

Refinements of Nash equilibrium in potential games

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We prove the existence of a pure-strategy trembling-hand perfect equilibrium in upper semicontinuous potential games, and we show that generic potential games possess pure-strategy strictly perfect and essential equilibria. We also establish a more powerful result: the set of maximizers of an upper semicontinuous potential contains a strategically stable set of pure-strategy Nash equilibria.

KEYWORDS. Discontinuous game, potential game, trembling-hand perfect equilibrium, stable set, essential equilibrium.

JEL CLASSIFICATION. C72.

1. INTRODUCTION

A strategic-form game is a potential game if the incentive of all players to change their strategy can be expressed in one global function, called the game's potential. Potential games have many applications in Economics and other disciplines (cf. [Rosenthal 1973](#), [Monderer and Shapley 1996](#), [Ostrovsky and Schwarz 2005](#), [Armstrong and Vickers 2001](#), [Myatt and Wallace 2009](#), inter alia). Potential games have a distinct computational advantage in that any maximizer of a potential is a pure-strategy Nash equilibrium. Therefore, the computation of an equilibrium is reduced to the solution of an optimization problem, thus obviating the need for computational fixed point theory. A Nash equilibrium need not maximize the potential function, so the set of maximizers provides a natural equilibrium selection device. Given the focus on the set of maximizers of a potential, it would be helpful to know if any maximizers are “robust” as Nash equilibria, and it is this issue that we study in this paper.

Structurally, we work with games in which each player's strategy set is a nonempty, compact metric space and for which there exists an upper semicontinuous potential. Consequently, we are working in a framework substantially more general than the case of finite strategy spaces, and the upper semicontinuity assumption adds extra flexibility in applications. Furthermore, the upper semicontinuity assumption allows us to use some basic machinery from variational analysis.¹

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We thank the co-editor and an anonymous referee for their comments.

¹The existence of pure-strategy trembling-hand perfect equilibria in general (possibly discontinuous) strategic-form games, requires more structure (cf. [Carbonell-Nicolau 2011c](#)).

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DOI: 10.3982/TE1178

We begin by proving the existence of a pure-strategy trembling-hand perfect equilibrium in upper semicontinuous potential games. In particular, [Theorem 1](#) proves that the set of maximizers of an upper semicontinuous potential contains a pure-strategy trembling-hand perfect equilibrium, defined according to an extension of the standard notion of perfection for finite strategic-form games (cf. [Selten 1975](#)) to infinite strategic-form games (cf. [Simon and Stinchcombe 1995](#), [Al-Najjar 1995](#), and [Carbonell-Nicolau 2011a, 2011b, 2011c, 2011d](#)). [Example 1](#) shows that this result is tight: assuming the existence of a maximizer for the potential (rather than imposing upper semicontinuity) need not imply even the existence of a trembling-hand perfect equilibrium.

In [Theorem 2](#), we establish a more powerful result: the set of maximizers of an upper semicontinuous potential contains a strategically stable set of pure-strategy Nash equilibria in the sense of [Kohlberg and Mertens \(1986\)](#). In [Example 2](#), we present a game without strictly perfect equilibria and for which the unique strategically stable set is a proper subset of the set of maximizers of the potential.

In the last section of the paper, we present results for generic games. We begin with [Proposition 1](#) showing that, in the class of games that admit an upper semicontinuous potential, the set of games whose potential has a *unique* maximizer is dense. If strategy sets are finite, we show in [Proposition 2](#) that the set of games whose potential has a unique maximizer is open and dense. This set, however, need not be open when strategy sets are not finite, as we demonstrate by means of an example.

Using the notion of essential equilibrium, we show in [Proposition 3](#) that in the class of games that admit an upper semicontinuous potential, there exists a dense, residual set of games for which every maximizer of the potential is a pure-strategy essential equilibrium, hence a strictly perfect equilibrium.

Finite potential games (i.e., games with finite strategy sets) are special cases of upper semicontinuous (in fact, continuous) games and all of the results in this paper are evidently applicable to the finite case. In [Proposition 2](#), we show that potential games with finite action spaces whose corresponding potentials have unique maximizers exhibit strong robustness properties. These properties complement the robustness properties of unique maximizers in finite potential games studied by [Hofbauer and Sorger \(1999\)](#) and [Ui \(2001\)](#), as discussed at the end of [Section 5](#).

The paper concludes with an economic application that illustrates the main results. We formulate a discontinuous, potential investment game for which the set of trembling-hand perfect equilibria that maximize the potential is a strict subset of the set of maximizers of the potential.

2. PRELIMINARIES

A *strategic-form game* is a tuple $G = (X_i, u_i)_{i=1}^N$, where N is a finite number of players, X_i is a nonempty set of actions for player i , and u_i is a real-valued payoff function defined on $X := \times_{i=1}^N X_i$. A game $G = (X_i, u_i)_{i=1}^N$ is a *compact metric game* if it satisfies the following assumptions.

- (i) Each X_i is a compact metric space.
- (ii) Each u_i is bounded and Borel measurable.

In this paper, we assume that *all* games are compact, metric games. These games will be referred to simply as games. We will, however, make further assumptions later regarding, e.g., continuity of the payoffs.

Throughout the paper, we will view a payoff profile $u = (u_1, \dots, u_2)$ as an element of the complete metric space $(B(X)^N, d)$, where $B(X)$ denotes the space of bounded real-valued functions on X and the metric $d: B(X)^N \times B(X)^N \rightarrow \mathbb{R}$ is defined by

$$d((f_1, \dots, f_N), (g_1, \dots, g_N)) := \sum_{i=1}^N \sup_{x \in X} |f_i(x) - g_i(x)|.$$

Let $X_{-i} := \times_{j \neq i} X_j$ for each i . Given i and $(x_i, x_{-i}) \in X_i \times X_{-i}$, we employ the standard convention and write $(x_1, \dots, x_N)$ in X as (x_i, x_{-i}) . As usual, X is endowed with the product metric topology.

2.1 Potential games

Given $G = (X_i, u_i)_{i=1}^N$, a map $P: X \rightarrow \mathbb{R}$ is a *potential* for G if for each i and every $x_{-i} \in X_{-i}$,

$$u_i(x_i, x_{-i}) - u_i(y_i, x_{-i}) = P(x_i, x_{-i}) - P(y_i, x_{-i}) \quad \text{for all } \{x_i, y_i\} \subseteq X_i.$$

DEFINITION 1. A game is a *potential game* if it admits a potential. A game is an *upper semicontinuous potential game* if it admits an upper semicontinuous potential.

Potential games possess an important and convenient feature: a maximizer of a potential function is a pure-strategy Nash equilibrium. Stability of certain equilibria can now be defined in terms of stability of optimizers and we will exploit this in our treatment of equilibrium refinements.

2.2 Perfect and strictly perfect equilibrium

If X_i is a compact metric space, let $\Delta(X_i)$ represent the set of regular Borel probability measures on X_i , endowed with the topology of weak convergence. Since each X_i is a compact metric space, it follows that the topology of weak convergence is metrizable and that $\Delta(X_i)$ is a compact metric space. In particular, a sequence in $\Delta(X_i)$ is weakly convergent if and only if the sequence is convergent with respect to the Prokhorov metric.

Next, extend u_i to $\Delta(X) := \times_{i=1}^N \Delta(X_i)$ in the usual manner by using Fubini's theorem (recall that u_i is bounded and Borel measurable) and defining

$$u_i(\mu) := \int_X u_i d(\mu_1 \otimes \dots \otimes \mu_N) = \int_{X_1} \dots \int_{X_N} u_i d\mu_1 \dots d\mu_N.$$

The usual *mixed extension* of G is the strategic-form game

$$\overline{G} = (\Delta(X_i), u_i)_{i=1}^N.$$

For each pure strategy $x_i \in X_i$, let $\nu_{\{x_i\}}^i$ denote the corresponding Dirac measure in $\Delta(X_i)$. For each $x = (x_1, \dots, x_N) \in X$, let

$$\nu(x) := (\nu_{\{x_1\}}^1, \dots, \nu_{\{x_N\}}^N)$$

and note that the mapping $\nu: X \rightarrow \Delta(X)$ is an embedding (Theorem 15.8 in [Aliprantis and Border 2006](#)). If $I \subseteq \widehat{N} := \{1, \dots, N\}$, $x_i \in X_i$ for each $i \in I$, $\mu_i = \nu_{\{x_i\}}^i$ for each $i \in I$, and $\mu_i \in \Delta(X_i)$ for each $i \notin I$, we will write

$$u_i(\mu) = u_i(x_I, \mu_{\widehat{N} \setminus I})$$

so that, as usual, $u_i(x_1, \dots, x_N) = u_i(\nu_{\{x_1\}}^1, \dots, \nu_{\{x_N\}}^N)$.

Let $\pi_X(u)$ denote the set of pure-strategy Nash equilibria of the game $G = (X_i, u_i)_{i=1}^N$ and let $\xi_X(u)$ denote the set of mixed-strategy Nash equilibria of G , i.e., the Nash equilibria of the mixed extension $\overline{G} = (\Delta(X_i), u_i)_{i=1}^N$.

Let $B_\varepsilon(x)$ denote the open ball centered at $x \in X$ with radius $\varepsilon > 0$ (defined with respect to the product metric on X) and let $B_\varepsilon^\Delta(\sigma)$ denote the open ball of radius ε centered at $\sigma \in \Delta(X)$ (defined with respect to the product Prokhorov metric on $\Delta(X)$).

Let $M_+(X_i)$ denote the set of all regular measures defined on the Borel sets in X_i . A measure $\mu_i \in M_+(X_i)$ is *strictly positive* if $\mu_i(U) > 0$ for every nonempty open set U in X_i . Let $M_{++}(X_i)$ denote the set of all strictly positive measures in $M_+(X_i)$, let $\widehat{\Delta}(X_i)$ denote the set of all strictly positive probability measures in $M_+(X_i)$, and let $\widehat{\Delta}(X) := \times_{i=1}^N \widehat{\Delta}(X_i)$. Given $\delta = (\delta_1, \dots, \delta_N) \in (0, 1)^N$ and $\mu = (\mu_1, \dots, \mu_N) \in \widehat{\Delta}(X)$, define $u_i^{(\delta, \mu)}: X \rightarrow \mathbb{R}$ as

$$u_i^{(\delta, \mu)}(x) := u_i((1 - \delta_1)\nu_{\{x_1\}}^1 + \delta_1\mu_1, \dots, (1 - \delta_N)\nu_{\{x_N\}}^N + \delta_N\mu_N).$$

Note that $u_i^{(\delta, \mu)}$ is bounded and Borel measurable as a consequence of Fubini's theorem. Let $G_{(\delta, \mu)}$ denote the game defined as

$$G_{(\delta, \mu)} := (X_i, u_i^{(\delta, \mu)})_{i=1}^N.$$

Using the notational convention established above, $\pi_X(u^{(\delta, \mu)})$ denotes the set of pure-strategy Nash equilibria of the game $G_{(\delta, \mu)} = (X_i, u_i^{(\delta, \mu)})_{i=1}^N$ and $\xi_X(u^{(\delta, \mu)})$ denotes the set of mixed-strategy Nash equilibria of $G_{(\delta, \mu)}$, i.e., the Nash equilibria of the mixed extension $\overline{G}_{(\delta, \mu)} = (\Delta(X_i), u_i^{(\delta, \mu)})_{i=1}^N$.

DEFINITION 2 ([Selten 1975](#)). A strategy profile $\sigma \in \xi(u)$ is a *trembling-hand perfect equilibrium* in $G = (X_i, u_i)_{i=1}^N$ if there exist sequences (δ^n) , (μ^n) , and (σ^n) such that $(0, 1)^N \ni \delta^n \rightarrow 0$, $\mu^n \in \widehat{\Delta}(X)$, $\sigma^n \rightarrow \sigma$, and $\sigma^n \in \xi_X(u^{(\delta^n, \mu^n)})$ for each n .

DEFINITION 3 ([Okada 1984](#)). A strategy profile $\sigma \in \xi_X(u)$ is a *strictly perfect equilibrium* in $G = (X_i, u_i)_{i=1}^N$ if for all sequences (δ^n) and (μ^n) such that $(0, 1)^N \ni \delta^n \rightarrow 0$ and $\mu^n \in \widehat{\Delta}(X)$, there exists a sequence (σ^n) satisfying $\sigma^n \in \xi_X(u^{(\delta^n, \mu^n)})$ for each n and $\sigma^n \rightarrow \sigma$.

Every strictly perfect equilibrium is a trembling-hand perfect equilibrium. For alternative, equivalent definitions of trembling-hand perfection, the reader is referred to [Carbonell-Nicolau \(2011d\)](#).

Throughout the paper, we will not generally distinguish between the profile $(x_1, \dots, x_N) \in X$ and the corresponding profile $(\nu_{\{x_1\}}^1, \dots, \nu_{\{x_N\}}^N) \in \Delta(X)$. Consequently, we will refer to $(x_1, \dots, x_N) \in X$ as a trembling-hand perfect or strictly perfect equilibrium when we mean that the corresponding profile $(\nu_{\{x_1\}}^1, \dots, \nu_{\{x_N\}}^N)$ is a trembling-hand perfect or strictly perfect equilibrium.

3. PERFECT EQUILIBRIUM

We begin with two results that are essential for the proof of [Theorem 1](#). Their proofs are relegated to the [Appendix](#). [Lemma 1](#) asserts that the perturbed game $G_{(\delta, \mu)}$ is an upper semicontinuous potential game if the original game G is an upper semicontinuous potential game, and that an upper semicontinuous potential for $G_{(\delta, \mu)}$ can be constructed in a natural way from an upper semicontinuous potential for G . [Lemma 2](#) states that the optimizer correspondence for upper semicontinuous functions exhibits continuity with respect to uniform perturbations of the objective function.

LEMMA 1. *Suppose that $G = (X_i, u_i)_{i=1}^N$ is an upper semicontinuous potential game with upper semicontinuous potential P and suppose that $(\delta, \mu) \in (0, 1)^N \times \widehat{\Delta}(X)$. For each $x = (x_1, \dots, x_N) \in X$, define $q_i^{x_i} \in \Delta(X_i)$ as*

$$q_i^{x_i} := (1 - \delta_i)\nu_{\{x_i\}}^i + \delta_i\mu_i.$$

Then $P^{(\delta, \mu)} : X \rightarrow \mathbb{R}$, defined as

$$P^{(\delta, \mu)}(x_1, \dots, x_N) := \int_X P dq_1^{x_1} \dots dq_N^{x_N},$$

is an upper semicontinuous potential for $G_{(\delta, \mu)}$. Therefore, $G_{(\delta, \mu)}$ has a pure-strategy Nash equilibrium, i.e., $\pi_X(u^{(\delta, \mu)}) \neq \emptyset$.

LEMMA 2. *Suppose that S is a metric space and suppose that (f^n) is a uniformly convergent sequence of upper semicontinuous real-valued functions on S with uniform limit f . If $x^n \in \arg \max_{x \in X} f^n(x)$ for each n and $x^n \rightarrow x$, then $x \in \arg \max_{x \in X} f(x)$.*

We now state our main result for trembling-hand perfect equilibria.

THEOREM 1. *Suppose that $G = (X_i, u_i)_{i=1}^N$ is an upper semicontinuous potential game with upper semicontinuous potential P . Then G possesses a pure-strategy trembling-hand perfect equilibrium in $\arg \max_{x \in X} P(x)$.*

PROOF. Suppose that P is an upper semicontinuous potential for the game $G = (X_i, u_i)_{i=1}^N$. Choose $\mu \in \widehat{\Delta}(X)$ and a sequence $\delta^n = (\delta_1^n, \dots, \delta_N^n) \in (0, 1)^N$ with $\delta^n \rightarrow 0$. Applying [Lemma 1](#), it follows that $P^{(\delta^n, \mu)}$ is an upper semicontinuous potential for

$G_{(\delta^n, \mu)}$, implying that $\arg \max_{x \in X} P^{(\delta^n, \mu)}(x) \neq \emptyset$ for each n . Applying [Lemma 5](#) in [Appendix A.1](#), we conclude that P is the uniform limit of the sequence $(P^{(\delta^n, \mu)})$. For each n , choose

$$x^n \in \arg \max_{x \in X} P^{(\delta^n, \mu)}(x).$$

Then x^n is a pure-strategy equilibrium in $G_{(\delta^n, \mu)}$, i.e., $\nu(x^n) \in \pi_X(u^{(\delta^n, \mu)})$. Since X is compact, there exists a subsequence (x^{n_k}) of (x^n) and a pure-strategy profile $x \in X$ such that $x^{n_k} \rightarrow x$. From [Lemma 2](#), we conclude that

$$x \in \arg \max_{y \in X} P(y),$$

from which it follows that x is a pure-strategy equilibrium in G , i.e., $\nu(x) \in \pi_X(u)$. Finally, note that since $\nu(x^{n_k}) \in \pi_X(u^{(\delta^{n_k}, \mu)})$, $\delta^{n_k} \rightarrow 0$ and $\nu: X \rightarrow \Delta(X)$ is an embedding (hence continuous), we conclude that $\nu(x^{n_k}) \rightarrow \nu(x)$, implying that x is a pure-strategy trembling-hand perfect equilibrium in G . $\square$

We conclude this section by noting that one cannot drop upper semicontinuity of the potential in the hypothesis of [Theorem 1](#). In fact, [Example 1](#) below presents a game whose potential has a unique maximizer and whose set of pure-strategy trembling-hand perfect equilibria is empty.

Assuming that the potential of G is upper semicontinuous ensures that the corresponding potentials for perturbed games of the form $G_{(\delta, \mu)}$ are upper semicontinuous, and this, in turn, guarantees the existence of a global maximizer for the potentials of the perturbations. Simply assuming that G admits a potential that can be maximized will generally *not* be sufficient for perturbations of the form $G_{(\delta, \mu)}$ to have potentials that attain a maximum. In fact, while the game G in [Example 1](#) (below) does not admit an upper semicontinuous potential, the game does admit a potential that attains a maximum in X . Nevertheless, no sequence $(G_{(\delta^n, \mu^n)})$ of perturbations (with $(0, 1)^N \ni \delta^n \rightarrow 0$ and $\mu^n \in \hat{\Delta}(X)$) can be obtained such that each $G_{(\delta^n, \mu^n)}$ admits a potential that can be maximized.

EXAMPLE 1. For each $k \geq 1$, let $\alpha^k = (k+1)/(k+2)$. Consider the game $G = (X_i, u_i)_{i=1}^2$, where $X_1 := \{1\} \cup \{\alpha^k : k \geq 1\}$, $X_2 := [0, 1]$, and $u_1 := u_2 := u$, where

$$u(x_1, x_2) := \begin{cases} 1 & \text{if } (x_1, x_2) = (1, 0) \\ 0 & \text{if } x_1 = 1 \text{ and } x_2 \neq 0 \\ \alpha^k & \text{if } (x_1, x_2) = (\alpha^k, 0) \\ \frac{1}{2} & \text{if } x_1 = \alpha^k \text{ and } x_2 \neq 0. \end{cases}$$

Note that each X_i is compact in the Euclidean metric topology. Since $u_1 = u_2$, it follows that G is a potential game with potential $P = u$. From [Lemma 2.7](#) in [Monderer and Shapley \(1996\)](#), it follows that if $\hat{P}$ is any other potential for G , then $\hat{P} = P + c = u + c$

for some constant c . The payoff function u is not upper semicontinuous since $(\alpha^n, 1) \rightarrow (1, 1)$ but

$$\limsup_{n \rightarrow \infty} u(\alpha^n, 1) = \frac{1}{2} > 0 = u(1, 1).$$

Therefore, G has a potential but no potential for G is upper semicontinuous. To see that no equilibrium in G is trembling-hand perfect, first observe that $x_2 = 0$ is a (strictly) dominant strategy for player 2 in G , implying that the pure-strategy profile $(x_1, x_2) = (1, 0)$ is the unique equilibrium in G . Furthermore, $(x_1, x_2) = (1, 0)$ is the unique maximizer of any potential function for G . Next, choose sequences (δ^n) and (μ^n) with $\delta^n \in (0, 1)^2$ and $\delta^n \rightarrow (0, 0)$, and $\mu^n \in \hat{\Delta}(X)$.

First, we claim that $x_2 = 0$ is also the unique best response (pure or mixed) of player 2 in the game $G_{(\delta^n, \mu^n)}$. To see this, note first that if $x_1 = 1$ and $0 < y_2 \leq 1$, then $u_2(x_1, y_2) = 0$ and $\frac{1}{2} < \alpha^k$ for every k so that

$$\begin{aligned} u_2((1 - \delta_1^n)\nu_{\{x_1\}}^1 + \delta_1^n\mu_1^n, 0) &= (1 - \delta_1^n)u_2(1, 0) + \delta_1^n u_2(\mu_1^n, 0) \\ &= (1 - \delta_1^n) + \delta_1^n \left[\sum_k u_2(\alpha^k, 0)\mu_1^n(\alpha^k) + u_2(1, 0)\mu_1^n(1) \right] \\ &= (1 - \delta_1^n) + \delta_1^n \left[\sum_k \alpha^k \mu_1^n(\alpha^k) + \mu_1^n(1) \right] \\ &> 0 + \delta_1^n \left[\sum_k \frac{1}{2} \mu_1^n(\alpha^k) + 0 \right] \\ &= (1 - \delta_1^n)u_2(1, y_2) + \delta_1^n \left[\sum_k u_2(\alpha^k, y_2)\mu_1^n(\alpha^k) + u_2(1, y_2)\mu_1^n(1) \right] \\ &= u_2((1 - \delta_1^n)\nu_{\{x_1\}}^1 + \delta_1^n\mu_1^n, y_2). \end{aligned}$$

If, on the other hand, $x_1 = \alpha^m$ for some m and $0 < y_2 \leq 1$, then $u_2(x_1, y_2) = \frac{1}{2} < \alpha^k$ for every k so that

$$\begin{aligned} u_2((1 - \delta_1^n)\nu_{\{x_1\}}^1 + \delta_1^n\mu_1^n, 0) &= (1 - \delta_1^n)u_2(\alpha^m, 0) + \delta_1^n u_2(\mu_1^n, 0) \\ &= (1 - \delta_1^n)\alpha^m + \delta_1^n \left[\sum_k u_2(\alpha^k, 0)\mu_1^n(\alpha^k) + u_2(1, 0)\mu_1^n(1) \right] \\ &= (1 - \delta_1^n)\alpha^m + \delta_1^n \left[\sum_k \alpha^k \mu_1^n(\alpha^k) + \mu_1^n(1) \right] \\ &> (1 - \delta_1^n)\frac{1}{2} + \delta_1^n \left[\sum_k \frac{1}{2} \mu_1^n(\alpha^k) + 0 \right] \end{aligned}$$

$$\begin{aligned}
&= (1 - \delta_1^n)u_2(\alpha_k, y_2) + \delta_1^n \left[\sum_k u_2(\alpha^k, y_2) \mu_1^n(\alpha^k) + u_2(1, y_2) \mu_1^n(1) \right] \\
&= u_2((1 - \delta_1^n) \nu_{\{x_1\}}^1 + \delta_1^n \mu_1^n, y_2).
\end{aligned}$$

From these observations, it follows that $x_2 = 0$ is the unique best response of player 2 in the game $G_{(\delta^n, \mu^n)}$. To complete the argument, we show that player 1 has no best response (pure or mixed) to $x_2 = 0$ in the game $G_{(\delta^n, \mu^n)}$, implying that the game $G_{(\delta^n, \mu^n)}$ does not have an equilibrium. First, note that

$$\begin{aligned}
u_1(\alpha^m, \mu_2^n) &= \int_{\{0\}} u_1(\alpha^m, y_2) d\mu_2^n + \int_{(0,1]} u_1(\alpha^m, y_2) d\mu_2^n \\
&= u_1(\alpha^m, 0) \mu_2^n(\{0\}) + \int_{(0,1]} \frac{1}{2} d\mu_2^n \\
&= \alpha^m \mu_2^n(\{0\}) + \frac{1}{2} \mu_2^n((0, 1]).
\end{aligned}$$

Therefore,

$$u_1(\alpha^m, \mu_2^n) \leq u_1(\alpha^{m+1}, \mu_2^n)$$

for each m , implying that

$$\begin{aligned}
&u_1((1 - \delta_1^n) \nu_{\{\alpha^m\}}^1 + \delta_1^n \mu_1^n, (1 - \delta_2^n) \nu_{\{0\}}^2 + \delta_2^n \mu_2^n) \\
&= (1 - \delta_1^n) u_1(\alpha^m, (1 - \delta_2^n) \nu_{\{0\}}^2 + \delta_2^n \mu_2^n) + \delta_1^n u_1(\mu_1^n, (1 - \delta_2^n) \nu_{\{0\}}^2 + \delta_2^n \mu_2^n) \\
&= (1 - \delta_1^n) [(1 - \delta_2^n) u_1(\alpha^m, 0) + \delta_2^n u_1(\alpha^m, \mu_2^n)] + \delta_1^n u_1(\mu_1^n, (1 - \delta_2^n) \nu_{\{0\}}^2 + \delta_2^n \mu_2^n) \\
&= (1 - \delta_1^n) [(1 - \delta_2^n) \alpha^m + \delta_2^n u_1(\alpha^m, \mu_2^n)] + \delta_1^n u_1(\mu_1^n, (1 - \delta_2^n) \nu_{\{0\}}^2 + \delta_2^n \mu_2^n) \\
&< (1 - \delta_1^n) [(1 - \delta_2^n) \alpha^{m+1} + \delta_2^n u_1(\alpha^{m+1}, \mu_2^n)] + \delta_1^n u_1(\mu_1^n, (1 - \delta_2^n) \nu_{\{0\}}^2 + \delta_2^n \mu_2^n) \\
&= (1 - \delta_1^n) [(1 - \delta_2^n) u_1(\alpha^{m+1}, 0) + \delta_2^n u_1(\alpha^{m+1}, \mu_2^n)] + \delta_1^n u_1(\mu_1^n, (1 - \delta_2^n) \nu_{\{0\}}^2 + \delta_2^n \mu_2^n) \\
&= u_1((1 - \delta_1^n) \nu_{\{\alpha^{m+1}\}}^1 + \delta_1^n \mu_1^n, (1 - \delta_2^n) \nu_{\{0\}}^2 + \delta_2^n \mu_2^n).
\end{aligned}$$

Therefore, $u_1^{(\delta^n, \mu^n)}(\alpha^m, 0) < u_1^{(\delta^n, \mu^n)}(\alpha^{m+1}, 0)$, implying that there does not exist an m such that α^m is best response to $x_2 = 0$ in $G_{(\delta^n, \mu^n)}$. Next, observe that

$$u_1(1, \mu_2^n) = \int_{\{0\}} u_1(1, y_2) d\mu_2^n + \int_{(0,1]} u_1(1, y_2) d\mu_2^n = u_1(1, 0) \mu_2^n(\{0\}) = \mu_2^n(\{0\}).$$

Furthermore, $(0, 1]$ is open in X_2 and $\mu_2^n \in \widehat{\Delta}(X_2)$, implying that $\mu_2^n((0, 1]) > 0$. Since $\mu_2^n((0, 1]) > 0$ and $\alpha^m \rightarrow 1$, there exists an $\widehat{m}$ such that

$$(1 - \delta_2^n)(1) + \delta_2^n \mu_2^n(\{0\}) < (1 - \delta_2^n) \alpha^{\widehat{m}} + \delta_2^n \alpha^{\widehat{m}} \mu_2^n(\{0\}) + \delta_2^n \frac{1}{2} \mu_2^n((0, 1]).$$

Rearranging this expression, we obtain

$$(1 - \delta_2^n)(1) + \delta_2^n \mu_2^n(\{0\}) < (1 - \delta_2^n) \alpha^{\widehat{m}} + \delta_2^n \left[\alpha^{\widehat{m}} \mu_2^n(\{0\}) + \frac{1}{2} \mu_2^n((0, 1]) \right],$$

and this implies that

$$\begin{aligned}
 u_1(1, (1 - \delta_2^n)\nu_{\{0\}}^2 + \delta_1^n\mu_2^n) &= (1 - \delta_2^n)u_1(1, 0) + \delta_2^n u_1(1, \mu_2^n) \\
 &= (1 - \delta_2^n)(1) + \delta_2^n \mu_2^n(\{0\}) \\
 &< (1 - \delta_2^n)\alpha^{\hat{m}} + \delta_2^n \left[\alpha^{\hat{m}} \mu_2^n(\{0\}) + \frac{1}{2} \mu_2^n((0, 1]) \right] \\
 &= (1 - \delta_2^n)\alpha^{\hat{m}} + \delta_2^n \left[\alpha^{\hat{m}} \mu_2^n(\{0\}) + \frac{1}{2} \mu_2^n((0, 1]) \right] \\
 &= (1 - \delta_2^n)u_1(\alpha^{\hat{m}}, 0) + \delta_2^n u_1(\alpha^{\hat{m}}, \mu_2^n) \\
 &= u_1(\alpha^{\hat{m}}, (1 - \delta_2^n)\nu_{\{0\}}^2 + \delta_1^n\mu_2^n).
 \end{aligned}$$

Therefore,

$$\begin{aligned}
 u_1^{(\delta^n, \mu^n)}(1, 0) &= \delta_1^n u_1(1, (1 - \delta_2^n)\nu_{\{0\}}^2 + \delta_1^n\mu_2^n) + (1 - \delta^n)u_1(\mu_1^n, (1 - \delta_2^n)\nu_{\{0\}}^2 + \delta_1^n\mu_2^n) \\
 &< \delta_1^n u_1(\alpha^{\hat{m}}, (1 - \delta_2^n)\nu_{\{0\}}^2 + \delta_1^n\mu_2^n) + (1 - \delta^n)u_1(\mu_1^n, (1 - \delta_2^n)\nu_{\{0\}}^2 + \delta_1^n\mu_2^n) \\
 &= u_1^{(\delta^n, \mu^n)}(\alpha^{\hat{m}}, 0),
 \end{aligned}$$

implying that $x_1 = 1$ is not a best response to $x_2 = 0$ in $G_{(\delta^n, \mu^n)}$. This proves that the game $G_{(\delta^n, \mu^n)}$ has no Nash equilibrium and we conclude that G has no trembling-hand perfect equilibrium. $\diamond$

4. STABLE SETS OF EQUILIBRIA

If $G = (X_i, u_i)_{i=1}^N$ is a potential game with potential P , then

$$\arg \max_{x \in X} P(x) \subseteq \pi_X(u),$$

i.e., every maximizer of P is a pure-strategy Nash equilibrium in G . Therefore, $\arg \max_{x \in X} P(x)$ defines a refinement of the set of equilibria. We have shown that $\arg \max_{x \in X} P(x)$ contains a pure-strategy trembling-hand perfect equilibrium, and it is our goal to provide a relationship between $\arg \max_{x \in X} P(x)$ and strategically stable sets.

DEFINITION 4 (Kohlberg and Mertens 1986). Suppose that $G = (X_i, u_i)_{i=1}^N$ is a game. A subset $E \subseteq \xi_X(u)$ is *KM pre-stable* if E is closed and the following condition is satisfied: for every open set U containing E , there exists a $\kappa > 0$ such that for every $\delta = (\delta_1, \dots, \delta_N)$ with $0 < \delta_i < \kappa$ and for every $\mu = (\mu_1, \dots, \mu_N)$ with $\mu_i \in \hat{\Delta}(X_i)$ for each i ,

$$\xi_X(u^{(\delta, \mu)}) \cap U \neq \emptyset.$$

A subset $E \subseteq \xi_X(u)$ is a *KM stable set* if E is a minimal (with respect to set inclusion) KM pre-stable set.

REMARK 1. As a consequence of Lemma 3 in the Appendix, an equilibrium $\sigma \in \xi(u)$ is strictly perfect if and only if the set $E = \{\sigma\}$ is a KM stable set.

In the next result, we show that the set of maximizers of the potential contains a KM stable set.

THEOREM 2. Suppose that $G = (X_i, u_i)_{i=1}^N$ is an upper semicontinuous potential game with upper semicontinuous potential P . Then

$$A := \left\{ (\nu_{\{x_1\}}^1, \dots, \nu_{\{x_N\}}^N) \in \Delta(X) : (x_1, \dots, x_N) \in \arg \max_{y \in X} P(y) \right\}$$

contains a KM stable set for G .

PROOF. For each $(\delta, \mu) \in (0, 1)^N \times \widehat{\Delta}(X)$, let $G_{(\delta, \mu)}$ be the game defined in Section 2.2 as

$$G_{(\delta, \mu)} = (X_i, u_i^{(\delta, \mu)})_{i=1}^N,$$

where $u_i^{(\delta, \mu)} : X \rightarrow \mathbb{R}$ is given by

$$u_i^{(\delta, \mu)}(x) := u_i((1 - \delta_1)\nu_{\{x_1\}}^1 + \delta_1\mu_1, \dots, (1 - \delta_N)\nu_{\{x_N\}}^N + \delta_N\mu_N).$$

Let A be defined as in the statement of the theorem and let

$$A^{(\delta, \mu)} := \left\{ (\nu_{\{x_1\}}^1, \dots, \nu_{\{x_N\}}^N) \in \Delta(X) : (x_1, \dots, x_N) \in \arg \max_{y \in X} P^{(\delta, \mu)}(y) \right\}.$$

To show that A is KM pre-stable, first note that $\arg \max_{y \in X} P(y)$ is closed (since P is upper semicontinuous) in X , implying that $\arg \max_{y \in X} P(y)$ is compact in X . Since each $\nu_i : X_i \rightarrow \Delta(X_i)$ is continuous (in fact, an embedding; see Theorem 15.8 in Aliprantis and Border 2006), it follows that $\nu : X \rightarrow \Delta(X)$ is continuous. Therefore, A is compact in $\Delta(X)$, hence is closed in $\Delta(X)$ since $\Delta(X)$ is a metric space. To complete the proof that A is KM pre-stable, note that $A^{(\delta, \mu)} \subseteq \pi_X(u^{(\delta, \mu)}) \subseteq \xi_X(u^{(\delta, \mu)})$, so it suffices to prove that for every open set U containing A , there exists a $\kappa > 0$ such that the following condition holds: for every $(\delta_1, \dots, \delta_N)$ with $0 < \delta_i < \kappa$ for each i and for every $(\mu_1, \dots, \mu_N)$ with $\mu_i \in \widehat{\Delta}(X_i)$ for each i ,

$$A^{(\delta, \mu)} \cap U \neq \emptyset.$$

To see this, suppose not. Then there exists an open set U containing A , and for each n , there exist numbers $0 < \delta_i^n < 1/n$ and probability measures $\mu_i^n \in \widehat{\Delta}(X_i)$ such that $A^{(\delta^n, \mu^n)} \cap U = \emptyset$. Since P is the uniform limit of the sequence $(P^{(\delta^n, \mu^n)})$ (apply Lemma 5 in Appendix A.1) and X is compact, we can apply the same argument as that used in the proof of Theorem 1 and conclude that there exists a subsequence $(P^{(\delta^{n_k}, \mu^{n_k})})$ and a sequence $x^k \in \arg \max_{x \in X} P^{(\delta^{n_k}, \mu^{n_k})}(x)$ such that $\nu(x^k) \rightarrow \nu(x)$ and $x \in \arg \max_{y \in X} P(y)$. This contradiction establishes the claim and we conclude that A is KM pre-stable.

To complete the proof, we show that A contains a minimal KM pre-stable set by applying Zorn's lemma in a standard way. Let $\mathcal{F}$ be defined as the collection of sets E of

Nash equilibria of G satisfying (i) $E \subseteq A$ and (ii) E is KM pre-stable in G . Next, suppose that $\mathcal{F}$ is ordered by set inclusion and suppose that $\mathcal{C}$ is a totally ordered subcollection of $\mathcal{F}$. The collection $\mathcal{C}$ has the finite intersection property. Therefore, $S := \bigcap \{E : E \in \mathcal{C}\}$ is compact and nonempty since each member of $\mathcal{C}$ is closed and A is compact. To show that S is KM pre-stable, suppose that U is open and $S \subseteq U$. Then there exists $E' \in \mathcal{C}$ such that $E' \subseteq U$. Otherwise, $\{E \setminus U : E \in \mathcal{C}\}$ is a collection of closed subsets of A satisfying the finite intersection property. This implies that $S \setminus U = \bigcap \{E \setminus U : E \in \mathcal{C}\} \neq \emptyset$, an impossibility. Since E' is KM pre-stable, it follows that S is KM pre-stable. The existence of a minimal KM pre-stable set in G contained in A now follows from Zorn's lemma. $\square$

While $\arg \max_{x \in X} P(x)$ contains a KM stable set, the next example shows that $\arg \max_{x \in X} P(x)$ itself need not be KM stable. This game is (trivially) continuous and also demonstrates that a continuous potential game need not have a strictly perfect equilibrium. In addition, the unique stable set of this game is a proper subset of the set of trembling-hand perfect equilibria, and the set of pure-strategy trembling-hand perfect equilibria is a proper subset of the set of maximizers of the potential.

EXAMPLE 2. Consider the finite two-player game G defined as

	L	C	R
T	1, 1	1, 1	0, 0
B	1, 1	0, 0	1, 1

The game G is a potential game and the value of the potential P at each strategy pair is indicated in the table.

	L	C	R
T	1	1	0
B	1	0	1

In this example,

$$\arg \max_{x \in X} P(x) = \{(T, L), (T, C), (B, L), (B, R)\}.$$

The set of trembling-hand perfect equilibria consists of strategy pairs in which player 2 chooses L and player 1 randomizes arbitrarily over T and B . However, the unique KM stable set for G is $\{(T, L), (B, L)\}$, which coincides with the set of pure-strategy trembling-hand perfect equilibria. Finally, we note that G has no strictly perfect equilibria. $\diamond$

To complete the discussion of strategic stability, we show that a strategically stable set contained in $\arg \max_{x \in X} P(x)$ consists of trembling-hand perfect pure-strategy equilibria.

THEOREM 3. Suppose that $G = (X_i, u_i)_{i=1}^N$ is a game with upper semicontinuous potential P . If $E \subseteq \arg \max_{x \in X} P(x)$ and if

$$S := \{(\nu_{\{x_1\}}^1, \dots, \nu_{\{x_N\}}^N) \in \Delta(X) : (x_1, \dots, x_N) \in E\}$$

is a KM stable set, then each element of E is a pure-strategy trembling-hand perfect equilibrium.

PROOF. Let S be as defined in the statement of the theorem and suppose that S is KM stable. If $|E| = 1$, then the one member of S is a strictly perfect equilibrium, hence a trembling-hand perfect equilibrium. So suppose that $|E| > 1$. Choose $\varepsilon > 0$ so that $S \setminus B_\varepsilon^\Delta(\nu(x)) \neq \emptyset$. Since S is KM stable and $S \setminus B_\varepsilon^\Delta(\nu(x))$ is closed and nonempty, it follows from minimality that $S \setminus B_\varepsilon^\Delta(\nu(x))$ is not KM pre-stable. Therefore, there exists an open set $U \subseteq \Delta(X)$ containing $S \setminus B_\varepsilon^\Delta(\nu(x))$ such that, for every k , there exist $0 < \delta_i^k < 1/k$ and $\mu_k \in \hat{\Delta}(X)$ such that $\xi_X(u^{(\delta^k, \mu^k)}) \cap U = \emptyset$. Next, note that $S \subseteq U \cup B_\varepsilon^\Delta(\nu(x))$ and $U \cup B_\varepsilon^\Delta(\nu(x))$ is open. Since S is pre-stable, it follows that $\xi_X(u^{(\delta^k, \mu^k)}) \cap [U \cup B_\varepsilon^\Delta(\nu(x))] \neq \emptyset$ for sufficiently large k . In particular, $\xi_X(u^{(\delta^k, \mu^k)}) \cap B_\varepsilon^\Delta(\nu(x)) \neq \emptyset$ for sufficiently large k and we conclude that x is trembling-hand perfect. $\square$

REMARK 2. Results on the existence of trembling-hand perfect equilibria and stable sets in general discontinuous infinite strategic-form games have been furnished elsewhere, and the reader may wonder whether [Theorem 1](#) and [Theorem 2](#) follow from extant results. Existence results regarding the existence of (pure and mixed) trembling-hand perfect equilibria and stable sets in strategic-form games (e.g., [Carbonell-Nicolau 2011a, 2011b, 2011c, 2011d](#)) require conditions stronger than the notion of better-reply security introduced in [Reny \(1999\)](#). As the following example demonstrates, there are upper semicontinuous potential games that fail even better-reply security, implying that upper semicontinuous potential games cannot be handled by known results.

The following definitions are needed for the formal definition of better-reply security. The *graph* of a metric game $G = (X_i, u_i)_{i=1}^N$ is the set

$$\Gamma_G := \{(x, \alpha) \in X \times \mathbb{R}^N : u_i(x) = \alpha_i, \text{ for all } i \in \{1, \dots, N\}\}.$$

The closure of Γ_G is denoted by $\bar{\Gamma}_G$. A metric game $G = (X_i, u_i)_{i=1}^N$ is *better-reply secure* if for every $(x, \alpha) \in \bar{\Gamma}_G$ such that x is not a Nash equilibrium of G , there exist i , $y_i \in X_i$, $\beta \in \mathbb{R}$, and a neighborhood $V_{x_{-i}}$ of x_{-i} such that

$$u_i(y_i, y_{-i}) \geq \beta > \alpha_i \quad \text{for all } y_{-i} \in V_{x_{-i}}.$$

Consider the two-player, upper semicontinuous potential game

$$G = ([0, 1], [0, 1], u_1, u_2),$$

where

$$u_1(x_1, x_2) := u_2(x_1, x_2) := P(x_1, x_2) \quad \text{for all } (x_1, x_2) \in [0, 1]^2$$

and where $P: [0, 1]^2 \rightarrow \mathbb{R}$ is defined by

$$P(x_1, x_2) := \begin{cases} 1 & \text{if } x_1 = 1 \text{ and } x_2 \in (0, 1] \\ 2 & \text{if } (x_1, x_2) = (1, 0) \\ 0 & \text{otherwise.} \end{cases}$$

Clearly, $((1, 1), (1, 1)) \in \bar{\Gamma}_G$ and $(1, 1)$ is not a Nash equilibrium of G . Moreover, for each i and each $y_i \in [0, 1]$, and for every neighborhood V of 1, there exists $y_{-i} \in V$ such that $u_i(y_i, y_{-i}) \leq 1$, implying that G is not better-reply secure.

The game G is quasiconcave (i.e., the map $x_i \mapsto u_i(x_i, x_{-i})$ defined on X_i is quasiconcave for every $x_{-i} \in X_{-i}$). Since the results on the existence of pure-strategy trembling-hand perfect equilibria proved in [Carbonell-Nicolau \(2011c\)](#) require conditions stronger than better-reply security and quasiconcavity, the current example also illustrates that our results do not follow from known results even if one confines attention to quasiconcave potential games.

5. RESULTS FOR GENERIC GAMES

Given the geometry of maximization, it is reasonable to conjecture that “most” upper semicontinuous functions have a unique maximizer. Therefore, it is also reasonable to conjecture that the potential function for “most” upper semicontinuous potential games will have a unique maximizer. In this final section of the paper, we examine this conjecture and a related genericity result in the case of finite and general potential games.

Recall that each X_i is a compact metric space. Define $\mathcal{P}(X)$ to be the set of payoff profiles $u = (u_1, \dots, u_N)$ such that $(X_i, u_i)_{i=1}^N$ is an upper semicontinuous potential game. We view $\mathcal{P}(X)$ as a subset of the metric space $(B(X)^N, d)$, as defined in [Section 2](#). Suppose that $u = (u_1, \dots, u_N) \in \mathcal{P}(X)$, let P be a potential for u , and define

$$\varphi_X(u) := \arg \max_{x \in X} P(x).$$

This definition is unambiguous since two potentials for u give rise to the same set of maximizers.

Recalling that $\pi_X(u)$ (resp. $\xi_X(u)$) denotes the set of pure-strategy Nash equilibria (resp. mixed-strategy Nash equilibria) for u , it is clear that $\varphi_X(u) \subseteq \pi_X(u) \subseteq \xi_X(u)$.

To begin, let

$$Y(X) := \{u \in \mathcal{P}(X) : |\varphi_X(u)| = 1\},$$

i.e., each $u \in Y(X)$ defines an upper semicontinuous potential game G for which any potential has a unique maximizer (which must be the same for all potentials representing a given game). The next result provides a sense in which a potential for “most” upper semicontinuous potential games has a unique maximizer where the word “most” is translated as “dense.” The proof of [Proposition 1](#) is given in the [Appendix](#).

PROPOSITION 1. *The set $Y(X)$ is dense in $\mathcal{P}(X)$. If $u \in Y(X)$, then $|\varphi_X(u)| = 1$, the profile $x^* \in \varphi_X(u)$ is a strictly perfect equilibrium, and $\{x^*\}$ is a singleton stable set.*

A stronger result would be obtained if we could also show that $Y(X)$ is open in $\mathcal{P}(X)$. However, the following example demonstrates that this stronger result is not generally true.

EXAMPLE 3. Let $X_1 := [-1, 1] =: X_2$ and

$$u_1(x_1, x_2) = u_2(x_1, x_2) = P(x_1, x_2) := -(x_1^2 + x_2^2).$$

Then $x = 0$ is the unique maximizer of the upper semicontinuous potential P . Now let $u_1^n(x_1, x_2) = u_2^n(x_1, x_2) = P^n(x_1, x_2)$, where

$$P^n(x_1, x_2) := \begin{cases} 0 & \text{if } -\frac{1}{n} \leq x_i \leq \frac{1}{n} \text{ for each } i \in \{1, 2\} \\ -(x_1^2 + x_2^2) & \text{otherwise.} \end{cases}$$

Note that for each n , P^n is an upper semicontinuous potential with infinitely many maximizers. In addition, $|P^n(x_1, x_2) - P(x_1, x_2)| \leq 2/n^2$ for each $x \in X$, implying that (P^n) is uniformly convergent with limit P . Therefore, (u^n) is uniformly convergent with limit u and, consequently, any open set containing u also contains a v with infinitely many maximizers. This establishes that $Y(X)$ is not open in $\mathcal{P}(X)$. $\diamond$

The strategy sets of Example 3 above are not finite. If each X_i is finite, then $Y(X)$ is both open and dense. Furthermore, $x^* \in Y(X)$ is a *strict equilibrium* in $(X_i, u_i)_{i=1}^N$, i.e., x_i^* is the unique (mixed strategy) best response to x_{-i}^* . Strictness is arguably the strongest refinement concept for finite strategic-form games. In particular, every strict equilibrium is regular, hence strictly perfect (see Section 2.5, particularly Corollary 2.5.3, in van Damme 1991). These observations are summarized in the next result.

PROPOSITION 2. Suppose that each X_i is finite. Then $Y(X)$ is open and dense in $\mathcal{P}(X)$. If $u \in Y(X)$, then $|\varphi_X(u)| = 1$, the profile $x^* \in \varphi_X(u)$ is a strict equilibrium, hence a strictly perfect equilibrium, and $\{x^*\}$ is a singleton stable set.

Since every open dense set is residual, Proposition 2 implies that $Y(X)$ is dense and residual when each X_i is finite. In the general case, we will show that $Y(X)$ can be replaced by a dense residual set Z of payoff profiles in $\mathcal{P}(X)$ with the property that, for each $u \in Z$, every element of $\varphi_X(u)$ is strictly perfect, hence a singleton stable set. To prove the genericity result for general compact metric strategy sets, we need the notion of *essential equilibrium*.

DEFINITION 5 (Wu and Jiang 1962). Suppose that $u \in \mathcal{P}(X)$. An equilibrium $\sigma \in \xi_X(u)$ is *essential* if the following condition is satisfied: for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\xi_X(v) \cap B_\varepsilon^\Delta(\sigma) \neq \emptyset$ whenever $v \in \mathcal{P}(X)$ and $d(v, u) < \delta$.

We note here that every essential equilibrium of an upper semicontinuous potential game is a strictly perfect equilibrium (see Lemma 7 in the Appendix). To show that all members of $\varphi_X(u)$ are essential for all u in a “topologically large” subset of $\mathcal{P}(X)$, we exploit the relationship between essentiality and lower hemicontinuity. In particular,

all members of $\varphi_X(u)$ are essential equilibria for any $u \in \mathcal{P}(X)$ at which the correspondence $\varphi: \mathcal{P}(X) \rightrightarrows X$ is lower hemicontinuous. The key result for establishing our genericity theorem is a classic result of Fort (1951) that, informally stated, says that an upper hemicontinuous correspondence is generically lower hemicontinuous. Fort's theorem has been used in a number of papers to establish genericity of essential equilibria and essential components of equilibria in strategic-form games (e.g., Zhou et al. 2007 and the references cited therein, and Carbonell-Nicolau 2010). As an application of Fort's theorem, we obtain the following result whose proof is relegated to the Appendix.

PROPOSITION 3. *There exists a dense, residual subset $Z \subseteq \mathcal{P}(X)$ such that $\varphi_X: \mathcal{P}(X) \rightrightarrows X$ is lower hemicontinuous at each $u \in Z$. If $u \in Z$, then each $x \in \varphi_X(u)$ is an essential equilibrium, hence a strictly perfect equilibrium, and $\{x\}$ is a singleton stable set.*

In this paper, we have studied the robustness of equilibria that are maximizers of an upper semicontinuous potential with respect to perturbations of strategy sets (strict perfection) and payoffs (essentiality). When a potential for a finite game has a unique maximizer, that maximizer is robust in both of these senses (since the unique maximizer is strictly perfect and strict). Several other authors have studied potential games with finite action spaces whose corresponding potentials have unique maximizers and have shown that this unique maximizer exhibits robustness properties that complement those presented in Proposition 2. Ui (2001) considers robustness with respect to incomplete information as defined in Kajii and Morris (1997a, 1997b). Formally, Ui shows that if $G = (X_i, u_i)_{i=1}^N$ is a finite potential game with potential P and if x^* is the unique maximizer of P , then x^* is robust to canonical elaborations. Informally, this means that in each incomplete information game (in the special class of canonical elaborations) sufficiently close to G , there exists a Bayes–Nash equilibrium whose outcome assigns nearly all probability mass to the pure-strategy profile x^* . Hofbauer and Sorger (1999) consider the class of finite, two-player symmetric potential games and define a special (continuous) symmetric potential function Q derived from the mixed extension of these finite games. Hofbauer and Sorger show that if x^* is the unique maximizer of Q , then x^* defines a symmetric mixed-strategy equilibrium of the underlying finite game and that x^* is dynamically robust. In particular, x^* is approachable along a perfect foresight path.

6. APPLICATION

The set of potential maximizers contains a KM stable set (Theorem 2). In general, KM stable sets contained in the set of potential maximizers need not coincide with the set of potential maximizers. In this section, we present an investment game with externalities that admits an upper semicontinuous potential and has the following property: the set of trembling-hand perfect equilibria in the set of potential maximizers is a strict subset of the set of potential maximizers.

There are two agents. Agent i 's endowment is $w_i > 0$. Each agent can invest in two projects. For each i , let a_i (resp. b_i) denote agent i 's investment level in project A (resp. project B). A vector of investments $(a, b) = ((a_1, b_1), (a_2, b_2))$ generates a payoff of $F(a, b)$ for both agents. Note that this setting allows for external effects across projects.

The agents simultaneously choose investment levels. Agent i 's investments (a_i, b_i) are selected from the set

$$X_i := \{(x, y) \in \mathbb{R}_+^2 : x + y \leq w_i\}.$$

Let $X := X_1 \times X_2$. We assume that F is upper semicontinuous. Unlike continuity, upper semicontinuity of F is a natural assumption when the projects' technology exhibits indivisibilities: investment levels above certain thresholds suffice to generate significantly higher returns. We consider the following investment game $G := (X_i, u_i)_{i=1}^N$, where for each i , $u_i: X \rightarrow \mathbb{R}$ is defined by

$$u_i((a_1, b_1), (a_2, b_2)) := F((a_1, b_1), (a_2, b_2)) + v_i(w_i - a_i - b_i),$$

where $v_i: \mathbb{R}_+ \rightarrow \mathbb{R}$ is strictly increasing.

Define $P: X \rightarrow \mathbb{R}$ by

$$P((a_1, b_1), (a_2, b_2)) := F((a_1, b_1), (a_2, b_2)) + \sum_{i=1}^2 v_i(w_i - a_i - b_i).$$

It is routine to verify that P is an upper semicontinuous potential for G . By [Theorem 2](#), therefore, $\arg \max_{x \in X} P(x)$ contains a KM stable set.

[Monderer and Shapley \(1996\)](#) already pointed out that the set of maximizers of a potential refines the set of Nash equilibria. The game G may well exhibit Nash equilibria that do not maximize the potential.

On the other hand, G may have maximizers that are not trembling-hand perfect. Let $\tau(u)$ represent the set of trembling-hand perfect equilibria of G . We provide an example illustrating that for the game G , one can have

$$\arg \max_{x \in X} P(x) \supsetneq \left(\arg \max_{x \in X} P(x) \right) \cap \tau(u). \quad (1)$$

Let $w_1 = w_2 = 1$. Let $v_i(x) := \sqrt{x}$ for all $x \in \mathbb{R}_+$ and for each i , and assume that F takes the form

$$F((a_1, b_1), (a_2, b_2)) := \begin{cases} a_1 + a_2 + b_1 + b_2 & \text{if } a_1 + a_2 \geq c \\ \frac{1}{2}a_1 + a_2 + b_1 + b_2 & \text{if } a_1 + a_2 < c, \end{cases}$$

where $c \in (0, 1)$. This technology can be given the following interpretation: both projects yield the same returns above a certain investment threshold c , but project A 's returns are lower for investment levels below c .

If $c \in (\frac{3}{4}, 1)$, then the strategy profile

$$\left(\left(\frac{3}{4}, 0 \right), \left(\frac{3}{4}, 0 \right) \right)$$

belongs to $\arg \max_{x \in X} P(x)$. In addition, if $c \in (\frac{3}{4}, 1)$, this profile is not trembling-hand perfect. To see this, choose $\epsilon \in (0, c - \frac{3}{4})$ and note that for each $(a_2, b_2) \in X_2$, and for every $z_1 \in (\frac{3}{4} - \epsilon, \frac{3}{4} + \epsilon)$ and $z_2 \in [0, \epsilon)$ with $z_1 + z_2 \leq 1$,

$$u_1((0, z_1 + z_2), (a_2, b_2)) \geq u_1((z_1, z_2), (a_2, b_2)).$$

In addition, observe that because $\frac{3}{4} - \epsilon > 0$, there exists $\beta > 0$ such that for each $(a_2, b_2) \in X_2$ with $a_2 \in [0, c - \frac{3}{4} - \epsilon)$, and for every $z_1 \in (\frac{3}{4} - \epsilon, \frac{3}{4} + \epsilon)$ and $z_2 \in [0, \epsilon)$ with $z_1 + z_2 \leq 1$,

$$u_1((0, z_1 + z_2), (a_2, b_2)) - u_1((z_1, z_2), (a_2, b_2)) \geq \beta.$$

This implies that if player 2 “trembles” at $(\frac{3}{4}, 0)$, player 1 will not choose a strategy in a neighborhood of $(\frac{3}{4}, 0)$, implying that $((\frac{3}{4}, 0), (\frac{3}{4}, 0))$ is not trembling-hand perfect.

More precisely, let (δ^n) be a sequence in $(0, 1)^2$ with $\delta^n \rightarrow 0$, and choose a sequence (μ^n) with $\mu^n \in \widehat{\Delta}(X)$ for each n . Let (σ^n) be a sequence satisfying $\sigma^n = (\sigma_1^n, \sigma_2^n) \rightarrow ((\frac{3}{4}, 0), (\frac{3}{4}, 0))$ and $\sigma^n \in \xi_X(u^{(\delta^n, \mu^n)})$ for each n .

Step 1. Since $\sigma^n \in \xi_X(u^{(\delta^n, \mu^n)})$ for each n , we claim that

$$u_1(z_1, (1 - \delta_2^n)\sigma_2^n + \delta_2^n\mu_2^n) = u_1(\sigma_1^n, (1 - \delta_2^n)\sigma_2^n + \delta_2^n\mu_2^n) \quad \text{for each } z_1 \in \text{supp}(\sigma_1^n).$$

This result is well known when payoffs are continuous. In our upper semicontinuous case, suppose that $z_1^* \in \text{supp}(\sigma_1^n)$ but

$$u_1(z_1^*, (1 - \delta_2^n)\sigma_2^n + \delta_2^n\mu_2^n) \leq u_1(\sigma_1^n, (1 - \delta_2^n)\sigma_2^n + \delta_2^n\mu_2^n) - \alpha$$

for some $\alpha > 0$. Since $z_1 \mapsto u_1(z_1, (1 - \delta_2^n)\sigma_2^n + \delta_2^n\mu_2^n)$ is upper semicontinuous, there exists an open set U^* in X_1 such that $z_1^* \in U^*$ and

$$\begin{aligned} u_1(z_1, (1 - \delta_2^n)\sigma_2^n + \delta_2^n\mu_2^n) &< u_1(z_1^*, (1 - \delta_2^n)\sigma_2^n + \delta_2^n\mu_2^n) + \alpha \\ &\leq u_1(\sigma_1^n, (1 - \delta_2^n)\sigma_2^n + \delta_2^n\mu_2^n) \quad \text{for all } z_1 \in U^*. \end{aligned}$$

This implies that $\sigma_1^n(U^*) = 0$. Therefore, $X_1 \setminus U^*$ is a closed set with $\sigma_1^n(X_1 \setminus U^*) = 1$, implying that $z_1^* \notin \text{supp}(\sigma_1^n)$, a contradiction. This establishes that

$$u_1(z_1, (1 - \delta_2^n)\sigma_2^n + \delta_2^n\mu_2^n) \geq u_1(\sigma_1^n, (1 - \delta_2^n)\sigma_2^n + \delta_2^n\mu_2^n) \quad \text{for all } z_1 \in \text{supp}(\sigma_1^n),$$

and since

$$u_1(z_1, (1 - \delta_2^n)\sigma_2^n + \delta_2^n\mu_2^n) \leq u_1(\sigma_1^n, (1 - \delta_2^n)\sigma_2^n + \delta_2^n\mu_2^n) \quad \text{for all } z_1 \in X_1,$$

the desired result is obtained.

Step 2. Next, we claim that there exists a subsequence $(\sigma_1^{n_k})$ and a sequence (z_1^k) such that $z_1^k \rightarrow (\frac{3}{4}, 0)$ and

$$z_1^k \in \text{supp}(\sigma_1^{n_k}) \quad \text{for each } k.$$

To see this, we need the following result from [Carbonell-Nicolau and McLean \(2013, Lemma 2\)](#): Let X be a compact metric space and suppose that (μ^n) is a sequence in $\Delta(X)$ weakly converging to $\mu \in \Delta(X)$. Then there exists a subsequence (μ^{n_k}) and a set $S \subseteq X$ such that $\text{supp}(\mu) \subseteq S$ and $(\text{supp}(\mu^{n_k}))$ is convergent in the Hausdorff metric topology with limit S . Applying this result, there exists a subsequence $(\sigma_1^{n_k})$ and a set

$S \subseteq X_1$ such that $(\frac{3}{4}, 0) \in S$ and $(\text{supp}(\mu^{n_k}))$ is convergent in the Hausdorff metric topology (see, e.g., Beer 1993) with limit S . Defining (the Kuratowski–Painlevé topological limit inferior of $\text{supp}(\sigma_1^{n_k}))$

$$\text{Li supp}(\sigma_1^{n_k}) := \{y \in X_1 : y^k \rightarrow y \text{ and } y^k \in \text{supp}(\sigma_1^{n_k}) \text{ for each } k\},$$

it follows from Corollary 5.1.11 and Theorem 5.2.6 in Beer (1993) that $\text{Li supp}(\sigma^{n_k}) = S$. Therefore, $(\frac{3}{4}, 0) \in \text{Li supp}(\sigma^{n_k})$ and the claim is proved.

Step 3. Since $z_1^k \rightarrow (\frac{3}{4}, 0)$, it follows that for all sufficiently large k , we can express z_1^k as $z_1^k = (z_{11}^k, z_{12}^k)$, where $z_{11}^k \in (\frac{3}{4} - \epsilon, \frac{3}{4} + \epsilon)$, $z_{12}^k \in [0, \epsilon)$ and $z_{11}^k + z_{12}^k \leq 1$. Defining $y_1^k := (0, z_{11}^k + z_{12}^k)$ and

$$C := \left\{ (x, y) \in \mathbb{R}_+^2 : x + y \leq 1 \text{ and } x \in \left[0, c - \frac{3}{4} - \epsilon\right] \right\},$$

we conclude that since C has a nonempty interior and $\mu^n \in \widehat{\Delta}(X)$ for each n ,

$$\begin{aligned} u_1(y_1^k, (1 - \delta_2^{n_k})\sigma_2^{n_k} + \delta_2^{n_k}\mu_2^{n_k}) - u_1(z_1^k, (1 - \delta_2^{n_k})\sigma_2^{n_k} + \delta_2^{n_k}\mu_2^{n_k}) \\ \geq \beta[(1 - \delta_2^{n_k})\sigma_2^{n_k}(C) + \delta_2^{n_k}\mu_2^{n_k}(C)] \\ > 0. \end{aligned}$$

Together with Step 1, this implies that for all large enough k ,

$$\begin{aligned} u_1(y_1^k, (1 - \delta_2^{n_k})\sigma_2^{n_k} + \delta_2^{n_k}\mu_2^{n_k}) &> u_1(z_1^k, (1 - \delta_2^{n_k})\sigma_2^{n_k} + \delta_2^{n_k}\mu_2^{n_k}) \\ &= u_1(\sigma_1^{n_k}, (1 - \delta_2^{n_k})\sigma_2^{n_k} + \delta_2^{n_k}\mu_2^{n_k}), \end{aligned}$$

contradicting the assumption that $\sigma^{n_k} \in \xi_X(u^{(\delta^{n_k}, \mu^{n_k})})$ for each k . Therefore, $((\frac{3}{4}, 0), (\frac{3}{4}, 0))$ is not trembling-hand perfect.

Because $((\frac{3}{4}, 0), (\frac{3}{4}, 0))$ is not trembling-hand perfect and maximizes the potential,

$$\arg \max_{x \in X} P(x) \neq \left(\arg \max_{x \in X} P(x) \right) \cap \tau(u).$$

Consequently, Theorem 1 implies the containment in (1). We note that the strategy profile $((0, \frac{3}{4}), (0, \frac{3}{4}))$ belongs to $\arg \max_{x \in X} P(x)$ and is trembling-hand perfect.

APPENDIX

A.1 Preliminary lemmata

First, we record a useful characterization of strict perfection.

LEMMA 3. *Let $G = (X_i, u_i)_{i=1}^N$ be a game and let $\sigma \in \xi_X(u)$ be a strategy profile. The following statements are equivalent.*

- (i) *The profile σ is a strictly perfect equilibrium in G .*

(ii) For every $\varepsilon > 0$, there exists a $\kappa > 0$ such that the following condition holds: if $0 < \delta_i < \kappa$ for each i and if $\mu \in \widehat{\Delta}(X)$, then $\xi(u^{(\delta, \mu)}) \cap B_\varepsilon^\Delta(\sigma) \neq \emptyset$.

LEMMA 4. If $G = (X_i, u_i)_{i=1}^N$ is a potential game with potential $P: X \rightarrow \mathbb{R}$, then P is bounded and Borel measurable.

PROOF. Suppose $G = (X_i, u_i)_{i=1}^N$ is potential game with potential $P: X \rightarrow \mathbb{R}$. Fix $\bar{x} = (\bar{x}_1, \dots, \bar{x}_n) \in X$. It is straightforward to verify that $P^*: X \rightarrow \mathbb{R}$, defined as

$$P^*(x) := P(x) - P(\bar{x}),$$

is also a potential for G . Writing

$$\begin{aligned} P^*(x_1, \dots, x_N) &= \sum_{i=1}^N [P(x_1, \dots, x_i, \bar{x}_{i+1}, \dots, \bar{x}_N) - P(x_1, \dots, x_{i-1}, \bar{x}_i, \dots, \bar{x}_N)] \\ &= \sum_{i=1}^N [u_i(x_1, \dots, x_i, \bar{x}_{i+1}, \dots, \bar{x}_N) - u_i(x_1, \dots, x_{i-1}, \bar{x}_i, \dots, \bar{x}_N)], \end{aligned}$$

it follows that P^* is bounded and measurable since each u_i is bounded and measurable. Consequently, P is bounded and measurable. $\square$

The following lemma is used in the proof of [Theorem 1](#).

LEMMA 5. Suppose that $G = (X_i, u_i)_{i=1}^N$ is an upper semicontinuous potential game with upper semicontinuous potential P . For every $\varepsilon > 0$, there exists a $\kappa \in (0, 1)$ such that the following condition holds: for every $(\delta_1, \dots, \delta_N)$ with $0 < \delta_i < \kappa$ for each i , and for every $(\mu_1, \dots, \mu_N)$ with $\mu_i \in \widehat{\Delta}(X_i)$ for each i ,

$$\sup_{x \in X} |P(x) - P^{(\delta, \mu)}(x)| < \varepsilon.$$

PROOF. Let $\widehat{N} = \{1, \dots, N\}$. Applying an induction argument, it follows that

$$P^{(\delta, \mu)}(z) = \sum_{I \subseteq \widehat{N}} \left[\prod_{i \in I} (1 - \delta_i) \prod_{i \in \widehat{N} \setminus I} \delta_i \right] P(z_I, \mu_{\widehat{N} \setminus I})$$

so that

$$P^{(\delta, \mu)}(z) = \left[\prod_{i \in \widehat{N}} (1 - \delta_i) \right] P(z) + \sum_{\substack{I \subseteq \widehat{N}: \\ I \neq \widehat{N}}} \left[\prod_{i \in I} (1 - \delta_i) \prod_{i \in \widehat{N} \setminus I} \delta_i \right] P(z_I, \mu_{\widehat{N} \setminus I}).$$

Let $M = \sup_{x \in X} |P(x)|$ (P is bounded by [Lemma 4](#)), choose $\varepsilon > 0$, and choose $\kappa \in (0, 1)$ so that

$$[1 - (1 - \kappa)^N] + \kappa(2^N - 1)M < \varepsilon.$$

If $I \neq \widehat{N}$, then there exists a $j \in \widehat{N} \setminus I$ such that

$$\prod_{i \in I} (1 - \delta_i) \prod_{i \in \widehat{N} \setminus I} \delta_i = \delta_j \left[\prod_{i \in I} (1 - \delta_i) \prod_{i \in \widehat{N} \setminus (I \cup \{j\})} \delta_i \right],$$

implying (since $0 < \delta_i < \kappa < 1$ for each i) that

$$\prod_{i \in I} (1 - \delta_i) \prod_{i \in \widehat{N} \setminus I} \delta_i = \delta_j \left[\prod_{i \in I} (1 - \delta_i) \prod_{i \in \widehat{N} \setminus (I \cup \{j\})} \delta_i \right] < \delta_j < \kappa.$$

Then for each $z \in X$, it follows that

$$\begin{aligned} |P(z) - P^{(\delta, \mu)}(z)| &\leq \left[1 - \prod_{i \in \widehat{N}} (1 - \delta_i) \right] |P(z)| + \sum_{\substack{I \subseteq \widehat{N}: \\ I \neq \widehat{N}}} \left[\prod_{i \in I} (1 - \delta_i) \prod_{i \in \widehat{N} \setminus I} \delta_i \right] |P(z_I, \mu_{\widehat{N} \setminus I})| \\ &\leq \left[\left[1 - \prod_{i \in \widehat{N}} (1 - \delta_i) \right] + \sum_{\substack{I \subseteq \widehat{N}: \\ I \neq \widehat{N}}} \left[\prod_{i \in I} (1 - \delta_i) \prod_{i \in \widehat{N} \setminus I} \delta_i \right] \right] M \\ &\leq [1 - (1 - \kappa)^N] + \kappa(2^N - 1) M < \varepsilon \end{aligned}$$

as desired. $\square$

A.2 Proof of Lemma 1

LEMMA 1. Suppose that $G = (X_i, u_i)_{i=1}^N$ is an upper semicontinuous potential game with upper semicontinuous potential P and suppose that $(\delta, \mu) \in (0, 1)^N \times \widehat{\Delta}(X)$. For each $x = (x_1, \dots, x_N) \in X$, define $q_i^{x_i} \in \Delta(X_i)$ as

$$q_i^{x_i} := (1 - \delta_i) \nu_{\{x_i\}}^i + \delta_i \mu_i.$$

Then $P^{(\delta, \mu)} : X \rightarrow \mathbb{R}$ defined as

$$P^{(\delta, \mu)}(x_1, \dots, x_N) := \int_X P dq_1^{x_1} \dots dq_N^{x_N}$$

is an upper semicontinuous potential for $G_{(\delta, \mu)}$. Therefore, $G_{(\delta, \mu)}$ has a pure-strategy Nash equilibrium, i.e., $\pi_X(u^{(\delta, \mu)}) \neq \emptyset$.

PROOF. Suppose that $G = (X_i, u_i)_{i=1}^N$ is an upper semicontinuous potential game with upper semicontinuous potential P and suppose that $(\delta, \mu) \in (0, 1)^N \times \widehat{\Delta}(X)$. Define $q_i^{x_i}$ and $P : X \rightarrow \mathbb{R}$ as in the statement of the lemma and let $q_{-i}^{x_{-i}} = \bigotimes_{j \neq i} q_j^{x_j}$. Given i , $z_{-i} \in X_{-i}$, and $\{x_i, y_i\} \subseteq X_i$, we have

$$\begin{aligned} &u_i((1 - \delta_i) \nu_{\{x_i\}}^i + \delta_i \mu_i, z_{-i}) - u_i((1 - \delta_i) \nu_{\{y_i\}}^i + \delta_i \mu_i, z_{-i}) \\ &= (1 - \delta_i) u_i(x_i, z_{-i}) + \delta_i u_i(\mu_i, z_{-i}) - (1 - \delta_i) u_i(y_i, z_{-i}) - \delta_i u_i(\mu_i, z_{-i}) \end{aligned}$$

$$\begin{aligned}
&= (1 - \delta_i)(u_i(x_i, z_{-i}) - u_i(y_i, z_{-i})) \\
&= (1 - \delta_i)(P(x_i, z_{-i}) - P(y_i, z_{-i})) \\
&= (1 - \delta_i)(P(x_i, z_{-i}) - P(y_i, z_{-i})) + \delta_i \left(\int_{X_i} P(\cdot, z_{-i}) d\mu_i - \int_{X_i} P(\cdot, z_{-i}) d\mu_i \right) \\
&= \left[(1 - \delta_i)P(x_i, z_{-i}) + \delta_i \int_{X_i} P(\cdot, z_{-i}) d\mu_i \right] \\
&\quad - \left[(1 - \delta_i)P(y_i, z_{-i}) + \delta_i \int_{X_i} P(\cdot, z_{-i}) d\mu_i \right] \\
&= P((1 - \delta_i)v_{\{x_i\}}^i + \delta_i\mu_i, z_{-i}) - P((1 - \delta_i)v_{\{y_i\}}^i + \delta_i\mu_i, z_{-i}).
\end{aligned}$$

Consequently,

$$\begin{aligned}
u_i^{(\delta, \mu)}(x_i, x_{-i}) - u_i^{(\delta, \mu)}(y_i, x_{-i}) &= \int_{X_{-i}} [u_i((1 - \delta_i)v_{\{x_i\}}^i + \delta_i\mu_i, \cdot) - u_i((1 - \delta_i)v_{\{y_i\}}^i + \delta_i\mu_i, \cdot)] dq_{-i}^{x_{-i}} \\
&= \int_{X_{-i}} [P((1 - \delta_i)v_{\{x_i\}}^i + \delta_i\mu_i, \cdot) - P((1 - \delta_i)v_{\{y_i\}}^i + \delta_i\mu_i, \cdot)] dq_{-i}^{x_{-i}} \\
&= P^{(\delta, \mu)}(x_i, x_{-i}) - P^{(\delta, \mu)}(y_i, x_{-i}),
\end{aligned}$$

implying that $P^{(\delta, \mu)}$ is a potential for $G_{(\delta, \mu)}$.

To see that $P^{(\delta, \mu)}$ is upper semicontinuous, suppose that

$$x^n = (x_1^n, \dots, x_N^n) \rightarrow (x_1, \dots, x_N) = x.$$

Then $q_i^{x_i^n} \rightarrow q_i^{x_i}$ in the topology of weak convergence on $\Delta(X_i)$. Consequently, $(q_1^{x_1^n}, \dots, q_N^{x_N^n}) \rightarrow (q_1^{x_1}, \dots, q_N^{x_N})$ in the product topology on $\Delta(X)$. Applying Theorem 1 in Glycopantis and Muir (2000) or Theorem 3.2 in Billingsley (1968) for example, we conclude that

$$q_1^{x_1^n} \otimes \dots \otimes q_N^{x_N^n} \rightarrow q_1^{x_1} \otimes \dots \otimes q_N^{x_N},$$

so applying Fubini's theorem and Theorem 15.5 in Aliprantis and Border (2006), we obtain

$$\begin{aligned}
\limsup_{n \rightarrow \infty} P^{(\delta, \mu)}(x_1^n, \dots, x_N^n) &= \limsup_{n \rightarrow \infty} \int_X P dq_1^{x_1^n} \dots dq_N^{x_N^n} \\
&= \limsup_{n \rightarrow \infty} \int_X P d(q_1^{x_1^n} \otimes \dots \otimes q_N^{x_N^n}) \\
&\leq \int_X P d(q_1^{x_1} \otimes \dots \otimes q_N^{x_N}) \\
&= \int_X P dq_1^{x_1} \dots dq_N^{x_N} \\
&= P^{(\delta, \mu)}(x_1, \dots, x_N).
\end{aligned}$$

Since $P^{(\delta, \mu)}$ is an upper semicontinuous potential for $G_{(\delta, \mu)}$, $P^{(\delta, \mu)}$ attains a maximum at a pure-strategy Nash equilibrium of $G_{(\delta, \mu)}$. $\square$

A.3 Proof of Lemma 2

We require a few basic results from variational analysis that we record here.

DEFINITION 6. Suppose that S is a metric space. A sequence (f^n) of real-valued functions on S is *hypoconvergent with hypolimit* f if for each $x \in S$, the following conditions hold.

- (i) There exists a sequence (z^n) such that $z^n \rightarrow x$ and

$$f(x) = \lim_{n \rightarrow \infty} f^n(z^n).$$

- (ii) For every sequence (x^n) such that $x^n \rightarrow x$, we have

$$\limsup_{n \rightarrow \infty} f^n(x^n) \leq f(x).$$

The next lemma is proved for $S \subseteq \mathbb{R}^k$ in Rockafellar and Wets (2009) (Proposition 7.15) and we include a simple direct proof when S is a metric space for the sake of completeness.

LEMMA 6. Suppose that S is a metric space and suppose that (f^n) is a uniformly convergent sequence of upper semicontinuous real-valued functions on S with uniform limit f . Then f is upper semicontinuous and (f^n) is hypoconvergent with hypolimit f .

PROOF. Choose $x \in S$ and suppose that (x^n) is convergent in S with limit x and choose $\varepsilon > 0$. Uniform convergence implies that there exists an m such that

$$|f^m(x) - f(x)| < \frac{1}{2}\varepsilon$$

for all $x \in X$. Since f^m is upper semicontinuous, there exists an $\hat{n}$ such that

$$f^m(x^n) < f(x) + \frac{1}{2}\varepsilon$$

whenever $n > \hat{n}$. Therefore, $n > \hat{n}$ implies that

$$f(x^n) - f(x) = [f(x^n) - f^m(x^n)] + [f^m(x^n) - f(x)] < \varepsilon$$

and we conclude that f is upper semicontinuous. Next, note that uniform convergence implies that for each $\varepsilon > 0$, there exists an n^* such that

$$|f^n(x^n) - f(x^n)| < \varepsilon$$

for all $n > n^*$. Therefore,

$$f^n(x^n) < f(x^n) + \varepsilon$$

whenever $n > n^*$. The upper semicontinuity of f implies that

$$\limsup_{n \rightarrow \infty} f^n(x^n) \leq \limsup_{n \rightarrow \infty} f(x^n) + \varepsilon \leq f(x) + \varepsilon$$

and it follows that

$$\limsup_{n \rightarrow \infty} f^n(x^n) \leq f(x).$$

Therefore, condition (ii) in the definition of hypoconvergence is satisfied. To show that condition (i) is satisfied, define $z^n = x$ for all n . Noting that uniform convergence implies pointwise convergence, it follows that

$$f(x) = \lim f^n(x) = \lim f^n(z^n),$$

proving that (f^n) is hypoconvergent with hypolimit f . $\square$

LEMMA 2. *Suppose that S is a metric space and suppose that (f^n) is a uniformly convergent sequence of upper semicontinuous real-valued functions on S with uniform limit f . If $x^n \in \arg \max_{x \in X} f^n(x)$ for each n and $x^n \rightarrow x$, then $x \in \arg \max_{x \in X} f(x)$.*

PROOF. Suppose that (f^n) is a uniformly convergent sequence of upper semicontinuous real-valued functions on S with uniform limit f , and suppose that $x^n \in \arg \max_{x \in X} f^n(x)$ for each n and $x^n \rightarrow x$. Applying Lemma 6, it follows that (f^n) is hypoconvergent with hypolimit f . Applying Theorems 5.3.5 and 5.3.6 in Beer (1993), we conclude that $x \in \arg \max_{x \in X} f(x)$. $\square$

A.4 Proof of Propositions 1, 2, and 3

We first state and prove a number of preliminary results.

LEMMA 7. *Suppose that $u \in \mathcal{P}(X)$. If $\sigma \in \xi_X(u)$ is an essential equilibrium in $G = (X_i, u_i)_{i=1}^N$, then σ is a strictly perfect equilibrium.*

PROOF. Suppose that $\sigma \in \xi_X(u)$ is an essential equilibrium in $G = (X_i, u_i)_{i=1}^N$. Fix $\varepsilon > 0$. Then there exists a $\theta > 0$ such that $\xi_X(v) \cap B_\varepsilon^\Delta(\sigma) \neq \emptyset$ whenever $v \in \mathcal{P}(X)$ and $d(v, u) < \theta$. Next, we can duplicate the proof of Lemma 5 (with u_i replacing P and $u_i^{(\delta, \mu)}$ replacing $P^{(\delta, \mu)}$) and conclude that there exists a $\kappa > 0$ such that the following condition holds for each player i : for every $(\delta_1, \dots, \delta_N)$ with $0 < \delta_i < \kappa$ for each i , and for every $(\mu_1, \dots, \mu_N)$ with $\mu_i \in \hat{\Delta}(X_i)$ for each i ,

$$\sup_{x \in X} |u_i(x) - u_i^{(\delta, \mu)}(x)| < \frac{\theta}{n}.$$

Since $u \in \mathcal{P}(X)$ admits at least one upper semicontinuous potential P , it follows that $u^{(\delta, \mu)} \in \mathcal{P}(X)$ since $P^{(\delta, \mu)}$ is an upper semicontinuous potential for $u^{(\delta, \mu)}$ as a consequence of Lemma 1. Consequently, $u^{(\delta, \mu)} \in \mathcal{P}(X)$ and $d(u^{(\delta, \mu)}, u) < \theta$ whenever $\mu \in \hat{\Delta}(X)$ and $0 < \delta_i < \kappa$ for each i . Therefore, $\xi_X(u^{(\delta, \mu)}) \cap B_\varepsilon^\Delta(\sigma) \neq \emptyset$, and we deduce from Lemma 3 that σ is strictly perfect. $\square$

PROPOSITION 1. *The set $Y(X)$ is dense in $\mathcal{P}(X)$. If $u \in Y(X)$, then $|\varphi_X(u)| = 1$, the profile $x^* \in \varphi_X(u)$ is a strictly perfect equilibrium, and $\{x^*\}$ is a singleton stable set.*

PROOF. Choose $u \in \mathcal{P}(X)$, a potential P for u , and $x^* \in \varphi_X(u)$. Next, define for each n a function $P^n : X \rightarrow \mathbb{R}$ as

$$P^n(x) := \begin{cases} P(x) & \text{if } x \neq x^* \\ P(x^*) + \frac{1}{n} & \text{if } x = x^*. \end{cases}$$

In addition, define for each i and n , a function $u_i^n : X \rightarrow \mathbb{R}$ as

$$u_i^n(x) := \begin{cases} u_i(x) & \text{if } x \neq x^* \\ u_i(x^*) + \frac{1}{n} & \text{if } x = x^*. \end{cases}$$

We claim that P^n is a potential for $(X_i, u_i^n)_{i=1}^N$. To see this, choose i , $x_{-i} \in X_{-i}$, and $\{x'_i, x''_i\} \subseteq X_i$. If $x_{-i} \neq x^*_{-i}$, then

$$\begin{aligned} u_i^n(x'_i, x_{-i}) - u_i^n(x''_i, x_{-i}) &= u_i(x'_i, x_{-i}) - u_i(x''_i, x_{-i}) \\ &= P(x'_i, x_{-i}) - P(x''_i, x_{-i}) \\ &= P^n(x'_i, x_{-i}) - P^n(x''_i, x_{-i}). \end{aligned}$$

If $x_{-i} = x^*_{-i}$, $x'_i = x^*_i$, and $x''_i \neq x^*_i$, then

$$\begin{aligned} u_i^n(x'_i, x_{-i}) - u_i^n(x''_i, x_{-i}) &= u_i(x^*_i, x^*_{-i}) + \frac{1}{n} - u_i(x''_i, x^*_{-i}) \\ &= P(x^*_i, x^*_{-i}) + \frac{1}{n} - P(x''_i, x^*_{-i}) \\ &= P^n(x^*_i, x^*_{-i}) - P^n(x''_i, x^*_{-i}) \\ &= P^n(x'_i, x_{-i}) - P^n(x''_i, x_{-i}). \end{aligned}$$

If $x_{-i} = x^*_{-i}$ and $x'_i = x^*_i = x''_i$, then

$$u_i^n(x'_i, x_{-i}) - u_i^n(x''_i, x_{-i}) = 0 = P^n(x'_i, x_{-i}) - P^n(x''_i, x_{-i}).$$

Furthermore, P^n is upper semicontinuous, and $|\varphi_X(u^n)| = 1$ since x^* is the unique maximizer of P^n . Since $u^n \in Y(X)$ for each n and (u^n) converges uniformly with limit u , we conclude that $Y(X)$ is dense in $\mathcal{P}(X)$. $\square$

Let $\text{USC}(X)$ denote the space of upper semicontinuous real-valued functions on $X = X_1 \times \cdots \times X_N$ and recall that $\mathcal{P}(X)$ is the set of payoff profiles $u = (u_1, \dots, u_N)$ such that $(X_i, u_i)_{i=1}^N$ is an upper semicontinuous potential game. Since a given potential game can be identified with an equivalence class of potentials that only differ by a constant, it will be convenient to specify a particular normalized potential with each $u \in \mathcal{P}(X)$. Fix $\bar{x} \in X$. For each $u \in \mathcal{P}(X)$, let $F(u) \in \text{USC}(X)$ denote the potential for u defined as

$$F(u)(x_1, \dots, x_N) = \sum_{i=1}^N [u_i(x_1, \dots, x_i, \bar{x}_{i+1}, \dots, \bar{x}_N) - u_i(x_1, \dots, x_{i-1}, \bar{x}_i, \dots, \bar{x}_N)].$$

Consequently,

$$\varphi_X(u) = \arg \max_{x \in X} F(u)(x).$$

We will suppress the dependence of F on $\bar{x}$ to lighten the notation.

LEMMA 8. *The mapping $F: \mathcal{P}(X) \rightarrow \text{USC}(X)$ is uniformly continuous.*

PROOF. Choose $\varepsilon > 0$, choose $0 < \delta < \frac{1}{2}\varepsilon$, and suppose that $\{u, v\} \subseteq \mathcal{P}(X)$ and $d(v, u) < \delta$. Then for each $(x_1, \dots, x_N) \in X$, we have

$$\begin{aligned} & |F(u)(x_1, \dots, x_N) - F(v)(x_1, \dots, x_N)| \\ & \leq \sum_{i=1}^N |u_i(x_1, \dots, x_i, \bar{x}_{i+1}, \dots, \bar{x}_N) - v_i(x_1, \dots, x_i, \bar{x}_{i+1}, \dots, \bar{x}_N)| \\ & \quad + \sum_{i=1}^N |v_i(x_1, \dots, x_{i-1}, \bar{x}_i, \dots, \bar{x}_N) - u_i(x_1, \dots, x_{i-1}, \bar{x}_i, \dots, \bar{x}_N)| \\ & \leq 2\delta \\ & < \varepsilon, \end{aligned}$$

so F is uniformly continuous. $\square$

LEMMA 9 (Fort 1951). *Suppose that S is a topological space and Y is a metric space. If the correspondence $\psi: S \rightrightarrows Y$ is nonempty-valued, compact-valued, and upper hemicontinuous, then ψ is lower hemicontinuous at all points in a residual subset of S .*

PROPOSITION 2. *Suppose that each X_i is finite. Then $Y(X)$ is open and dense in $\mathcal{P}(X)$. If $u \in Y(X)$, then $|\varphi_X(u)| = 1$, the profile $x^* \in \varphi_X(u)$ is a strict equilibrium, hence a strictly perfect equilibrium, and $\{x^*\}$ is a singleton stable set.*

PROOF. Given Proposition 1, we must show that $Y(X)$ is open in $\mathcal{P}(X)$. Choose $u \in \mathcal{P}(X)$ and suppose that $\{x^*\} = \varphi_X(u)$. Since X is finite and x^* is the unique maximizer of the potential function $F(u)$, there exists $\varepsilon > 0$ such that $F(u)(x^*) - F(u)(x) \geq \varepsilon$ for all $x \neq x^*$. Applying Lemma 8, there exists a $\delta > 0$ such $|F(u)(x) - F(v)(x)| < \frac{1}{3}\varepsilon$ for all $x \in X$ whenever $v \in \mathcal{P}(X)$ and $d(v, u) < \delta$. We claim that $|\varphi_X(v)| = 1$ if $d(v, u) < \delta$, proving that $Y(X)$ is open. To see this, choose v satisfying $d(v, u) < \delta$. It suffices to show that x^* is the unique maximizer for $F(v)$. If $x \in X$, then

$$\begin{aligned} F(v)(x^*) - F(v)(x) &= [F(v)(x^*) - F(u)(x^*)] \\ & \quad + [F(u)(x^*) - F(u)(x)] + [F(u)(x) - F(v)(x)] \\ & > \left(-\frac{1}{3}\varepsilon\right) + \varepsilon + \left(-\frac{1}{3}\varepsilon\right) \\ &= \frac{1}{3}\varepsilon \end{aligned}$$

and we conclude that x^* is the unique maximizer for $F(v)$. If $u \in Y(X)$ and $\varphi_X(u) = \{x^*\}$, then x^* is a strict equilibrium in $(X_i, u_i)_{i=1}^N$, i.e., x_i^* is the unique (mixed strategy) best response to x_{-i}^* . Applying Corollary 2.5.3, Theorem 2.5.5, Corollary 2.4.5, and Theorem 2.4.3 in van Damme (1991), we conclude that x^* is a strictly perfect equilibrium and a KM stable singleton set. $\square$

PROPOSITION 3. *There exists a dense, residual subset $Z \subseteq \mathcal{P}(X)$ such that $\varphi_X : \mathcal{P}(X) \rightrightarrows X$ is lower hemicontinuous at each $u \in Z$. If $u \in Z$, then each $x \in \varphi_X(u)$ is an essential equilibrium, hence a strictly perfect equilibrium, and $\{x\}$ is a singleton stable set.*

PROOF. Obviously, $\varphi_X(u) = \arg \max_{x \in X} F(u)(x)$ is nonempty and compact for each $u \in \mathcal{P}(X)$. Next, we claim that the correspondence $\varphi_X : \mathcal{P}(X) \rightrightarrows X$ is upper hemicontinuous. Since X is compact, it suffices to show that φ_X has a closed graph. To see this, suppose that $u^n \rightarrow u$, $x^n \rightarrow x$, and $x^n \in \varphi_X(u^n) = \arg \max_{y \in X} F(u^n)(y)$ for each n . Applying Lemma 8, it follows that $F(u^n) \rightarrow F(u)$ uniformly on X , so from Lemma 2 we conclude that $x \in \varphi_X(u)$. Applying Lemma 9 to the upper hemicontinuous correspondence φ_X , there exists a residual subset $Z \subseteq \mathcal{P}(X)$ such that $\varphi_X : \mathcal{P}(X) \rightrightarrows X$ is lower hemicontinuous at each $u \in Z$. Suppose that $u \in Z$ and $x \in \varphi_X(u)$, and choose $\varepsilon > 0$. We must show that there exists a $\delta > 0$ such that $\xi_X(v) \cap B_\varepsilon^\Delta(\nu(x)) \neq \emptyset$ whenever $v \in \mathcal{P}(X)$ and $d(v, u) < \delta$. Since each $\nu_i : X_i \rightarrow \Delta(X_i)$ is an embedding, it follows that $\nu : X \rightarrow \Delta(X)$ is continuous and injective. Since $\nu(x) \in \nu(\varphi_X(u)) \cap B_\varepsilon^\Delta(\nu(x))$ and ν is injective, it follows that $x \in \nu^{-1}[\nu(\varphi_X(u)) \cap B_\varepsilon^\Delta(\nu(x))] = \varphi_X(u) \cap \nu^{-1}[B_\varepsilon^\Delta(\nu(x))]$. Since $B_\varepsilon^\Delta(\nu(x))$ is open in $\Delta(X)$ and ν is continuous, and since φ_X is lower hemicontinuous at u , we conclude that there exists a $\delta > 0$ such that $\varphi_X(v) \cap \nu^{-1}[B_\varepsilon^\Delta(\nu(x))] \neq \emptyset$ whenever $v \in \mathcal{P}(X)$ and $d(v, u) < \delta$. Therefore, ν injective implies that

$$\emptyset \neq \nu(\varphi_X(v) \cap \nu^{-1}[B_\varepsilon^\Delta(\nu(x))]) \subseteq \nu(\varphi_X(v)) \cap B_\varepsilon^\Delta(\nu(x)) \subseteq \xi_X(v) \cap B_\varepsilon^\Delta(\nu(x))$$

whenever $v \in \mathcal{P}(X)$ and $d(v, u) < \delta$. It follows that every member of $\varphi_X(u)$ is an essential equilibrium and hence a strictly perfect equilibrium (Lemma 7). To complete the proof, we show that Z is dense. From Lemma 6, we conclude that $\mathcal{P}(X)$ is a closed subset of the Banach space $[B(X)]^N$, implying that $\mathcal{P}(X)$ is a complete metric space. Therefore, Z is dense as a consequence of the Baire category theorem. $\square$

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