can exclusively characterize PS. Therefore, Theorem 1 offers a new perspective on this mechanism for an appealing domain of problems that subsumes the original BM setting. Consider problems in which the total supply of real objects is less than or equal to the number of agents and all objects are acceptable. In this case, all random assignments are nonwasteful. Hence, a random assignment is ordinally fair if and only if it is the PS outcome.

Corollary 1. *In an environment in which all objects are acceptable and the total quota of the objects does not exceed the number of agents, a mechanism is ordinally fair if and only if it is* PS.

This result is important as it shows that ordinal fairness offers a non-algorithmic definition of PS. In the matching literature, almost all mechanisms are defined through algorithmic procedures that have useful properties. This is in contrast to some well known mechanisms in other contexts such as the Walrasian market mechanism, whose conceptual definition preceded any of its algorithmic (or fixed-point) constructions. As far as we are aware, ours is the first "redefinition" of a matching mechanism based on a single property.

6. Second characterization of probabilistic serial

Sd-efficiency and sd-envy-freeness are among the most appealing properties of mechanisms. Our second result characterizes PS through these two fundamental properties together with a mild robustness condition.

Theorem 2. *A mechanism is sd-efficient, sd-envy-free, and weakly invariant if and only if it is* PS.[16]

Before giving a formal proof of Theorem 2, we provide an illustration in Example 1 of our proof strategy for the necessity part. Although the actual proof is more subtle, much of the intuition behind the proof can be grasped from this simple example.

Example 1. Suppose that there are three agents $N = \{1, 2, 3\}$ and three objects $A = \{a, b, c\}$ each with unit quota. Consider preferences $\succ = ((abc), (abc), (bca))$ and $\succ_3^* = (bac)$. Then the PS outcomes for $\succ$ and $(\succ_3^*, \succ_{-3})$ are given as

$$\mathrm{PS}(\succ) = \mathrm{PS}(\succ_3^*, \succ_{-3}) = \begin{array}{cccc} \hline & a & b & c \\ \hline 1 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 2 & \frac{1}{2} & \frac{1}{6} & \frac{1}{3} \\ 3 & 0 & \frac{2}{3} & \frac{1}{3} \\ \hline \end{array}.$$

This follows from the SEA as follows: Agents 1 and 2 initially start eating a and agent 3 starts eating b under either profile. At time $\tau = \frac{1}{2}$, object a is fully consumed, giving $\frac{1}{2}$ share of a to agents 1 and 2; thus, all agents continue with b. At this point only $\frac{1}{2}$ of b is available and this remainder is equally shared among the three agents giving $\frac{1}{2} + \frac{1}{6} = \frac{2}{3}$ share of b to agent 3 and $\frac{1}{6}$ share of b to agents 1 and 2. Finally, in the remaining time each agent consumes $\frac{1}{3}$ of c.

We demonstrate that if a mechanism ϕ is sd-efficient, sd-envy-free, and weakly invariant, then $\phi(\succ) = \mathrm{PS}(\succ)$. We show this for each object following the order in which objects are exhausted in SEA. In this example, for both $\succ$ and $(\succ_3^*, \succ_{-3})$, object a is first exhausted at time $\frac{1}{2}$, object b is second at time $\frac{2}{3}$, and then object c is third at time 1. That is, we show $\phi_a(\succ) = \mathrm{PS}_a(\succ)$ first and then $\phi_b(\succ) = \mathrm{PS}_b(\succ)$. Note that in this case, $\phi_c(\succ) = \mathrm{PS}_c(\succ)$ immediately follows from those two equalities and the feasibility constraint of random assignments.

[16]The sd-efficiency requirement in Theorem 2 can be weakened to the following condition (*2-sd-efficiency*): *For all* $\succ \in \mathbf{P}^N$, *there exists no* $P \neq \phi(\succ)$ *such that* P *stochastically dominates* $\phi(\succ)$ *at* $\succ$ *and* $|\{i \in N \mid P_i \neq \phi_i(\succ)\}| \leq 2$. See HH for more on this point.

Let $P=\phi(\succ)$. First consider P_a. By sd-efficiency, $p_{3,a}=0$. Then sd-envy-freeness (and non-wastefulness) implies $p_{1,a}=p_{2,a}=\frac{1}{2}$. Next, we examine the assignment of object b, where we invoke weak invariance in addition to sd-efficiency and sd-envy-freeness. To determine P_b, consider $P^*=\phi(\succ_3^*,\succ_{-3})$. For agents 1 and 3 not to envy each other at $\succ^*$, we need $p^*_{1,a}+p^*_{1,b}=p^*_{3,b}$. Similarly, $p^*_{2,a}+p^*_{2,b}=p^*_{3,b}$ must also hold. Furthermore, by a similar argument to the case of P_a, it is easy to see $P^*_a=(\frac{1}{2},\frac{1}{2},0)$. Hence, $P^*_b=(\frac{1}{6},\frac{1}{6},\frac{2}{3})$. Then by weak invariance, $p_{3,b}=p^*_{3,b}=\frac{2}{3}$. Finally, by sd-envy-freeness at $\succ$, we have $p_{1,b}=p_{2,b}=\frac{1}{6}$, i.e., $P_b=\mathrm{PS}_b(\succ)$ as claimed. ◊

To prove the necessity part of Theorem 2, we first order objects according to the time they are exhausted in SEA. Then we argue by induction on this order that an sd-efficient, sd-envy-free, and weakly invariant mechanism ϕ assigns each agent each object with exactly the same probability as PS. As we did above, we manipulate the preferences such that the order in which objects are exhausted in SEA is unaffected while sd-efficiency and sd-envy-freeness have enough bite to pin down the assignment probabilities under ϕ. The above example is simple enough that we needed to manipulate only one agent's preferences, focusing only on one object. However, for the proof to apply in general, one needs to iteratively consider several agents and several objects. Therefore, our general proof warrants careful and tedious construction of new preference profiles while keeping track of assignment probabilities.

Before the proof, we introduce some useful notation. For each $\succ\in\mathbf{P}^N$ and $a\in A$, let $\tau_\succ(a)=\min_{j\in N}F(\succ_j,a,\mathrm{PS}_j(\succ))$. If object a is exhausted in SEA under $\succ$, $\tau_\succ(a)$ represents the time when it is exhausted. If it is not exhausted, $\tau_\succ(a)$ is set to 1. Now fix a complete strict order $\gtrdot$ on A independent of $\succ$ and relabel the objects as $a^1(\succ),\dots,a^{|A|}(\succ)\in A$ so that (i) $\tau_\succ(a^1(\succ))\le\cdots\le\tau_\succ(a^{|A|}(\succ))$ and (ii) $\tau_\succ(a^k(\succ))=\tau_\succ(a^{k+1}(\succ))$ implies $a^k(\succ)\lessdot a^{k+1}(\succ)$. Given $\succ\in\mathbf{P}^N$, let $A^0(\succ)=\varnothing$ and $A^s(\succ)=\{a^1(\succ),\dots,a^s(\succ)\}$. Let $\overline{A'}=A\setminus A'$ for all $A'\subseteq A$.

Proof of Theorem 2. ($\Leftarrow$) PS is *sd-efficient* and *sd-envy-free*. It thus suffices to show that PS is *weakly invariant*. Let $\succ\in\mathbf{P}^N$, $i\in N$, $\succ'_i\in\mathbf{P}$, and $a^*\in A$, and assume $U(\succ_i,a^*)=U(\succ'_i,a^*)$ and $\succ_i|_{U(\succ_i,a^*)}=\succ'_i|_{U(\succ'_i,a^*)}$. Until time $\tau=F(\succ_i,a^*,\mathrm{PS}_i(\succ))$, SEA under $(\succ'_i,\succ_{-i})$ works in exactly the same way as under $\succ$. If $\tau=1$, this implies $\mathrm{PS}(\succ)=\mathrm{PS}(\succ'_i,\succ_{-i})$. If $\tau<1$, any $a\in U(\succ_i,a^*)$ is exhausted by time τ under $(\succ'_i,\succ_{-i})$ as well as under $\succ$. Therefore, $\mathrm{PS}_{i,a}(\succ'_i,\succ_{-i})=\mathrm{PS}_{i,a}(\succ)$ for all $a\in U(\succ_i,a^*)$.

($\Rightarrow$) Suppose that ϕ is *sd-efficient*, *sd-envy-free*, and *weakly invariant*. We prove by induction on $s\in\{0,\dots,|A|\}$ that for each $\succ\in\mathbf{P}^N$, $\phi(\succ)|_{A^s(\succ)}=\mathrm{PS}(\succ)|_{A^s(\succ)}$. It is obvious for $s=0$, as $A^s(\succ)=\varnothing$. Fix $s\ge1$. Assume as our inductive assumption that for all $\succ\in\mathbf{P}^N$, $\phi(\succ)|_{A^{s-1}(\succ)}=\mathrm{PS}(\succ)|_{A^{s-1}(\succ)}$.

It suffices to prove from the inductive assumption that for all $\succ\in\mathbf{P}^N$ and $i\in N$, $\phi_{i,a^s(\succ)}(\succ)=\mathrm{PS}_{i,a^s(\succ)}(\succ)$. Fix arbitrary $\succ\in\mathbf{P}^N$. We have two cases, depending on whether $a^s(\succ)$ is exhausted in SEA under $\succ$.

If $a^s(\succ)$ is not fully exhausted, $\mathrm{PS}(\succ)$ is the only assignment that satisfies the inductive assumption and *non-wastefulness*; hence, $\phi(\succ)=\mathrm{PS}(\succ)$.

Alternatively, if $a^s(\succ)$ is fully exhausted in SEA under $\succ$, to prove the inductive hypothesis at s, it suffices to show that

$$\text{for all } i \in N, \quad \phi_{i,a^s(\succ)}(\succ) \geq \mathrm{PS}_{i,a^s(\succ)}(\succ). \tag{1}$$

This claim follows from the following reasoning. Recall that $\sum_{i\in N} \mathrm{PS}_{i,a^s(\succ)}(\succ) = q_{a^s(\succ)}$ as $a^s(\succ)$ is exhausted under SEA. Therefore, the inequality (1) implies that for all $i \in N$, $\phi_{i,a^s(\succ)}(\succ) = \mathrm{PS}_{i,a^s(\succ)}(\succ)$, for otherwise the feasibility constraint $\sum_{i\in N} \phi_{i,a^s(\succ)}(\succ) \leq q_{a^s(\succ)}$ is violated.

Our strategy to show (1) is as follows. We construct a sequence of preference profiles $\succ^0, \dots, \succ^T$ (where T is defined later) and show that $\phi(\succ^T)$ is wasteful if (1) does not hold, which in turn leads to a contradiction.

First, we introduce some more notation. For any $\succ' \in \mathbf{P}^N$, let $a_i^*(\succ')$ be i's favorite object in $\overline{A^{s-1}(\succ')}$, i.e., $a_i^*(\succ') \succeq_i b$ for all $b \in \overline{A^{s-1}(\succ')}$. Also, let

$$N^s(a, \succ') = \{i \in N \mid a_i^*(\succ') = a\}.$$

Observe that by the definition of SEA, for all $i \in N$,

$$F(\succ'_i, a_i^*(\succ'), \mathrm{PS}_i(\succ')) = \tau_{\succ'}(a_i^*(\succ')). \tag{2}$$

Define $\mathbf{Q}_i$ as the set of $\succ'_i \in \mathbf{P}$ such that (i) $U(\succ_i, a_i^*(\succ)) = U(\succ'_i, a_i^*(\succ))$ and (ii) $\succ_i|_{U(\succ_i,a_i^*(\succ))} = \succ'_i|_{U(\succ_i,a_i^*(\succ))}$. That is, it is the set of preference relations whose rankings coincide with $\succ_i$ down to $a_i^*(\succ)$. Finally, let $\mathbf{Q}^N = \prod_{i\in N} \mathbf{Q}_i$.

Claim 1. *For all $\succ' \in \mathbf{Q}^N$, the following statements hold:*

(i) For all $r \in \{1, \dots, s\}$, $a^r(\succ) = a^r(\succ')$.

(ii) For all $r \in \{1, \dots, s\}$, $\tau_\succ(a^r(\succ)) = \tau_{\succ'}(a^r(\succ))$.

(iii) We have $\mathrm{PS}(\succ)|_{A^{s-1}(\succ)} = \mathrm{PS}(\succ')|_{A^{s-1}(\succ')} = \phi(\succ)|_{A^{s-1}(\succ)} = \phi(\succ')|_{A^{s-1}(\succ')}$.

(iv) For all $i \in N$ and $r \in \{1, \dots, s-1\}$, if $a_i^*(\succ) \succ'_i a^r(\succ)$, then $\phi_{i,a^r(\succ)}(\succ') = 0$.

Proof. Note that SEA under $\succ'$ works in exactly the same way as under $\succ$ until time $\tau = \tau_\succ(a^s(\succ))$. This implies parts (i) and (ii). Also, $A^{s-1}(\succ) = A^{s-1}(\succ')$ and $\mathrm{PS}(\succ)|_{A^{s-1}(\succ)} = \mathrm{PS}(\succ')|_{A^{s-1}(\succ')}$. By the inductive assumption, $\phi(\succ)|_{A^{s-1}(\succ)} = \mathrm{PS}(\succ)|_{A^{s-1}(\succ)}$ and $\phi(\succ')|_{A^{s-1}(\succ')} = \mathrm{PS}(\succ')|_{A^{s-1}(\succ')}$. These equalities imply part (iii). Note that $a_i^*(\succ) \succ'_i a^r(\succ)$ implies $\mathrm{PS}_{i,a^r(\succ)}(\succ') = 0$ by the definition of $a^r(\cdot)$ and part (i). Thus, part (iv) is a special case of part (iii). □

By Claim 1, the variables $a^1(\cdot), \dots, a^s(\cdot)$, $a_i^*(\cdot)$, and $A^1(\cdot), \dots, A^s(\cdot)$ remain constant on $\mathbf{Q}^N$. As we modify the original preference profile $\succ$ only within $\mathbf{Q}^N$, we omit the arguments of these variables for simplicity. Also, $N^s(a, \succ')$ remains constant for all $\succ' \in \mathbf{Q}^N$, so we simply write $N^s(a)$.

Let

$$B = \{a_i^* \mid i \in N\}.$$

We order the elements of B as follows: For each $b \in B$, define

$$\tau^{\max}(b) = \frac{q_b + \sum_{i\in N^s(b)}[F(\succ'_i, b, \text{PS}_i(\succ')) - \text{PS}_{i,b}(\succ')]}{|N^s(b)|} \tag{3}$$

for all $\succ' \in \mathbf{Q}^N$. For each $b \in B$, $\tau^{\max}(b)$ is uniquely defined, since by Claim 1, $\sum_{i\in N^s(b)}[F(\succ'_i, b, \text{PS}_i(\succ')) - \text{PS}_{i,b}(\succ')]$ stays constant across all $\succ' \in \mathbf{Q}^N$. This term represents the hypothetical maximum time for object b to be exhausted in SEA, which is the case if no agent $i \notin N^s(b)$ ever eats b and each agent $i \in N^s(b)$ continues eating b even after time 1. Then we order the elements of B as $b^1, \dots, b^{|B|}$ so that $b^1 = a^s$ and $\tau^{\max}(b^t) \le \tau^{\max}(b^{t+1})$ for each $t \in \{1, \dots, |B|-1\}$.[17]

Next, we construct a sequence $\{\succ^t\}_{t=0}^{|B|}$ in $\mathbf{Q}^N$. For each $t \in \{1, \dots, |B|\}$, let

$$B^t = \{b^1, \dots, b^t\} \quad \text{and} \quad M^t = N^s(b^1) \cup \dots \cup N^s(b^t).$$

Set $\succ^0 = \succ$ and let $\succ^t = (\succ^*_{M^t}, \succ_{-M^t})$ for each $t \ge 1$, where $\succ^*_i \in \mathbf{Q}_i$ is defined as[18,19]

$$\succ^*_i = (\succ_i|_{U(\succ_i, b^1)}, \succ_i|_{A^{s-1}\setminus U(\succ_i, b^1)}, b^2, \dots, b^{|B|}, \succ_i|_{\overline{A^{s-1}}\setminus B})$$
$$\text{for all } i \in N^s(b^1)$$
$$\succ^*_i = (\succ_i|_{U(\succ_i, b^t)}, \succ_i|_{A^{s-1}\setminus U(\succ_i, b^t)}, b^1, \dots, b^{t-1}, b^{t+1}, \dots, b^{|B|}, \succ_i|_{\overline{A^{s-1}}\setminus B})$$
$$\text{for all } t \ge 2, i \in N^s(b^t).$$

It is important in this construction that for each i and $t = 0, \dots, |B|$, there exists $a_{i,t} \in A$ such that $U(\succ^*_i, a_{i,t}) = A^{s-1} \cup \{a^*_i\} \cup B^t$, and, in particular, if also $i \in M^t$, we have $a^*_i \in B^t$ and, hence, $U(\succ^*_i, a_{i,t}) = A^{s-1} \cup B^t$. (For $t = 0$, define $B^0 = \varnothing$.)

We show that if (1) does not hold, then $\phi_{i,b^t}(\succ^t) < \text{PS}_{i,b^t}(\succ^t)$ for all $t \in \{1, \dots, T\}$ and $i \in N^s(b^t)$ (Claim 8), where

$$T = \min\bigl(\{t \in \{1, \dots, |B|-1\} \mid F(\succ^t_j, b^{t+1}, \phi_j(\succ^t)) = 1 \text{ for all } j \in M^{t+1}\} \cup \{|B|\}\bigr). \tag{4}$$

To this end, we prove some auxiliary results (Claims 2–7). We say that $b \in B$ is *undersupplied to $i \in N^s(b)$ at $\succ' \in \mathbf{Q}^N$* if $\phi_{i,b}(\succ') < \text{PS}_{i,b}(\succ')$, which can be equivalently written as $F(\succ'_i, b, \phi_i(\succ')) < F(\succ'_i, b, \text{PS}_i(\succ')) = \tau_{\succ'}(b)$, where the latter equality follows from (2).

Claim 2. *For all $t \in \{1, \dots, T\}$, where T is defined in (4), if b^t is undersupplied to some agent in $N^s(b^t)$ at $\succ^{t-1}$, then it is undersupplied to all agents in $N^s(b^t)$ at $\succ^t$.*

[17]By the definition of a^s and the assumption that a^s is fully exhausted in SEA, the entire probability share of a^s is exhausted by the agents in $N^s(a^s)$ during SEA under $\succ$, i.e., $\tau^{\max}(a^s) = \tau_\succ(a^s)$. Then if $\tau^{\max}(b^t) < \tau^{\max}(a^s)$ for some $t > 1$, it follows that $\tau_\succ(b^t) < \tau_\succ(a^s)$, which contradicts the definition of a^s. Thus, $a^s \in \arg\min_{b\in B} \tau^{\max}(b)$.

[18]For each $\succ \in \mathbf{Q}^N$ and $N' \subseteq N$, we write $\succ_{N'} = (\succ_i)_{i\in N'}$ and $\succ_{-N'} = (\succ_i)_{i\in N\setminus N'}$. If $\succ$ is written as $(\succ'_{N'}, \succ''_{-N'})$, then $\succ$ is such that $\succ_i = \succ'_i$ if $i \in N'$ and $\succ_i = \succ''_i$ otherwise.

[19]Recall that we associate a vector $(c_1, \dots, c_n)$ with the preference relation $\succ_i$ such that $c_1 \succ_i \dots \succ_i c_n$. For example, $\succ^*_i = (\succ_i|_{\{b^2\}}, \succ_i|_{\{a\}}, b^1, b^3, \succ_i|_{\{e\}}) = (c, a, b, d, e)$ if $A = \{a, b, c, d, e\}$, $A^{s-1} = \{a\}$, $B = \{b, c, d\}$, $(b^1, b^2, b^3) = (b, c, d)$, and $\succ_i = (c, e, d, b, a)$.

Proof. To begin, note that for any $N' \subseteq N^s(b^t)$, if b^t is undersupplied to some $i \in N^s(b^t) \setminus N'$ at $\succ' = (\succ^*_{N'}, \succ^{t-1}_{-N'})$, then it is so also at $\succ'' = (\succ^*_{N' \cup \{i\}}, \succ^{t-1}_{-(N' \cup \{i\})})$. This is simply because ϕ and PS are both *weakly invariant*, and, thus, $\text{PS}_{i,b^t}(\succ') = \text{PS}_{i,b^t}(\succ'')$ and $\phi_{i,b^t}(\succ') = \phi_{i,b^t}(\succ'')$. (Recall that the rankings of $\succ^{t-1}_i = \succ_i$ and $\succ^*_i$ coincide down to $b^t = a^*_i$.)

Next, we show that for any $N' \subseteq N^s(b^t)$, if b^t is undersupplied to some $i \in N'$ at $\succ' = (\succ^t_{N'}, \succ^{t-1}_{-N'})$, then it is undersupplied to all $j \in N^s(b^t)$ at $\succ'$. By the definition of $\succ'_i = \succ^t_i = \succ^*_i$, there exists $a \in A$ such that $U(\succ'_i, a) = A^{s-1} \cup \{b^t\}$. Let j be an arbitrary member of $N^s(b^t)$. First, as $U(\succ'_j, b^t) \subseteq U(\succ'_i, a)$, $F(\succ'_j, b^t, \phi_j(\succ')) \leq F(\succ'_i, a, \phi_j(\succ'))$. Second, by *sd-envy-freeness*, $F(\succ'_i, a, \phi_j(\succ')) \leq F(\succ'_i, a, \phi_i(\succ'))$. Third, by the assumption that b^t is undersupplied to i, $F(\succ'_i, a, \phi_i(\succ')) < F(\succ'_i, a, \text{PS}_i(\succ')) = F(\succ'_i, b^t, \text{PS}_i(\succ')) = \tau_{\succ'}(b^t)$, where the first equality follows from Claim 1(iv) and the second equality follows from (2). Combining these three inequalities, we obtain $F(\succ'_j, b^t, \phi_j(\succ')) < \tau_{\succ'}(b^t)$.

Therefore, if b^t is undersupplied to $i \in N^s(b^t)$ at $\succ^{t-1}$, we can expand N' from $N' = \varnothing$ to $N' = N^s(b^t)$ by repeatedly applying the above two arguments so that b^t is undersupplied to any $j \in N^s(b^t)$ at $\succ^t = (\succ^*_{N^s(b^t)}, \succ^{t-1}_{-N^s(b^t)})$. This completes the proof of the claim. □

Claim 3. *For all* $t \in \{0, \ldots, |B| - 1\}$, $\tau_{\succ^t}(b^1) \leq \cdots \leq \tau_{\succ^t}(b^{t+1})$ *and* $\tau_{\succ^t}(b^{t+1}) \leq \tau_{\succ^t}(b^u)$ *for all* $u \in \{t+2, \ldots, |B|\}$.

Proof. We argue by induction on t. For $t = 0$, as $b^1 = a^s$ and $\succ^0 = \succ$, we have $\tau_{\succ^0}(b^1) \leq \tau_{\succ^0}(b^u)$ for all u by the definition of a^s. Fix $t \in \{1, \ldots, |B|\}$. Assume the claim is true for $t-1$ as our inductive assumption. By the definition of $\succ^t$, SEA under $\succ^t$ works in exactly the same way as under $\succ^{t-1}$ until time $\tau^* = \tau_{\succ^{t-1}}(b^t)$. In particular, for all $b \in B^t$, $\tau_{\succ^{t-1}}(b) = \tau_{\succ^t}(b) \leq \tau^*$. Also, b^{t+1} is not exhausted before time τ^* under $\succ^{t-1}$ (and hence under $\succ^t$), for otherwise the claim does not hold for $t-1$, contrary to the inductive assumption. Therefore, $\tau_{\succ^t}(b^1) \leq \cdots \leq \tau_{\succ^t}(b^{t+1})$. It remains to show that $\tau_{\succ^t}(b^{t+1}) \leq \tau_{\succ^t}(b^u)$ for all $u \in \{t+2, \ldots, |B|\}$. Suppose, to the contrary, that $\tau_{\succ^t}(b^u) < \tau_{\succ^t}(b^{t+1})$ for some $u > t+1$. Without loss of generality, suppose $b^u \in \arg\min_{b \in B \setminus B^t} \tau_{\succ^t}(b)$. Then it follows from the description of SEA that b^u is eaten away only by the agents in $N^s(b^u)$ under $\succ^t$. Hence, $\tau_{\succ^t}(b^u) = \tau^{\max}(b^u)$, where $\tau^{\max}(\cdot)$ is given by (3). However, $\tau_{\succ^t}(b^{t+1}) \leq \tau^{\max}(b^{t+1}) \leq \tau^{\max}(b^u)$, where the second inequality follows by the construction of the sequence $b^1, \ldots, b^{|B|}$. This in turn implies $\tau_{\succ^t}(b^{t+1}) \leq \tau_{\succ^t}(b^u)$, which is a contradiction. □

Claim 4. *For all* $t \in \{1, \ldots, |B|\}$ *and* $i \in M^t$, $F(\succ^t_i, b^t, \text{PS}_i(\succ^t)) = \tau_{\succ^t}(b^t)$.

Proof. Let $t \in \{1, \ldots, |B|\}$. Fix $j \in N^s(b^t)$. Equation (2) implies $F(\succ^t_j, b^t, \text{PS}_j(\succ^t)) = \tau_{\succ^t}(b^t)$. Fix $u < t$ and $i \in N^s(b^u)$. In SEA under $\succ^t$, by Claim 3 for all $v \in \{u, \ldots, t\}$, object b^v is not fully exhausted before b^{v-1}. Thus, by the construction of $\succ^t_i$, once b^u is fully exhausted, agent i will turn to object b^{u+1} since objects in $A^{s-1} \setminus U(\succ^t_i, b^u)$ have already been exhausted. Then he will turn to $b^{u+2}, \ldots,$ and then to b^t in SEA under $\succ^t$.

Thus, at time $\tau_{\succ^t}(b^t)$, i has just finished consuming an object b^v with $v \leq t$ such that $\tau_{\succ^t}(b^v) = \tau_{\succ^t}(b^t)$ and, hence, if $b^v \neq b^t$, i cannot consume any of the objects $b^{v+1}, \dots, b^t$. Thus, $F(\succ_i^t, b^t, \mathrm{PS}_i(\succ^t)) = \tau_{\succ^t}(b^t)$. □

Claim 5. *For all $t \in \{1, \dots, |B|\}$ and $i, j \in M^t$, if $P \in \mathcal{R}$ is sd-envy-free at $\succ^t$, then $\sum_{a \in A^{s-1} \cup B^t} p_{i,a} = \sum_{a \in A^{s-1} \cup B^t} p_{j,a}$.*

Proof. Let $i, j \in M^t$. Thus, we have $a_i^*, a_j^* \in B^t$. Hence, by the construction of $\succ^t$, there exist $a_i, a_j \in A$ such that $U(\succ_i^t, a_i) = U(\succ_j^t, a_j) = A^{s-1} \cup B^t$. Thus, $\sum_{a \in A^{s-1} \cup B^t} p_{i,a} = F(\succ_i^t, a_i, P_i) = F(\succ_j^t, a_j, P_i) \leq F(\succ_j^t, a_j, P_j) = \sum_{a \in A^{s-1} \cup B^t} p_{j,a}$, where the inequality follows from *sd-envy-freeness*. Switching i and j, we obtain the opposite inequality. Thus, we have the desired equality. □

Claim 6. *For all $t \in \{1, \dots, T\}$, where T is defined as in (4), $\sum_{i \in M^t} \sum_{b \in B^{t-1}} \phi_{i,b}(\succ^t) \leq \sum_{i \in M^t} \sum_{b \in B^{t-1}} \mathrm{PS}_{i,b}(\succ^t)$.*

Proof. We consider two cases. First, suppose that $\tau_{\succ^t}(b^{t-1}) < 1$. In this case, by Claim 3 and the construction of $\succ^t$, all objects in B^{t-1} are exhausted by the agents in M^{t-1} in SEA under $\succ^t$. That is, $\sum_{i \in M^{t-1}} \mathrm{PS}_{i,b}(\succ^t) = \sum_{i \in M^t} \mathrm{PS}_{i,b}(\succ^t) = q_b$ for all $b \in B^{t-1}$, and the desired inequality immediately follows from feasibility. Second, suppose that $\tau_{\succ^t}(b^{t-1}) = 1$. In this case, $F(\succ_i^t, b^{t-1}, \mathrm{PS}_i(\succ^t)) = 1$ for all $i \in M^t$. By Claim 1, this implies $\sum_{b \in B^{t-1}} \mathrm{PS}_{i,b}(\succ^t) = 1 - \sum_{a \in A^{s-1}} \phi_{i,a}(\succ^t)$ for all $i \in M^t$. Therefore, if $\sum_{i \in M^t} \sum_{b \in B^{t-1}} \phi_{i,b}(\succ^t) > \sum_{i \in M^t} \sum_{b \in B^{t-1}} \mathrm{PS}_{i,b}(\succ^t)$, there must exist $j \in M^{t-1}$ such that $\sum_{a \in A} \phi_{j,a}(\succ^t) > 1$, which is a contradiction. □

Claim 7. *For all $t \in \{1, \dots, T\}$, where T is defined as in (4), if b^t is undersupplied to all agents in $N^s(b^t)$ at $\succ^t$, then $\sum_{a \in A^{s-1} \cup B^t} \phi_{i,a}(\succ^t) < \tau_{\succ^t}(b^t)$ for all $i \in M^t$.*

Proof. We consider two cases. First, suppose that $\phi_{k,b^t}(\succ^t) > 0$ for some $k \in M^{t-1}$ and fix arbitrary $j \in N^s(b^t)$. Then, by *sd-efficiency*, $\phi_{j,b}(\succ^t) = 0$ for all $b \in B^{t-1}$ because $b \succ_k^t b^t$ and $b^t \succ_j^t b$ by construction of $\succ^t$. Since b^t is undersupplied to j at $\succ^t$, $\sum_{a \in A^{s-1} \cup B^t} \phi_{j,a}(\succ^t) = F(\succ_j^t, b^t, \phi_j(\succ^t)) < \tau_{\succ^t}(b^t)$. Then, by Claim 5, $\sum_{a \in A^{s-1} \cup B^t} \phi_{i,a}(\succ^t) < \tau_{\succ^t}(b^t)$ for all $i \in M^t$.

Second, suppose that $\phi_{k,b^t}(\succ^t) = 0$ for all $k \in M^{t-1}$. This implies $\sum_{i \in M^t} \phi_{i,b^t}(\succ^t) < \sum_{i \in M^t} \mathrm{PS}_{i,b^t}(\succ^t)$, because b^t is undersupplied to all agents in $N^s(b^t)$. Also, recall that by Claim 1, $\phi_{i,a}(\succ^t) = \mathrm{PS}_{i,a}(\succ^t)$ for all $i \in M^t$ and $a \in A^{s-1}$. These arguments together with Claim 6 and (2) imply

$$\sum_{i \in M^t} \sum_{b \in A^{s-1} \cup B^t} \phi_{i,b}(\succ^t) < \sum_{i \in M^t} \sum_{b \in A^{s-1} \cup B^t} \mathrm{PS}_{i,b}(\succ^t) = |M^t| \tau_{\succ^t}(b^t).$$

Then Claim 5 implies that for all $i \in M^t$,

$$\sum_{b \in A^{s-1} \cup B^t} \phi_{i,b}(\succ^t) = |M^t|^{-1} \sum_{k \in M^t} \sum_{b \in A^{s-1} \cup B^t} \phi_{k,b}(\succ^t) < \tau_{\succ^t}(b^t).$$

□

Claim 8. *Suppose that a^s is undersupplied to some agent in $N^s(a^s)$ at $\succ$. Then, for each $t \in \{1, \ldots, T\}$, b^t is undersupplied to all agents in $N^s(b^t)$ at $\succ^t$, where T is defined as in (4).*

Proof. We argue by induction on t. For $t = 1$, it is immediate from Claim 2. We assume that b^t is undersupplied to all agents in $N^s(b^t)$ at $\succ^t$, where $t < T$. By Claim 2, we need to show that b^{t+1} is undersupplied to some agent in $N^s(b^{t+1})$ at $\succ^t$. Let $P = \phi(\succ^t)$ and $P' = \mathrm{PS}(\succ^t)$. We consider three cases, where Cases 1 and 2 are not mutually exclusive.

Case 1. For some $i \in M^t$, $\sum_{a\in A^{s-1}\cup B^{t+1}} p_{i,a} < \tau_{\succ^t}(b^{t+1})$. Then, for all $j \in N^s(b^{t+1})$, by *sd-envy-freeness* and $U(\succ^t_j, b^{t+1}) \subseteq U(\succ^t_i, b^{t+1})$, we have $F(\succ^t_j, b^{t+1}, P_j) \leq F(\succ^t_i, b^{t+1}, P_j) \leq F(\succ^t_i, b^{t+1}, P_i) \leq \sum_{a\in A^{s-1}\cup B^{t+1}} p_{i,a} < \tau_{\succ^t}(b^{t+1})$. Thus, b^{t+1} is undersupplied to j at $\succ^t$.

Case 2. We have $\tau_{\succ^t}(b^{t+1}) = 1$. Since $t < T$, there exists $i \in M^{t+1}$ such that $F(\succ^t_i, b^{t+1}, P_i) < 1 = \tau_{\succ^t}(b^{t+1})$. If $i \in M^t$, then this case reduces to Case 1. Otherwise $i \in N^s(b^{t+1})$, to whom b^{t+1} is undersupplied at $\succ^t$.

Case 3. We have $\tau_{\succ^t}(b^{t+1}) < 1$ and for all $i \in M^t$, $\sum_{a\in A^{s-1}\cup B^{t+1}} p_{i,a} \geq \tau_{\succ^t}(b^{t+1})$. Then, since b^t is undersupplied to all agents in $N^s(b^t)$ at $\succ^t$, it follows from Claim 7 that for all $i \in M^t$, $\sum_{a\in A^{s-1}\cup B^t} p_{i,a} < \tau_{\succ^t}(b^t)$. Thus, by our assumption,

for all $i \in M^t$,

$$p_{i,b^{t+1}} = \sum_{a\in A^{s-1}\cup B^{t+1}} p_{i,a} - \sum_{a\in A^{s-1}\cup B^{t}} p_{i,a} > \tau_{\succ^t}(b^{t+1}) - \tau_{\succ^t}(b^t) = p'_{i,b^{t+1}}, \tag{5}$$

where the last equality follows from the fact that by Claim 3, in SEA agent $i \in M^t$ turns to eating b^{t+1} at time $\tau_{\succ^t}(b^t)$ until $\tau_{\succ^t}(b^{t+1})$.

Since $\tau_{\succ^t}(b^{t+1}) < 1$, b^{t+1} is fully consumed at $\succ^t$ in SEA, i.e., $\sum_{i\in N} p'_{i,b^{t+1}} = q_{b^{t+1}}$. Also, since $\tau_{\succ^t}(b^{t+1}) \leq \tau_{\succ^t}(b^u)$ for all $u > t+1$ by Claim 3, any agent $i \in N \setminus M^{t+1}$ does not eat b^{t+1} in SEA, i.e., $p'_{i,b^{t+1}} = 0$. Thus,

$$\sum_{i\in M^{t+1}} p'_{i,b^{t+1}} = q_{b^{t+1}}. \tag{6}$$

Therefore, it follows from (5) and (6) that $\sum_{i\in N^s(b^{t+1})} p_{i,b^{t+1}} \leq q_{b^{t+1}} - \sum_{i\in M^t} p_{i,b^{t+1}} < q_{b^{t+1}} - \sum_{i\in M^t} p'_{i,b^{t+1}} = \sum_{i\in M^{t+1}} p'_{i,b^{t+1}} - \sum_{i\in M^t} p'_{i,b^{t+1}} = \sum_{i\in N^s(b^{t+1})} p'_{i,b^{t+1}}$. Thus, for some $i \in N^s(b^{t+1})$, we have $p_{i,b^{t+1}} < p'_{i,b^{t+1}}$. That is, b^{t+1} is undersupplied to i at $\succ^t$. □

Finally, we are ready to derive a contradiction if (1) does not hold.

For notational simplicity, let

$$P = \phi(\succ^T) \quad \text{and} \quad P' = \mathrm{PS}(\succ^T).$$

Define

$$B^* = \{b \in B \setminus B^T \mid \exists i \in N^s(b) \text{ and } a \in \overline{U(\succ^T_i, b)} \text{ s.t. } p_{i,a} > 0\}$$

and

$$N^* = \bigcup_{b\in B^*} N^s(b).$$

Suppose (1) does not hold. Then there exists some $k \in N$ such that $\phi_{k,a^s}(\succ) < \mathrm{PS}_{k,a^s}(\succ)$. As $\mathrm{PS}_{k,a^s}(\succ) > 0$, $k \in N^s(a^s)$, i.e., a^s is undersupplied to k at $\succ$. Our objective is to show that $\sum_{i\in N^*}\sum_{a\in A} p_{i,a} > |N^*|$, which is a contradiction because $\sum_{i\in N^*}\sum_{a\in A} p_{i,a} \le |N^*|$ by the definition of a random assignment.

Step 1. We show for all $i \in M^T$, $\sum_{a\in A^{s-1}\cup B^T} p_{i,a} < \sum_{a\in A^{s-1}\cup B^T} p'_{i,a} = \tau_{\succ^T}(b^T)$ and, thus, $\sum_{a\in B^T} p_{i,a} < \sum_{a\in B^T} p'_{i,a}$. Fix $i \in M^T$. First, by Claim 1(iii) and (iv), $\sum_{a\in A^{s-1}\cup B^T} p'_{i,a} = F(\succ_i^T, b^T, P'_i)$. Second, Claim 4 implies $F(\succ_i^T, b^T, P'_i) = \tau_{\succ^T}(b^T)$. Third, as a^s is undersupplied to agent $k \in N^s(a^s)$ at $\succ$, Claim 8 implies that b^T is undersupplied to all agents in $N^s(b^T)$, which in turn implies by Claim 7 that $\sum_{a\in A^{s-1}\cup B^T} p_{i,a} < \tau_{\succ^T}(b^T)$. These three statements imply $\sum_{a\in A^{s-1}\cup B^T} p_{i,a} < \sum_{a\in A^{s-1}\cup B^T} p'_{i,a}$. Finally, Claim 1(iii) implies $\sum_{a\in B^T} p_{i,a} < \sum_{a\in B^T} p'_{i,a}$.

Step 2. We show $p_{i,a} = 0$ for all $i \in N \setminus (M^T \cup N^*)$ and $a \in \overline{U(\succ_i^T, a_i^*)}$. Suppose $i \in N \setminus M^T$ and $a \in \overline{U(\succ_i^T, a_i^*)}$. Then if $p_{i,a} > 0$, we have $a_i^* \in B^*$ and, thus, $i \in N^*$. Therefore, $p_{i,a}$ must be 0 if $i \notin N^*$.

Step 3. We show that there exist $i^* \in N^*$ and $b \in B^T$ such that $p_{i^*,b} > 0$. First note that $\sum_{i\in N} p_{i,b} = q_b$ for all $b \in B^T$, since P is *non-wasteful*, and for all $i \in M^T$, the objects in $A^{s-1} \cup B^T$ are ranked highest under $\succ_i^T$ and $\sum_{a\in A^{s-1}\cup B^T} p_{i,a} < \sum_{a\in A^{s-1}\cup B^T} p'_{i,a} \le 1$ by Step 1. There exist $i^* \in N \setminus M^T$ and $b \in B^T$ such that $p_{i^*,b} > 0$, for otherwise Step 1 implies $\sum_{a\in B^T} q_a = \sum_{i\in M^T}\sum_{a\in B^T} p_{i,a} < \sum_{i\in M^T}\sum_{a\in B^T} p'_{i,a} \le \sum_{a\in B^T} q_a$, which is a contradiction. Observe that $b \in \overline{U(\succ_{i^*}^T, a_{i^*}^*)}$. Thus, $i^* \in N^*$ by Step 2.

Step 4. We show (i) $T < |B|$, (ii) for all $i \in M^{T+1}$, $F(\succ_i^T, b^{T+1}, P_i) = 1$, and (iii) $b^{T+1} \in \overline{B^*}$. Step 3 implies $N^* \ne \varnothing$. This in turn implies $T < |B|$, because $N \setminus M^T = \varnothing$ when $T = |B|$. Therefore, for all $i \in M^{T+1}$, $F(\succ_i^T, b^{T+1}, P_i) = 1$ by the definition of T. In particular, $F(\succ_i^T, b^{T+1}, P_i) = 1$ for all $i \in N^s(b^{T+1})$, which implies $b^{T+1} \in \overline{B^*}$.

Step 5. We show that for all $a \in B^*$, $\sum_{i\in N^*} p_{i,a} = q_a$. Fix $a \in B^*$. Observe that for all $i \in N \setminus (M^T \cup N^*)$, $a \in \overline{U(\succ_i^T, a_i^*)}$ and, therefore, $p_{i,a} = 0$ by Step 2. Moreover, for all $i \in M^T$, $a \in \overline{U(\succ_i^T, b^{T+1})}$ by Step 4(iii) and the construction of $\succ_i^T$, and, therefore, by Step 4(ii), $p_{i,a} = 0$. Thus, $\sum_{i\in N^*} p_{i,a} = \sum_{i\in N} p_{i,a}$. Alternatively, there exists some $i \in N^*$ with $a = a_i^*$ and some $b \in \overline{U(\succ_i^T, a)}$ such that $p_{i,b} > 0$ by the definition of B^* and N^*. Hence, $\sum_{j\in N^*} p_{j,a} = \sum_{j\in N} p_{j,a} = q_a$, where the second equality follows from the *non-wastefulness* of P.

Step 6. We show for all $u > T$, $\tau_{\succ^T}(b^u) = 1$:

$$
\begin{aligned}
q_{b^{T+1}} &\ge \sum_{i\in M^{T+1}} p_{i,b^{T+1}} \\
&= \sum_{i\in M^T} (1 - F(\succ_i^T, b^T, P_i)) + \sum_{i\in N^s(b^{T+1})} \Bigl(1 - \sum_{a\in A^{s-1}} p'_{i,a}\Bigr) \\
&\qquad \text{by Step 4(ii) and Claim 1(iii)} \\
&\ge \sum_{i\in M^T} \Bigl(1 - \sum_{a\in A^{s-1}\cup B^T} p_{i,a}\Bigr) + \sum_{i\in N^s(b^{T+1})} \Bigl(1 - \sum_{a\in A^{s-1}} p'_{i,a}\Bigr) \\
&\qquad \text{since } U(\succ_i^T, b^T) \subseteq A^{s-1} \cup B^T
\end{aligned}
$$

$$> \sum_{i \in M^T} \Big(1 - \sum_{a \in A^{s-1} \cup B^T} p'_{i,a}\Big) + \sum_{i \in N^s(b^{T+1})} \Big(1 - \sum_{a \in A^{s-1}} p'_{i,a}\Big) \quad \text{by Step 1}$$

$$\geq \sum_{i \in M^{T+1}} p'_{i,b^{T+1}}.$$

No agent $i \in N \setminus M^{T+1}$ ever eats b^{T+1} in SEA under $\succ^T$, as $a_i^* \in B \setminus B^{T+1}$ is not exhausted before b^{T+1} by Claim 3. Thus, $q_{b^{T+1}} > \sum_{i \in N} p'_{i,b^{T+1}}$, i.e., b^{T+1} is not fully exhausted in SEA. Hence, $\tau_{\succ^T}(b^{T+1}) = 1$. Again by Claim 3, for all $u > T$, $\tau_{\succ^T}(b^u) = 1$.

Step 7. We finally show $\sum_{i \in N^*} \sum_{a \in A} p_{i,a} > |N^*|$. First, $\sum_{a \in A} p_{i,a} \geq \sum_{a \in B^*} p_{i,a} + \sum_{a \in A^{s-1}} p_{i,a}$ for all $i \in N^*$, and the inequality is strict if $i = i^*$ by Step 3. Thus $\sum_{i \in N^*} \sum_{a \in A} p_{i,a} > \sum_{i \in N^*} \sum_{a \in B^*} p_{i,a} + \sum_{i \in N^*} \sum_{a \in A^{s-1}} p_{i,a}$. The first summation on the right-hand side equals $\sum_{a \in B^*} q_a$ by Step 5 and the second summation equals $\sum_{i \in N^*} \sum_{a \in A^{s-1}} p'_{i,a}$ by Claim 1(iii). Therefore,

$$\begin{aligned}
\sum_{i \in N^*} \sum_{a \in A} p_{i,a} &> \sum_{a \in B^*} q_a + \sum_{i \in N^*} \sum_{a \in A^{s-1}} p'_{i,a} \\
&\geq \sum_{i \in N^*} F(\succ_i^T, a_i^*, P'_i) \\
&= \sum_{i \in N^*} \tau_{\succ^T}(a_i^*) \quad \text{by (2)} \\
&= |N^*| \quad \text{by } a_i^* \in B \setminus B^T \text{ for all } i \in N^* \text{ and Step 6.}
\end{aligned}$$

This completes the proof. □

We finally consider a special case of our model that assumes the existence of the null object (i.e., an object that is always abundant in supply) and provide an interesting corollary of Theorem 2 for this case. The null object represents an agent's outside option that depends on the specific context, i.e., the option of not being assigned a *real* object from A. This special case of the model could be of important practical relevance since it gives rise to some natural preference misrepresentations that may arise in practice.

For example, in many real-world assignment procedures, authorities often cap the number of objects that agents can include in their preference lists.[20] Even without caps, it could be unrealistic and impractical to expect agents to evaluate and list all of their acceptable objects, especially when the assignment problem involves a large number of objects.[21] Given that agents may need to shorten their preference lists, truncated lists

[20]For instance, freshmen at the University of Pennsylvania may list up to eight choices in their campus-housing applications (http://www.business-services.upenn.edu/housing/assets/pdf/brochures/freshman.pdf; retrieved on November 15, 2010). See Haeringer and Klijn (2009), Calsamiglia et al. (2010), and Pathak and Sönmez (2013) for more examples and implications of caps.

[21]In the context of school choice, more than 500 programs participate in the New York City high school match (Abdulkadiroğlu et al. 2005). It is possible that hundreds of programs are acceptable to some students, but it is highly unlikely that they list all of their acceptable schools. In fact, Boston Public Schools encourage families to list at least five school choices ("more is better") when registering

would be among the most natural and likely preference reports to observe in practice. The resulting assignment is potentially volatile, depending on whether agents truncate their lists or not, which would be unfavorable for authorities and potentially unfavorable for agents as well. Hence, robustness against truncations is a desirable property of a mechanism. This property is implied by weak invariance. As it turns out, for individually rational mechanisms, the converse is also true. We introduce formally these next.

Let us denote the null object by $\varnothing$. A preference relation $\succ_i'$ is called a *truncation* of $\succ_i$ if $U(\succ_i', \varnothing) \subseteq U(\succ_i, \varnothing)$ and $\succ_i|_{U(\succ_i',\varnothing)} = \succ_i'|_{U(\succ_i',\varnothing)}$ (Roth and Rothblum 1999). That is, truncation $\succ_i'$ is obtained from $\succ_i$ by shrinking the list of acceptable objects while preserving the relative rankings of those objects that remain acceptable. The following axiom asks that the probability with which an agent i receives an (real) object a stays the same whenever his preferences are truncated, provided that a remains acceptable after the truncation.

Definition 3. A mechanism ϕ is *weakly truncation robust* if for all $\succ \in \mathbf{P}^N$, $i \in N$, and $a \in A$, $\phi_{i,a}(\succ) = \phi_{i,a}(\succ_i', \succ_{-i})$ whenever $a \succ_i' \varnothing$ and $\succ_i'$ is a truncation of $\succ_i$.[22]

Definition 4. A mechanism ϕ is *individually rational* if for all $\succ \in \mathbf{P}^N$, $i \in N$, and $a \in A$, $\phi_{i,a}(\succ) = 0$ whenever $\varnothing \succ_i a$.

By definition, if $\succ_i'$ is a truncation of $\succ_i$ and $a \succ_i' \varnothing$, then the rankings of the two preferences coincide down to a. Therefore, weak invariance immediately implies weak truncation robustness. The converse statement is also true for individually rational mechanisms.[23]

Proposition 1. *Suppose that the null object exists. A mechanism is weakly truncation robust if it is weakly invariant. The converse is true if the mechanism is individually rational.*

Proof. To see the first part, note that if $\succ_i'$ is a truncation of $\succ_i$ and $a \succ_i' \varnothing$, then $U(\succ_i', a) = U(\succ_i, a)$ and $\succ_i'|_{U(\succ_i',a)} = \succ_i|_{U(\succ_i',a)}$. To show the second part, suppose that a

(http://www.bostonpublicschools.org/node/169). Also, San Francisco Unified School District warns in boldface that "[p]arents who do not list up to 7 choices run a higher risk of getting assigned to a school they did not request" (http://portal.sfusd.edu/template/default.cfm?page=policy.placement.process). This suggests that some families may not list the maximum number of choices even when that number is small. Even though some of them might actually have a smaller number of acceptable schools than the maximum, still others might shorten their preference lists owing to a host of other reasons including various costs involved in the application process. All the web pages were retrieved on November 15, 2010.

[22]In the context of deterministic assignments, Ehlers and Klaus (2009) propose an axiom called *truncation invariance*. It requires all agents' assignments to remain the same as a result of agent i's truncation, as long as the object that agent i obtains before the truncation remains acceptable. Truncation invariance would appear stronger than weak truncation robustness: The former imposes the invariance restriction for all objects, whereas the latter only for a particular one. They are, in fact, incomparable, because the former restricts the class of truncations but the latter does not.

[23]The following is an example of a mechanism that is weakly truncation robust but not weakly invariant. For any $\succ \in \mathbf{P}^N$, let $\phi(\succ) = P$ if there exist two distinct $i, j \in N$ such that $\succ_i = \succ_j$ and $\varnothing \succeq_i a$ for all $a \in A$, and let $\phi(\succ) = P'$ otherwise, where P and P' are two arbitrary but distinct random assignments.

mechanism ϕ satisfies *individual rationality* and *weak truncation robustness*. Fix $a \in A$, $i \in N$, and $\succ \in \mathbf{P}^N$. Let $\succ_i'$ be an arbitrary preference such that $U(\succ_i, a) = U(\succ_i', a)$ and $\succ_i|_{U(\succ_i,a)} = \succ_i'|_{U(\succ_i,a)}$. If $\varnothing \succ_i a$ (and thus $\varnothing \succ_i' a$), then $\phi_{i,a}(\succ) = \phi_{i,a}(\succ_i', \succ_{-i}) = 0$ by *individual rationality*. If $a \succ_i \varnothing$ (and thus $a \succ_i' \varnothing$), let $\succ_i''$ be a truncation of $\succ_i$ such that $U(\succ_i'', \varnothing) = U(\succ_i, a) \cup \{\varnothing\}$. Then $\succ_i''$ is also a truncation of $\succ_i'$ and, thus, $\phi_{i,a}(\succ_i, \succ_{-i}) = \phi_{i,a}(\succ_i'', \succ_{-i}) = \phi_{i,a}(\succ_i', \succ_{-i})$ by *weak truncation robustness*. If $a = \varnothing$, then $\succ_i$ is a truncation of $\succ_i'$ and, thus, $\phi_{i,b}(\succ) = \phi_{i,b}(\succ_i', \succ_{-i})$ for each $b \succ_i \varnothing$. Hence, by *individual rationality*, $\phi_{i,\varnothing}(\succ_i, \succ_{-i}) = 1 - \sum_{b \succ_i \varnothing} \phi_{i,b}(\succ) = 1 - \sum_{b \succ_i \varnothing} \phi_{i,b}(\succ_i', \succ_{-i}) = \phi_{i,\varnothing}(\succ_i', \succ_{-i})$. □

It follows from Proposition 1 that weak truncation robustness can replace weak invariance in Theorem 2 if the null object is present.

Corollary 2. *Suppose that the null object exists. A mechanism is sd-efficient, sd-envy-free, and weakly truncation robust if and only if it is* PS.[24]

7. Concluding remarks

Finally, we establish the logical independence of the axioms in Theorems 1 and 2. We start with Theorem 1. An ordinally fair but wasteful mechanism is the following. When the total quota of objects exceeds the number of agents,[25] consider the following strategy: Fix $q_a' \leq q_a$ for all $a \in A$ such that $\sum_{a \in A} q_a' = |N|$. The PS mechanism that assigns objects according to the artificial quota vector $(q_a')_{a \in A}$ is ordinally fair but wasteful. Alternatively, a simple (deterministic) serial dictatorship is a non-wasteful but ordinally unfair mechanism.

The independence of the axioms in Theorem 2 can be shown as follows. The PS mechanism with an artificial quota vector defined above is sd-envy-free and weakly invariant but sd-inefficient. A serial dictatorship is an sd-efficient and weakly invariant mechanism that induces sd-envy. The mechanism in Example 2 is sd-efficient and sd-envy-free, but not weakly invariant.

Example 2. Suppose $N = \{1, 2, 3\}$, $A = \{a, b, c\}$, and $q_a = q_b = q_c = 1$. Define preference profile $\succ^* = ((abc), (abc), (bca))$. Let mechanism ϕ be such that

$\phi(\succ^*) =$

	a	b	c
1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$
2	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$
3	0	$\frac{1}{3}$	$\frac{2}{3}$

[24] As in HH's Theorem 2, by slightly modifying the proof, we can weaken sd-envy-freeness to the condition $\sum_{a \succ_i \varnothing} \phi_{i,a}(\succ) \geq \sum_{a \succ_i \varnothing} \phi_{j,a}(\succ)$ *for all* $\succ \in \mathbf{P}^N$ *and* $i, j \in N$.

[25] If the total quota of objects is equal to the number of agents, we have an assignment problem with perfect supply. Thus, non-wastefulness holds vacuously.

and for all $\succ \neq \succ^*$, $\phi(\succ) = \text{PS}(\succ)$. Then $\phi(\succ)$ is sd-efficient and sd-envy-free for all $\succ$. Note that[26]

$$\text{PS}_{3,b}(\underbrace{(abc), (abc), (bca)}_{=\succ^*}) = \text{PS}_{3,b}(\underbrace{(abc), (abc)}_{=\succ^*_{-3}}, (bac)) = \tfrac{2}{3}.$$

However, the above definition of ϕ violates weak invariance because

$$\phi_{3,b}(\underbrace{(abc), (abc), (bca)}_{=\succ^*}) = \tfrac{1}{3} \neq \tfrac{2}{3} = \phi_{3,b}(\underbrace{(abc), (abc)}_{=\succ^*_{-3}}, (bac)).$$

◊

One may wonder whether sd-efficiency can be weakened to non-wastefulness in Theorem 2. The answer is negative: Suppose that the total quota of objects is equal to the number of agents. Then the uniform mechanism, which assigns $\phi_{i,a}(\succ) = q_a/|N|$ for all $i \in N$, $a \in A$, and $\succ \in \mathbf{P}^N$, is non-wasteful, sd-envy-free, and weakly invariant, but sd-inefficient. We can construct a counterexample in a similar spirit even if the null object exists.

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[26]See Example 1, where it explains how the PS outcome for these problems are found.

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Submitted 2011-5-27. Final version accepted 2013-1-4. Available online 2013-1-4.