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# Stochastic stability in monotone economies

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This paper extends a family of well known stability theorems for monotone economies to a significantly larger class of models. We provide a set of general conditions for existence, uniqueness, and stability of stationary distributions when monotonicity holds. The conditions in our main result are both necessary and sufficient for global stability of monotone economies that satisfy a weak mixing condition introduced in the paper. Through our analysis, we develop new insights into the nature and causes of stability and instability.

Keywords. Stability, monotonicity, stationary equilibria.

JEL classification. C62, C63.

## 1. Introduction

The stability results for monotone economies developed in Hopenhayn and Prescott (1992, Theorem 2) have become a standard tool for analysis of dynamics and stationary equilibria. For example, Huggett (1993) used their results to study asset distributions in incomplete-market economies with infinitely lived agents. The same results were applied to variants of Huggett's model with features such as habit formation, endogenous labor supply, capital accumulation, and international trade (Díaz et al. 2003, Joseph and Weitzenblum 2003, Pijoan-Mas 2006, Marcet et al. 2007). They were used to study the classical one-sector optimal growth model by Hopenhayn and Prescott (1992), a stochastic endogenous growth model by de Hek (1999), and a small open economy by Chatterjee and Shukayev (2012). They have been used in a wide range of overlapping generations (OLG) models with features such as credit rationing (Aghion and Bolton 1997, Piketty 1997), human capital (Owen and Weil 1998, Lloyd-Ellis 2000, Cardak 2004, Couch and Morand 2005, Hidalgo-Cabrillana 2009), international trade (Ranjan 2001, Das 2006), nonconcave production (Morand and Reffett 2007), and occupational choice

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(Lloyd-Ellis and Bernhardt 2000, Antunes and Cavalcanti 2007). Other well known applications include variants of Hopenhayn and Rogerson's (1993) model of job turnover (Cabrales and Hopenhayn 1997, Samaniego 2008) and Hopenhayn's (1992) model of entry and exit (Cooley and Quadrini 2001, Samaniego 2006).[1]

While Hopenhayn and Prescott's stability results have proved to be a useful set of tools, there are important economic models to which they do not apply. One problem is that they assume compactness of the state space, a condition that fails to hold in many applications. A typical example is macroeconomic models where exogenous productivity follows a first order autoregressive process with unbounded shocks. A restriction to compact state spaces also causes difficulties if, for example, we are studying models of the wealth (or income or firm size) distribution and our interest centers on whether the stationary distribution follows a power law, or if we wish to analyze dynamics of asset prices in a model where tail events can have large impacts on portfolio returns. Furthermore, since compactness of the state space usually requires that the shocks that perturb the state variables must themselves be bounded, it also precludes the use of some standard probability distributions that are routinely used in applications, such as the normal, log normal, exponential, Pareto, Cauchy, gamma, and $t$-distributions. In summary, a restriction to compact state spaces forces researchers to make modeling assumptions for technical rather than economic or empirical reasons, and impinges on their ability to address important economic questions.

In this paper, we show that it is possible to significantly weaken the conditions of earlier monotone stability results. We begin by introducing a mixing condition called *order reversing* that is weaker than the monotone mixing condition used by Hopenhayn and Prescott. We also relax the restriction that the state space be compact and order bounded. In this setting, we obtain general conditions for monotone, order reversing processes to attain global stability. The conditions are also necessary, and, hence, we are able to fully characterize global stability for monotone economies that satisfy this very weak mixing condition.

Our discussion of mixing extends a long line of earlier results, as the general concept of mixing plays a key role in the theory of stability of stochastic systems. In essence, mixing refers to movement of the state variable through "most" parts of the state space. For example, irreducibility of finite Markov chains is a classical mixing concept, the definition of which is that any point in the state space can eventually be visited from any other point. Models with a low degree of mixing can become trapped in certain regions of the state space. In such a setting, initial conditions can have permanent effects. In terms of stationary outcomes, the permanent effect of initial conditions can lead to multiple stationary distributions in distinct "absorbing" subsets of the state space. Such outcomes violate the definition of global stability.

[1]It should be noted here that Hopenhayn and Prescott's results were preceded by similar results in Bhattacharya and Lee (1988). Hopenhayn and Prescott's results were obtained independently, rely on different techniques, and provide a useful separate treatment of existence. On the other hand, Bhattacharya and Lee use a more general notion of mixing and show exponential convergence rates in their stability results. More details are given in the literature review.

On the other hand, when mixing is strong, the state travels widely through the state space regardless of where it starts and, as a result, the effects of initial conditions tend to die out. Because mixing reduces the importance of initial conditions, it tends to make initial differences smaller over time. On a mathematical level, smaller differences can be translated into smaller *distances* in some appropriate metric. For this reason, mixing properties tend to be related to contraction mapping arguments (because contractions are operators that map distinct points closer together). In fact, at least for the Euclidean case, the existence, uniqueness and stability results of Hopenhayn and Prescott (1992) can all be obtained simultaneously via Banach's contraction mapping theorem (see Bhattacharya and Lee 1988).

These results are simple, elegant, and powerful, and, for infinite state spaces, the monotone mixing conditions are often easier to check and more likely to be satisfied than classical irreducibility conditions. At the same time, there is a sense in which the strength of these results is also their weakness: Strong results usually require strong assumptions, and this case is no exception. In particular, the uniform contraction rate present in the Banach contraction theorem requires that some minimal positive rate of mixing occurs from any point in the state space. This works well in compact state spaces, where the minimum is usually attained at the extremities of the state space. But when the state space is not compact, the same approach tends to break down.

In this paper, we develop contraction-type arguments driven by our weak mixing assumption, but without requiring the uniformity of the previous results. Without uniformity, Banach's theorem does not apply, so we develop a new fixed point result that gives existence, uniqueness, and stability by combining a weak notion of contraction with order-theoretic and topological constructs. Doing so frees us from the more restrictive compactness and uniform mixing assumptions found in Hopenhayn and Prescott (1992).

Some of the benefits of weakening these assumptions were discussed above. To put these ideas in a more applied light, suppose that we have a model with unbounded shocks and, as a result, the state space is unbounded. It is possible to truncate these shocks, thereby creating a version of the model with a compact state space. One immediate problem is that we are approximating in an ad hoc manner, and this approximation may change qualitative and quantitative features of the model. A second problem is that the stability problem might now be significantly harder, because we have reduced the amount of mixing in the model.[2] A third problem is that estimation might be more difficult because the shock distribution, which determines the likelihood function, has been transformed from a standard to a nonstandard distribution. A fourth problem is that certain questions become more difficult to address, such as whether large shocks

[2]As an example of how truncation might interfere with mixing, consider a model where, in the absence of shocks, the dynamics would yield multiple locally stable steady states (see., e.g., Azariadis and Drazen 1990). When shocks are present, these points will still be locally attracting "on average." Unless the shocks are sufficiently large, the state might not be able to escape from their basin of attraction. (Stokey et al. 1989, p. 381, provide an intuitive description of this idea.) In this case, initial conditions have permanent effects and global stability fails because of insufficient mixing.

are destabilizing or whether the tails of the stationary distribution have certain properties. For all of these reasons, it may be preferable to work with the original model. As we show below, this can be done in a natural and convenient way.

Our results are illustrated in two applications: a model of renewable resource exploitation and an overlapping generations model of the wealth distribution. In both applications, we illustrate situations where the conditions of our theorem are satisfied while those of previous results are not. In fact, no current theory from the literature on Markov processes can be used to obtain stability in these cases.

Concerning related literature, the stability of monotone economic models with the Markov property has been studied by Razin and Yahav (1979), Stokey et al. (1989), Hopenhayn and Prescott (1992), Bhattacharya and Majumdar (2001), and Szeidl (2012). Studies of monotone Markov theory in the mathematical literature include Dubins and Freedman (1966), Yahav (1975), Bhattacharya and Lee (1988), Heikkila and Salonen (1996), Chueshov (2002), and Bhattacharya et al. (2010).

The studies of Yahav (1975), Razin and Yahav (1979), Stokey et al. (1989), and Hopenhayn and Prescott (1992) all use a certain monotone mixing condition suitable for compact, order bounded state spaces. As clarified below, our order reversing condition is weaker than this monotone mixing condition. The papers by Dubins and Freedman (1966), Bhattacharya and Lee (1988), Bhattacharya and Majumdar (2001), and Bhattacharya et al. (2010) analyze stability in the monotone setting via a mixing condition called *splitting*. Our order reversing condition is also weaker than splitting. At the same time, the literature on splitting contains important results not treated in this paper.

The paper by Szeidl (2012) is, like our paper, a direct extension of the Hopenhayn–Prescott stability results for monotone economies. It studies processes that satisfy a certain "weak mixing" condition. Our order reversing condition is weaker than this weak mixing condition, and the main stability results in Szeidl's paper are special cases of Theorems 1 and 2 below. Nonetheless, Szeidl's paper contains many thoughtful arguments, and his weak mixing condition can be viewed as a useful way to establish our concept of order reversing.

The work by Chueshov (2002) is a contribution to random dynamical systems theory. It permits unbounded state spaces, but requires continuity throughout, and uses a set of sufficient conditions not directed toward economic applications. Finally, Heikkila and Salonen (1996) provide some extensions to the existence component of Hopenhayn and Prescott's results that are applicable in noncompact state spaces, but do not treat global stability.

The rest of the paper is structured as follows. Section 2 reviews some basic definitions and introduces the concept of order reversing. Section 3 states the main results and compares them to the existing literature. Section 4 gives applications and Section 5 concludes. Proofs can be found in the Technical Appendix.

## 2. Preliminaries

At each time $t = 0, 1, \ldots$, the state of the economy is described by a point $X_t$ in topological space $S$. The space $S$ is equipped with its Borel sets $\mathscr{B}_S$ and a closed partial order $\leq$.

An order interval of $S$ is a set of the form $[a, b] := \{x \in S : a \leq x \leq b\}$. A function $f : S \to \mathbb{R}$ is called *increasing* if $f(x) \leq f(y)$ whenever $x \leq y$. A subset $B$ of $S$ is called *order bounded* if there exists an order interval $[a, b] \subset S$ with $B \subset [a, b]$. In addition, $B$ is called *increasing* if its indicator function $\mathbb{1}_B$ is increasing, and called *decreasing* if $\mathbb{1}_B$ is decreasing.

To simplify terminology, we often use the word "distribution" to mean "probability measure on $(S, \mathscr{B}_S)$." The set of all probability measures on $(S, \mathscr{B}_S)$ will be denoted by $\mathscr{P}_S$. We let $\mathrm{cb}\, S$ denote the continuous bounded functions from $S$ to $\mathbb{R}$ and let $\mathrm{ib}\, S$ denote the set of increasing bounded measurable functions from $S$ to $\mathbb{R}$. We adopt the standard definitions of convergence in distribution and stochastic domination: Given sequence $\{\mu_n\}_{n=0}^{\infty}$ in $\mathscr{P}_S$, we say that $\mu_n$ *converges to* $\mu$ and write $\mu_n \to \mu_0$ if $\int h\, d\mu_n \to \int h\, d\mu_0$ for all $h \in \mathrm{cb}\, S$. We say that $\mu_2$ *stochastically dominates* $\mu_1$ and write $\mu_1 \preceq \mu_2$ if $\int h\, d\mu_1 \leq \int h\, d\mu_2$ for all $h \in \mathrm{ib}\, S$.

Following Hopenhayn and Prescott (1992), we assume that $S$ is a normally ordered Polish space.[3] Hopenhayn and Prescott assume in addition that $S$ is compact, with least element $a$ and greatest element $b$. (A point $a$ is called a *least element* of $S$ if $a \in S$ and $a \leq x$ for all $x \in S$. A point $b$ is called a *greatest element* of $S$ if $b \in S$ and $x \leq b$ for all $x \in S$.) Since we wish to include more general state spaces such as $\mathbb{R}^n$, we make the weaker assumption that a subset of $S$ is compact if and only if it is closed and order bounded. This is obviously the case in Hopenhayn and Prescott's setting, where all subsets of $S$ are order bounded and any closed subset is compact. It also holds for $S = \mathbb{R}^n$ with its standard partial order, since order boundedness is then equivalent to boundedness. In addition, it holds in common state spaces such as $\mathbb{R}^n_+$ or $\mathbb{R}^n_{++}$, or in any set of the form $I_1 \times \cdots \times I_n \subset \mathbb{R}^n$, where each $I_i$ is an open, closed, half-open, or half-closed interval in $\mathbb{R}$.[4]

### 2.1 *Markov properties*

Throughout the paper, we suppose that the model under consideration is time-homogeneous and Markovian. The dynamics of such a model can be summarized by a stochastic kernel $Q$, where $Q(x, B)$ represents the probability that the state moves from $x \in S$ to $B \in \mathscr{B}_S$ in one unit of time. As usual, we require that $Q(x, \cdot) \in \mathscr{P}_S$ for each $x \in S$ and that $Q(\cdot, B)$ is measurable for each $B \in \mathscr{B}_S$. For each $t \in \mathbb{N}$, let $Q^t$ be the $t$th order kernel, defined by

$$Q^1 := Q, \qquad Q^t(x, B) := \int Q^{t-1}(y, B) Q(x, dy) \quad (x \in S, B \in \mathscr{B}_S).$$

The value $Q^t(x, B)$ represents the probability of transitioning from $x$ to $B$ in $t$ steps.

Here and below, $(\Omega, \mathscr{F}, \mathbb{P})$ denotes a fixed probability space on which all random variables are defined, and $\mathbb{E}$ is the corresponding expectations operator. Given $\mu \in \mathscr{P}_S$

[3] A *Polish space* is a separable and completely metrizable topological space. The space $(S, \leq)$ is *normally ordered* if, given any closed increasing set $I$ and closed decreasing set $D$ with $I \cap D = \varnothing$, there exists an $f$ in $\mathrm{ib}\, S \cap \mathrm{cb}\, S$ such that $f(x) = 0$ for all $x \in D$ and $f(x) = 1$ for all $x \in I$.

[4] A simple example that does not satisfy our assumptions is $S = (0, 1) \cup (2, 3)$. In this case, the order interval $[0.5, 2.5]$ is closed and order bounded but not compact.

and stochastic kernel $Q$, an $S$-valued stochastic process $\{X_t\}_{t\in\mathbb{Z}_+}$ is called $(Q,\mu)$-*Markov* if $X_0$ has distribution $\mu$ and $Q(x,\cdot)$ is the conditional distribution of $X_{t+1}$ given $X_t = x$.[5] If $\mu$ is the distribution $\delta_x \in \mathscr{P}_S$ concentrated on $x \in S$, we call $\{X_t\}$ $(Q,x)$-*Markov*. We call $\{X_t\}$ $Q$-*Markov* if $\{X_t\}$ is $(Q,\mu)$-Markov for some $\mu \in \mathscr{P}_S$.

Example 1. Many economic models result in processes for the state variables represented by nonlinear, vector-valued stochastic difference equations. As a generic example, consider the $S$-valued process

$$X_{t+1} = F(X_t, \xi_{t+1}), \quad \{\xi_t\} \overset{\text{IID}}{\sim} \phi, \tag{1}$$

where $\{\xi_t\}$ takes values in $Z \subset \mathbb{R}^m$, the function $F : S \times Z \to S$ is measurable, and $\phi$ is a probability measure on the Borel sets of $Z$. Let $Q_F$ be the kernel

$$Q_F(x,B) := \mathbb{P}\{F(x,\xi_t) \in B\} = \phi\{z \in Z : F(x,z) \in B\}. \tag{2}$$

Then $\{X_t\}$ in (1) is $Q_F$-Markov.[6] ♢

For each $Q$, we define two operators, sometimes called the left and the right Markov operators. The left Markov operator maps $\mu \in \mathscr{P}_S$ into $\mu Q \in \mathscr{P}_S$, where

$$(\mu Q)(B) := \int Q(x,B)\mu(dx) \quad (B \in \mathscr{B}_S).$$

The right Markov operator maps bounded measurable function $h : S \to \mathbb{R}$ into bounded measurable function $Qh$, where

$$(Qh)(x) := \int h(y)Q(x,dy) \quad (x \in S).$$

The interpretation of the left Markov operator $\mu \mapsto \mu Q$ is that it shifts the distribution for the state forward by one time period. In particular, if $\{X_t\}$ is $(Q,\mu)$-Markov, then $\mu Q^t$ is the distribution of $X_t$. The interpretation of the right Markov operator $h \mapsto Qh$ is that $(Q^t h)(x)$ is the expectation of $h(X_t)$ given $X_0 = x$. If $Q_F$ is the kernel in (2), then $(Q_F h)(x) = \int h[F(x,z)]\phi(dz)$. Also, given any $x \in S$, $B \in \mathscr{B}_S$, and $t \in \mathbb{N}$, the $t$th order kernel and the left and right Markov operators are related by $Q^t(x,B) = (\delta_x Q^t)(B) = (Q^t \mathbb{1}_B)(x)$. Here $\mathbb{1}_B$ is the indicator function of $B$.

A sequence $\{\mu_n\} \subset \mathscr{P}_S$ is called *tight* if, for all $\epsilon > 0$, there exists a compact $K \subset S$ such that $\mu_n(K \setminus S) \le \epsilon$ for all $n$. A stochastic kernel $Q$ is called *bounded in probability* if the sequence $\{Q^t(x,\cdot)\}_{t\ge 0}$ is tight for all $x \in S$. If $\mu^* \in \mathscr{P}_S$ and $\mu^* Q = \mu^*$, then $\mu^*$ is called *stationary* (or *invariant*) for $Q$. If $Q$ has a unique stationary distribution $\mu^*$ in $\mathscr{P}_S$ and, in addition, $\mu Q^t \to \mu^*$ as $t \to \infty$ for all $\mu \in \mathscr{P}_S$, then $Q$ is called *globally stable*. In this case, $\mu^*$ is naturally interpreted as the long-run equilibrium of the economic system. If $\mu^*$ is

[5]More formally, $\mathbb{P}[X_{t+1} \in B | \mathscr{F}_t] = Q(X_t, B)$ almost surely for all $B \in \mathscr{B}_S$, where $\mathscr{F}_t$ is the $\sigma$-algebra generated by the history $X_0, \dots, X_t$.

[6]Although the process (1) is only first order, models that include higher order lags of the state and shock process can be rewritten in the form of (1) by redefining the state variables.

stationary, then any $(Q, \mu^*)$-Markov process $\{X_t\}$ is strict-sense stationary with $X_t \sim \mu^*$ for all $t$.

If $\mu \in \mathscr{P}_S$ and $\mu Q \preceq \mu$, then $\mu$ is called *excessive*. If $\mu \preceq \mu Q$, then $\mu$ is called *deficient*. If $Q$ satisfies $\mu Q \preceq \mu' Q$ whenever $\mu \preceq \mu'$, then $Q$ is called *increasing*.[7] It is, in fact, sufficient to check that $Q(x, \cdot) \preceq Q(x', \cdot)$ whenever $x \leq x'$. A third equivalent condition is that $Qh \in \mathrm{ib}\, S$ whenever $h \in \mathrm{ib}\, S$. If, on the other hand, $Qh \in \mathrm{cb}\, S$ whenever $h \in \mathrm{cb}\, S$, then $Q$ is called *Feller*.

Remark 1. Let $Q$ be an increasing stochastic kernel. If $A$ is an increasing set, then $x \mapsto Q(x, A)$ is increasing. If $A$ is a decreasing set, then $x \mapsto Q(x, A)$ is decreasing.

Remark 2. If $S$ has a least element $a$, then $\delta_a$ is deficient for any kernel $Q$, because $\delta_a \preceq \mu$ for every $\mu \in \mathscr{P}_S$ and, hence, $\delta_a \preceq \delta_a Q$. Similarly, if $S$ has a greatest element $b$, then $\delta_b$ is excessive for $Q$.

Remark 3. Let $F$ and $Q_F$ be as in Example 1. If $x \mapsto F(x, z)$ is increasing, then $Q_F$ is increasing. If $x \mapsto F(x, z)$ is continuous, then $Q_F$ is Feller.

### 2.2 *Order reversing*

Next we introduce our order-theoretic mixing condition. Let $Q$ be a stochastic kernel on $S$. We call $Q$ *order reversing* if, for any given $x$ and $x'$ in $S$ with $x \geq x'$, and any independent $Q$-Markov processes $\{X_t\}$ and $\{X'_t\}$ starting at $x$ and $x'$, respectively, there exists a $t \in \mathbb{N}$ with $\mathbb{P}\{X_t \leq X'_t\} > 0$. In other words, there exists a point in time at which the initial ordering is reversed with positive probability.

Example 2. Suppose we are studying a model of household wealth dynamics. Informally, the model is order reversing if, for two households receiving idiosyncratic shocks from the same distribution, it is the case that, regardless of the initial ranking of the two households according to wealth, the probability that their relative wealth positions will be reversed at some point in time is strictly positive. $\Diamond$

We make three preliminary comments on the definition. First, in verifying order reversing, it is clearly sufficient to check the existence of a $t$ with $\mathbb{P}\{X_t \leq X'_t\} > 0$ for *arbitrary* pair $x, x' \in S$. Often this is just as easy, and much of the following discussion proceeds accordingly. Second, once $x$ and $x'$ are chosen, there are many pairs of independent $Q$-Markov processes $\{X_t\}$ and $\{X'_t\}$ starting at $x$ and $x'$, respectively, just as there are many random variables that have a given distribution $F$. It is enough to check that there exists a $t \in \mathbb{N}$ with $\mathbb{P}\{X_t \leq X'_t\} > 0$ for any one of these pairs $\{X_t\}$ and $\{X'_t\}$, because

[7] Many examples of models with increasing kernels were given in the Introduction. Other examples not discussed there include various infinite horizon optimal growth models with features such as irreversible investment, renewable resources, distortions, and capital-dependent utility. Increasing kernels are also found in stochastic OLG models in addition to those mentioned previously, such as models with limited commitment, and in a variety of stochastic games. See, for example, Olson (1989), Amir (2002), Gong et al. (2010), Balbus et al. (2012), and Mirman et al. (2008).

all such pairs have the same joint distribution. Third, it is not entirely clear from the definition given above that order reversing is a property of $Q$ alone. This fact is clarified in the Technical Appendix, where we give an alternative, more formal, definition.

In Remark 4 below, we show that for any increasing kernel $Q$, order reversing is weaker than the monotone mixing condition (MMC) used in Hopenhayn and Prescott (1992). For increasing kernels, order reversing is also weaker than the splitting condition used by Bhattacharya and Majumdar (2001), the "weak mixing" condition used by Szeidl (2012), and the "order mixing" condition used by Kamihigashi and Stachurski (2012). The proofs are quite straightforward, and details are available from the authors.

Remark 4. Let $S$ be a compact metric space with least element $a$ and greatest element $b$, and let $Q$ be an increasing kernel on $S$. In this setting, $Q$ is said to satisfy the MMC whenever

$$\exists \bar{x} \in S \text{ and } k \in \mathbb{N} \quad \text{such that} \quad Q^k(a, [\bar{x}, b]) > 0 \text{ and } Q^k(b, [a, \bar{x}]) > 0. \tag{3}$$

Under these conditions, $Q$ is order reversing: If we start independent $Q$-Markov processes $\{X_t^a\}$ and $\{X_t^b\}$ at $a$ and $b$, respectively, then (3) implies the order reversal $X_k^b \leq X_k^a$ occurs at time $k$ with positive probability. Since $Q$ is increasing, closer initial conditions only make this event more likely.[8]

Remark 5. To see that order reversing is strictly weaker than the MMC, consider the stochastic kernel $Q(x, B) = \mathbb{P}\{\rho x + \xi_t \in B\}$ on $S = \mathbb{R}$ associated with the linear Gaussian model

$$X_{t+1} = \rho X_t + \xi_{t+1}, \quad \{\xi_t\} \overset{\text{IID}}{\sim} N(0, 1). \tag{4}$$

The MMC cannot be applied here, because $S = \mathbb{R}$ and, hence, the state possesses neither a least nor a greatest element. On the other hand, $Q$ is order reversing. To see this, fix $(x, x') \in \mathbb{R}^2$ and take a second $Q$-Markov process $X'_{t+1} = \rho X'_t + \xi'_{t+1}$, where $X'_0 = x'$, $X_0 = x$, and $\{\xi_t\}$ and $\{\xi'_t\}$ are independent and identically distributed (IID), standard normal, and independent of each other. The condition $\mathbb{P}\{X_t \leq X'_t\} > 0$ is satisfied with $t = 1$, because

$$\mathbb{P}\{X_1 \leq X'_1\} = \mathbb{P}\{\rho x + \xi_1 \leq \rho x' + \xi'_1\} = \mathbb{P}\{\xi_1 - \xi'_1 \leq \rho(x' - x)\}.$$

Since $\xi_1 - \xi'_1$ is Gaussian, this probability is strictly positive.

## 3. Results

We can now state our main results, which concern stability of increasing, order reversing stochastic kernels.

[8] To be precise, let $\bar{x}$ and $k$ be as in (3). Fix $x, x' \in S$, and let $\{X_t\}$ and $\{X'_t\}$ be independent, $(Q, x)$-Markov, and $(Q, x')$-Markov, respectively. By independence and $\{X_k \leq \bar{x} \leq X'_k\} \subset \{X_k \leq X'_k\}$, we have $\mathbb{P}\{X_k \leq \bar{x}\}\mathbb{P}\{\bar{x} \leq X'_k\} = \mathbb{P}\{X_k \leq \bar{x} \leq X'_k\} \leq \mathbb{P}\{X_k \leq X'_k\}$. But $\mathbb{P}\{\bar{x} \leq X'_k\} = Q^k(x, [a, \bar{x}])$ and $\mathbb{P}\{X_k \leq \bar{x}\} = Q^k(x, [\bar{x}, b])$ are strictly positive by (3) and Remark 1. Hence, $Q$ is order reversing.

### 3.1 *Global stability*

Our first result extends Hopenhayn and Prescott's stability theorem to a broader class of models. It also characterizes the set of increasing order reversing kernels that are globally stable. The proof is given in the Technical Appendix.

THEOREM 1. *Let $Q$ be a stochastic kernel on $S$ that is both increasing and order reversing. Then $Q$ is globally stable if and only if*

*(i) $Q$ is bounded in probability and*

*(ii) $Q$ has either a deficient or an excessive distribution.*

REMARK 6. In terms of sufficient conditions for global stability, the order reversing assumption cannot be omitted, even for *existence* of a stationary distribution. In particular, there exist increasing kernels that are bounded in probability and possess an excessive or deficient distribution, but have no stationary distribution.[9] On the other hand, regarding necessity, neither monotonicity nor order reversal is used in the proof. Global stability alone implies conditions (i) and (ii).

To see that the conditions of Theorem 1 are weaker than those of Hopenhayn and Prescott's stability theorem (Hopenhayn and Prescott 1992, Theorem 2), suppose as they do that $S$ is a compact metric space with least element $a$ and greatest element $b$, and that $Q$ is an increasing kernel that satisfies the MMC. The conditions of Theorem 1 then hold. First, $Q$ is increasing by assumption. Second, $Q$ is order reversing, as shown in Remark 4. Third, $Q$ is bounded in probability, since $S$ is compact and, hence, $\{Q^t(x, \cdot)\}$ is always tight. Fourth, $Q$ has a deficient distribution because $S$ has a least element (see Remark 2).

To see that the conditions of Theorem 1 are strictly weaker than those of Hopenhayn and Prescott, consider the linear Gaussian model (4) with $\rho \in [0, 1)$. Here the Gaussian shocks force us to choose the state space $S = \mathbb{R}$, which is not compact, and the Hopenhayn–Prescott theorem in its original formulation cannot be applied. On the other hand, all the conditions of Theorem 1 are satisfied.[10] (Of course this is an extremely simple example. Nontrivial applications are presented in Section 4.)

Regarding the proof of Theorem 1, boundedness in probability and existence of an excessive or deficient distribution generalize Hopenhayn and Prescott's assumption that $S$ is compact and has a least and greatest element. As Hopenhayn and Prescott show, if $S$ is compact and has a least and greatest element, then the Knaster–Tarski fixed point theorem implies that every increasing stochastic kernel has a stationary distribution.

[9] An example is the kernel $Q$ associated with the deterministic process on $S = \mathbb{R}_+$ defined by $X_{t+1} = 1/2 + \sum_{n=0}^{\infty} \mathbb{1}\{n \le X_t < n+1\}(n + (X_t - n)/2)$. It is easy to check that $X_{t+1} > X_t$ with probability 1, and, hence, $X_{t+1}$ and $X_t$ can never have the same distribution. On the other hand, $Q$ is increasing, bounded in probability (because each interval $[n, n+1)$ is absorbing), and has the deficient distribution $\delta_0$ (cf. Remark 2).

[10] That the model is order reversing was shown in Remark 4. Monotonicity follows from Remark 3. Boundedness in probability is shown below. For existence of a $\mu$ with $\mu \preceq \mu Q$, we can take $\mu = N(0, (1 - \rho^2)^{-1})$.

Adding the MMC then yields uniqueness and global stability. In our setting, the same arguments cannot be applied. As Remark 6 shows, our mixing condition is needed even for existence. Our proof of Theorem 1 is more akin to a contraction mapping argument than to the Knaster–Tarski fixed point theorem.

We make two final comments. First, even when the conditions of Theorem 1 hold, they may not be trivial to verify. In Section 3.2, we provide a variety of techniques for checking the conditions. Further illustration is given in the applications. Second, there is no continuity requirement in Theorem 1. However, in many applications, the kernel $Q$ will have the Feller property (see Remark 3). If $Q$ is Feller, then condition (ii) can be omitted. Since this result is likely to be useful, we state it as a second theorem.

Theorem 2. *Let $Q$ be increasing, order reversing, and Feller. Then $Q$ is globally stable if and only if $Q$ is bounded in probability.*

### 3.2 *Verifying the conditions*

Theorem 1 requires that $Q$ is increasing, order reversing, bounded in probability, and possesses an excessive or deficient distribution. A sufficient condition for $Q$ to be increasing was given in Remark 3. In this section, we present a number of sufficient conditions for the remaining properties.

3.2.1 *Checking boundedness in probability* Boundedness in probability is a standard condition in the Markov process literature. As is well known, if $Q$ is a stochastic kernel on either $S = \mathbb{R}^n$ or $S = \mathbb{R}^n_+$, then $Q$ is bounded in probability whenever $\sup_t \mathbb{E}\|X_t\| < \infty$ for any $(Q, x)$-Markov process $\{X_t\}$. (The norm $\|\cdot\|$ can be any norm on $\mathbb{R}^n$.) For example, it is easy to show by this method that the process (4) is bounded in probability whenever $|\rho| < 1$. More systematic approaches to establishing boundedness in probability can be found in Meyn and Tweedie (2009, Chapter 12).

3.2.2 *Finding excessive and deficient distributions* Condition (ii) of Theorem 1 requires existence of either an excessive or a deficient distribution. If $S$ has a least element or a greatest element, then the condition always holds (see Remark 2). However, there are many settings where $S$ has neither ($S = \mathbb{R}^n$ and $S = \mathbb{R}^n_{++}$ are obvious examples), and the existence is harder to verify. In this case, one can work more carefully with the definition of the model to construct excessive and deficient distributions. One example is Zhang (2007), who constructs such distributions for the stochastic optimal growth model. However, it is useful to have a more systematic method that is relatively straightforward to check in different applications. To this end, we provide the following result. In the result, the statement $Q \preceq Q'$ means that $\mu Q \preceq \mu Q'$ for all $\mu \in \mathscr{P}_S$.

Proposition 1. *Let $Q$ be a stochastic kernel on $S$. If there exists another kernel $Q'$ such that $Q'$ is Feller, bounded in probability, and $Q \preceq Q'$ (resp., $Q' \preceq Q$), then $Q$ has an excessive (resp., deficient) distribution.*

An illustration of how the proposition can be used is given in Section 4.1.

3.2.3 *Checking the order reversing property* In this section, we give sufficient conditions for order reversing. To state them, we introduce two new definitions: We call kernel $Q$ on $S$ *upward reaching* if, given any $(Q, x)$-Markov process $\{X_t\}$ and $c$ in $S$, there exists a $t \in \mathbb{N}$ such that $\mathbb{P}\{X_t \geq c\} > 0$. We call $Q$ *downward reaching* if, given any $(Q, x)$-Markov process $\{X_t\}$ and $c$ in $S$, there exists a $t \in \mathbb{N}$ such that $\mathbb{P}\{X_t \leq c\} > 0$. For example, the linear Gaussian process in (4) is both upward and downward reaching: If we fix $x, c$ in $S = \mathbb{R}$ and take $t = 1$, then $\mathbb{P}\{X_1 \leq c\} = \mathbb{P}\{\rho x + \xi_1 \leq c\} = \mathbb{P}\{\xi_1 \leq c - \rho x\}$. This term is strictly positive because the support of $\xi_t$ is all of $\mathbb{R}$. Hence, $Q$ is downward reaching. The proof of upward reaching is similar.

Proposition 2. *Suppose that $Q$ is bounded in probability. If $Q$ is either upward or downward reaching, then $Q$ is order reversing.*

It follows that the statements in Theorems 1 and 2 remain valid if order reversing is replaced by either upward or downward reaching.

## 4. Applications

We now turn to more substantial applications of the results described above.

### 4.1 *Optimal exploitation of a renewable resource*

Consider an elementary model of renewable resource exploitation, where a single planner maximizes $\mathbb{E} \sum_{t=0}^{\infty} \beta^t u(c_t)$ subject to $y_{t+1} = \xi_t f(y_t - c_t)$. Here $y_t$ is the stock of the resource, $c_t$ is consumption, all variables are nonnegative, and $\{\xi_t\} \overset{\text{IID}}{\sim} \phi$. For simplicity, we assume that $u$ is bounded with $u' > 0$, $u'' < 0$, and $u'(0) = \infty$. The growth function $f$ for the resource is assumed to satisfy $f(0) = 0$, $f' > 0$, $f'(0) = \infty$, and $f'(\infty) = 0$. Since $f$ is biologically determined, we do not assume it is concave. To study dynamics, we take $y_t$ as the state variable and consider the optimal process $y_{t+1} = \xi_t f(y_t - \sigma(y_t))$, where $\sigma(\cdot)$ is an optimal consumption policy. Let $Q$ be the corresponding stochastic kernel. For the state space, we take $S = (0, \infty)$. Zero is deliberately excluded from $S$ so that any stationary distribution on $S$ is automatically nontrivial. Models similar to the one described above have been studied by various authors, including Nishimura and Stachurski (2005), Kamihigashi (2007), and Mitra and Roy (2006).

Regarding the shock process $\{\xi_t\}$, we permit the occurrence of arbitrarily bad shocks. In the natural resource setting, large negative shocks can take the form of a sudden introduction of pollutants (e.g., oil spills), the arrival of invasive species, disease, extreme droughts, earthquakes, fires, storms, and floods. Such low-probability events can have catastrophic environmental and financial consequences. The importance of modeling these left-tail events has been highlighted in a number of recent studies, including Clarke and Reed (1994), Yin and Newman (1996), Brock and Carpenter (2010), and Weitzman (2011). For our purposes, we will assume that $\mathbb{P}\{\xi_t \leq z\} > 0$ for all $z \in S$, and that $\mathbb{E}\xi_t < \infty$ and $\mathbb{E}(1/\xi_t) < \infty$.

For this model, one difficulty for stability analysis is that $f$ is not concave and, hence, the optimal policy may be discontinuous. As a result, the stochastic kernel $Q$ is not

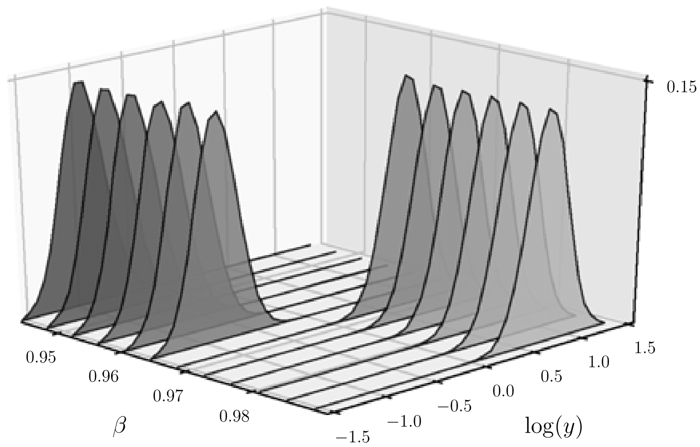


FIGURE 1. Stationary distributions as a function of $\beta$.

Feller. Moreover, without additional assumptions, the MMC does not apply, $Q$ is not irreducible, the splitting condition fails, the model is not an expected contraction, the state space is unbounded, and the standard Harris recurrence conditions are not satisfied.[11] On the other hand, Theorem 1 can easily be applied: $Q$ is still increasing and bounded in probability (see, e.g., Nishimura and Stachurski 2005). Existence of an excessive distribution can be established using Proposition 1.[12] Moreover, the process is downward reaching (and, hence, order reversing; cf. Proposition 2) because if $y_0$ and $\bar{y}$ in $S$ are given, then

$$\mathbb{P}\{y_1 \leq \bar{y}\} = \mathbb{P}\big\{\xi_1 f(y_0 - \sigma(y_0)) \leq \bar{y}\big\} = \mathbb{P}\big\{\xi_1 \leq \bar{y}/f(y_0 - \sigma(y_0))\big\} > 0.$$

Hence, Theorem 1 applies and $Q$ is globally stable.

Figure 1 shows a collection of stationary distributions for $\log y_t$, each one corresponding to a different value of the discount factor $\beta$.[13] For this model, a sudden shift in the optimal harvest policy occurs around $\beta = 0.965$. As a result, a very small difference in the patience of the agent can lead to a large difference in the steady state population of the stock.

[11] For a discussion of irreducibility and Harris recurrence, see Meyn and Tweedie (2009). On the splitting condition, see, e.g., Bhattacharya and Lee (1988) or Bhattacharya and Majumdar (2001).

[12] Since $f' > 0$ and $f'(\infty) = 0$, we can choose positive constants $\alpha, \beta$ with $\alpha \mathbb{E}\xi_t < 1$ and $f(x) \leq \alpha x + \beta$. Now take $G(x, z) := z(\alpha x + \beta)$, so that $F(x, z) := zf(x - \sigma(x)) \leq zf(x) \leq G(x, z)$. Letting $Q_F$ and $Q_G$ be the corresponding kernels, the last inequality implies $Q_F \preceq Q_G$. It can be shown that $Q_G$ is both bounded in probability and Feller (for details, see the working paper version, Kamihigashi and Stachurski 2011), so Proposition 1 applies.

[13] The utility function is $u(x) = 1 - \exp(-\theta x^\gamma)$ and production is $f(x) = x^\alpha \ell(x)$, where $\ell$ is the logistic function $\ell(x) = a + (b - a)/(1 + \exp(-c(x - d)))$. The parameters are $a = 1$, $b = 2$, $c = 20$, $d = 1$, $\theta = 0.5$, $\gamma = 0.9$, and $\alpha = 0.5$. The discount factor $\beta$ ranges from 0.945 to 0.99. The shock is log normal $(-0.1, 0.2)$. For details on the calculations, including full justification of consistency, see the working paper version (Kamihigashi and Stachurski 2011).

### 4.2 *Wealth distribution dynamics*

Next we consider an OLG model of wealth distribution. Following the existing literature, we introduce persistence in inequality by assuming that old agents provide financial support to their child (cf., e.g., Antunes and Cavalcanti 2007, Cardak 2004, Couch and Morand 2005, Lloyd-Ellis 2000, Lloyd-Ellis and Bernhardt 2000, Owen and Weil 1998, Piketty 1997, Ranjan 2001). There are idiosyncratic shocks to endowments and production, but no aggregate uncertainty. Unlike much of the literature, we assume that the shocks and, hence, the state space of the model are unbounded. Permitting unbounded shocks in the wealth distribution allows for the investigation of issues of significant current interest for economists. For example, numerous studies have found that wealth holdings across households are strongly concentrated in the upper tail and also relatively concentrated in the left tail (see the survey of Davies and Shorrocks 2000). This leads naturally to modeling with heavy-tailed (and, in particular, unbounded) distributions, such as Pareto or other power law distributions (e.g., Levy and Levy 2003 or Benhabib et al. 2011). Here we simply assume generic unbounded shocks and leave the connection to fat tails for future research.

In the model, agents live for two periods and consume only when old. Households consist of one old agent and one child. There is a unit mass of such households indexed by $i \in [0, 1]$. In each period $t$, the old agent of household $i$ provides financial support $b_t^i$ to her child. The child has the option to become an entrepreneur, investing one unit of the consumption good in a "project" and receiving stochastic output $\theta + \eta_{t+1}^i$ in period $t+1$. Let $k_{t+1}^i \in \{0, 1\}$ be young agent $i$'s investment in the project. If the remainder $b_t^i - k_{t+1}^i$ is positive, then she invests this quantity at the world risk-free rate $R$. If it is negative, then she borrows $k_{t+1}^i - b_t^i$ at the same risk-free rate. Independent of her investment choice, she receives an endowment of $e_{t+1}^i$ units of the consumption good when old. Suppressing the $i$ superscript to simplify notation, her wealth at the beginning of period $t+1$ is, therefore,

$$w_{t+1} = (\theta + \eta_{t+1})k_{t+1} - R(k_{t+1} - b_t) + e_{t+1}. \tag{5}$$

We assume that

$$e_{t+1} = \rho e_t + \epsilon_{t+1}, \quad 0 < \rho < 1. \tag{6}$$

The idiosyncratic shocks $\{\eta_t\}$ and $\{\epsilon_t\}$ are taken to be IID and nonnegative, and $\epsilon_t$ satisfies $\mathbb{P}\{\epsilon_t > z\} > 0$ for any $z \geq 0$. (For example, $\epsilon_t$ might be log normal.) We also assume that $R < \theta$, which implies that becoming an entrepreneur is always profitable, even ex post, and every agent would choose to do so absent additional constraint. Due to a credit market imperfection, however, each agent may borrow only up to a fraction $\lambda \in (0, 1)$ of $\theta + \rho e_t$, the minimum possible value of her old-age income (cf., e.g., Matsuyama 2004). That is,

$$R(k_{t+1} - b_t) \leq \lambda(\theta + \rho e_t). \tag{7}$$

As becoming an entrepreneur is always profitable, young agents do so whenever feasible, implying

$$k_{t+1} = \kappa(b_t, e_t) := \mathbb{1}\{R(1 - b_t) \leq \lambda(\theta + \rho e_t)\}.$$

(Here $\mathbb{1}\{\cdot\}$ is an indicator function.) Let $c_{t+1}$ denote consumption at $t+1$. It is common in the literature on wealth distribution to assume that each agent derives utility from her own consumption and financial support to her child. Following this approach, we assume that young agents maximize $\mathbb{E}_t[c_{t+1}^{1-\gamma} b_{t+1}^{\gamma}]$ subject to (5), (7), and the budget constraint $c_{t+1} + b_{t+1} = w_{t+1}$. Regarding the parameter $\gamma$, we assume that $\gamma R < 1$. Maximization of $c_{t+1}^{1-\gamma} b_{t+1}^{\gamma}$ subject to the budget constraint implies that $b_{t+1} = \gamma w_{t+1}$. Combining this equality, (5), and (6), we obtain

$$b_{t+1} = \gamma[(\theta + \eta_{t+1} - R)\kappa(b_t, e_t) + R b_t + \rho e_t + \epsilon_{t+1}]. \tag{8}$$

Together, (6) and (8) define a Markov process with state vector $X_t := (b_t, e_t)$ taking values in state space $S := \mathbb{R}^2_+$. Let $Q$ denote the corresponding stochastic kernel.[14]

Recalling that $R < \theta$, $\rho \in (0, 1)$ and $\eta_{t+1} \geq 0$, and observing that $\kappa(b_t, e_t)$ is increasing in $(b_t, e_t)$, we can see from (6) and (8) that $(b_{t+1}, e_{t+1})$ is increasing in $(b_t, e_t)$ when the values of the shocks are held fixed. Hence, $Q$ is increasing (cf. Remark 3). On the other hand, (8) is discontinuous in $(b_t, e_t)$, so $Q$ is not Feller.

As far as we are aware, no existing Markov process theory can be used to show that $Q$ is globally stable unless additional conditions are imposed. In contrast, global stability can be obtained in a straightforward way from Theorem 1. To begin, let $m_\eta := \mathbb{E}\eta_t$ and $m_\epsilon := \mathbb{E}\epsilon_t$. To see that $Q$ is bounded in probability, we can take expectations of (6) and iterate backward to obtain

$$\mathbb{E}e_t \leq m_\epsilon/(1-\rho) + \rho^t e_0 \leq m_\epsilon/(1-\rho) + e_0 =: \bar{e} \tag{9}$$

for all $t$. In addition, it follows from (8) and (9) that

$$\mathbb{E}b_{t+1} \leq \gamma[\theta + m_\eta - R + R\mathbb{E}b_t + \bar{e}].$$

Using $\gamma R < 1$ and iterating backward, we obtain the bound

$$\mathbb{E}b_t \leq \gamma[\theta + m_\eta - R + \bar{e}]/(1-\gamma R) + b_0 \tag{10}$$

for all $t$. Together, (9) and (10) imply that $Q$ is bounded in probability.[15] Since $\mathbb{P}\{\epsilon_t > z\} > 0$ for any $z \geq 0$, and since both $b_t$ and $e_t$ can be made arbitrarily large by choosing $\epsilon_t$ sufficiently large (see (6) and (8)), it follows that $Q$ is upward reaching and thus order reversing by Proposition 2. In view of these results and Theorem 1, $Q$ will be globally stable whenever it has a deficient or excessive distribution. Since $(0, 0)$ is a least element for $S$, Remark 2 implies that $Q$ has a deficient distribution and we conclude that $Q$ is globally stable.

Figure 2 shows smoothed histograms that represent the marginal stationary distribution of wealth at two different values of $\lambda$, computed by simulation.[16] The shift in

[14] We do not exclude $(0, 0)$ from the state space since it is not an absorbing state.

[15] The function $V(b, e) = |b| + |e|$ is a norm on $\mathbb{R}^2$. Equations (9) and (10) yield $\sup_t \mathbb{E}[V(b_t, e_t)] \leq \sup_t \mathbb{E}[b_t] + \sup_t \mathbb{E}[e_t] < \infty$, implying boundedness in probability. See Section 3.2.1.

[16] The values of $\lambda$ are 0.57 and 0.58. The other parameters are $\gamma = 0.2$, $R = 1.05$, $\theta = 1.1$, and $\rho = 0.9$. The shock $\epsilon$ is log normal with parameters $\mu = -3$ and $\sigma = 0.1$. The shock $\eta$ is beta with shape parameters 3, 10. For full details on the calculations, see the working paper version (Kamihigashi and Stachurski 2011).

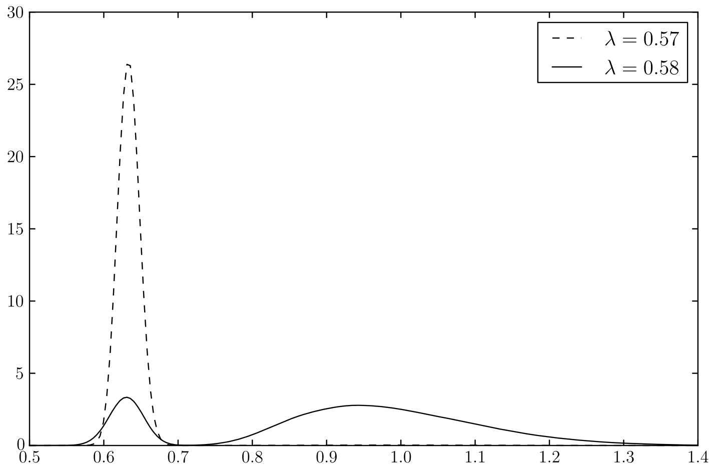


Figure 2. Stationary distribution of wealth.

the densities shows how the distribution of wealth in the stationary equilibrium can be highly sensitive to the value of the borrowing constraint parameter $\lambda$.

## 5. Conclusion

The methods for analyzing stability of monotone processes developed by Hopenhayn and Prescott (1992) and several other authors have become an important tool in economic modeling. In this paper, we introduced a new and very weak mixing condition defined in terms of order, and characterized global stability for monotone models that satisfy our condition. Two applications were discussed.

## Technical Appendix

Before proving Theorem 1, we need some additional results and notation. To begin, let $Q$ be any stochastic kernel on $S$, let $x \in S$, and let $S$-valued stochastic process $\{X_t\}$ be $(Q, x)$-Markov. The joint distribution of $\{X_t\}$ over the sequence space $S^\infty$ will be denoted by $\mathbf{P}_x^Q$. For example, $\mathbf{P}_x^Q\{X_t \in B\} = Q^t(x, B)$ for any $B \subset S$, and $\mathbf{P}_x^Q \bigcup_{t=0}^\infty \{X_t \in B\}$ is the probability that the process ever enters $B$. The symbol $\mathbf{E}_x^Q$ represents the expectations operator corresponding to $\mathbf{P}_x^Q$. For given kernel $Q$, we say that Borel set $B \subset S$ is

- *strongly accessible* if $\mathbf{P}_x^Q \bigcup_{t=0}^\infty \{X_t \in B\} = 1$ for all $x \in S$ and
- *$C$-accessible* if, for all compact $K \subset S$, there exists an $n \in \mathbb{N}$ with $\inf_{x \in K} Q^n(x, B) > 0$.

The following lemma is fundamental to our results, although the proofs are delayed to maintain continuity.

Lemma A.1. *Let $B$ be a Borel subset of $S$. If $Q$ is bounded in probability and $B$ is $C$-accessible, then $B$ is strongly accessible.*

It is helpful to provide a second definition of order reversing. To do so, let

$$\mathbb{G} := \operatorname{graph}(\leq) := \{(y, y') \in S \times S : y \leq y'\},$$

so that $y \leq y'$ if and only if $(y, y') \in \mathbb{G}$. Also, let $Q$ be a stochastic kernel on $S$, and consider the product kernel $Q \times Q$ on $S \times S$ defined by

$$(Q \times Q)((x, x'), A \times B) = Q(x, A)Q(x', B)$$

for $(x, x') \in S \times S$ and $A, B \in \mathscr{B}_S$.[17] The product kernel represents the stochastic kernel of the joint process $\{(X_t, X_t')\}$ when $\{X_t\}$ and $\{X_t'\}$ are independent $Q$-Markov processes. Using this notation, $Q$ is order reversing if and only if

$$\forall x, x' \in S \text{ with } x' \leq x, \exists t \in \mathbb{N} \quad \text{such that} \quad (Q \times Q)^t((x, x'), \mathbb{G}) > 0. \tag{11}$$

This second definition emphasizes the fact that order reversing is a property of the kernel $Q$ alone (taking $S$ and $\leq$ as given). Condition (11) can alternatively be written as

$$\forall x, x' \in S \text{ with } x' \leq x, \exists t \geq 0 \quad \text{such that} \quad \mathbf{P}^{Q\times Q}_{x,x'}\{X_t \leq X_t'\} > 0,$$

where $\{X_t\}$ and $\{X_t'\}$ are independent of each other and are $(Q, x)$-Markov and $(Q, x')$-Markov, respectively. Following Kamihigashi and Stachurski (2012), $Q$ is called *order mixing* if $\mathbf{P}^{Q\times Q}_{x,x'} \bigcup_{t=0}^{\infty}\{X_t \leq X_t'\} = 1$ for all $x, x' \in S$. Put differently, $Q$ is order mixing if $\mathbb{G}$ is strongly accessible for the product kernel $Q \times Q$.

Lemma A.2. *If $Q$ is bounded in probability on $S$, then so is $Q \times Q$ on $S \times S$.*

Lemma A.3. *If $Q$ is increasing and bounded in probability, then $\{\mu Q^t\}$ is tight for all $\mu \in \mathscr{P}_S$.*

Lemma A.4. *If $Q$ is increasing and order reversing, then $\mathbb{G}$ is $C$-accessible for $Q \times Q$.*

Proofs are given at the end of this appendix.

Let us now turn to the proof of Theorem 1. The proof proceeds as follows: First we show that under the conditions of the theorem, $Q$ is order mixing. Using order mixing, we then go on to prove existence of a stationary distribution and global stability.

Lemma A.5. *If $Q$ is increasing, bounded in probability, and order reversing, then $Q$ is order mixing.*

Proof. To show that $Q$ is order mixing, we need to prove that $\mathbb{G}$ is strongly accessible for $Q \times Q$ under the conditions of Theorem 1. Since $Q$ is bounded in probability, $Q \times Q$ is also bounded in probability (Lemma A.2), and, hence, by Lemma A.1, it suffices to show that $\mathbb{G}$ is $C$-accessible for $Q \times Q$. This follows from Lemma A.4. □

[17] Sets of the form $A \times B$ with $A, B \in \mathscr{B}_S$ provide a semi-ring in the product $\sigma$-algebra $\mathscr{B}_S \otimes \mathscr{B}_S$ that also generates $\mathscr{B}_S \otimes \mathscr{B}_S$. Defining the probability measure $Q((x, x'), \cdot)$ on this semi-ring uniquely defines $Q((x, x'), \cdot)$ on all of $\mathscr{B}_S \otimes \mathscr{B}_S$. See, e.g., Dudley (2002, Theorem 3.2.7).

We now prove global stability, making use of order mixing. In the sequel, we define $\operatorname{icb} S$ to be the bounded, increasing, and continuous functions from $S$ to $\mathbb{R}$ (i.e., $\operatorname{icb} S = \operatorname{ib} S \cap \operatorname{cb} S$). To simplify notation, we will also use inner product notation to represent integration, so that

$$\langle \mu, h \rangle := \int h(x)\mu(dx) \quad \text{for } \mu \in \mathscr{P}_S \text{ and } h \in \operatorname{ib} S \cup \operatorname{cb} S.$$

It is well known (see, e.g., Stokey et al. 1989, p. 219) that the left and right Markov operators are adjoint, in the sense that, for any such $h$ and any $\mu \in \mathscr{P}_S$, we have $\langle \mu, Qh \rangle = \langle \mu Q, h \rangle$.

We will make use of the following results, which are proved at the end of this appendix.

Lemma A.6. *Let $\mu, \mu', \mu_n \in \mathscr{P}_S$. Then*

*(i) $\mu \preceq \mu'$ if and only if $\langle \mu, h \rangle \leq \langle \mu', h \rangle$ for all $h \in \operatorname{icb} S$*

*(ii) $\mu = \mu'$ if and only if $\langle \mu, h \rangle = \langle \mu', h \rangle$ for all $h \in \operatorname{icb} S$, and*

*(iii) $\mu_n \to \mu$ if and only if $\{\mu_n\}$ is tight and $\langle \mu_n, h \rangle \to \langle \mu, h \rangle$ for all $h \in \operatorname{icb} S$.*

Proof of Theorem 1. We begin by showing that if $Q$ is globally stable, then conditions (i) and (ii) of the theorem hold. Regarding condition (i), fix $x \in S$. Global stability implies that $\{\mu Q^t\}$ is convergent for each $\mu \in \mathscr{P}_S$ and, hence, $\{Q^t(x, \cdot)\} = \{\delta_x Q^t\}$ is convergent. Since convergent sequences are tight (Dudley 2002, Proposition 9.3.4) and $x \in S$ was arbitrary, we conclude that $Q$ is bounded in probability, and condition (i) is satisfied. Condition (ii) is trivial, because global stability implies existence of a stationary distribution, and every stationary distribution is both deficient and excessive.

Next we show that if $Q$ is increasing, order reversing, and conditions (i) and (ii) of Theorem 1 hold, then $Q$ has at least one stationary distribution. By Lemma A.5, $Q$ is order mixing and, hence, by Kamihigashi and Stachurski (2012, Theorem 3.1), for any $\nu$ and $\nu'$ in $\mathscr{P}_S$, we have

$$\lim_{t\to\infty} |\langle \nu Q^t, h \rangle - \langle \nu' Q^t, h \rangle| = 0 \quad \forall h \in \operatorname{ib} S. \tag{12}$$

By condition (ii) of Theorem 1, there exists a $\mu \in \mathscr{P}_S$ that is either excessive or deficient. In what follows, we will assume it is deficient, since the excessive case only changes the direction of inequalities. Since $\mu$ is deficient, we have $\mu \preceq \mu Q$. Since $Q$ is increasing, we can iterate on this inequality to establish that the sequence $\{\mu Q^t\}$ is monotone increasing in $\preceq$. By condition (i) of Theorem 1 and Lemma A.3, the sequence $\{\mu Q^t\}$ is also tight.

By Prohorov's theorem (Dudley 2002, Theorem 11.5.4), tightness implies existence of a subsequence of $\{\mu Q^t\}$ converging to some $\psi^* \in \mathscr{P}_S$. Since $\{\mu Q^t\}$ is $\preceq$-increasing, it follows that, for any given $h \in \operatorname{icb} S$, the entire sequence $\langle \mu Q^t, h \rangle$ converges up to $\langle \psi^*, h \rangle$. Because $\{\mu Q^t\}$ is tight, part (iii) of Lemma A.6 implies that $\mu Q^t \to \psi^*$.

In addition to $\mu Q^t \to \psi^*$, we also have $\mu Q^t \preceq \psi^*$ for all $t \geq 0$, because for any $h \in \text{icb}\, S$ and $t \geq 0$ we have

$$\langle \mu Q^t, h \rangle \leq \sup_{t \geq 0} \langle \mu Q^t, h \rangle = \lim_{t \to \infty} \langle \mu Q^t, h \rangle = \langle \psi^*, h \rangle.$$

The inequality $\mu Q^t \preceq \psi^*$ now follows from part (i) of Lemma A.6.

Next, we claim that $\psi^* \preceq \psi^* Q$. To see this, pick any $h \in \text{icb}\, S$. Since $\mu Q^t \preceq \psi^*$ for all $t$ and since $Qh \in \text{ib}\, S$,

$$\langle \mu Q^t, Qh \rangle \leq \langle \psi^*, Qh \rangle = \langle \psi^* Q, h \rangle.$$

Using this inequality and the fact that $h \in \text{cb}\, S$, we obtain

$$\langle \psi^*, h \rangle = \lim_{t \to \infty} \langle \mu Q^{t+1}, h \rangle = \lim_{t \to \infty} \langle \mu Q^t, Qh \rangle \leq \langle \psi^* Q, h \rangle.$$

Hence $\langle \psi^*, h \rangle \leq \langle \psi^* Q, h \rangle$ for all $h \in \text{icb}\, S$, and $\psi^* \preceq \psi^* Q$ as claimed. Iterating on this inequality, we obtain $\psi^* \preceq \psi^* Q^t$ for all $t$.

To summarize our results so far, we have $\mu Q^t \preceq \psi^* \preceq \psi^* Q \preceq \psi^* Q^t$ for all $t \geq 0$ and, hence,

$$\langle \mu Q^t, h \rangle \leq \langle \psi^*, h \rangle \leq \langle \psi^* Q, h \rangle \leq \langle \psi^* Q^t, h \rangle \quad \text{for all } h \in \text{icb}\, S.$$

Applying (12), we obtain $\langle \psi^*, h \rangle = \langle \psi^* Q, h \rangle$ for all $h \in \text{icb}\, S$. By Lemma A.6, this implies that $\psi^* = \psi^* Q$. In other words, $\psi^*$ is stationary for $Q$.

It remains to show that $Q$ is globally stable. Fixing $\nu \in \mathscr{P}_S$ and applying (12) again, we have

$$\langle \nu Q^t, h \rangle \to \langle \psi^*, h \rangle \quad \forall h \in \text{ib}\, S. \tag{13}$$

Since $\text{icb}\, S \subset \text{ib}\, S$ and $\{\nu Q^t\}$ is tight (cf. Lemma A.3), this implies that $\nu Q^t \to \psi^*$ (Lemma A.6, part (iii)). Finally, uniqueness is also immediate, because if $\nu$ is also stationary, then by (13), we have $\langle \nu, h \rangle = \langle \psi^*, h \rangle$ for all $h \in \text{icb}\, S$. By Lemma A.6, we then have $\nu = \psi^*$. □

Proof of Theorem 2. Under the conditions of the theorem, $Q$ is order mixing, as proved in Lemma A.5. In addition, boundedness in probability and the Feller property guarantee the existence of a stationary distribution by the Krylov–Bogolubov theorem (Meyn and Tweedie 2009, Proposition 12.1.3 and Lemma D.5.3). Given existence of a stationary distribution $\psi^*$, the proof that $Q$ is globally stable is now identical to the proof of the same claim given for Theorem 1 (see the preceding paragraph). □

Proof of Proposition 1. Suppose that $Q'$ is Feller and bounded in probability with $Q' \preceq Q$. By the Krylov–Bogolubov theorem (Meyn and Tweedie 2009, Proposition 12.1.3 and Lemma D.5.3), $Q'$ has at least one stationary distribution $\mu$. For this $\mu$, we have $\mu = \mu Q' \preceq \mu Q$. In other words, $\mu$ is deficient for $Q$. A similar argument shows that if $Q'$ is Feller and bounded in probability with $Q \preceq Q'$, then $Q$ has an excessive distribution. □

Proof of Proposition 2. Let $Q$ be bounded in probability. Suppose first that $Q$ is upward reaching. Pick any $(x, x') \in S \times S$. Let $\{X_t\}$ and $\{X'_t\}$ be independent, and $(Q, x)$-Markov and $(Q, x')$-Markov, respectively. We need to prove existence of a $k \in \mathbb{N}$ such that $\mathbb{P}\{X_k \leq X'_k\} > 0$. Since $Q$ is bounded in probability, there exists a compact $C \subset S$ with $\mathbb{P}\{X_t \in C\} > 0$ for all $t \geq 0$. Since compact sets are assumed to be order bounded, we can take an order interval $[a, b]$ of $S$ with $C \subset [a, b]$. For this $a, b$, we have $\mathbb{P}\{a \leq X_t \leq b\} > 0$ for all $t \geq 0$. As $Q$ is upward reaching, there is a $k \in \mathbb{N}$ such that $\mathbb{P}\{b \leq X'_k\} > 0$. Using independence, we now have

$$\mathbb{P}\{X_k \leq X'_k\} \geq \mathbb{P}\{X_k \leq b \leq X'_k\} = \mathbb{P}\{X_k \leq b\}\mathbb{P}\{b \leq X'_k\} > 0,$$

as was to be shown. The proof for the downward reaching case is similar. □

Finally, we complete the proof of all remaining lemmas stated in this section.

Proof of Lemma A.1. Let $B$ be a $C$-accessible subset of $S$. To prove the lemma, it suffices to show that $\mathbf{P}^Q_x \bigcup_t \{X_t \in B\} = 1$ whenever $\{Q^t(x, \cdot)\}$ is tight. To this end, fix $x \in S$ and assume that $\{Q^t(x, \cdot)\}$ is tight. Let $\tau := \inf\{t \geq 0 : X_t \in B\}$. Evidently we have $\bigcup_{t=0}^{\infty} \{X_t \in B\} = \{\tau < \infty\}$. Thus, we need to show that $\mathbf{P}^Q_x\{\tau < \infty\} = 1$.

Fix $\epsilon > 0$. Since $\{Q^t(x, \cdot)\}$ is tight, there exists a compact set $C$ such that

$$\inf_t \mathbf{P}^Q_x\{X_t \in C\} = \inf_t Q^t(x, C) \geq 1 - \epsilon.$$

Since $B$ is $C$-accessible, there exists an $n \in \mathbb{N}$ and $\delta > 0$ such that $\inf_{y \in C} Q^n(y, B) \geq \delta$. For $t \in \mathbb{N}$, define $p_t := \mathbf{P}^Q_x\{\tau \leq tn\}$. We wish to obtain a relationship between $p_t$ and $p_{t+1}$. To this end, note that

$$\begin{aligned}
\mathbb{1}\{\tau \leq (t+1)n\} &= \mathbb{1}\{\tau \leq tn\} + \mathbb{1}\{\tau > tn\}\mathbb{1}\{\tau \leq (t+1)n\} \\
&\geq \mathbb{1}\{\tau \leq tn\} + \mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{(t+1)n} \in B\} \\
&\geq \mathbb{1}\{\tau \leq tn\} + \mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{tn} \in C\}\mathbb{1}\{X_{(t+1)n} \in B\}.
\end{aligned}$$

Taking expectations yields

$$p_{t+1} \geq p_t + \mathbf{E}^Q_x \mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{tn} \in C\}\mathbb{1}\{X_{(t+1)n} \in B\}.$$

We estimate the last expectation as

$$\begin{aligned}
\mathbf{E}^Q_x \mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{tn} \in C\}\mathbb{1}\{X_{(t+1)n} \in B\} &= \mathbf{E}^Q_x\big[\mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{tn} \in C\}\mathbf{E}^Q_x[\mathbb{1}\{X_{(t+1)n} \in B\}|\mathscr{F}_{tn}]\big] \\
&= \mathbf{E}^Q_x[\mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{tn} \in C\}Q^n(X_{tn}, B)] \\
&\geq \mathbf{E}^Q_x \mathbb{1}\{\tau > tn\}\mathbb{1}\{X_{tn} \in C\}\delta \\
&= \mathbf{E}^Q_x(1 - \mathbb{1}\{\tau \leq tn\})\mathbb{1}\{X_{tn} \in C\}\delta \\
&= \mathbf{E}^Q_x \mathbb{1}\{X_{tn} \in C\}\delta - \mathbf{E}^Q_x \mathbb{1}\{\tau \leq tn\}\mathbb{1}\{X_{tn} \in C\}\delta
\end{aligned}$$

$$
\begin{aligned}
&\geq (1-\epsilon)\delta - \mathbf{E}_x^Q \mathbb{1}\{\tau \leq tn\}\delta \\
&= (1-\epsilon)\delta - p_t\delta; \\
\therefore \quad p_{t+1} &\geq p_t + (1-\epsilon)\delta - p_t\delta = (1-\delta)p_t + (1-\epsilon)\delta.
\end{aligned}
$$

The unique, globally stable fixed point of $q_{t+1} = (1-\delta)q_t + (1-\epsilon)\delta$ is $1-\epsilon$, so $1-\epsilon \leq \lim_{t\to\infty} p_t = \mathbf{P}_x^Q\{\tau < \infty\} \leq 1$. Since $\epsilon$ was arbitrary, we obtain $\mathbf{P}_x^Q\{\tau < \infty\} = 1$. □

Proof of Lemma A.2. Fix $x, x' \in S$ and $\epsilon > 0$. Since $Q$ is bounded in probability, we can choose compact sets $C$ and $C'$ such that

$$
\begin{aligned}
&Q^t(x, C) \geq (1-\epsilon)^{1/2} \quad \text{and} \quad Q^t(x', C') \geq (1-\epsilon)^{1/2} \quad \text{for all } t; \\
\therefore \quad &(Q \times Q)^t((x, x'), C \times C') = Q^t(x, C)Q^t(x', C') \geq 1-\epsilon \quad \text{for all } t.
\end{aligned}
$$

Since $C \times C'$ is compact in the product space, $Q \times Q$ is bounded in probability. □

Proof of Lemma A.3. Fix $\mu \in \mathscr{P}_S$ and $\epsilon > 0$. Since individual elements of $\mathscr{P}_S$ are tight (Dudley 2002, Theorem 11.5.1), we can choose a compact set $C_\mu \subset S$ with $\mu(C_\mu) \geq 1-\epsilon$. By assumption, we can take an order interval $[a, b]$ of $S$ with $C_\mu \subset [a, b]$. For this $a, b$, we have

$$
\mu([a, b]^c) = \mu(S \setminus [a, b]) \leq \epsilon. \tag{14}
$$

By hypothesis, $\{Q^t(x, \cdot)\}$ is tight for all $x \in S$, so we choose compact subsets $C_a$ and $C_b$ of $S$ with $Q^t(a, C_a) \geq 1-\epsilon$ and $Q^t(b, C_b) \geq 1-\epsilon$ for all $t$. Since $C_a \cup C_b$ is also compact, we can take an order interval $[\alpha, \beta]$ of $S$ with $C_a \cup C_b \subset [\alpha, \beta] \subset S$. We then have $Q^t(a, [\alpha, \beta]) \geq 1-\epsilon$ and $Q^t(b, [\alpha, \beta]) \geq 1-\epsilon$ for all $t$. Letting $I_\alpha := \{x \in S : x \geq \alpha\}$ and $D_\beta := \{x \in S : x \leq \beta\}$ leads to

$$
Q^t(a, I_\alpha) \geq 1-\epsilon \quad \text{and} \quad Q^t(b, D_\beta) \geq 1-\epsilon \quad \text{for all } t. \tag{15}
$$

In view of Remark 1 and (15), we have

$$
a \leq x \quad \Longrightarrow \quad Q^t(x, I_\alpha) \geq Q^t(a, I_\alpha) \geq 1-\epsilon,
$$

and, by a similar argument,

$$
x \leq b \quad \Longrightarrow \quad Q^t(x, D_\beta) \geq Q^t(b, D_\beta) \geq 1-\epsilon.
$$

Since $[\alpha, \beta] := \{x \in S : \alpha \leq x \leq \beta\} = I_\alpha \cap D_\beta$, we have

$$
Q^t(x, [\alpha, \beta]^c) = Q^t(x, D_\beta^c \cup I_\alpha^c) \leq 2 - Q^t(x, D_\beta) - Q^t(x, I_\alpha).
$$

This leads to the estimate

$$
a \leq x \leq b \quad \Longrightarrow \quad Q^t(x, [\alpha, \beta]^c) \leq 2\epsilon. \tag{16}
$$

Combining (14) and (16), we now have

$$\begin{aligned}\mu Q^t([\alpha,\beta]^c) &= \int Q^t(x,[\alpha,\beta]^c)\mu(dx)\\ &= \int_{[a,b]} Q^t(x,[\alpha,\beta]^c)\mu(dx) + \int_{[a,b]^c} Q^t(x,[\alpha,\beta]^c)\mu(dx)\\ &\leq \int_{[a,b]} 2\epsilon\mu(dx) + \mu([\alpha,\beta]^c) \leq 3\epsilon.\end{aligned}$$

Since $[\alpha,\beta]$ is compact and $t$ is arbitrary, we conclude that $\{\mu Q^t\}$ is tight. □

Proof of Lemma A.4. Let $C$ be any compact subset of $S\times S$. We need to prove existence of an $n\in\mathbb{N}$ and $\delta>0$ such that $(Q\times Q)^n((x,x'),\mathbb{G})\geq\delta$ whenever $(x,x')\in C$. To do so, we introduce the function

$$\psi_n(x,x') := (Q\times Q)^n((x,x'),\mathbb{G}) = \mathbf{P}^{Q\times Q}_{x,x'}\{X_n\leq X'_n\},$$

where $(X_n,X'_n)$ is $(Q\times Q,(x,x'))$-Markov. Intuitively, since $Q$ is increasing, the event $\{X_n\leq X'_n\}$ becomes less likely as $x$ rises and $x'$ falls, and, hence, $\psi_n(x,x')$ is decreasing in $x$ and increasing in $x'$ for each $n$. A routine argument confirms this is the case.

Since $C\subset S\times S$ is compact, we can take an order interval $[a,b]$ of $S$ with $C\subset [a,b]\times[a,b]$.[18] Moreover, since $Q$ is order reversing, we can take an $n\in\mathbb{N}$ such that $\delta:=\psi_n(b,a)>0$. Observe that

$$\begin{aligned}&(x,x')\in C \implies (x,x')\in[a,b]\times[a,b] \implies x\leq b \text{ and } x'\geq a;\\ \therefore\quad &(x,x')\in C \implies (Q\times Q)^n((x,x'),\mathbb{G}) = \psi_n(x,x')\geq\psi_n(b,a)=\delta.\end{aligned}$$

In other words, $\mathbb{G}$ is $C$-accessible for $Q\times Q$. □

Proof of Lemma A.6. The statement $\mu\preceq\mu'$ if and only if $\langle\mu,h\rangle\leq\langle\mu',h\rangle$ for all $h\in\text{icb}\,S$ holds for every normally ordered space, as shown by Whitt (1980, Theorem 2.6). Moreover, since $\preceq$ is a partial order on $\mathscr{P}_S$ (Kamae and Krengel 1978, Theorem 2) and, hence, antisymmetric, it follows that $\mu=\mu'$ if and only if $\langle\mu,h\rangle=\langle\mu',h\rangle$ for all $h\in\text{icb}\,S$. Regarding the third assertion of the lemma, observe first that if $\mu_n\to\mu$, then since $S$ is Polish, the sequence $\{\mu_n\}$ is tight (Dudley 2002, Theorem 11.5.3). The statement $\langle\mu_n,h\rangle\to\langle\mu,h\rangle$ whenever $h\in\text{icb}\,S$ is obvious. To prove the converse, suppose that $\{\mu_n\}$ is tight and $\langle\mu_n,h\rangle\to\langle\mu,h\rangle$ for all $h\in\text{icb}\,S$. Take any subsequence $\{\mu_n\}_{n\in\mathbb{N}_1}$ of $\{\mu_n\}$. By tightness and Prohorov's theorem (Dudley 2002, Theorem 11.5.4), this subsequence has a subsubsequence converging to some $\nu\in\mathscr{P}_S$:

$$\exists\mathbb{N}_2\subset\mathbb{N}_1 \quad\text{such that}\quad \lim_{n\in\mathbb{N}_2}\langle\mu_n,h\rangle=\langle\nu,h\rangle \quad\text{for all } h\in\text{cb}\,S.$$

[18] To see this, let $K$ be a compact subset of $S$ with $C\subset K\times K$. (Such a $K$ can be obtained by projecting $C$ onto the first and second axis, and defining $K$ as the union of these projections.) Since $K$ is order bounded in $S$ by assumption, we just choose $a,b\in S$ with $K\subset[a,b]$.

Since $\langle \mu_n, h \rangle \to \langle \mu, h \rangle$ for all $h \in \text{icb}\, S$, we now have $\lim_{n \in \mathbb{N}_2} \langle \mu_n, h \rangle = \langle \nu, h \rangle = \langle \mu, h \rangle$ for all $h \in \text{icb}\, S$ and, hence, $\nu = \mu$. We have now shown that every subsequence of $\{\mu_n\}$ has a sub-subsequence converging to $\mu$ and, hence, the entire sequence also converges to $\mu$. □

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