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The transfer problem: A complete characterization

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The transfer problem refers to the possibility that a donor country could end up better off after giving away some resources to another country. The simplest version of that problem can be formulated in a two consumer exchange economy with fixed total resources. The existence of a transfer problem at some equilibrium is known to be equivalent to instability in the case of two goods. This characterization is extended to an arbitrary number of goods by showing that a transfer problem exists at a (regular) equilibrium if and only if this equilibrium has an index value equal to -1. Samuelson's conjecture that there is no transfer problem at tatonnement stable equilibria is therefore true for any number of goods.

Keywords. Transfer problem, regular equilibrium, index value.
JEL classification. D51, F20.

1. Introduction

Does a country's utility necessarily decrease when that country gives away some resources to another country? This problem is known in trade theory as the transfer problem and has led to a substantial literature. One aspect of the transfer problem is the characterization under simple assumptions (no trade impediments such as transportation costs and tariffs in particular) of those equilibria at which the donor country can improve its utility when giving away resources. The simplest model in which the transfer problem can be studied in the case of an arbitrary number of goods is the exchange model with two consumers and fixed total resources.[1] In the case of multiple equilibria, any one of the equilibria that do not give the highest utility level to consumer 1 can be improved by selecting one of those that yields a higher utility level. This trivial solution makes sense only if the equilibrium selection map is permitted to be discontinuous.

This formulation of the transfer problem requires that the equilibrium be regular with an associated locally continuous (in fact smooth) equilibrium selection map. The following results are then known: there are examples of economies that have regular equilibria with a transfer problem, i.e., such that a consumer (country) can be better off

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[1] See, for example, Samuelson (1952).

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by giving away some resources (Leontief 1936); there is no transfer problem at tatonnement stable equilibria (Samuelson 1947, footnote p. 29, and Samuelson 1952); tatonnement stability is not only sufficient but also necessary to prevent a transfer problem in the case of two goods (Balasko 1978). Using the theory of smooth economies, I show in this paper that, at a regular equilibrium and for the equilibrium selection map defined by that equilibrium, there is a transfer problem if and only if the index value of that equilibrium is equal to -1. The utility level of the donor country then increases for any gifts that remain small enough to stay in the domain of the equilibrium selection map. This property extends to an arbitrary number of goods the characterization given in Balasko (1978) for the case of two goods.

Section 2 of this paper is devoted to the main assumptions, definitions, and notation. The geometric or dual formulation of the exchange model in the price–income space (limited to the case of two consumers, fixed total resources, and an arbitrary number of goods) occupies Section 3. This formulation is then used in Section 4 for a complete characterization by their index value of the regular equilibria that feature a transfer problem. Concluding comments end this paper with Section 5.

2. Definitions, assumptions, and notation

There are $\ell \geq 2$ goods. The commodity space is $\mathbb{R}^\ell$ and $X = \mathbb{R}^\ell_{++}$ denotes the strictly positive orthant of that space. The price vector $p = (p_1, \dots, p_\ell) \in X$ (all prices are strictly positive) is normalized by the numeraire assumption $p_\ell = 1$. Let $\bar{p} = (p_1, \dots, p_{\ell-1}) \in \mathbb{R}^{\ell-1}_{++}$ denote the first $\ell - 1$ coordinates of the normalized price vector $p \in S$. This gives $p = (\bar{p}, 1)$. The set of numeraire normalized prices is denoted by $S = \mathbb{R}^{\ell-1}_{++} \times \{1\}$.

There are two consumers (or countries). Consumer i, with $1 \leq i \leq 2$, is endowed with the goods bundle $\omega_i \in \mathbb{R}^\ell_{++}$. The endowment vector $\omega = (\omega_1, \omega_2) \in \mathbb{R}^\ell_{++} \times \mathbb{R}^\ell_{++}$ defines an economy. Total resources, equal to the vector $r = \omega_1 + \omega_2$, are fixed. Let $\Omega = \{\omega = (\omega_1, \omega_2) \in X^2 \mid \omega_1 + \omega_2 = r\}$ denote the endowment or parameter space.

Consumer i's preferences are represented by a utility function $u_i : X \to \mathbb{R}$ that satisfies the following assumptions that are standard in this kind of literature: (i) smoothness; (ii) smooth monotonicity, i.e., $Du_i(x_i) \in X$ for $x_i \in X$, where $Du_i(x_i)$ is the gradient vector defined by the first-order derivatives of u_i; (iii) smooth strict quasi-concavity, namely, the restriction of the quadratic form defined by the Hessian matrix $D^2u_i(x_i)$ to the tangent hyperplane to the indifference surface $\{y_i \in X \mid u_i(y_i) = u_i(x_i)\}$ through x_i is negative definite; (iv) the indifference surface $\{y_i \in X \mid u_i(y_i) = u_i(x_i)\}$ is closed in $\mathbb{R}^\ell$ for all $x_i \in X$. The utility function u_i is extended to $x_i = 0$ by setting $u_i(0) = \inf_{x_i \in X} u_i(x_i)$.

Consumer i's demand function is the map $f_i : S \times \mathbb{R}_{++} \to X$, where $f_i(p, w_i)$ is the unique solution to the problem of maximizing the utility $u_i(x_i)$ subject to the budget constraint $p \cdot x_i \leq w_i$. The demand function f_i is smooth and satisfies Walras' law (namely the identity $p \cdot f_i(p, w_i) = w_i$). Its (numeraire normalized) Slutsky matrix is negative definite. (For details, see, for example, Balasko 2011, Chapter 2.)

Consumer i's indirect utility function is defined by $\hat{u}_i(p, w_i) = u_i(f_i(p, w_i))$.

So as to define equilibrium, we introduce the vector $z(p, \omega) \in \mathbb{R}^\ell = f_1(p, p \cdot \omega_1) + f_2(p, p \cdot \omega_2) - r$ that represents the excess demand associated with the pair

$(p, \omega) \in S \times \Omega$. The pair $(p, \omega) \in S \times \Omega$ is an equilibrium if

$$z(p, \omega) = 0, \tag{1}$$

equality known as the equilibrium equation, is satisfied. The equilibrium manifold is the subset E of $S \times \Omega$ defined by (1).

Let $\bar{z}(p, \omega) \in \mathbb{R}^{\ell-1}$ denote the vector defined by the first $\ell - 1$ coordinates of the vector $z(p, \omega) \in \mathbb{R}^{\ell}$ for $(p, \omega) \in S \times \Omega$. It follows from the identity $p \cdot z(p, \omega) = 0$ (a consequence of Walras' law satisfied by individual demand functions) that the equation (1) is equivalent to $\bar{z}(p, \omega) = 0 \in \mathbb{R}^{\ell-1}$.

By definition, the equilibrium $(p, \omega) \in E$ is regular if the $(\ell-1) \times (\ell-1)$ Jacobian matrix $J(p, \omega) = D\bar{z}(p, \omega)/D\bar{p}$ is invertible. The index of the regular equilibrium $(p, \omega) \in E$ is then equal to $+1$ (resp. -1) if the sign of $(-1)^{\ell-1} \det J(p, \omega)$ is positive (resp. negative). An index value of $+1$ is related to tatonnement stability in the following sense: local stability (for Walras tatonnement) is identified, roughly speaking, with the Jacobian matrix $J(p, \omega)$ having eigenvalues with strictly negative real parts; the product of these eigenvalues is equal to $\det J(p, \omega)$. Therefore, a tatonnement stable equilibrium always has an index value equal to $+1$. The converse is not true.

Locally defined equilibrium selection maps are associated with regular equilibria. More specifically, $(p, \omega) \in E$ is a regular equilibrium. It is then possible to apply the implicit function theorem to the equation $\bar{z}((\bar{p}, 1), \omega) = 0$, where the unknown is the vector $\bar{p} \in \mathbb{R}^{\ell-1}_{++}$. Then there exists a neighborhood U of ω, a neighborhood $V \subset E$ of the equilibrium $(p, \omega) \in E$, and a smooth map $s: U \to S$ such that the map $\sigma: U \to V$ defined by $\sigma(\omega') = (s(\omega'), \omega')$ is a diffeomorphism between U and V (i.e., a smooth bijection with a smooth inverse map). For a neighborhood U of ω that is small enough, the map $\sigma: U \to V$ (resp. $s: U \to S$) depends only on the regular equilibrium $(p, \omega) \in E$. The map $\sigma: U \to V$ (resp. $s: U \to S$) is known as the local equilibrium selection map (resp. local equilibrium price selection map) associated with the regular equilibrium $(p, \omega) \in E$. For open sets U that are small enough, the maps σ and s are determined by the (regular) equilibrium $(p, \omega) \in E$. For details, see Balasko (2011, Proposition 7.2).

We now have all the necessary ingredients for a rigorous formulation of the transfer problem. Let us start with $\omega = (\omega_1, \omega_2)$ and let $\omega' = (\omega'_1, \omega'_2)$ in Ω be two endowment vectors (or economies). By definition, consumer 1 gives away some resources when the economy moves from ω to ω' if inequality $\omega'_1 \lneqq \omega_1$ (i.e., $\omega'_1 \leq \omega_1$ and $\omega'_1 \neq \omega_1$) is satisfied.

Definition 1. There is a transfer problem at the regular equilibrium (p, ω) if there exists an endowment vector $\omega' = (\omega'_1, \omega'_2) \in U$ with $\omega'_1 \lneqq \omega_1$ such that

$$u_1(f_1(s(\omega'), s(\omega') \cdot \omega'_1)) > u_1(f_1(s(\omega), s(\omega) \cdot \omega_1)), \tag{2}$$

where $s: U \to S$ is the local equilibrium price selection map associated with the regular equilibrium $(p, \omega) \in E$.

By definition, the transfer problem requires only the existence of one endowment vector ω' with $\omega'_1 \lneqq \omega_1$ such that inequality (2) is satisfied.

Remark. Definition 1 requires the equilibrium $(p, \omega) \in E$ to be regular. This restriction is minor, since the set of regular equilibria is an open subset with a full measure of the equilibrium manifold E by Balasko (1992) or Balasko (2011, Proposition 8.10).

3. The geometric approach to the transfer problem

The price–income space is the set $S \times \mathbb{R}^2_{++}$ that consists of the triplets (p, w_1, w_2), where w_1 and w_2 denote the wealth of consumers 1 and 2, respectively. With total resources r fixed, the set $H(r)$ is the subset of $S \times \mathbb{R}^2_{++}$ defined by the linear equation $w_1 + w_2 = p \cdot r$. This is a hyperplane of dimension ℓ. This set is known as the ambient space. A set of coordinates for the ambient space $H(r)$ is given by the ℓ-tuple $(p_1, \dots, p_{\ell-1}, w_1) = (\bar{p}, w_1) \in \mathbb{R}^\ell_{++}$. Then w_2 is determined by the formula $w_2 = p \cdot r - w_1$.

By definition, the section manifold $B(r)$ is the subset of $H(r)$ that consists of the points $b = (p, w_1, w_2)$ that satisfy

$$f_1(p, w_1) + f_2(p, w_2) = r.$$

This set is a smooth submanifold of $H(r)$ of dimension $m - 1$ by Proposition 5.4.1 of Balasko (2009). Therefore, for $m = 2$, the section manifold is just a smooth curve. Here is a direct proof of that property. The section manifold $B(r)$ is, in fact, closely related to the contract curve of the Edgeworth box, i.e., the set of Pareto optima associated with the fixed total resources r. Let $P(r)$ denote that set. A Pareto optimum then results from the maximization of the second consumer's utility $u_2(x_2)$ subject to the first consumer's utility constraint $u_1(x_1) = u_1$ with $u_1 \in [u_1(0), u_1(r)]$, the total resources being fixed and equal to $r \in \mathbb{R}^\ell_{++}$. Let $x(u_1) = (x_1(u_1), x_2(u_1))$ be the Pareto optimum that solves that constrained maximization problem and let $p(u_1) \in S$ denote the (numeraire normalized) price vector that supports that Pareto optimum $x(u_1)$. For $u_1(0) < u_1 < u_1(r)$, the price vector $p(u_1) \in S$ is collinear with the two gradient vectors $Du_1(x_1(u_1))$ and $Du_2(x_2(u_1))$; for u_1 equal to $u_1(0)$ (resp. $u_1(r)$), the price vector $p(u_1(0)) \in S$ is collinear with $Du_2(r)$ (resp. $p(u_1(r))$ with $Du_1(r)$). The set of Pareto optima is generated by varying consumer 1's utility level u_1 between $u_1(0)$ and $u_1(r)$. This defines a smooth curve with two end points: the allocations $(0, r)$ and $(r, 0)$.

The section manifold now comes in with the observation that the point $M(u_1) = (p(u_1), p(u_1) \cdot x_1(u_1), p(u_1) \cdot x_2(u_1))$ in the price–income space $H(r)$ belongs to $B(r)$ and, conversely, any point of $B(r)$ is associated with a unique utility level $u_1 \in [u_1(0), u_1(r)]$ for the first consumer. The utility u_1 parameterizes not only the contract curve $P(r)$, but also the section manifold $B(r)$. The section manifold $B(r)$ is, therefore, a smooth curve with two end points: the points $M_0 = (p(u_1(0)), 0, p(u_1(0)) \cdot r)$ and $M_1 = (p(u_1(r)), p(u_1(r)) \cdot r, 0)$. See Figure 1.

By definition, the positive orientation of the curve $B(r)$ corresponds to increasing values of the parameter u_1. The curve $B(r)$ is separated by the point $M(u_1)$ into two connected pieces: the arc $\widehat{M_0M(u_1)}$ and the arc $\widehat{M(u_1)M_1}$. Equilibrium allocations that belong to the arc $\widehat{M_0M(u_1)}$ (resp. $\widehat{M(u_1)M_1}$) yield utility levels for consumer 1 lower (resp. higher) than u_1.

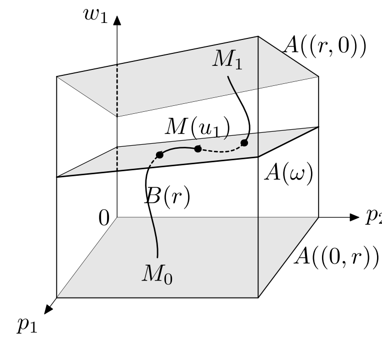

FIGURE 1. The submanifolds $A(\omega)$ and $B(r)$.

The derivative of the map $u_1 \to M(u_1) \in H(r)$ is denoted by $t(u_1)$. It represents a vector that is tangent to the curve $B(r)$ at the point $M(u_1)$. The direction defined by the vector $t(u_1)$ corresponds to increasing utility levels for consumer 1 along the curve $B(r)$ in a neighborhood of the point $M(u_1)$.

The budget hyperplane $A(\omega)$ associated with the endowment vector $\omega \in \Omega$ is the subset of $H(r)$ defined by $w_1 = p \cdot \omega_1$ in the coordinate system $(\bar{p}, w_1)$. In what follows, only the part of the budget hyperplane $A(\omega)$ that is defined for $\bar{p} \in \mathbb{R}^{\ell-1}_{++}$ (i.e., for strictly positive prices) is considered.

One sees readily that $(p, \omega) \in S \times \Omega$ is an equilibrium if and only if the point $b = (p, p \cdot \omega_1, p \cdot \omega_2) \in H(r)$ belongs to the intersection $B(r) \cap A(\omega)$.

The study of the equilibrium equation (1) is equivalent to the study of the intersection of the curve $B(r)$ with the budget hyperplane $A(\omega)$ when ω is varied in Ω. The curve $B(r)$ captures all the nonlinearities of equilibrium equation (1). In addition, the curve $B(r)$ does not depend at all on the endowment vector $\omega \in \Omega$. This feature will come in handy in the study of the transfer problem.

Let π_j be the vector in $\mathbb{R}^{\ell-1}$ with coordinates equal to 0 except for the jth, which is equal to 1. In the coordinate system $(\bar{p}, w_1)$ for $H(r)$, let $e_j(\omega) = (\pi_j, \omega_1^j)$. The (affine) hyperplane $A(\omega)$ is parallel to the vector subspace generated by the $\ell - 1$ vectors $e_1, \dots, e_{\ell-1}$. The base $(e_1, e_2, \dots, e_{\ell-1})$ then defines the positive orientation of $A(\omega)$.

Let $b = (p, p \cdot \omega_1, p \cdot \omega_2) \in H(r)$ be the point in the price–income space that is associated with the equilibrium $(p, \omega) \in E$. Let $u_1(p, \omega) = u_1(f_1(p, p \cdot \omega_1))$ be the utility of consumer 1 at the equilibrium allocation $x = (f_1(p, p \cdot \omega_1), f_2(p, p \cdot \omega_2))$. This yields $b = M(u_1(p, \omega))$.

Regularity of the equilibrium $(p, \omega) \in E$ is equivalent to the transversality of the smooth submanifolds $B(r)$ and $A(\omega)$ at b. The ℓ vectors $e_1(\omega), \dots, e_{\ell-1}(\omega)$ and $t(u_1(p, \omega))$ are then linearly independent in $H(r)$ and the determinant

$$\Delta(p, \omega) = \det(e_1(\omega), \dots, e_{\ell-1}(\omega), t(u_1(p, \omega)))$$

is not equal to 0.

LEMMA 1. *The index number of the regular equilibrium* $(p, \omega) \in E$ *is equal to* $+1$ *(resp.* -1*) if* $\Delta(p, \omega)$ *is positive (resp. negative).*

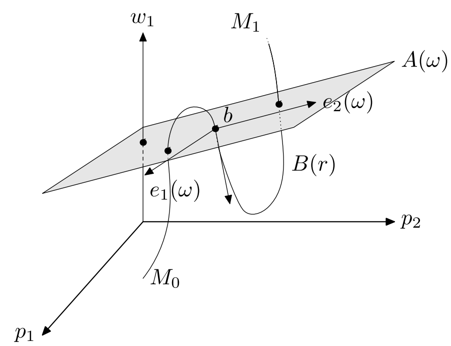

FIGURE 2. Orientation of $A(\omega)$ and $B(r)$ at b.

PROOF. It is possible to show directly, but after somewhat tedious and lengthy computations, that $\Delta(p, \omega)$ has the sign opposite to $\det(D\bar{z}(p, \omega)/D\bar{p})$ for any regular equilibrium $(p, \omega) \in E$, which would prove the lemma.

The following short proof avoids any computation by exploiting the connectedness of the curve $B(r)$ through its parameterization by consumer 1's utility level $u_1 \in [u_1(0), u_1(r)]$. Let $b(u_1) = (p(u_1), w_1(u_1), w_2(u_1))$ be the point of the curve $B(r)$ parameterized by u_1. This gives $M(u_1) = b(u_1)$. Define $\omega_1(u_1) = f_1(p(u_1), w_1(u_1))$, $\omega_2(u_1) = f_2(p(u_1), w_2(u_1))$, and $\omega(u_1) = (\omega_1(u_1), \omega_2(u_1))$. Walras' law for individual demands implies $w_1(u_1) = p(u_1) \cdot \omega_1(u_1)$ and $w_2 = p \cdot \omega_2(u_1)$. The pair $(p(u_1), \omega(u_1))$ is, therefore, an equilibrium and, actually, a no-trade equilibrium since $\omega_i(u_1) = f_i(p(u_1), p(u_1) \cdot \omega_i(u_1))$ for $i = 1, 2$. In addition, this equilibrium is regular since every no-trade equilibrium is regular by Balasko (1975) or Balasko (2011, Proposition 8.2).

The budget hyperplane $A(\omega(u_1))$ depends continuously on u_1. Therefore, the function $u_1 \to \delta(u_1) = \Delta(p(u_1), \omega(u_1))$ is also continuous.

The function $\delta(u_1)$ is different from 0 for all $u_1 \in [u_1(0), u_1(r)]$ since every no-trade equilibrium is regular. Therefore, it suffices to check the sign of this function for any particular value of u_1. A good candidate is $u_1 = u_1(0)$. Then $\omega(u_1) = (0, r)$ and the vector $e_j(\omega(u_1))$ is equal to $(\pi_j, 0)$ for $1 \leq j \leq \ell - 1$. The point M_0 is in the horizontal hyperplane $A(0, r)$, a hyperplane with equation $w_1 = 0$ in the $(\bar{p}, w_1)$ coordinate system. Furthermore, the tangent vector $t(u_1)$ to the curve $B(r)$ at $u_1 = u_1(0)$ necessarily points upward. This implies that the ℓth coordinate of the vector $t(u_1)$ is greater than or equal to 0 for $u_1 = u_1(0)$ and cannot be equal to 0 because of the transversality property. This proves the strict inequality $\delta(u_1(0)) > 0$. $\square$

Figure 2 shows an example of a negative index number at the intersection point b of $A(\omega)$ and $B(r)$ for $\ell = 3$ goods.

REMARK. Lemma 1 can be reformulated as saying that the index number of the regular equilibrium $(p, \omega) \in E$ is the same thing as the intersection number in the sense of Guillemin and Pollack (1974, p. 96) at the intersection $b = (p, p \cdot \omega_1, p \cdot \omega_2)$ (a point also denoted by $M(u_1)$ in the earlier sections) of the submanifolds $A(\omega)$ and $B(r)$.

Remark. If the endowment vector $\omega \in \Omega$ is regular (i.e., all equilibria $(p, \omega) \in E$ associated with ω are regular), there is only a finite number of equilibria (Debreu 1970). It has been shown by Dierker (1972) that the sum of the indices over all these equilibria is an invariant equal to $+1$ (Dierker 1972). This number is the same thing as the oriented intersection number of the submanifolds $B(r)$ and $A(\omega)$ as defined in Guillemin and Pollack (1974, p. 107).

It follows from the value equal to $+1$ of the oriented intersection number that the number of equilibria of a regular economy is odd. With this number equal to $2n+1$, $n+1$ equilibria have an index equal to $+1$ and n have an index equal to -1.

4. Application to the transfer problem

The key issue is the relation between the transfer problem and the intersection number of $A(\omega)$ and $B(r)$ at their intersection point b that corresponds to the regular equilibrium $(p, \omega) \in E$.

We now reformulate the property for consumer 1 of giving away some resources between ω and ω' as a property of the hyperplanes $A(\omega)$ and $A(\omega')$.

By definition, the hyperplane $A(\omega')$ lies below the hyperplane $A(\omega)$ if the strict inequality $p \cdot \omega'_1 < p \cdot \omega_1$ is satisfied for any $p \in S$. Note that the relative positions of $A(\omega)$ and $A(\omega')$ are considered only above the price set S. It then can be defined as follows.

Lemma 2. *The hyperplane $A(\omega')$ lies below the hyperplane $A(\omega)$ if and only if $\omega'_1 \lneqq \omega_1$.*

Proof. The condition is equivalent to $p \cdot (\omega'_1 - \omega_1) < 0$ for any $p \in S$. This readily implies that all coordinates of $\omega'_1 - \omega_1$ are less than or equal to 0 and at least one of them is strictly negative. □

We now reformulate the (local) equilibrium price selection map of Definition 1 within the setup defined by the curve $B(r)$ and the hyperplane $A(\omega)$. The equilibrium $(p, \omega) \in E$ is regular if the curve $B(r)$ and the hyperplane $A(\omega)$ intersect transversally at the point $b = (p, w_1, w_2)$, where $w_1 = p \cdot \omega_1$ and $w_2 = p \cdot \omega_2$. Let u_1 be consumer 1's utility level such that $M(u_1) = b$. The vector $t(u_1)$, which is by definition tangent to the curve $B(r)$ at the point b, is not contained in the hyperplane $A(\omega)$ because of transversality.

For ω' sufficiently close to ω and some sufficiently small neighborhood of b, transversality at b implies that the intersection $B(r) \cap A(\omega')$ contains a unique point b' in that neighborhood of b. This construction defines a map $\omega' \to b' = b(\omega')$ that satisfies $b(\omega) = b$. The composition of that map with the projection $b' = (p', w'_1, w'_2) \to p' \in S$ is the (local) equilibrium price selection map considered in Definition 1.

The regular equilibria that feature a transfer problem can now be characterized as follows.

Theorem. *The regular equilibrium $(p, \omega) \in E$ features a transfer problem if and only if its index is equal to -1.*

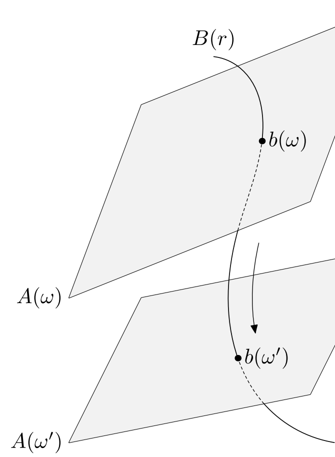

FIGURE 3. Orientation at the intersection and the transfer problem.

PROOF. Let $b(\omega) = (p, p \cdot \omega_1, p \cdot \omega_2) \in B(r) \cap A(\omega)$. Let $u_1 = u_1(f_1(p, p \cdot \omega_1))$. With the notation of earlier sections, this becomes $b(\omega) = M(u_1)$. The arc $\widehat{M(u_1)M_1}$ (resp. $\widehat{M_0M(u_1)}$) out of the curve $B(r)$ consists of the points of $B(r)$ that are parameterized by utility levels greater than or equal to u_1 (resp. less than or equal to u_1). See Figure 1.

If the intersection number of $B(r)$ and $A(\omega)$ at $b(\omega)$ is equal to -1, there is a neighborhood $V \subset H(r)$ of the point $b(\omega) = M(u_1)$ such that the points of the intersection $\widehat{M(u_1)M_1} \cap V$ are below the hyperplane $A(\omega)$. Similarly, the points of the intersection $\widehat{M_0M(u_1)} \cap V$ are above $A(\omega)$.

Let the open neighborhood U be the domain of the local equilibrium price selection map $s: U \to S$ defined at the regular equilibrium $(p, \omega) \in E$ as in Definition 1. For $\omega' \in U$ and $\omega' \lneqq \omega$, the hyperplane $A(\omega')$ is below $A(\omega)$. The point $b(\omega')$ is, therefore, below the hyperplane $A(\omega)$ and, therefore, belongs to the path $\widehat{M(u_1)M_1}$. Consumer 1's utility $u_1(f_1(s(\omega'), s(\omega') \cdot \omega_1'))$ is, therefore, strictly higher than $u_1 = u_1(f_1(s(\omega), s(\omega) \cdot \omega_1))$. See Figure 3.

The same line of reasoning shows that if the intersection number of $B(r)$ and $A(\omega)$ at $b(\omega)$ is equal to $+1$, then the strict inequality

$$u_1(f_1(s(\omega'), s(\omega') \cdot \omega_1')) < u_1(f_1(s(\omega), s(\omega) \cdot \omega_1))$$

is satisfied for any $\omega' \in U$ with $\omega_1' \lneqq \omega_1$. □

The set of regular equilibria is partitioned into path-connected components and the index is constant over each one of these components. It follows from Balasko (2012) that the equilibria with an index value equal to $+1$ belong to just one such path-connected component: the component that contains the set of no-trade equilibria. This implies that the transfer problem can exist only for sufficiently large volumes of trade.

5. Concluding comments

By transferring resources from one country to another, the goal is generally to make the receiving country better off. It follows from the theorem of this paper that quirks in the market mechanism render that goal impossible to achieve at equilibria with an index number equal to -1. Samuelson's intuition was that the causes for this misbehavior of competitive markets were somehow related to those that create instability. This intuition is correct since an equilibrium with index number -1 is unstable. Having an index value equal to $+1$ does not imply stability, but at those equilibria, the behavior of competitive markets does not interfere with the goal of the donor to make the receiver better off. This makes the concept of index value equal to $+1$ a possible substitute to the concept of stability. This is obviously true for the case of two consumers. The general case of an arbitrary number of consumers justifies further research.

From the perspective of comparative statics, it is noteworthy that the transfer problem can be observed only for sufficiently large volumes of trade.

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