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Strategy proofness and Pareto efficiency in quasilinear exchange economies

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In this paper, we revisit a longstanding question on the structure of strategy-proof and Pareto-efficient social choice functions (SCFs) in classical exchange economies (Hurwicz 1972). Using techniques developed by Myerson in the context of auction design, we show that in a specific quasilinear domain, every Pareto-efficient and strategy-proof SCF that satisfies non-bossiness and a mild continuity property is dictatorial. The result holds for an arbitrary number of agents, but the two-person version does not require either the non-bossiness or the continuity assumptions. It also follows that the dictatorship conclusion holds on any superset of this domain. We also provide a minimum consumption guarantee result in the spirit of Serizawa and Weymark (2003).

Keywords. Exchange economies, strategy proofness, Pareto efficiency, dictatorship.

JEL classification. C72, D51, D71.

1. Introduction

The problem of allocating goods among a set of agents who have preferences over these goods is a familiar one in economic theory. A minimal and uncontroversial requirement for such allocations is *Pareto efficiency*. However, so as to satisfy this requirement, knowledge of agents' preferences is essential. If these preferences are private information, they must be elicited from agents. Consequently, the allocation procedure or social choice function (SCF) must also be *incentive-compatible*; that is, the SCF must induce agents to reveal their preferences truthfully. The most attractive incentive-compatibility

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requirement is *strategy proofness*; if an SCF is strategy-proof, then no agent can benefit by lying, irrespective of her beliefs regarding the announcements of other agents. Strategy proofness is, however, a stringent requirement. According to the well known Gibbard–Satterthwaite theorem (Gibbard 1973 and Satterthwaite 1975), an SCF defined over an unrestricted domain with a range of at least three alternatives is strategy-proof only if it is *dictatorial*. A dictatorial SCF is a trivial SCF that always picks the best outcome for a given agent.

In many contexts, it is natural to assume that agent preferences are *restricted*. In such cases, the dictatorship result may not hold. An extensive literature has developed that investigates the structure of strategy-proof SCFs in models with single-peaked preferences, quasilinear preferences with money as a numeraire good, and so on. In this paper, we consider another well known restricted domain model, that of a *classical exchange economy*. There is a fixed amount of m goods, $m \geq 2$, that have to be distributed among n agents $n \geq 2$. Agent preferences are defined over bundles of m goods that are assumed to be *strictly increasing*, *continuous*, and *strictly convex*. Although there are several papers on this model, there is as yet no comprehensive characterization of strategy-proof and Pareto-efficient SCFs on the domain that consists of all such preferences (see the literature review below). The goal of our paper is to investigate these questions using techniques adapted from auction design theory. One of our results is an almost complete answer to the characterization question referred to earlier.

1.1 *Literature review*

The classic paper on incentive compatibility in exchange economies is Hurwicz (1972). It shows that there do not exist strategy-proof, efficient, and individually rational SCFs in a two-agent world. Dasgupta et al. (1979) also consider the two-agent case, and show that every efficient and strategy-proof SCF is dictatorial when the domain of agent preferences is the set of all (strictly) convex and monotone orderings. Zhou (1991) extends this result to the case of classical preferences, i.e., those that are (strictly) convex, monotone, and continuous. There are several variants of this result (all for two agents) for restricted domains, for example, Schummer (1996) for linear preferences, Ju (2003) for classical quasilinear and constant elasticity of substitution (CES), and Hashimoto (2008) for Cobb–Douglas preferences. There are significant difficulties involved in extending these results to the case of an arbitrary number of agents. Zhou (1991) conjectured that efficient dictatorial SCFs in the case of $n \geq 3$ must be *inversely dictatorial*.[1] Kato and Ohseto (2002) have shown by means of an example that the conjecture is false. If the domain is nonclassical, it is possible to construct SCFs that are strategy-proof, Pareto-efficient, and nondictatorial. For instance, Nicoló (2004) shows that in the domain of Leontief preferences, fixed-price trading rules are strategy-proof and Pareto-efficient.

Serizawa (2006) and Serizawa and Weymark (2003) provide results for strategy-proof and Pareto-efficient social choice functions for general n. The latter shows that every

[1] An SCF is *inversely dictatorial* if there exists an agent whose allocation is zero at every preference profile.

strategy-proof and efficient SCF violates a *minimum consumption guarantee* (MCG) assumption. In particular, for every $\epsilon > 0$, there exists a profile and an agent whose allocation is less than ϵ in terms of the Euclidean norm. Although this result is illuminating, it is far from being a characterization. In particular, it says nothing about the value of a Pareto-efficient and strategy-proof SCF at an arbitrary profile.

Serizawa (2006) proves a dictatorship result by strengthening strategy proofness to *effective pairwise strategy proofness*. Effective pairwise strategy proofness requires pairs of agents not to have a "self-enforcing manipulation" in addition to strategy proofness. A manipulation by a pair of agents is *self-enforcing* if it does not decrease the utility of either agents in the pair, increases utility of at least one, and neither of the agents has an incentive to betray his partner. Note that effective pairwise strategy proofness, like notions such as group strategy proofness, requires coordination between agents. However, if information is private, it is not clear how this coordination takes place.

We note that strategy proofness, Pareto efficiency, and individual rationality are incompatible in classical exchange economies. Hurwicz (1972) demonstrates this for two-good, two-agent models. Serizawa (2002) extends this result to an arbitrary numbers of agents and goods for all domains that include all homothetic preferences. Two other papers that deal with the $n \geq 2$ agents case are Barberà and Jackson (1995) and Satterthwaite and Sonnenschein (1981). These papers explore the implications of strategy proofness and additional axioms (not including Pareto efficiency) in classical exchange economies.

1.2 *Our results and contribution*

The starting point of our analysis differs from that in the existing literature. We consider a domain of classical *quasilinear preferences* of the kind

$$u_i(x_{i1}, \dots, x_{im}; \theta_i) = \theta_i\{\sqrt{x_{i1}} + \cdots + \sqrt{x_{im-1}}\} + x_{im}, \quad \theta_i > 0.$$

Our main goal is to show that strategy proofness and Pareto efficiency imply dictatorship in this domain (in the presence of some additional assumptions when there are three or more agents). So as to provide an insight into this result, we note that some implications of strategy proofness in such domains are well understood from auction design theory. For instance, in a two-good version of the same model, where $u_i(x_i, y_i; \theta_i) = \theta_i\sqrt{x_i} + y_i$ with $\theta_i > 0$, strategy proofness requires agent i's allocation of good x_i to be a weakly increasing function of θ_i for every $(n-1)$-tuple θ_{-i} of types of other agents (see, for example, Myerson 1981).[2] Moreover the allocation of good y_i is determined by the revenue equivalence theorem up to a constant that depends only on θ_{-i}. Now consider the imposition of Pareto efficiency. If the allocations of all agents are *interior*, the allocation of good x is independent of that of good y and is obtained as a solution to the problem of maximizing $\sum_i \theta_i\sqrt{x_i}$ subject to the resource constraints. The allocation of good y to agent i can thus be thought of as a Vickrey–Groves–Clarke (VCG) payment. Since Pareto efficiency will also require the amounts of good y across agents

[2]We show that for our purposes, an m-good model can be reduced to a two-good model.

to "add up," standard arguments about the "nonbalancedness" of VCG mechanisms will also imply that achieving it will be impossible.

A general argument is less straightforward because there is no reason to assume that allocations are interior. If agents are constrained in the amount of goods they consume, the characterization of Pareto-efficient allocations in the previous paragraph is no longer valid. Consequently, strategy-proof SCFs can no longer be identified with VCG mechanisms and the earlier line of argumentation breaks down. However, we are able to show that in the case where there are at least three agents, strategy proofness and efficiency in conjunction with *non-bossiness* and *continuity* imply dictatorship.[3] If there are two agents, the non-bossiness and continuity assumptions are redundant. An important observation is that all dictatorship results extend in a straightforward way to all supersets of this domain including the domain of all classical preferences. Using our methods, we also provide an elementary proof of the MCG result of Serizawa and Weymark (2003).

We believe that our techniques may be useful in characterizing strategy-proof SCFs on domains where all preferences are not quasilinear but contain "some" that are. The idea would be to use quasilinear methods to completely describe strategy-proof SCFs on the quasilinear subdomain and extend the result to the larger domain. Another aspect of our contribution deserves mention. Our domain is a "single-crossing" domain in the following sense: a pair of indifference curves of two preference orderings for an agent in (x_i, y_i) space can intersect at most once. This is the familiar Spence–Mirrlees single-crossing property, which is the cornerstone of the screening literature in the theory of asymmetric information. Although such domains are important from a theoretical standpoint, they have not received a great deal of attention in the strategy proofness literature. An exception is Saporiti (2009), who considers domains over a finite number of alternatives.[4] An important observation is that single-crossing domains are not closed with respect to Maskin monotonic transformations. Loosely speaking, a Maskin monotonic transformation allows the indifference curve of an agent to be "bent upward" at any interior commodity bundle; see Remark 1 and footnote 9 for a definition and further elaboration. Existing papers on strategy proofness in economic domains use these transformations in a central way (see Remark 1 again). Consequently, they do not cover single-crossing domains: we believe that ours is the first result on such domains.

Recently we have become aware of a related paper by Takeshi Momi (Momi 2013) written contemporaneously with ours. This paper extends the results in Serizawa and Weymark (2003) and Serizawa (2006) to a full-fledged dictatorship result. It considers the homothetic domain (like the earlier papers) and shows that every strategy-proof, efficient, and non-bossy SCF for an arbitrary number of agents is dictatorial. The results in our papers are independent of each other because the domains considered are very different: the homothetic domain is not a single-crossing domain. However, the generalization of the Momi (2013) result to the entire domain of classical preferences

[3] An SCF is *non-bossy* if each agent is unable to affect the allocation of others whenever his change in preference does not affect his own allocation.

[4] See also Goswami (2011b) for another recent contribution to the literature on strategy proofness on a single-crossing domain.

is marginally stronger than our counterpart (Theorem 3). This is because we require a continuity assumption on SCFs on our subdomain. We are unable to determine whether this assumption can be dispensed with.

Our paper is organized as follows. In Section 2, we describe our model, while Section 3 introduces and discusses quasilinear domains. Results are contained in Section 4, while Section 5 concludes.

2. Notation and definitions

We consider an exchange economy with the set of agents $I = \{1, 2, \dots, n\}$ and the set of goods $M = \{1, 2, \dots, m\}$. We assume that $n \geq 2$ and $m \geq 2$. Let the fixed total endowment of good j be denoted by Ω_j and let the total endowment vector be denoted $\Omega = (\Omega_1, \Omega_2, \dots, \Omega_m)$. We assume $\Omega_j > 0$ for all $j \in M$. The set of feasible allocations is the set $\Delta = \{(x_{i1}, \dots, x_{im}) \mid x_{ij} \geq 0 \text{ for all } j \in M \text{ and } i \in I, \text{ and } \sum_{i \in I} x_{ij} = \Omega_j \text{ for all } j \in M\}$.

A preference ordering for agent i, R_i, is a complete, reflexive, and transitive ordering of the elements of $\Re^m_+$. We say that R_i is *classical* if it is (a) continuous, (b) strictly monotonic in $\Re^m_{++}$, and (c) the upper contour sets are strictly convex in $\Re^m_{++}$.[5] The asymmetric component of R_i is P_i. Let $\mathbb{D}^c$ denote the set of classical orderings. A preference profile R is an n-tuple $R \equiv (R_1, R_2, \dots, R_n) \in [\mathbb{D}^c]^n$. We let R_{-i} denote the $(n-1)$-tuple $R_{-i} \equiv (R_1, \dots, R_{i-1}, R_{i+1}, \dots, R_n) \in [\mathbb{D}^c]^{n-1}$.

An *admissible* domain $\mathbb{D}$ is a subset of $\mathbb{D}^c$. A *social choice function* (SCF) is a map $F : [\mathbb{D}]^n \to \Delta$. We will let $F_i(R_i, R_{-i})$ denote the allocation to agent i at the profile (R_i, R_{-i}) under the SCF F.

Definition 1. An SCF F is *manipulable* by agent i at profile R via $R'_i \in \mathbb{D}$ if $F_i(R'_i, R_{-i})\, P_i\, F_i(R)$. It is *strategy-proof* if it is not manipulable by any agent at any profile. Equivalently, F is strategy-proof if $F_i(R)\, R_i\, F_i(R'_i, R_{-i})$ for all $R_i, R'_i \in \mathbb{D}$, for all $R_{-i} \in [\mathbb{D}]^{n-1}$, and for all $i \in I$.

In the usual strategic voting model, an agent's preference ordering is private information and F represents the mechanism designer's objectives. If F is strategy-proof, all agents have dominant-strategy incentives to reveal their private information truthfully.

Definition 2. An allocation $x \in \Delta$ is *Pareto-efficient* at profile R if there does not exist another allocation $x' \in \Delta$ such that $x'_i\, R_i\, x_i$ for all $i \in I$ and $x'_j\, P_j\, x_j$ for some $j \in I$.

Let $\mathrm{PE}(R)$ denote the set of Pareto-efficient allocations at R.

Definition 3. An SCF F is *Pareto-efficient* if $F(R) \in \mathrm{PE}(R)$ for all $R \in [\mathbb{D}]^n$.

[5] For a preference ordering R_i and a vector $x \in \Re^m_+$, the upper contour set of R_i at x is $\mathrm{UC}(R_i, x) = \{z \in \Re^m_+ \mid z\, R_i\, x\}$. Similarly, the lower contour set of R_i at x is $\mathrm{LC}(R_i, x) = \{z \in \Re^m_+ \mid x\, R_i\, z\}$. A preference ordering R_i is continuous if $\mathrm{UC}(R_i, x)$ and $\mathrm{LC}(R_i, x)$ are both closed for all $x \in \Re^m_+$. A preference ordering R_i is strictly convex if $\mathrm{UC}(R_i, x)$ is strictly convex for all $x \in \Re^m_{++}$. For $x, z \in \Re^m_+$, by $x > z$, we mean $x_k \geq z_k$ for all $k \in M$ and $x_k > z_k$ for some k. A preference ordering is strictly monotonic in $\Re^m_{++}$ if $x > z$ implies $x\, P_i\, z$.

Definition 4. An SCF F is *non-bossy* if, for all $R_i, R'_i \in \mathbb{D}$, $R_{-i} \in [\mathbb{D}]^{n-1}$ and $i \in I$,

$$[F_i(R_i, R_{-i}) = F_i(R'_i, R_{-i})] \quad \Rightarrow \quad [F(R_i, R_{-i}) = F(R'_i, R_{-i})].$$

The non-bossiness axiom was introduced by Satterthwaite and Sonnenschein (1981). If an SCF is non-bossy, an agent who is unable to change her allocation by a unilateral deviation from a preference profile is also unable to change the allocation of any other agent by the same deviation. The axiom is particularly useful in restricting the class of strategy-proof SCFs in environments where an agent can be indifferent across outcomes. It has been widely used in the literature.[6]

An important and familiar SCF is *dictatorship*.

Definition 5. An SCF F is *dictatorial* if there exists an agent i such that for all $R \in [\mathbb{D}]^n$,

$$F_i(R) = \Omega.$$

A dictatorial SCF gives all resources to the same agent at all preference profiles. It is of course both strategy-proof and Pareto-efficient, but ethically unsatisfactory. Serizawa and Weymark (2003) introduce a condition that ensures that all agents receive a minimal bundle of goods.

Definition 6. An SCF F satisfies the *minimum consumption guarantee (MCG)* axiom if there exists an $\epsilon > 0$ such that for all profiles $R \in [\mathbb{D}]^n$ and all $i \in I$,

$$\|F_i(R)\| \geq \epsilon.$$[7]

3. Quasilinear domains

Quasilinear preferences are preference orderings that can be represented by utility functions of the form $u_i(x) = v_i(x_{i1}, \dots, x_{im-1}) + x_{im}$.[8] These preferences are widely used in economic theory. In this paper, we restrict attention to a small subclass of quasilinear preferences that are represented by utility functions of the form

$$u_i(x_i, y_i; \theta_i) = \theta_i(\sqrt{x_{i1}} + \cdots + \sqrt{x_{i,m-1}}) + y_i, \tag{1}$$

where $\theta_i > 0$. Observe that for each i and θ_i, the preferences are symmetric across the first $m-1$ goods. Note also that these preferences are classical.

For notational convenience, we denote the mth good by y. The set of preferences in (1) is denoted by $\mathbb{D}^q$. Note that all preferences from $\mathbb{D}^q$ are represented by a parameter θ_i and a preference profile in $[\mathbb{D}^q]^n$ is represented by an n-tuple $\theta \equiv (\theta_1, \theta_2, \dots, \theta_n)$.

[6]See, for example, Pápai (2000), Svensson (1999), and Barberà and Jackson (1995). For a review, see Barberà (2011).

[7]Here $\|\cdot\|$ denotes the Euclidean norm.

[8]A preference ordering R_i is represented by the utility function $u_i : \Re^m_+ \to \Re$ if, for all $x, x' \in \Re^m_+$, $x \, R_i \, x' \Leftrightarrow u_i(x) \geq u_i(x')$.

In one of our results, we shall refer to some specific subdomains of $\mathbb{D}^q$ that we now describe. Let $\alpha > 0$. The domain $\overline{\mathbb{D}}^q_\alpha$ consists of all preference orderings that can be represented by utility functions of the form $u_i(x_i, y_i; \theta_i) = \theta_i(\sqrt{x_{i1}} + \cdots + \sqrt{x_{i,m-1}}) + y_i$, where $\theta_i \geq \alpha$. Similarly, $\underline{\mathbb{D}}^q_\alpha$ consists of all preference orderings that can be represented by utility functions of the form $u_i(x_i, y_i; \theta_i) = \theta_i(\sqrt{x_{i1}} + \cdots + \sqrt{x_{i,m-1}}) + y_i$, where $0 < \theta_i \leq \alpha$.

An SCF F defined over domains $\mathbb{D}^q$, $\overline{\mathbb{D}}^q_\alpha$, and $\underline{\mathbb{D}}^q_\alpha$ can be represented by maps $F: \Re^n_{++} \to \Delta$, $F: [\alpha, \infty)^n \to \Delta$, and $F: (0, \alpha]^n \to \Delta$, respectively. The continuity of F can, therefore, be defined straightforwardly.

Definition 7. Let $\mathbb{D}$ be any domain such that $\mathbb{D}^q \subset \mathbb{D}$. An SCF $F: [\mathbb{D}]^n \to \Delta$ is *q-continuous* if the restriction of F to $[\mathbb{D}^q]^n$ is continuous.

We emphasize that q-continuity is a relatively mild requirement because it imposes conditions only on the restriction of an SCF to the domain $\mathbb{D}^q$.

Remark 1. The domain $\mathbb{D}^q$ is “narrow” in a specific technical sense: it is a *single-crossing domain*. As mentioned in the Introduction, single-crossing domains are important in information economics, but have not been adequately analyzed in mechanism design. Exceptions are Saporiti (2009), who considers a model with finite set of alternatives, and Goswami (2011b), who looks at one-dimensional (i.e., two-good) models. Single-crossing domains present serious difficulties for mechanism design in economic environments; specifically, they do not permit Maskin monotonic (MM) transformations.[9] These transformations are central to the characterization arguments in other domains in the literature (see, for example, Momi 2013, Serizawa and Weymark 2003, Serizawa 2006, Zhou 1991, Barberà and Jackson 1995), but are not admissible in $\mathbb{D}^q$. To see this, observe that if θ'_i is an MM transformation of θ_i, the gradient vectors of the two indifference curves at a commodity bundle must be the same. However, this is impossible unless $\theta'_i = \theta_i$.

4. Results

The results in the paper are as follows.

Theorem 1. *Let $\mathbb{D}$ be an arbitrary domain such that $\mathbb{D}^q \subset \mathbb{D}$. Let $F: [\mathbb{D}]^n \to \Delta$ be a strategy-proof and Pareto-efficient SCF. Then F does not satisfy MCG.*

Theorem 2. *Let $\mathbb{D}$ be such that either $\underline{\mathbb{D}}^q_\alpha \subset \mathbb{D}$ or $\overline{\mathbb{D}}^q_\alpha \subset \mathbb{D}$ for some $\alpha > 0$. Let $F: [\mathbb{D}]^2 \to \Delta$ be a strategy-proof and Pareto-efficient SCF. Then F is dictatorial.*

Theorem 3. *Let $\mathbb{D}$ be an arbitrary domain with $\mathbb{D}^q \subset \mathbb{D}$. Let $F: [\mathbb{D}]^n \to \Delta$ be a strategy-proof, Pareto-efficient, non-bossy, and q-continuous SCF. Then F is dictatorial.*

[9] Let R_i and R'_i be two admissible preference orderings for i, and let x_i be a commodity bundle. We say that R'_i is an MM transformation of R_i at x_i if (i) $\mathrm{UC}(R'_i, x_i) \subset \mathrm{UC}(R_i, x_i)$, and (ii) $y_i \in \mathrm{UC}(R'_i, x_i)$ and $y_i \neq x_i$ implies $y_i \, P_i \, x_i$. In other words, all commodity bundles that are weakly worse than x_i under R_i are also weakly worse than x_i under R'_i. Equivalently, the indifference curve for R'_i through x_i is obtained by “bending upward” (at x_i) the indifference curve of R_i through x_i.

Theorem 1 is an MCG result for quasilinear domains, while Theorems 2 and 3 are dictatorship results.

This section is organized as follows. All the results depend critically on the structure of Pareto-efficient allocations in the domain $\mathbb{D}^q$. This is outlined in the next subsection together with an argument that allows us to reduce an m-good problem to a two-good problem. The next subsection proves preliminary results on strategy proofness in the two-good quasilinear model following Myerson (1981). The next three subsections prove and discuss each of the three results.

4.1 *Pareto efficiency in* $\mathbb{D}^q$

As we observed earlier, a profile in the domain $\mathbb{D}^q$ is a nonnegative n-tuple of real numbers θ. The set of Pareto-efficient allocations at a profile will be denoted by $\mathrm{PE}(\theta)$. Allocations in this set satisfy the property below.

Proposition 1. *If $(x^*(\theta), y^*(\theta)) \in \mathrm{PE}(\theta)$, then the following statements hold:*

(i) For all $i \in I$, if $x^*_{ij}(\theta) = 0$ for some $j \in \{1, \dots, m-1\}$, then $x^*_{ij}(\theta) = 0$ for all $j \in \{1, \dots, m-1\}$.

(ii) For all $i \in I$, if $x^*_{ij}(\theta) > 0$ for some $j \in \{1, \dots, m-1\}$, then $x^*_{ij}(\theta)/x^*_{ij'}(\theta) = \Omega_j/\Omega_{j'}$ for all $j' \in \{1, \dots, m-1\}$.

The proof of the result is contained in the Appendix.

Thus every Pareto-efficient allocation has the feature that every agent i receives all goods from 1 through $m-1$ in fixed proportions independently of θ_i. This suggests a reduction of the problem from an m-good to a two-good model. The utility of agent i from a Pareto-efficient allocation $(x^*_1, \dots, x^*_{m-1}, y)$ in the m-good model is

$$u_i(x^*_{i1}, \dots, x^*_{im-1}, y^*_i; \theta_i) = \theta_i\left[1 + \sum_{j \in M\setminus\{1\}} \sqrt{\frac{\Omega_j}{\Omega_1}}\right]\sqrt{x^*_{i1}} + y^*_i.$$

Now consider a two-good model with goods x_1 and y with endowments Ω_1 and Ω_m, respectively. Since $\theta_i[1 + \sum_{j\in M\setminus\{1\}} \sqrt{\Omega_j/\Omega_1}]$ is a positive real number, it follows that the allocation (x^*_1, y^*) is Pareto-efficient in the two-good economy in the domain $\mathbb{D}^q$ for the profile δ, where $\delta_i = \theta_i[1 + \sum_{j\in M\setminus\{1\}} \sqrt{\Omega_j/\Omega_1}]$. Now consider an SCF F that is strategy-proof and Pareto-efficient in the m-good economy. We can construct a two-good SCF $\bar{F}$ from F as, for every m-good profile θ,

$$[F(\theta) = (x_1, \dots, x_{m-1}, y)] \quad \Rightarrow \quad [\bar{F}(\delta) = (x_1, y)],$$

where δ is defined as above. By our earlier arguments, $\bar{F}$ is Pareto-efficient. It is easily verified that $\bar{F}$ is strategy-proof. For every strategy-proof and Pareto-efficient SCF in the m-good model, there is an "equivalent" (in the sense above) strategy-proof and Pareto-efficient SCF in the two-good model. Henceforth, we restrict attention to the two-good model and the results generalize in an obvious way to the m-good case.

Remark 2. Serizawa and Weymark (2003), Serizawa (2002), and Momi (2013) reduce the problem to a two-agent characterization problem in smooth homothetic domains. They do so by fixing $n-1$ agents' preferences at the same preference ordering. This can be done because Pareto-efficiency in a smooth homethetic domain requires agents' allocations to be proportional to each other. A similar reduction to a two-agent problem is not possible in quasilinear domains because of the non-interiority of Pareto-efficient allocations in such domains.

In what follows, we consider two goods x and y, and utility functions of the form $u_i(x_i, y_i; \theta_i) = \theta_i\sqrt{x_i} + y_i$. For each profile $\theta \in \Re^n_{++}$, $\{(x_i^*(\theta), y_i^*(\theta))\}_{i\in I} \in \Re^{2n}_+$ represents an allocation in $\mathrm{PE}(\theta)$. Without loss of generality, we set the total endowments of both goods to 1.

The proposition below provides necessary conditions for allocations to be Pareto-efficient in the two-good model.

Proposition 2. *If* $(x^*(\theta), y^*(\theta)) \in \mathrm{PE}(\theta)$, *then for all* $i \in I$, *the following statements hold:*

P1. If $x_i^*(\theta) < \theta_i^2/\sum_{k\in I}\theta_k^2$, *then* $y_i^*(\theta) = 0$.

P2. If $x_i^*(\theta) > \theta_i^2/(\theta_i^2 + \min_{k\neq i}\theta_k^2)$, *then* $y_i^*(\theta) = 1$.

The proof of the proposition is contained in the Appendix.

The following fact about Pareto efficiency in quasilinear domains is well known: if x^* solves $\max_{x_1,\dots,x_n}\sum_{i\in I}\theta_i\sqrt{x_i}$ subject to the resource constraint on x, then any allocation of good y together with x^* is Pareto-efficient. For instance, $(\theta_1^2/\sum_{k\in I}\theta_k^2, \dots, \theta_n^2/\sum_{k\in I}\theta_k^2)$ solves $\max_{x_1,\dots,x_n}\sum_{i\in I}\theta_i\sqrt{x_i}$ subject to $\sum_{i\in I}x_i = 1$. We say that agent i is *constrained* at θ if $x_i(\theta) < \theta_i^2/\sum_{k\in I}\theta_k^2$. According to condition P1, a constrained agent must not get a positive amount of good y. According to P2, any agent i whose x_i exceeds a certain bound must get the entire amount of good y.

4.2 *Strategy proofness in* $\mathbb{D}^q$

Lemmas 1 and 2 below are straightforward extensions of familiar results from auction design. Their proofs are omitted and can be found in Goswami (2011a).

Henceforth, we shall denote $F(\theta)$ by the pair $(x(\theta), y(\theta))$ for all θ. The allocation to agent i at θ will be denoted by $F_i(\theta) \equiv (x_i(\theta), y_i(\theta))$.

Lemma 1. *Let* $F: [\mathbb{D}^q]^n \to \Delta$ *be strategy-proof. For all i, for all* θ_i, θ_i' *with* $\theta_i' > \theta_i$, *and all* θ_{-i}, *one of the following two conditions holds:*

(a) $x_i(\theta_i', \theta_{-i}) = x_i(\theta_i, \theta_{-i})$ *and* $y_i(\theta_i', \theta_{-i}) = y_i(\theta_i, \theta_{-i})$

(b) $x_i(\theta_i', \theta_{-i}) > x_i(\theta_i, \theta_{-i})$ *and* $y_i(\theta_i', \theta_{-i}) < y_i(\theta_i, \theta_{-i})$.

LEMMA 2. *Let $F : [\mathbb{D}^q]^n \to \Delta$ be strategy-proof. For all i, for all $\theta_i \in [a_i, b_i] \subset \Re_{++}$, and all θ_{-i},*

$$u_i(F_i(\theta_i, \theta_{-i}); \theta_i) = u_i(F_i(a_i, \theta_{-i}); a_i) + \int_{a_i}^{\theta_i} x_i(t_i, \theta_{-i})^{1/2}\, dt_i.$$

If F is strategy-proof, the allocation of x_i to agent i must be weakly increasing in her type θ_i (Lemma 1). Lemma 2 is an expression of the revenue equivalence principle. The allocation of good x_i to agent i at profile (θ_i, θ_{-i}) determines the allocation of y_i to the agent at that profile up to a constant that depends only on θ_{-i}.

Suppose all agents in the coalition S, $|S| \geq 2$, are unconstrained in some neighborhood. In addition, all agents in the complementary coalition $I \setminus S$ get zero amounts of both goods. An application of Lemma 2 will imply that the allocation of good y to agents in S will correspond to VCG payments. Furthermore, Pareto efficiency will also require these VCG payments to add up to 1 (the aggregate endowment of good y), i.e., the VCG payments will have to "balance." Proposition 3 below shows this to be impossible in the domain $\mathbb{D}^q$. The difficulty of balancing VCG transfers in auction theory or achieving full efficiency in public good environments is well known (see Hurwicz and Walker 1990).

PROPOSITION 3. *Let $F : [\mathbb{D}^q]^n \to \Delta$ be strategy-proof. There does not exist $S \subset I$, $|S| \geq 2$, and a neighborhood $N_\epsilon(\theta')$ such that $x_i(\theta) = \theta_i^2 / \sum_{k\in S} \theta_k^2$ for all $i \in S$ and $\sum_{i\in S} y_i(\theta) = 1$ for all $\theta \in N_\epsilon(\theta')$.*[10]

PROOF. Suppose the proposition is false; i.e., there exists $S \subset I$ with $|S| \geq 2$ and $N_\epsilon(\theta')$ such that for all $\theta \in N_\epsilon(\theta')$, $x_i(\theta) = \theta_i^2 / \sum_{k\in S} \theta_k^2$ for all $i \in S$ and $\sum_{i\in S} y_i(\theta) = 1$.

Applying Lemma 2, it follows that for each $\theta \in N_\epsilon(\theta')$ and $i \in S$,

$$u_i(F_i(\theta_i, \theta_{-i}); \theta_i) = u_i(F_i(a_i, \theta_{-i}); a_i) + \int_{a_i}^{\theta_i} \left[\frac{t_i^2}{\sum_{k\in S\setminus\{i\}} \theta_k^2 + t_i^2}\right]^{1/2} dt_i, \tag{2}$$

where $(a_1, a_2, \ldots, a_n) \in N_\epsilon(\theta')$. Replacing $u_i(F_i(\theta_i, \theta_{-i}); \theta_i)$ with $\theta_i[\theta_i^2 / \sum_{k\in S} \theta_k^2]^{1/2} + y_i(\theta_i, \theta_{-i})$ in the left hand side (LHS) of (2) and letting

$$h_i(\theta_{-i}) \equiv a_i\left[\frac{a_i^2}{a_i^2 + \sum_{k\in S\setminus\{i\}} \theta_k^2}\right]^{1/2} + y_i(a_i, \theta_{-i})$$

in the right hand side (RHS), we obtain,

$$\theta_i\left[\frac{\theta_i^2}{\sum_{k\in S} \theta_k^2}\right]^{1/2} + y_i(\theta_i, \theta_{-i}) = h_i(\theta_{-i}) + \int_{a_i}^{\theta_i} \left[\frac{t_i^2}{\sum_{k\in S\setminus\{i\}} \theta_k^2 + t_i^2}\right]^{1/2} dt_i. \tag{3}$$

Summing (3) across i and noting that $\sum_{i\in S} y_i(\theta) = 1$, we obtain,

$$\left[\sum_{i\in S} \theta_i^2\right]^{1/2} + 1 - \sum_{i\in S} \int_{a_i}^{\theta_i} \frac{t_i}{[\sum_{k\in S\setminus\{i\}} \theta_k^2 + t_i^2]^{1/2}}\, dt_i = \sum_{i\in S} h_i(\theta_{-i}). \tag{4}$$

[10]We have $N_\epsilon(\theta') = \{\theta : \|\theta - \theta'\| < \epsilon\}$.

Solving for the integrals in the LHS of (4) and simplifying further, we get

$$(1-|S|)\Big[\sum_{i\in S}\theta_i^2\Big]^{1/2}+1+\sum_{i\in S}\Big[\sum_{k\in S\setminus\{i\}}\theta_k^2+a_i^2\Big]^{1/2}=\sum_{i\in S}h_i(\theta_{-i}). \tag{5}$$

The LHS of (5) is an infinitely differentiable function in $\Re_{++}^{|S|}$. Notice that its $|S|$th order cross-partial derivative is $c(|S|)(-1)^{(|S|)}(\prod_{i\in S}\theta_i)(\sum_{i\in S}\theta_i^2)^{-(2|S|-1)/2}$, where $c(|S|)$ is a constant that is not equal to zero for any value of $|S|$.[11] However, the $|S|$th order cross-partial derivative of the RHS of (5) vanishes at all θ. We have a contradiction. □

A restatement of Proposition 3 will be more convenient for our purpose.

Definition 8. The SCF $F:\mathbb{D}^n\to\Delta$ satisfies S-interiority for $S\subset I$ with $|S|\geq 2$ if there exists $N_\epsilon(\theta')$ such that for all $\theta\in N_\epsilon(\theta')$, we have $x_i(\theta), y_i(\theta)>0$ for all $i\in S$ and $x_i(\theta)=y_i(\theta)=0$ for all $i\notin S$.

Application of P1 in Proposition 2 and Proposition 3 immediately yields the following result.

Proposition 4. *Let $F:[\mathbb{D}^q]^n\to\Delta$ be strategy-proof and Pareto-efficient. Then F does not satisfy S-interiority for any S.*

Proposition 4 will be critical in the proofs of Theorems 2 and 3.

4.3 *Minimum consumption guarantees*

In this subsection, we prove Theorem 1, which is an MCG result on the domain $\mathbb{D}^q$. It is independent of the result in Serizawa and Weymark (2003), and considerably easier to prove. In addition, our arguments are constructive: we are able to identify some profiles where MCG fails. Of course, it implies the result for every superset of $\mathbb{D}^q$ such as the domain of all classical preferences. Note that Theorem 1 is valid for all n, but does not require either the non-bossiness or the continuity axiom.

Proof of Theorem 1. It is sufficient to prove the result for a strategy-proof and Pareto-efficient SCF $F:[\mathbb{D}^q]^n\to\Delta$. Let F be such an SCF. We first establish the following result.

Lemma 3. *Let θ be a profile such that $x_i(\theta)<\theta_i^2/\sum_{k\in I}\theta_k^2$, i.e., agent i is constrained. Then $y_i(\theta_i',\theta_{-i})<\theta_i$ whenever $\theta_i'<\theta_i$.*

Proof. Suppose not. That is, suppose $\theta_i'<\theta_i$, but $y_i(\theta_i',\theta_{-i})\geq\theta_i$. Now, by strategy proofness and the fact that $y_i(\theta_i,\theta_{-i})=0$, we have

$$\theta_i x_i(\theta_i,\theta_{-i})^{1/2}\geq\theta_i x_i(\theta_i',\theta_{-i})^{1/2}+y_i(\theta_i',\theta_{-i}).$$

[11]The LHS of (5) is the sum of two functions. The sth cross-partial derivative of the first term $(1-|S|)\times[\sum_{i\in S}\theta_i^2]^{1/2}+1$ is nonzero, while that of the second term $\sum_{i\in S}[\sum_{k\in S\setminus\{i\}}\theta_k^2+a_i^2]^{1/2}$ is zero.

Hence,

$$\theta_i[x_i(\theta_i, \theta_{-i})^{1/2} - x_i(\theta_i', \theta_{-i})^{1/2}] \geq y_i(\theta_i', \theta_{-i}).$$

Since $\theta_i > 0$,

$$[x_i(\theta_i, \theta_{-i})^{1/2} - x_i(\theta_i', \theta_{-i})^{1/2}] \geq \frac{y_i(\theta_i', \theta_{-i})}{\theta_i} \geq 1. \tag{6}$$

Since $x_i(\theta_i, \theta_{-i}), x_i(\theta_i', \theta_{-i}) \leq 1$, the inequality in (6) can be satisfied only if $x_i(\theta_i, \theta_{-i}) = 1$. However, $x_i(\theta) < \theta_i^2 / \sum_{k \in I} \theta_k^2 < 1$, leading to a contradiction.

Returning to the proof of Theorem 1, choose any ϵ such that $0 < \epsilon < \sqrt{2}$. We show the existence of an agent i and a profile (θ_i', θ_{-i}) such that $\|(x_i(\theta_i', \theta_{-i}), y_i(\theta_i', \theta_{-i}))\| < \epsilon$.

Consider the open set $O = \prod_{j=1}^{N}(0, \epsilon/\sqrt{2})$. By Proposition 3 and P1 in Proposition 2, we know that there is a profile $(\theta_i, \theta_{-i}) \in O$ and an agent i such that $y_i(\theta_i, \theta_{-i}) = 0$ and $x_i(\theta_i, \theta_{-i}) < \theta_i^2 / \sum_{k \in I} \theta_k^2$. By Lemma 3 and the choice of ϵ, we have that for any $\theta_i' < \theta_i$, $y_i(\theta_i', \theta_{-i}) < \theta_i < \epsilon/\sqrt{2} < 1$. Applying P2 in Proposition 2, we infer that $x_i(\theta_i', \theta_{-i}) \leq \theta_i'^2 / (\theta_i'^2 + \min_{j \neq i} \theta_j^2)$ for all $\theta_i' < \theta_i$. Observe that the RHS of this inequality converges to zero as θ_i' converges to zero. Hence, $\lim_{\theta_i \to 0} x_i(\theta_i, \theta_{-i}) = 0$. Therefore, there exists $\theta_i' < \theta_i$ such that $x_i(\theta_i', \theta_{-i}) < \epsilon/\sqrt{2}$ and $y_i(\theta_i', \theta_{-i}) < \epsilon/\sqrt{2}$. Hence,

$$\sqrt{(x_i(\theta_i', \theta_{-i}))^2 + (y_i(\theta_i', \theta_{-i}))^2} < \sqrt{\left(\frac{\epsilon}{\sqrt{2}}\right)^2 + \left(\frac{\epsilon}{\sqrt{2}}\right)^2} = \epsilon.$$

That is, $\|(x_i(\theta_i', \theta_{-i}), y_i(\theta_i', \theta_{-i}))\| < \epsilon$. ◁

□

Remark 3. Is it possible to directly relate MCG failure and dictatorship in particular domains? This appears to be a difficult question. Our argument for proving Theorem 1 uses a particular sequence of profiles along which the allocation of *one* agent approaches zero. Nothing can be said about the allocations of other agents: even the identity of the agent whose allocation is vanishing can change along the sequence. More importantly, nontrivial arguments are required to ensure that the limit profile belongs to the domain.

4.4 *Dictatorship in the* $n = 2$ *case*

The proof of Theorem 2 is contained in the Appendix. Below, we show by means of an example that the result does not hold if the set of types of the agents is both bounded above and bounded below away from zero.

Example 1. Let $\mathbb{D}^* \subset \mathbb{D}^q$ be such that $\inf_\alpha\{\mathbb{D}^*\} = \gamma > 0$ and $\sup_\alpha\{\mathbb{D}^*\} = \beta < \infty$. Pick agent i. Note that

$$\sup_\alpha\{\theta_i^2/(\theta_i^2 + \theta_j^2) \mid (\theta_i, \theta_j) \in [\mathbb{D}^*]^2\} = \beta^2/(\gamma^2 + \beta^2) < \infty$$

and

$$\inf_\alpha\{\theta_i^2/(\theta_i^2 + \theta_j^2) \mid (\theta_i, \theta_j) \in [\mathbb{D}^*]^2\} = \gamma^2/(\gamma^2 + \beta^2) > 0$$

for agent i. Define $F(\theta) = ((\gamma^2/(\gamma^2+\beta^2), 0), (\beta^2/(\gamma^2+\beta^2), 1))$ for all $\theta \in [\mathbb{D}^*]^2$. This SCF is trivially strategy-proof because it is constant. It is also Pareto-efficient, which follows from Proposition 1. ♢

Remark 4. When preferences are nonclassical, it is possible to construct a two-person nondictatorial, strategy-proof and Pareto-efficient SCF; see Nicoló (2004).

4.5 *Dictatorship in the $n \geq 3$ case*

This case is different from the two-agent case because strategy-proof and Pareto-efficient SCFs need not be dictatorial, as shown in Kato and Ohseto (2002). We have already shown that in quasilinear domains, a strategy-proof and Pareto-efficient SCF (for an arbitrary number of agents) must satisfy a highly restrictive property: at least one agent must receive a zero amount of good y in every neighborhood of profiles. If there are three or more agents, the identity of this agent could depend on the announcements of the other agents. This increases the possible complexity in the behavior of a strategy-proof SCF very dramatically. However, by imposing certain familiar regularity assumptions on SCFs, we are able to recover the dictatorship result.

Proof of Theorem 3. Let $F : [\mathbb{D}^q]^n \to \Delta$ be strategy-proof, Pareto-efficient, non-bossy, and q-continuous. We will show that F is dictatorial. We will first establish two lemmas.

Lemma 4. *Let θ be an arbitrary profile and let $S = \{j \in I \mid y_j(\theta) > 0\}$. Let $i \notin S$ be such that $x_i(\theta) > 0$. There exists θ_i^* and a neighborhood $N_\epsilon(\theta_i^*, \theta_{-i})$ such that for all $\theta' \in N_\epsilon(\theta_i^*, \theta_{-i})$, we have $y_k(\theta') > 0$ for all $k \in S \cup \{i\}$.*

Proof. Let θ, i, and S be as specified in the statement of Lemma 4. By P2 in Proposition 2, we know that $x_i(\theta) \leq \theta_i^2/(\theta_i^2 + \min_{k\neq i} \theta_k^2)$. Consider a decreasing sequence $\theta_i^r \to 0$ as $r \to \infty$. By Lemma 1, $x_i(\theta_i^r, \theta_{-i}) \leq x_i(\theta_i, \theta_{-i})$. Suppose $x_i(\theta_i^r, \theta_{-i}) = x_i(\theta_i, \theta_{-i})$ for all r. Clearly $y_i(\theta_i^r, \theta_{-i}) = y_i(\theta_i, \theta_{-i}) = 0$; otherwise i will manipulate. Observe that $(\theta_i^r)^2/((\theta_i^r)^2 + \min_{k\neq i} \theta_k^2) \to 0$ as $r \to \infty$. Therefore, $x_i(\theta_i^r, \theta_{-i}) > (\theta_i^r)^2/((\theta_i^r)^2 + \min_{k\neq i} \theta_k^2)$, while $y_i(\theta_i^r, \theta_{-i}) = 0$ for r large enough. This contradicts P2 in Proposition 2. Hence, $x_i(\theta_i^r, \theta_{-i}) < x_i(\theta_i, \theta_{-i})$ for r large enough, which by strategy proofness also implies $y_i(\theta_i^r, \theta_{-i}) > 0$ for r large enough. Let $\bar{\theta}_i = \inf_r\{\theta_i^r : y_i(\theta_i^r, \theta_{-i}) = 0\}$. Since $F_i(\bar{\theta}_i, \theta_{-i}) = F_i(\theta_i, \theta_{-i})$, the non-bossiness of F implies that $F(\bar{\theta}_i, \theta_{-i}) = F(\theta_i, \theta_{-i})$. By the q-continuity of F, there exists $\theta_i^* < \bar{\theta}_i$ and a neighborhood $N_\epsilon(\theta_i^*, \theta_{-i})$ such that for all θ' in the neighborhood, $y_k(\theta') > 0$ for all $k \in S \cup \{i\}$. ◁

Lemma 5. *Let θ be an arbitrary profile and let $S = \{j \in I \mid y_j(\theta) > 0\}$. There exists a neighborhood $N_\epsilon(\theta')$ and $S' \subset I$ with $S \subset S'$ such that for all $\tilde{\theta}$ in the neighborhood, we have $x_i(\tilde{\theta}), y_i(\tilde{\theta}) > 0$ for all $i \in S'$ and $\sum_{i\in S'} x_i(\tilde{\theta}) = \sum_{i\in S'} y_i(\tilde{\theta}) = 1$.*

Proof. Let θ be an arbitrary profile and let $S = \{j \in I \mid y_j(\theta) > 0\}$. If $\sum_{i\in S} x_i(\theta) = 1$ in a neighborhood of θ, then Lemma 5 follows by the q-continuity of F. To see this, note that by Pareto efficiency, only agents S are allocated positive amounts, whereas those outside S get zero of both the goods. By q-continuity and finiteness of agents, we will find a profile neighborhood around θ such that all the agents from the set S obtain a positive amount of good y.

Hence, consider an $i \notin S$ and $x_i(\theta) > 0$, but $y_i(\theta) = 0$. Applying Lemma 4, it follows that there exists a profile θ' and a neighborhood $N_\epsilon(\theta')$ such that $y_k(\theta'') > 0$ for all $k \in S \cup \{i\}$ for all θ'' in this neighborhood. Suppose there exists an agent i' with $i' \notin S \cup \{i\}$ such that $x_{i'}(\theta'') > 0$ and $y_{i'}(\theta'') = 0$ for some θ'' in this neighborhood. Applying Lemma 4 again, we can find another neighborhood such that for all profiles θ in this neighborhood, $y_k(\theta) > 0$ for all $k \in S \cup \{i, i'\}$. By P1 in Proposition 2, $x_k(\theta) > 0$ for all $k \in S \cup \{i, i'\}$. Proceeding in this way and using the fact that the number of agents is finite, the desired conclusion obtains.

We show that F is dictatorial. To see this, suppose that there exist θ and $S \subset I$ with $|S| \geq 2$ such that $y_i(\theta) > 0$ for all $i \in S$. By Lemma 5, there exist a neighborhood $N_\epsilon(\theta')$ and a set of agents S' with $S \subset S'$ with $x_i(\tilde{\theta}), y_i(\tilde{\theta}) > 0$ for all $i \in S'$ and $\sum_{i \in S'} x_i(\tilde{\theta}) = \sum_{i \in S'} y_i(\tilde{\theta}) = 1$ for all $\tilde{\theta} \in N_\epsilon(\theta')$. However, this implies that F satisfies S'-interiority, contradicting Proposition 4. Therefore, $|S| = 1$ for all profiles. By Pareto efficiency, this implies that for all θ, there exists an agent i such that $F_i(\theta) = (1, 1)$. A simple argument using non-bossiness establishes that F is dictatorial.

Finally, let $\mathbb{D}^q \subset \mathbb{D}$ and $F : [\mathbb{D}]^n \to \Delta$ be strategy-proof, Pareto-efficient, non-bossy, and q-continuous. We know from our earlier arguments that F restricted to the domain $\mathbb{D}^q$ is dictatorial. Let i be the dictator. Pick an arbitrary profile $R \in \mathbb{D}^n$. If $F_i(R_i, \theta_{-i}) \neq (1, 1)$, i will manipulate F at (R_i, θ_{-i}) via θ_i. Note also that for all $j \neq i$, strategy proofness implies $F_j(R_i, R_j, \theta_{-i,-j}) = (0, 0)$. By non-bossiness, $F_i(R_i, R_j, \theta_{-i,-j}) = (1, 1)$. By repeating this argument, it follows that $F_i(R) = (1, 1)$ so that i is the dictator for F. $\triangleleft$

$\square$

Remark 5. There are open questions relating to our non-bossiness and q-continuity assumptions. A reasonable conjecture is that strategy proofness and Pareto efficiency imply the *extreme valuedness* of F for the domain $\mathbb{D}^q$, i.e., at all profiles, there exists an agent who receives the entire allocation of all goods. Momi (2013) has shown that q-continuity is not required for the dictatorship result for classical preferences. He also proved an extreme-valuedness result for classical preferences for the case $n = 3$. Proving results without non-bossiness in the general case is clearly an important objective.

5. Conclusion

In this paper, we have analyzed the structure of strategy-proof and Pareto-efficient social choice functions in classical exchange economies. Our methodological contribution is to focus on a small class of quasilinear domains and use techniques developed in auction design. This approach enables us to prove dictatorship results for arbitrary numbers of agents in a fairly straightforward way. If the number of agents is more than 2, we require the SCFs to satisfy a continuity requirement as well as the non-bossiness assumption. An important open question is whether these assumptions, and the non-bossiness assumption, in particular, can be dispensed with.

Appendix

Proof of Proposition 1. Let $\{x_i^*(\theta), y_i^*(\theta)\}_{i=1}^N$ be a Pareto-efficient allocation. Fix an agent i. We first show that if $x_{ij}^*(\theta) = 0$ for some $j \in \{1, \dots, m-1\}$, then $x_{ij'}^*(\theta) = 0$ for all $j' \in \{1, \dots, m-1\}$. Suppose this false, i.e., $x_{ij}^*(\theta) = 0$ but $x_{ij'}^*(\theta) > 0$ for some $j' \in \{1, \dots, m-1\}$. We argue that this allocation is not Pareto-efficient. There must exist an agent i' with an allocation $(x_{i'}^*(\theta), y_{i'}^*(\theta))$ and $x_{i'j}^*(\theta) > 0$. For agents i and i', define $\Omega_j^{(i,i')} \equiv x_{ij}^*(\theta) + x_{i'j}^*(\theta) > 0$ and $\Omega_{j'}^{(i,i')} \equiv x_{ij'}^*(\theta) + x_{i'j'}^*(\theta) > 0$. Fix the allocation of the other agents and other goods, and consider the set of Pareto-efficient allocations in the Edgeworth box of agents i and i' with total endowments of j and j' being $\Omega_j^{(i,i')}$ and $\Omega_{j'}^{(i,i')}$, respectively. In this box, Pareto-efficient points lie on the diagonal. By fixing agent i''s *utility level* at $\theta_{i'}[x_{i'j}^{*1/2}(\theta) + x_{i'j'}^{*1/2}(\theta)]$, agent i can be made better off than at $x_{ij}^*(\theta) = 0$, $x_{ij'}^*(\theta) > 0$. Hence, the initial allocation cannot be Pareto-efficient.

To complete the proof of the proposition, consider the following optimization problem for agent i:

$$\max_{\{x_i, y_i\}_{i=1}^N} \left[\theta_i \sum_{j=1}^{m-1} x_{ij}^{1/2} + y_i\right]$$

$$\text{subject to} \quad \left[\theta_k \sum_{j=1}^{m-1} x_{jk}^{1/2} + y_k\right] \geq \bar{u}_k \quad \forall k \in N \setminus \{i\}$$

$$\sum_{i \in N} x_{ij} = \Omega_j \quad \forall j \in \{1, \dots, m-1\}, \qquad \sum_{i \in N} y_i = \Omega_m$$

$$x_{ij} \geq 0 \quad \forall i \in N, \forall j \in \{1, \dots, m-1\} \quad \text{and} \quad y_i \geq 0 \quad \forall i \in N.$$

If agent i is the only agent who obtains positive amounts of the first $(m-1)$ goods, then we are done. So let $T \subset N$ (with $|T| > 1$ and $i \in T$) be the set of agents who obtain positive amounts of the first $(m-1)$ goods. Since for any pair of agents in T, the marginal rate of substitution between any two goods j and j' (from the first $(m-1)$ goods) must be equal, we get $(x_{ij}^*(\theta))^{1/2}/(x_{ij'}^*(\theta))^{1/2} = (x_{i'j}^*(\theta))^{1/2}/(x_{i'j'}^*(\theta))^{1/2}$. Hence,

(A) $x_{ij}^*(\theta)/x_{ij'}^*(\theta) = x_{i'j}^*(\theta)/x_{i'j'}^*(\theta)$ for all $i' \in T \setminus \{i\}$

(B) $\sum_{i' \in T} x_{i'j}^*(\theta) = \Omega_j$ for all $j \in \{1, \dots, m-1\}$.

Using (A) and (B), we get

$$\frac{\sum_{i' \in T} x_{i'j}^*(\theta)}{\sum_{i' \in T} x_{i'j'}^*(\theta)} = \frac{\Omega_j}{\Omega_{j'}} \quad \Rightarrow \quad \frac{x_{ij}^*(\theta) + x_{ij}^*(\theta)(\sum_{i' \in T \setminus \{i\}} x_{i'j'}^*(\theta)/x_{ij'}^*(\theta))}{\sum_{i' \in T} x_{i'j'}^*(\theta)} = \frac{\Omega_j}{\Omega_{j'}}$$

$$\Rightarrow \quad \frac{x_{ij}^*(\theta)}{x_{ij'}^*(\theta)} = \frac{\Omega_j}{\Omega_{j'}}. \qquad \square$$

Proof of Proposition 2. We proceed in four steps.

Step 1. Consider a two-agent economy with agents i and j, and an arbitrary total endowment. We prove the following result. For a fixed profile $\theta \in [\mathbb{D}^q]^2$, if $y_i^*(\theta) > 0$, then $x_i^*(\theta) > 0$.

A Pareto-efficient allocation is a solution to the optimization problem

$$\max_{x_i, y_i} \theta_i x_i^{1/2} + y_i$$

$$\text{subject to} \quad \theta_j(\Omega_x - x_i)^{1/2} + \Omega_y - y_i \geq \bar{u}_j, \quad x_i \geq 0 \text{ and } y_i \geq 0,$$

where $\bar{u}_j$ is a nonnegative number. Now note that by strict monotonicity of the objective function, the maximum is achieved at an allocation (x_i^*, y_i^*) such that $\theta_j(\Omega_x - x_i^*)^{1/2} + \Omega_y - y_i^* = \bar{u}_j$. This constraint can be rewritten as $y_i = \Omega_y - \bar{u}_j + \theta_j(\Omega_x - x_i)^{1/2}$. This is a strictly decreasing function of x_i. Also the level sets of the objective function are strictly decreasing with $\lim_{x_i \to 0}(dy_i/dx_i) = -\infty$. The derivative of the function $y_i = \Omega_y - \bar{u}_j + \theta_j(\Omega_x - x_i)^{1/2}$ exists for all $x_i < \Omega_x$. From this, it can be argued that the level set of the objective function that meets the constraint at $x_i = 0$ must cut the constraint from below. Thus, $y_i^*(\theta) > 0$ and $x_i^*(\theta) = 0$ cannot be a Pareto-efficient allocation. Hence, the result follows.

Step 2. Consider the n-agent economy and suppose $(x^*(\theta), y^*(\theta)) \in \text{PE}(\theta)$. Fix an agent i. If $y_i^*(\theta) > 0$, then $x_i^*(\theta) > 0$.

Suppose not, i.e., let $(x^*(\theta), y^*(\theta))$ be a Pareto-efficient allocation with $y_i^*(\theta) > 0$ and $x_i^*(\theta) = 0$. Let $x_{i'}(\theta)^* > 0$. Let agents i and i' share $\Omega_1^{(i,i')}$ and $\Omega_2^{(i,i')}$ of good x and good y, respectively. Fix the allocation of the other agents. The utility functions of agents i and i' are now of the form $\theta_i x_i(\theta)^{1/2} + y_i(\theta)$ and $\theta_{i'} x_{i'}(\theta)^{1/2} + y_{i'}(\theta)$, respectively. However, from Step 1, we know that Pareto-efficient allocations in the two-agent, two-good model are such that if $x_i^*(\theta) = 0$, then $y_i^*(\theta) = 0$. Therefore, by keeping agent i''s utility level fixed at $\theta_{i'}(x_{i'}^*(\theta))^{1/2} + y_{i'}^*(\theta)$, agent i can be made better off with a positive amount of good x. This contradicts our assumption that $(x^*(\theta), y^*(\theta))$ is Pareto-efficient. This proves Step 2.

Step 3. If $(x^*(\theta), y^*(\theta)) \in \text{PE}(\theta)$ and for agent $i \in I$, $y_i^*(\theta) > 0$, then $x_i^*(\theta) \geq \theta_i^2 / \sum_{k \in S} \theta_k^2$, where $S = \{k \in I \mid x_k^*(\theta) > 0\}$.

Let $(x^*(\theta), y^*(\theta)) \in \text{PE}(\theta)$ be such that S $(\subset I)$ is the set of all agents who are allocated a positive amount of good x. Let S' $(\subset I)$ be the agents who are allocated a positive amount of good y. By Step 2, $S' \subset S$. Let $i \in S'$. The Lagrangian for agent i's optimization problem is

$$L = u_i(x_i, y_i; \theta_i) + \sum_{k \in I \setminus \{i\}} \alpha_k[-\bar{u}_k + u_k(x_k, y_k; \theta_k)]$$
$$+ \sum_{k \in I}(\beta_{k1} x_i + \beta_{k2} y_i) + \gamma_1\left(1 - \sum_{k \in I} x_k\right) + \gamma_2\left(1 - \sum_{k \in I} y_k\right),$$

where each α_k, β_{k1}, β_{k2}, γ_1, and γ_2 is a Lagrange multiplier. The first order conditions and complementary slackness conditions are

$$\frac{\partial L}{\partial x_i} = \frac{\theta_i}{2x_i^{1/2}} + \beta_{i1} - \gamma_1 = 0 \tag{7}$$

$$\frac{\partial L}{\partial x_k} = \frac{\theta_k \alpha_k}{2x_k^{1/2}} + \beta_{k1} - \gamma_1 = 0 \quad \forall k \in S \setminus \{i\} \tag{8}$$

$$\frac{\partial L}{\partial y_i} = 1 + \beta_{i2} - \gamma_2 = 0 \tag{9}$$

$$\frac{\partial L}{\partial y_k} = \alpha_k + \beta_{k2} - \gamma_2 = 0 \quad \forall k \in S' \setminus \{i\} \tag{10}$$

$$\frac{\partial L}{\partial y_k} = \alpha_k + \beta_{k2} - \gamma_2 \leq 0 \quad \forall k \in S \setminus \{S'\} \tag{11}$$

$$\sum_{i \in I} x_i = 1, \qquad \sum_{i \in I} y_i = 1 \tag{12}$$

$$\beta_{k1} x_k = 0, \qquad \beta_{k2} y_k = 0 \quad \forall k \in I \tag{13}$$

$$\alpha_k \geq 0 \quad \forall k \in S \setminus \{i\} \tag{14}$$

$$\beta_{ij} \geq 0 \quad \forall i \in I, \forall j \in \{1, 2\}. \tag{15}$$

From (13) and $y_i(\theta) > 0$, it follows that $\beta_{i2} = 0$ and, hence, using (9) we get $\gamma_2 = 1$. Since $\gamma_2 = 1$, from (10) and (11), we get $\alpha_k + \beta_{k2} = 1$ $\forall k \in S \setminus \{i\}$. By (14) and (15), we obtain $0 \leq \alpha_k \leq 1$ $\forall k \in S \setminus \{i\}$. Since by assumption $x_k > 0$ for all $k \in S$, we have $\beta_{k1} = 0$ for all $k \in S$. Now from (7) and (8), we have

$$\frac{\theta_i}{2x_i^{1/2}} = \frac{\alpha_k \theta_k}{2x_k^{1/2}} \quad \forall k \in S \setminus \{i\}.$$

By squaring both sides and simplifying, we obtain

$$\frac{\theta_i^2}{x_i} = \frac{\alpha_k^2 \theta_k^2}{x_k} \quad \forall k \in S \setminus \{i\}.$$

Hence,

$$x_k(\theta) = x_i(\theta) \frac{\alpha_k^2 \theta_k^2}{\theta_i^2} \quad \forall k \in S \setminus \{i\}.$$

From (12), we obtain

$$x_i(\theta) + x_i(\theta) \sum_{k \in S \setminus \{i\}} \frac{\alpha_k^2 \theta_k^2}{\theta_i^2} = 1.$$

Since $\alpha_k \leq 1$ $\forall k \in S \setminus \{i\}$,

$$x_i^*(\theta) = \frac{\theta_i^2}{\theta_i^2 + \sum_{k \in S \setminus \{i\}} \alpha_k^2 \theta_k^2} \geq \frac{\theta_i^2}{\sum_{k \in S} \theta_k^2}.$$

This proves Step 3.

Let $(x^*(\theta), y^*(\theta))$ be a Pareto-efficient allocation at θ. If $S' = \{k \in I \mid y_k^*(\theta) > 0\}$ and $S = \{k \in I \mid x_k^*(\theta) > 0\}$, then Step 2 implies $S' \subset S$. Also Step 3 implies $x_i^*(\theta) \geq \theta_i^2 / \sum_{k \in S} \theta_k^2 \geq \theta_i^2 / \sum_{k \in I} \theta_k^2$ for all $i \in S'$. Therefore, $y_i^*(\theta) > 0$ implies $x_i^*(\theta) \geq \theta_i^2 / \sum_{k \in I} \theta_k^2$, which is equivalent to condition P1 of this proposition.

Step 4. Let $(x^*(\theta), y^*(\theta)) \in \mathrm{PE}(\theta)$. If $x_i^*(\theta) > \theta_i^2 / (\theta_i^2 + \min_{k \neq i} \theta_k^2)$ for some $i \in I$, then $y_i^*(\theta) = 1$.

Suppose not, i.e., suppose $x_i^*(\theta) > \theta_i^2/(\theta_i^2 + \min_{k\neq i}\theta_k^2)$ and $y_i^*(\theta) < 1$. Therefore, there is at least one agent i' ($\neq i$) such that $y_{i'}^*(\theta) > 0$. By Step 2, $x_{i'}^*(\theta) > 0$. Solving the optimization problem in Step 3 for agent i', we obtain $\alpha_i \leq 1$. Suppose agent i' is the only agent other than i who obtains a positive allocation of good x. Since $\alpha_i \leq 1$, $x_i^*(\theta) = \alpha_i^2\theta_i^2/(\alpha_i^2\theta_i^2 + \theta_{i'}^2) \leq \theta_i^2/(\theta_i^2 + \theta_{i'}^2) \leq \theta_i^2/(\theta_i^2 + \min_{k\neq i}\theta_k^2)$. The equality follows from the optimization in Step 3 for agent i'. Note that if we allow more agents to obtain positive amounts of x, then the denominator in the fraction $\alpha_i^2\theta_i^2/(\alpha_i^2\theta_i^2 + \theta_{i'}^2)$ will increase. As a result, the allocation to agent i of good x will decrease further. Hence, we have a contradiction to our assumption that $x_i^*(\theta) > \theta_i^2/(\theta_i^2 + \min_{k\neq i}\theta_k^2)$. This proves Step 4 and condition P2 of the proposition. □

Proof of Theorem 2. We only consider the case $\underline{\mathbb{D}}_\alpha^q \subset \mathbb{D}$. The proof for the case $\overline{\mathbb{D}}_\alpha^q \subset \mathbb{D}$ is similar and can be found in Goswami (2011a).

Let $I = \{i, j\}$, $\alpha > 0$, and $F : [\underline{\mathbb{D}}_\alpha^q]^2 \to \Delta$ be a strategy-proof and Pareto-efficient SCF. We will show that either agent i or agent j is a dictator.

The following claim is an important intermediate step.

Claim 1. *Let $F : [\underline{\mathbb{D}}_\alpha^q]^2 \to \Delta$ be a strategy-proof and Pareto-efficient SCF. Consider a profile $(\theta_i^*, \theta_j^*) \in [\underline{\mathbb{D}}_\alpha^q]^2$ such that $x_i(\theta_i^*, \theta_j^*) = (\theta_i^*)^2/((\theta_i^*)^2 + (\theta_j^*)^2)$ and $y_i(\theta_i^*, \theta_j^*) = 0$. Then there exists $[a_i, b_i] \times [a_j, b_j] \subset [\underline{\mathbb{D}}_\alpha^q]^2$ such that $x_j(\theta_i, \theta_j) = \theta_j^2/(\theta_i^2 + \theta_j^2)$ for all $(\theta_i, \theta_j) \in [a_i, b_i] \times [a_j, b_j]$.*

Proof. We proceed in three steps.

Step 1. Given $x_i(\theta_i^*, \theta_j^*) = (\theta_i^*)^2/((\theta_i^*)^2 + (\theta_j^*)^2)$ and $x_j(\theta_i^*, \theta_j^*) = (\theta_j^*)^2/((\theta_i^*)^2 + (\theta_j^*)^2)$, we can choose $0 < a_i < b_i < \theta_i^*$ such that $1 > y_i(a_i, \theta_j^*) > y_i(b_i, \theta_j^*) > 0$ and $x_i(\theta_i, \theta_j^*) = \theta_i^2/(\theta_i^2 + (\theta_j^*)^2)$ for all $(\theta_i, \theta_j^*) \in [a_i, b_i] \times \{\theta_j^*\}$.

To establish Step 1, we use two observations.

O1. For all $\theta_i \in (0, \theta_i^*)$, $y_i(\theta_i, \theta_j^*) \in (0, 1]$.

O2. There exists $\theta_i' \in (0, \theta_i^*)$ such that $0 < y_i(\theta_i', \theta_j^*) < 1$.

To establish O1, suppose there exists a $\theta_i \in (0, \theta_i^*)$ such that $y_i(\theta_i, \theta_j^*) = 0$. By Lemma 1(a), it follows that $x_i(\theta_i^*, \theta_j^*) = (\theta_i^*)^2/((\theta_i^*)^2 + (\theta_j^*)^2) = x_i(\theta_i, \theta_j^*) > \theta_i^2/(\theta_i^2 + (\theta_j^*)^2)$. The inequality follows because $t_i^2/(t_i^2 + (\theta_j^*)^2)$ is increasing in t_i. But $x_i(\theta_i, \theta_j^*) > \theta_i^2/(\theta_i^2 + (\theta_j^*)^2)$ along with $y_i(\theta_i, \theta_j^*) = 0$ is a violation of P2 in Proposition 2. To establish O2, suppose $y_i(\theta_i, \theta_j^*) = 1$ for all $\theta_i < \theta_i^*$. Then, by Lemma 1(a), $x_i(\theta_i, \theta_j^*) = c < (\theta_i^*)^2/((\theta_i^*)^2 + (\theta_j^*)^2)$ for all $\theta_i < \theta_i^*$. Since $t_i^2/(t_i^2 + (\theta_j^*)^2)$ is continuous and increasing in t_i, there exists $\theta_i''' \in (0, \theta_i^*)$ such that $(\theta_i''')^2/((\theta_i''')^2 + (\theta_j^*)^2) = c$. But then for all $\theta_i \in (\theta_i''', \theta_i^*)$, $x_i(\theta_i, \theta_j^*) = c < (\theta_i)^2/((\theta_i)^2 + (\theta_j^*)^2)$ and, given $y_i(\theta_i, \theta_j^*) = 1$, we have a violation of P2 in Proposition 2.

From O1, O2, and Lemma 1, it follows that $y_i(\theta_i, \theta_j^*) = 1 - y_j(\theta_i, \theta_j^*) \in (0, 1)$ for all $\theta_i \in [\theta_i', \theta_i^*)$ and, by Proposition 2, $x_i(\theta_i, \theta_j^*) = (\theta_i)^2/((\theta_i)^2 + (\theta_j^*)^2)$ for all $\theta_i \in [\theta_i', \theta_i^*)$. By setting $\theta_i' = a_i$ and picking any $b_i \in (\theta_i', \theta_i^*)$, we get $0 < a_i < b_i < \theta_i^*$, $x_i(a_i, \theta_j^*) =$

$(a_i)^2/((a_i)^2 + (\theta_j^*)^2) < x_i(b_i, \theta_j^*) = (b_i)^2/((b_i)^2 + (\theta_j^*)^2)$, and, hence, by O1, O2, and Lemma 1(b), we get $1 > y_i(a_i, \theta_j^*) > y_i(b_i, \theta_j^*) > 0$.

Step 2. For each $\hat{\theta}_i \in [a_i, b_i]$, there exists $\hat{\theta}_j^* \in (0, \theta_j^*)$ such that $x_j(\hat{\theta}_i, \theta_j) = \theta_j^2/(\hat{\theta}_i^2 + \theta_j^2)$ and $y_j(\hat{\theta}_i, \theta_j) \in (0, 1)$ for all $(\hat{\theta}_i, \theta_j) \in \{\hat{\theta}_i\} \times [\hat{\theta}_j^*, \theta_j^*]$.

The proof of Step 2 is similar to O2 of Step 1 and, hence, is omitted. For any $\theta_i \in [a_i, b_i]$, define $\theta_j^* - \epsilon(\theta_i) \equiv \sup\{\theta_j \mid y_j(\theta_i, \theta_j) = 1\}$. What Step 2 guarantees is that if for some $\theta_i \in [a_i, b_i]$, this supremum is attained, then $\theta_j^* - \epsilon(\theta_i) < \theta_j^*$ and, hence, $\epsilon(\theta_i) > 0$.

Step 3. Consider θ_i' and θ_i'' in $[a_i, b_i]$ such that $\theta_i' < \theta_i''$, and assume that the suprema $\theta_j^* - \epsilon(\theta_i')$ and $\theta_j^* - \epsilon(\theta_i'')$ are attained. Then $\theta_j^* - \epsilon(\theta_i') \leq \theta_j^* - \epsilon(\theta_i'')$.

Suppose Step 3 is false, that is $\theta_i' < \theta_i''$ and $\theta_j^* - \epsilon(\theta_i') > \theta_j^* - \epsilon(\theta_i'')$. At the profile $(\theta_i', \theta_j^* - \epsilon(\theta_i'))$, $y_j(\theta_i', \theta_j^* - \epsilon(\theta_i')) = 1$ and, hence, $y_i(\theta_i', \theta_j^* - \epsilon(\theta_i')) = 0$. Given $y_i(\theta_i', \theta_j^* - \epsilon(\theta_i')) = 0$, for Pareto efficiency, $x_i(\theta_i', \theta_j^* - \epsilon(\theta_i')) \leq (\theta_i')^2/((\theta_i')^2 + (\theta_j^* - \epsilon(\theta_i'))^2)$.

Now, consider $(\theta_i'', \theta_j^* - \epsilon(\theta_i'))$. Lemma 1 guarantees that $y_j(\theta_i'', \theta_j^* - \epsilon(\theta_i')) \in (0, 1)$. To see this, note that from Step 1 we know that $y_j(\theta_i'', \theta_j^*) \in (0, 1)$, $\theta_j^* - \epsilon(\theta_i'') \equiv \sup\{\theta_j \mid y_j(\theta_i'', \theta_j) = 1\}$, and, by assumption, $\theta_j^* - \epsilon(\theta_i') > \theta_j^* - \epsilon(\theta_i'')$. Thus, for $(\theta_i'', \theta_j^* - \epsilon(\theta_i'))$, $y_i(\theta_i'', \theta_j^* - \epsilon(\theta_i')) = 1 - y_j(\theta_i'', \theta_j^* - \epsilon(\theta_i')) \in (0, 1)$, and by Proposition 2, $x_i(\theta_i'', \theta_j^* - \epsilon(\theta_i')) = (\theta_i'')^2/((\theta_i'')^2 + (\theta_j^* - \epsilon(\theta_i'))^2)$. Comparing the allocations of i for $(\theta_i', \theta_j^* - \epsilon(\theta_i'))$ and $(\theta_i'', \theta_j^* - \epsilon(\theta_i'))$, we first find that $y_i(\theta_i'', \theta_j^* - \epsilon(\theta_i')) > 0 = y_i(\theta_i', \theta_j^* - \epsilon(\theta_i'))$. Second, since $t_i^2/(t_i^2 + (\theta_j^* - \epsilon(\theta_i'))^2)$ is increasing in t_i, $x_i(\theta_i'', \theta_j^* - \epsilon(\theta_i')) = (\theta_i'')^2/((\theta_i'')^2 + (\theta_j^* - \epsilon(\theta_i'))^2) > (\theta_i')^2/((\theta_i')^2 + (\theta_j^* - \epsilon(\theta_i'))^2) \geq x_i(\theta_i', \theta_j^* - \epsilon(\theta_i'))$ (since $\theta_i'' > \theta_i'$). Thus, $F_i(\theta_i'', \theta_j^* - \epsilon(\theta_i')) > F_i(\theta_i', \theta_j^* - \epsilon(\theta_i'))$ and i manipulates at $(\theta_i', \theta_j^* - \epsilon(\theta_i'))$ via θ_i''. Hence, Step 3 follows.

From Steps 1, 2, and 3, it follows that if $\sup\{\theta_j \mid y_j(b_i, \theta_j) = 1\}$ is attained, then $x_j(\theta_i, \theta_j) = \theta_j^2/(\theta_i^2 + \theta_j^2)$ for all $(\theta_i, \theta_j) \in [a_i, b_i] \times [\theta_j^* - \epsilon(b_i), \theta_j^*]$. By setting $a_j \equiv \theta_j^* - \epsilon(b_i)$ and $b_j \equiv \theta_j^*$, Claim 1 follows. If $\sup\{\theta_j | y_j(b_i, \theta_j) = 1\}$ is not attained, then $x_i(\theta_i, \theta_j) = \theta_i^2/(\theta_i^2 + \theta_j^2)$ and $x_j(\theta_i, \theta_j) = \theta_j^2/(\theta_i^2 + \theta_j^2)$ for all $\theta_j < \theta_j^*$ and all $\theta_i \in [a_i, b_i]$. By setting $b_j = \theta_j^*$ and picking any $a_j \in (0, \theta_j^*)$, Claim 1 follows. $\triangleleft$

Observe that if an allocation for the profile (θ_i^*, θ_j^*) specified in Claim 1 holds, then its consequence leads to a violation of Proposition 3. Consider a profile (θ_i', θ_j) such that $0 < x_i(\theta_i', \theta_j) = c < (\theta_i')^2/((\theta_i')^2 + \theta_j^2)$ so that $y_i(\theta_i', \theta_j) = 0$. Since $t_i^2/(t_i^2 + \theta_j^2)$ is continuous and increasing in t_i, there exists $\theta_i < \theta_i'$ such that $c = \theta_i^2/(\theta_i^2 + \theta_j^2)$. If $x_i(\theta_i, \theta_j) < c$, then, by Lemma 1(b), $y_i(\theta_i, \theta_j) > 0$ and, hence, by P1 in Proposition 2, $x_i(\theta_i, \theta_j) \geq c$, which is not possible. Hence, we must have $x_i(\theta_i, \theta_j) = c = \theta_i^2/(\theta_i^2 + \theta_j^2)$ and $y_i(\theta_i, \theta_j) = 0$ so that we are in the realm of Claim 1. Thus, if we start from a profile (θ_i', θ_j), where agent i is constrained and getting a positive amount of good x, then there exists a profile (θ_i, θ_j) for which the allocation satisfies the properties of the allocation specified in Claim 1. Therefore, for Claim 1 not to contradict Proposition 3, it must be true that if agent i is constrained, he gets zero amounts of both goods, i.e., j is a dictator in F. A standard strategy proofness argument (which we provide below) implies that j is a dictator for any domain $\mathbb{D}$, where $\overline{\mathbb{D}}_\alpha^q \subset \mathbb{D}$ or $\underline{\mathbb{D}}_\alpha^q \subset \mathbb{D}$.

Pick an arbitrary profile $R \in [\mathbb{D}]^2$. Let $(\theta_i, \theta_j) \in [\underline{\mathbb{D}}^q_\alpha]^2$. If $F_j(R_j, \theta_i) \neq (1,1)$, j will manipulate F at (R_j, θ_i) via θ_j. If $F_i(R) \neq (0,0)$, i will manipulate F at (R_j, θ_i) via R_i. Therefore, j is a dictator in F. □

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