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Budget balance, fairness, and minimal manipulability

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A common real-life problem is to fairly allocate a number of indivisible objects and a fixed amount of money among a group of agents. Fairness requires that each agent weakly prefers his consumption bundle to any other agent's bundle. In this context, fairness is incompatible with budget balance and nonmanipulability (Green and Laffont 1979). Our approach here is to weaken or abandon nonmanipulability. We search for the rules that are minimally manipulable among all fair and budget-balanced rules. First, we show for a given preference profile, all fair and budget-balanced rules are either (all) manipulable or (all) nonmanipulable. Hence, measures based on counting profiles where a rule is manipulable or considering a possible inclusion of profiles where rules are manipulable do not distinguish fair and budget-balanced rules. Thus, a "finer" measure is needed. Our new concept compares two rules with respect to their degree of manipulability by counting for each profile the number of agents who can manipulate the rule. Second, we show that maximally preferred fair allocation rules are the minimally (individually and coalitionally) manipulable fair and budget-balanced allocation rules according to our new concept. Such rules choose allocations with the maximal number of agents for whom the utility is maximized among all fair and budget-balanced allocations.

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1. Introduction

We consider the allocation of indivisible objects and a fixed amount of money among a set of agents through a mechanism (Alkan et al. 1991, Svensson 1983, Tadenuma and Thomson 1991). The important criterion in this literature is fairness (or envy-freeness), meaning that each agent should like his own consumption bundle (consisting of an object and a monetary compensation) at least as well as that of anyone else.

When analyzing this type of allocation problem, fairness is often coupled with other properties. One such property is nonmanipulability, which guarantees that no agent can gain by strategic misrepresentation. Another one is budget balance, saying that the sum of monetary compensations should equal the fixed amount of money. A famous result by Green and Laffont (1979) shows that there exists no allocation mechanism that is nonmanipulable, fair, and budget-balanced. In this paper, we will weaken or abandon nonmanipulability and offer results that facilitate the comparison of fair and budget-balanced mechanisms according to their level of manipulability when preferences are represented by quasi-linear utility functions.[1]

One way of evaluating the degree of manipulability of a mechanism (e.g., Aleskerov and Kurbanov 1999, Kelly 1988, 1993, Maus et al. 2007a, 2007b) is the idea of counting the number of preference profiles at which a given mechanism is manipulable. A second direction (Pathak and Sönmez 2013) relies on comparing the sets of preference profiles on which any two mechanisms are manipulable. Previous papers have investigated a number of different problems, including voting rules, matching mechanisms, and school choice mechanisms. However, we are not aware of any study with attention to fair and budget-balanced rules.[2]

Our first main result shows for a given preference profile that all fair and budget-balanced rules are either (all) manipulable or (all) nonmanipulable. Therefore, measures based on counting profiles where a rule is manipulable and/or considering the inclusion of profiles where a rule is manipulable do not distinguish fair and budget-balanced allocation rules. With respect to those measures, all fair and budget-balanced allocation rules are equally manipulable. The above-mentioned measures of minimal manipulability are “coarse” in the sense that preference profiles are categorized as manipulable (for all fair and budget-balanced rules) or nonmanipulable (for all fair and budget-balanced rules). For this reason, none of the existing measures is satisfactory when evaluating rules in our context.

In resolving this problem, we introduce a new “finer” measure of minimal manipulability. Because this measure cannot be based solely on the preference domain, a natural approach is to compare two rules via the number of agents who can manipulate the rule at a given preference profile. Then a rule is minimally manipulable (with respect to agents counting) if, for each preference profile, the number of manipulating agents

[1] In the early literature (e.g., Moulin 1980), the primary focus was on restricting the preference domain under which a mechanism is nonmanipulable.

[2] Subsequent to this paper, Fujinaka and Wakayama (2011) and Andersson et al. (2012) adopted a fundamentally different approach by searching for the fair and budget-balanced allocation rules that minimize the maximal manipulation possibilities (defined in terms of an agent’s utility gain from manipulation) across agents.

is smaller than or equal to the number of manipulating agents at an arbitrary fair and budget-balanced allocation rule. This guarantees that the minimally manipulable rule is nonmanipulable whenever there exists a nonmanipulable rule. The main feature of (global) nonmanipulability is respected as much as possible in the sense that the ultimate goal of our new notion is to have zero manipulating agents at each preference profile. Our second main result shows that "maximally preferred" fair allocation rules are agents-counting-minimally manipulable among all fair and budget-balanced allocation rules. Roughly, speaking those rules choose allocations with the maximal number of agents for whom the utility is maximized among all fair and budget-balanced allocations.

We further show that any fair and budget-balanced allocation rule that is not maximally preferred is strongly more manipulable with respect to agents counting than a maximally preferred fair allocation rule. We also show that these results are robust with respect to coalitional manipulations. In the same vein as before, when comparing two mechanisms we count the number of coalitions that can manipulate at a given profile. Again, maximally preferred fair allocation rules are least coalitionally manipulable among all fair and budget-balanced allocation rules. Finally, when comparing two rules with respect to inclusion of the agents who can manipulate the rule at a profile (à la Pathak and Sönmez 2013), we show that preferred fair allocation rules are minimally manipulable among all fair and budget-balanced allocation rules. Such rules choose, for any profile, an arbitrary agent k and then select the allocations that maximize agent k's utility among all fair and budget-balanced allocations. Here it is possible that the same agent k is chosen for any profile.

An alternative approach to ours is to abandon budget balance. A complete characterization of the class of fair and nonmanipulable allocation rules has been provided by Andersson and Svensson (2008), Sun and Yang (2003), and Svensson (2009). Any such rule fixes a maximal compensation for each object, and for any profile, a "maximal" fair allocation is chosen without exceeding the fixed compensations for any object. As a result, the allocation rules in this class violate budget balance. However, in many fair allocation problems, budget balance is a necessary requirement and nonmanipulability must be abandoned. Even though this type of problem has been considered previously by, e.g., Tadenuma and Thomson (1993), Aragones (1995), Haake et al. (2002), Klijn (2000), Abdulkadiroğlu et al. (2004), Azacis (2008), and Velez (2011), two issues have not been investigated. First, although it is known that each fair and budget-balanced allocation rule is manipulable at some preference profile, a characterization of the preference profiles where successful misrepresentations are possible is missing. Second, there is a large class of fair and budget-balanced allocation rules but it is not known exactly which rules are "minimally" or "least" manipulable. Our paper addresses these two issues.

The paper is organized as follows. In Section 2, we introduce assignment with compensation, and fair and budget-balanced allocation rules. In Section 3, our first main result shows that for a given preference profile, all fair and budget-balanced rules are either (all) manipulable or (all) nonmanipulable. In Section 4, we discuss different measures of the degree of manipulability of rules. We show that measures that compare different rules via profiles counting or profiles inclusion cannot be used to distinguish among fair and budget-balanced allocation rules. Then we introduce our new criterion of minimal manipulability by counting at each profile the number of agents who can manipulate. In Section 5, we define k-preferred fair allocations. We show that k-preferred fair allocations always exist and that all agents are indifferent between all k-preferred fair allocations. We then introduce components and maximally preferred fair allocation rules. Our second main result shows that maximally preferred fair allocation rules are agents-counting-minimally manipulable among all fair and budget-balanced allocation rules. We show that the same result holds if we compare two rules by counting the number of coalitions that can manipulate the rule at the profile. Finally, we show that when comparing rules with respect to inclusion of the set of agents who can manipulate, preferred fair allocation rules are agents-inclusion-minimally manipulable among all fair and budget-balanced allocation rules. All technical results and proofs omitted in the main text are relegated to the Appendix.

2. Assignment with compensations

Let $N=\{1,\dots,n\}$ and $M=\{1,\dots,m\}$ denote the set of agents and objects, respectively. The number of agents and objects are assumed to coincide, i.e., $|N|=|M|$.[3] Each agent $i\in N$ consumes exactly one object $j\in M$ together with some amount of money. A consumption bundle is a pair $(j,x_j)\in M\times\mathbb{R}$, where x_j is the monetary compensation received when consuming object j. An allocation (a,x) is a list of $|N|$ consumption bundles, where $a:N\to M$ is a mapping assigning object a_i to agent $i\in N$ and where $x\in\mathbb{R}^M$ (or $x:M\to\mathbb{R}$) assigns the amount x_j of money for the object $j\in M$. An allocation (a,x) is *feasible* if $a_i\neq a_j$ whenever $i\neq j$ for $i,j\in N$, and $\sum_{j\in M}x_j\leq 0$.[4] If $\sum_{j\in M}x_j=0$, then the allocation (a,x) satisfies *budget balance*. Let $\mathcal{A}$ denote the set of feasible and budget-balanced allocations.

Each agent $i\in N$ has preferences over consumption bundles (j,x_j), which are represented by continuous utility functions $u_i:M\times\mathbb{R}^M\to\mathbb{R}$. We will write $u_{ij}(x)$ instead of $u_i(j,x)$ to denote the utility of agent $i\in N$ when consuming object $j\in M$ and receiving compensation x_j in the distribution vector x. The utility function u_i is assumed to be quasi-linear and strictly increasing (or monotonic) in money, i.e.,

$$u_{ij}(x)=v_{ij}+x_j \quad \text{for some } v_{ij}\in\mathbb{R}.$$

A list of utility functions $u=(u_i)_{i\in N}$ is a preference profile. We also adopt the notational convention of writing $u=(u_C,u_{-C})$ for $C\subseteq N$. The set of preference profiles with utility functions having the above properties is denoted by $\mathcal{U}$.

Let $u\in\mathcal{U}$ and (a,x) be a feasible allocation. Then (a,x) is *efficient* if there exists no feasible allocation (b,y) such that $u_{ib_i}(y)\geq u_{ia_i}(x)$ for all $i\in N$ with strict inequality holding for some $i\in N$. Obviously, if (a,x) is efficient, then (a,x) is budget-balanced.

[3] If $|N|>|M|$, then we simply add $|N|-|M|$ null objects with zero value for all agents.

[4] All our results remain true if the budget constraint is replaced by $\sum_{j\in M}x_j\leq x_0$ for an arbitrary constant $x_0\in\mathbb{R}$.

Throughout we focus on feasible allocations satisfying budget balance.[5] For convenience, in the following, "allocation" stands for "feasible allocation satisfying budget balance."

The important concept in this literature is fairness, which corresponds to envy-freeness (Foley 1967). It says that each agent weakly prefers his consumption bundle to any other agent's bundle.

Definition 1. For a given profile $u \in \mathcal{U}$, an allocation (a, x) is *fair* if $u_{ia_i}(x) \geq u_{ia_j}(x)$ for all $i, j \in N$. Let $F(u)$ denote the set of fair allocations for a given profile $u \in \mathcal{U}$.

Under fairness, for feasible allocations, efficiency is equivalent to budget balance.[6]

An *allocation rule* is a nonempty correspondence φ choosing for each profile $u \in \mathcal{U}$ a nonempty set of allocations, $\varphi(u) \subseteq \mathcal{A}$, such that (i) $u_{ib_i}(y) = u_{ia_i}(x)$ for all $i \in N$ and all $(a, x), (b, y) \in \varphi(u)$, and (ii) for all $(a, x) \in \varphi(u)$ and all $(b, y) \in \mathcal{A}$, if $u_{ib_i}(y) = u_{ia_i}(x)$ for all $i \in N$, then $(b, y) \in \varphi(u)$. Hence, (i) the various allocations in the set $\varphi(u)$ are utility equivalent (*essentially single-valuedness*) and (ii) any allocation, which is utility equivalent to an allocation in $\varphi(u)$, belongs to $\varphi(u)$ (*Pareto indifference*). Alternatively, we may consider *essentially single-valued* allocation rules (which do not necessarily satisfy (ii)) or *single-valued* allocation rules choosing for each profile $u \in \mathcal{U}$ a unique allocation. All our results remain unchanged for (essentially) single-valued allocation rules.[7]

An allocation rule φ is called *fair (and budget-balanced)* if for any profile $u \in \mathcal{U}$, $\varphi(u) \subseteq F(u)$.

3. Manipulability and nonmanipulability

Our first main result will determine the (non-)manipulation possibilities of fair allocation rules.

Definition 2. An allocation rule φ is *manipulable* at a profile $u \in \mathcal{U}$ by an agent $i \in N$ if there exists a profile $(\hat{u}_i, u_{-i}) \in \mathcal{U}$ and two allocations $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(\hat{u}_i, u_{-i})$ such that $u_{ib_i}(y) > u_{ia_i}(x)$. If the allocation rule φ is not manipulable by any agent at profile $u \in \mathcal{U}$, then φ is *nonmanipulable* at profile $u \in \mathcal{U}$.

Since allocation rules may choose sets of allocations, one may alternatively employ a more conservative notion of manipulability: φ is strongly manipulable at a profile $u \in \mathcal{U}$ by an agent $i \in N$ if there exists a profile $(\hat{u}_i, u_{-i}) \in \mathcal{U}$ such that $u_{ib_i}(y) > u_{ia_i}(x)$ for all $(a, x) \in \varphi(u)$ and all $(b, y) \in \varphi(\hat{u}_i, u_{-i})$. From Svensson (2009, Proposition 3 and its proof), it follows that for any fair allocation rule φ and any profile $u \in \mathcal{U}$, φ is strongly

[5] When budget balance is relaxed to $\sum_{j \in M} x_j \leq 0$, then general nonmanipulability results are possible; see, e.g., Andersson and Svensson (2008), Sun and Yang (2003), or Svensson (2009).

[6] This is due to the fact that any fair allocation must assign the objects efficiently.

[7] Details can be found in Andersson et al. (2010a).

manipulable at profile $u \in \mathcal{U}$ by $i \in N$ if and only if φ is manipulable at profile $u \in \mathcal{U}$ by $i \in N$.[8] Hence, we may use the conservative notion of manipulability instead of ours.

It is well known (Green and Laffont 1979) that any fair and budget-balanced rule φ is manipulable for some profile $u \in \mathcal{U}$. Even though we are primarily interested in manipulation by individuals, it will be interesting to formulate our main results in terms of manipulation by coalitions. We adopt the following version of coalitional manipulability and coalitional nonmanipulability.[9] As usual, a coalition is a nonempty subset of N.

Definition 3. An allocation rule φ is *(coalitionally) manipulable* at a profile $u \in \mathcal{U}$ by a coalition $C \subseteq N$ if there are a profile $(\hat{u}_C, u_{-C}) \in \mathcal{U}$ and two allocations $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(\hat{u}_C, u_{-C})$ such that $u_{ib_i}(y) > u_{ia_i}(x)$ for all $i \in C$. If the allocation rule φ is not manipulable by any coalition at profile u, then φ is *coalitionally nonmanipulable* at profile $u \in \mathcal{U}$.

Our first main result shows that a fair and budget-balanced allocation rule is nonmanipulable at a profile if and only if all fair and budget-balanced allocation rules are nonmanipulable at this profile. Furthermore, the same equivalence holds when considering coalitional nonmanipulability instead of individual nonmanipulability.

Theorem 1. *Let φ and ψ be two arbitrary fair and budget-balanced allocation rules. Then φ is (coalitionally) nonmanipulable at profile $u \in \mathcal{U}$ if and only if ψ is (coalitionally) nonmanipulable at profile $u \in \mathcal{U}$.*

4. Minimal manipulability

Fairness, budget balance, and (global) nonmanipulability are incompatible (Green and Laffont 1979). Our approach is to weaken or abandon nonmanipulability.[10] A natural question is whether there is a "minimally (or least) manipulable" allocation rule among all fair and budget-balanced rules. Several recent contributions[11] use a notion of the degree of manipulability to compare the ease of manipulation in allocation mechanisms that are known to be manipulable. The common feature is that these results (except for Theorem 4 in Pathak and Sönmez 2013) use measures for the degree of manipulability that are based on the preference domain.

To define the various notions of minimal manipulability, given an allocation rule φ, let $\mathcal{U}^\varphi \subseteq \mathcal{U}$ denote the subset of preference profiles at which φ is manipulable (by some

[8]Note that this equivalence does not hold in general: for instance, let $\succ_i$ denote agent i's preference where $a \succ_i b \succ_i c$ for three alternatives a, b, and c. Suppose that the rule chooses b when i reports $\succ_i$ and the rule chooses a and c when i reports a preference $\succ_i'$ (where a and c are indifferent and preferred to b). Then the rule is manipulable at $\succ_i$ by agent i but the rule is not strongly manipulable at $\succ_i$ by agent i.

[9]Again, in the same vein as above, we may use a more conservative notion of coalitional manipulability where all deviating agents are strictly better off after the deviation for any of the chosen allocations. This would not change any of our results.

[10]Several papers weaken or abandon budget balance (Sun and Yang 2003, Andersson and Svensson 2008, and Svensson 2009).

[11]See, e.g., Aleskerov and Kurbanov (1999), Kelly (1988, 1993), Maus et al. (2007a, 2007b), or Pathak and Sönmez (2013).

agent). In addition, let $P^{\varphi}(u)$ denote the set of agents who can manipulate the allocation rule φ at profile $u \in \mathcal{U}$.

In Definitions 4–7, we make weak comparisons of two rules and "more" stands for "weakly more" (like "preferred" stands for "weakly preferred").

Definition 4 (Profiles counting). Let φ and ψ be two allocation rules.

(a) φ is profiles-counting-more manipulable than ψ if $|\mathcal{U}^{\varphi}| \geq |\mathcal{U}^{\psi}|$.

(b) φ and ψ are profiles-counting-equally manipulable if $|\mathcal{U}^{\varphi}| = |\mathcal{U}^{\psi}|$.

Note that any two rules can be compared regarding their manipulability with respect to profiles counting. The following partial comparison was proposed by Pathak and Sönmez (2013).

Definition 5 (Profiles inclusion). Let φ and ψ be two allocation rules.

(a) φ is profiles-inclusion-more manipulable than ψ if $\mathcal{U}^{\varphi} \supseteq \mathcal{U}^{\psi}$.

(b) φ and ψ are profiles-inclusion-equally manipulable if $\mathcal{U}^{\varphi} = \mathcal{U}^{\psi}$.

Note that if φ is profiles-inclusion-more manipulable than ψ, then φ is profiles-counting-more manipulable than ψ. However, neither of these measures can be used to distinguish fair and budget-balanced allocation rules with respect to their degree of manipulability.

Proposition 1. *Let φ and ψ be two fair and budget-balanced allocation rules. Then (i) φ and ψ are profiles-counting-equally manipulable, and (ii) φ and ψ are profiles-inclusion-equally manipulable.*

Proof. By Theorem 1, both $\mathcal{U}^{\varphi} = \mathcal{U}^{\psi}$ and $|\mathcal{U}^{\varphi}| = |\mathcal{U}^{\psi}|$, which yields the desired conclusion. □

Since all fair and budget-balanced rules are equally manipulable if the measure is based only on the cardinality and/or set inclusions of subsets in the preference domain, a finer notion is needed. Note that an equivalent way of stating (global) nonmanipulability (or "strategy-proofness") is the following. Allocation rule φ is (globally) nonmanipulable if

$$|P^{\varphi}(u)| = 0 \quad \text{for all } u \in \mathcal{U}. \tag{1}$$

Given the fact that (1) never can be satisfied for fair and budget-balanced rules and the above insights, it is natural to search for rules where $|P^{\varphi}(u)|$ is minimized for each profile $u \in \mathcal{U}$. This guarantees that the rule is nonmanipulable whenever a nonmanipulable rule exists for a specific profile, and that the core idea of (global) nonmanipulability is respected as much as possible.

Definition 6 (Agents counting). Let φ and ψ be two allocation rules. Then φ is agents-counting-more manipulable than ψ if $|P^{\varphi}(u)| \geq |P^{\psi}(u)|$ for all $u \in \mathcal{U}$.

The corresponding notion with respect to inclusion was introduced by Pathak and Sönmez (2013).

Definition 7 (Agents inclusion). Let φ and ψ be two allocation rules. Then φ is agents-inclusion-more manipulable than ψ if $P^{\varphi}(u) \supseteq P^{\psi}(u)$ for all $u \in \mathcal{U}$.

While it is clear that these measures are partial comparisons of allocation rules, the following diagram shows the relations among the various measures of the degree of manipulability: For any two allocation rules φ and ψ, we have[12]

φ is agents-inclusion-more manipulable than ψ

$\Rightarrow$ φ is agents-counting-more manipulable than ψ

$\Rightarrow$ φ is profiles-inclusion-more manipulable than ψ

$\Rightarrow$ φ is profiles-counting-more manipulable than ψ.

The relations between the different concepts are general and do not depend on our specific model.

Note that Definitions 4–7 (weakly) compare two rules with respect to their manipulability. Naturally, any of these concepts would strongly compare two rules φ and ψ if φ is comparable to ψ but ψ is not comparable to φ. In other words, under a strong comparison, Definition 4(a) requires a strict inequality for some profile, Definition 5(a) requires a strict inclusion for some profile, Definition 6 requires a strict inequality for some profile, and Definition 7 requires a strict inclusion for some profile. Actually, as the careful reader may check, Pathak and Sönmez's (2013) second concept makes (only) a strong comparison in the vein of Definition 7 but requires, in addition, $\mathcal{U}^{\varphi} \supsetneq \mathcal{U}^{\psi}$. Of course, again by Theorem 1, in this sense no two fair and budget-balanced rules would be strongly comparable.

5. Agent k-preferred allocation rules

The concept of agent k-preferred allocations will play an important role. At these allocations, agent k's utility is maximized among all fair and budget-balanced allocations.

Definition 8. Let $k \in N$ and $u \in \mathcal{U}$. Allocation $(a, x) \in F(u)$ is *agent k-preferred* if it maximizes the utility of agent k in $F(u)$, i.e., $u_{ka_k}(x) \geq u_{kb_k}(y)$ for all $(b, y) \in F(u)$. Let $\psi^k(u) \subseteq F(u)$ denote the set of all fair and budget-balanced allocations that are agent *k-preferred* at profile u.

Thus, at an agent k-preferred allocation, agent k's utility is maximized among all fair and budget-balanced allocations. Our next result establishes (i) the existence of a k-preferred (fair and budget-balanced) allocation for all $k \in N$ and all $u \in \mathcal{U}$, and (ii) ψ^k

[12] In showing $\mathcal{U}^{\varphi} \supseteq \mathcal{U}^{\psi}$ for the second implication, note that for any $u \in \mathcal{U}^{\varphi}$, we have $0 = |P^{\varphi}(u)| \geq |P^{\psi}(u)| \geq 0$. Thus, both $|P^{\psi}(u)| = 0$ and $u \in \mathcal{U}^{\psi}$.

is an allocation rule, i.e., that for any profile, all agents are indifferent between all agent k-preferred fair allocations. The allocation rule ψ^k will be called the *agent k-preferred fair allocation rule* henceforth.

Theorem 2. *Let $k \in N$.*

(i) For each profile $u \in \mathcal{U}$, there exists an agent k-preferred allocation in $F(u)$, i.e., $\psi^k(u) \neq \varnothing$.

(ii) ψ^k is an allocation rule.

The corollary below will follow from the proof of Theorem 1.

Corollary 1. *(i) ψ^k cannot be manipulated by agent k at any profile $u \in \mathcal{U}$.*

(ii) For any two distinct agents $i, j \in N$, there exists no fair and budget-balanced allocation rule φ such that neither i nor j can manipulate φ at any profile $u \in \mathcal{U}$.

Corollary 1 has the same flavor as the corresponding results in two-sided matching (with men and women): (i) for any agent there exists a stable matching rule that is not manipulable by this agent at any profile, and (ii) there is no stable matching rule that cannot be manipulated by at least one man and at least one woman (Ma 1995).

We next introduce a more demanding notion, namely components. A component is a set of agents such that there exist fair and budget-balanced allocations that are preferred for all agents in the component and there is no superset of the component having the same property.

Given $G \subseteq N$ and $u \in \mathcal{U}$, let $\psi^G(u) = \bigcap_{k \in G} \psi^k(u)$.

Definition 9. Let $u \in \mathcal{U}$ and $G \subseteq N$. The set G is a *component* at u if $\psi^G(u) \neq \varnothing$ and there exists no $G \subsetneq G' \subseteq N$ such that $\psi^{G'}(u) \neq \varnothing$. Let $\mathcal{G}(u)$ denote the set of all components at u.

The next result states an important characteristic of components, namely that if agent k belongs to a component G, then all agent k-preferred allocations are also preferred for all agents belonging to G.

Lemma 1. *Let $u \in \mathcal{U}$. If $k \in G \in \mathcal{G}(u)$, then $\psi^k(u) = \psi^G(u)$.*

By Lemma 1, $\mathcal{G}(u)$ induces a partition of N because for any $G', G'' \in \mathcal{G}(u)$ with $k \in G' \cap G''$, we have $\psi^k(u) = \psi^{G'}(u) = \psi^{G''}(u)$ and $G' \cup G'' \in \mathcal{G}(u)$ (and, hence, $G' = G''$). Thus, for any $k \in N$, there exists a unique $G \in \mathcal{G}(u)$ with $k \in G$.

In determining the least manipulable fair and budget-balanced allocation rules, for agent k-preferred fair allocation rules, not only the preference profile, but also the selection of $k \in N$ may influence the manipulability possibilities. In the search for the agents-counting-minimally manipulable fair and budget-balanced allocation rules, it is important to select the right $k \in N$ for any given profile $u \in \mathcal{U}$. For this reason, the selection of agent k will be endogenously determined by the profile $u \in \mathcal{U}$. The general idea is first to select a component with maximal cardinality, then select some agent k belonging to this component, and, finally, select the set of agent k-preferred fair allocations.

Let

$$\bar{\mathcal{G}}(u) = \{G \in \mathcal{G}(u) : |G| \geq |G'| \text{ for all } G' \in \mathcal{G}(u)\}$$

denote the set of components with maximal cardinality. Let

$$\bar{G}(u) = \bigcup_{G \in \bar{\mathcal{G}}(u)} G$$

denote the union of all components with maximal cardinality.

A *(component) selection* is a function $\kappa : \mathcal{U} \to N$. The preferred fair allocation rule ϕ^κ based on $\kappa : \mathcal{U} \to N$ is defined as follows: for all $u \in U$, $\phi^\kappa(u) = \psi^{\kappa(u)}(u)$. In other words, a preferred fair allocation rule selects for each u an agent $\kappa(u)$ and chooses all $\kappa(u)$-preferred fair allocations. Note that (i) by Theorem 2, ϕ^κ is a well defined allocation rule and (ii) by Lemma 1, equivalently, κ chooses for all $u \in \mathcal{U}$ the component $G \in \mathcal{G}(u)$ such that $\kappa(u) \in G$ (and $\phi^\kappa(u) = \psi^G(u)$). Furthermore, we will say that an allocation rule φ is a *preferred* fair allocation rule if there exists a selection κ such that for all $u \in \mathcal{U}$, we have $\varphi(u) = \phi^\kappa(u)$.

A maximally preferred fair allocation rule chooses for each profile (i) a component G with maximal cardinality, (ii) some agent k belonging to G, and (iii) all agent k-preferred fair allocations. Note that different ks may be selected for different profiles. A *maximal (component) selection* is a function $\kappa : \mathcal{U} \to N$ such that for all $u \in \mathcal{U}$, we have $\kappa(u) \in \bar{G}(u)$. The maximally preferred fair allocation rule ϕ^κ is the preferred fair allocation rule based on κ. Again by Lemma 1, equivalently, κ chooses for all $u \in \mathcal{U}$ a component (with maximal cardinality) $G \in \bar{\mathcal{G}}(u)$ such that $\kappa(u) \in G$. Furthermore, we will say that an allocation rule φ is a *maximally preferred* fair allocation rule if there exists a maximal selection κ such that for all $u \in \mathcal{U}$, we have $\varphi(u) = \phi^\kappa(u)$. Note that the function κ is a systematic selection from $\bar{\mathcal{G}}(u)$. The meaning of "systematic selection" is that there is a well defined rule for selecting k. This rule can be arbitrary and all our results hold independently of this rule. For example, the rule could be based on a randomized selection from $\bar{G}(u)$ or simply the k with the lowest or highest index in $\bar{G}(u)$.

Our second main result establishes that maximally preferred fair allocation rules are agents-counting-minimally manipulable among all fair and budget-balanced allocation rules. Here a rule φ is agents-counting-minimally manipulable among all fair and budget-balanced allocation rules if for any fair and budget-balanced allocation rule φ', φ' is agents-counting-more manipulable than φ.

Theorem 3. *Let φ be an arbitrary fair and budget-balanced allocation rule. Then φ is agents-counting-minimally manipulable among all fair and budget-balanced allocation rules if and only if φ is a maximally preferred fair allocation rule.*

By Theorem 3, any fair and budget-balanced allocation rules can be compared to a maximally preferred fair allocation rule via agents-counting manipulability, and any fair and budget-balanced allocation rule that is not maximally preferred fair is strongly agents-counting-more manipulable (with a strict inequality for some profile in Definition 6) than any maximally preferred fair allocation rule. Note that except for degenerate preference profiles where N is the unique component, the set of fair and budget-balanced allocations is a continuum. Thus, any rule that chooses for some nondegenerate preference profile an allocation that is not preferred is strongly agents-counting-more manipulable than any maximally preferred fair allocation rule. Hence, the comparison is often strict.

In checking the robustness of Theorem 3, we consider the degree of coalitional manipulability. Using the same arguments as above, by Theorem 1, it is in general impossible to define a fair and budget-balanced rule to be less coalitionally manipulable than some other fair and budget-balanced rule if the measure is based only on the cardinality and/or set inclusions of subsets in the preference domain. Let $Q^{\varphi}(u)$ denote the coalitions $C \subseteq N$ that can manipulate the allocation rule φ at profile $u \in \mathcal{U}$. We adopt the following notion.

Definition 10. Let φ and ψ be two allocation rules. Then φ is coalitions-counting-more manipulable than ψ if $|Q^{\varphi}(u)| \geq |Q^{\psi}(u)|$ for all $u \in \mathcal{U}$.

The following result states that maximally preferred fair allocation rules are coalitions-counting-minimally manipulable among all fair and budget-balanced allocation rules. This can be seen as an extension of Theorem 3 from minimal individual manipulability to minimal coalitional manipulability, i.e., that Theorem 3 is robust with respect to coalitional manipulations.

Theorem 4. *Let φ be an arbitrary fair and budget-balanced allocation rule. Then φ is coalitions-counting-minimally manipulable among all fair and budget-balanced allocation rules if and only if φ is a maximally preferred fair allocation rule.*

Finally we will establish that preferred fair allocation rules are agents-inclusion-minimally manipulable among all fair and budget-balanced allocation rules.[13] We show that any fair and budget-balanced allocation rule is agents-inclusion-more manipulable than some preferred fair allocation rule. Here a rule φ is agents-inclusion-minimally manipulable among all fair and budget-balanced allocation rules if there exists no fair and budget-balanced allocation rule $\varphi' \neq \varphi$ such that φ is agents-inclusion-more manipulable than φ'.

Theorem 5. *Let φ be an arbitrary fair and budget-balanced allocation rule. Then φ is agents-inclusion-minimally manipulable among all fair and budget-balanced allocation rules if and only if φ is a preferred fair allocation rule.*

[13] The careful reader may note that Theorem 5 is the only new result that is not included in Andersson et al. (2010a).

Similar as above for agents-counting-minimal manipulability, Theorem 5 is robust with respect to coalitional manipulability (by considering inclusions of the set of coalitions that can manipulate the rule at a profile).

By Theorem 5, any fair and budget-balanced allocation rule that is not preferred fair is strongly agents-inclusion-more manipulable (with a strict inclusion for some profile in Definition 7) than some preferred fair allocation rule. A direct consequence of Theorem 5 is that for any $k \in N$, the agent k-preferred fair allocation rule is agents-inclusion-minimally manipulable among all fair and budget-balanced allocation rules.

Corollary 2. *Let $k \in N$. Then ψ^k is agents-inclusion-minimally manipulable.*

Obviously, by Corollary 1, for distinct $k, i \in N$, ψ^k and ψ^i cannot be compared with respect to agent-inclusion-more manipulability (because for all $u \in \mathcal{U}$, $k \notin P^{\psi^k}(u)$ and $i \notin P^{\psi^i}(u)$).

Appendix

The following lemmas are two well known properties of fair allocations (see, e.g., Svensson 2009): first, if two allocations are fair at a given profile, then one may interchange both the assignment of objects and the monetary distribution without losing fairness. Obviously, this result holds for fair allocations satisfying budget balance.

Lemma 2. *Suppose that allocations (a, x) and (b, y) are fair at profile $u \in \mathcal{U}$. Then allocations (a, y) and (b, x) are also fair at profile $u \in \mathcal{U}$.*

Second, for fair allocation rules, a unique distribution of money is chosen for any given preference profile.[14]

Lemma 3. *Let φ be a fair allocation rule and $u \in \mathcal{U}$. If $(a, x), (b, y) \in \varphi(u)$, then $x = y$.*

Proof. Since $(a, x), (b, y) \in \varphi(u)$, we have $u_{ia_i}(x) = u_{ib_i}(y)$ for all $i \in N$. By fairness, $u_{ia_i}(x) \geq u_{ib_i}(x)$. Thus, $u_{ib_i}(y) \geq u_{ib_i}(x)$ and $y_{b_i} \geq x_{b_i}$. Similarly, we obtain $x_{b_i} \geq y_{b_i}$. Hence, $x = y$, the desired conclusion. □

We proceed as follows. First, showing our main results requires a structural analysis with respect to indifferences at fair and budget-balanced allocations. We show that for any agent k and any fair and budget-balanced allocation, agent k's utility is maximized among all fair and budget-balanced allocations if and only if the allocation is agent k-linked: any agent can be linked to agent k through a sequence of agents (an indifference chain) whereby any agent in this sequence is indifferent between his consumption bundle and the bundle received by the next agent in the sequence.

An indifference component at an allocation is a set of agents such that any two agents can be linked through an indifference chain in this set at this allocation. We

[14] To make the presentation self-contained, we include the proof of Lemma 3 (which follows, for instance, from Lemma 3 in Alkan et al. (1991)).

show that for a given profile, indifference components are invariant among all fair and budget-balanced allocations, i.e., if G is an indifference component at a fair and budget-balanced allocation, then G is an indifference component at *all* fair and budget-balanced allocations. Therefore, if a fair and budget-balanced allocation is agent k-linked and k belongs to the indifference component G, then this allocation is agent i-linked for all agents i belonging to G (and the utility of agent i is maximized among all fair and budget-balanced allocations). Therefore, the set of indifference components and the set of components coincide. In Lemma 5, we show that indifference components are related to "isolated groups" (Definition 14 below) in the following way at fair allocations: if $N - G$ is an isolated group with maximal cardinality at a fair allocation, then G is an indifference component.

Second, for any fair and budget-balanced allocation rule and any preference profile, we characterize the set of agents and the set of coalitions who can profitably manipulate the rule at this profile: (i) if a group G is isolated at a chosen allocation, then any coalition contained in G can manipulate the rule at this profile, and (ii) if the rule chooses k-linked fair allocations at this profile, then no coalition containing agent k can manipulate the rule at this profile. The (non-)manipulability results Theorem 1 and Corollary 1 then follow easily.

Third, we show our minimal manipulability results Theorem 3, Theorem 4, and Theorem 5.

A.1 *Agent k-linked allocations*

It is well established that the possibility for agents to manipulate a fair allocation rule depends on the structure of the indifference relations at the allocation(s) chosen by the rule.[15] Below we introduce the concepts of indifference chains and agent k-linked (fair) allocations.

Definition 11. Let $(a, x) \in \mathcal{A}$.

(i) For any $i, j \in N$, we write $i \rightarrow_{(a,x)} j$ if $u_{ia_i}(x) = u_{ia_j}(x)$.

(ii) An *indifference chain* at allocation (a, x) consists of a tuple of distinct agents $g = (i_0, i_1, \ldots, i_k)$ such that $i_0 \rightarrow_{(a,x)} i_1 \rightarrow_{(a,x)} \cdots \rightarrow_{(a,x)} i_k$.

Note that $i \rightarrow_{(a,x)} j$ means that agent i is indifferent between his consumption bundle and agent j's consumption bundle, and agent i is directly linked via indifference to agent j at allocation (a, x). An indifference chain at an allocation is simply a sequence of agents such that any agent in the sequence is indifferent between his bundle and the bundle of the agent following him in the sequence. Indifference chains indirectly link agents via indifference in a sequence of directly linked agents.

The following concept of agent k-linked allocations will be useful.

[15] See, for example, Andersson and Svensson (2008), Andersson et al. (2010b), or Mishra and Talman (2010) for theoretical results, and Sankaran (1994) or Mishra and Parkes (2009) for efficient procedures to calculate allocations with the maximal number of indifference relations. Similar observations have previously also been made by, e.g., Dubey (1982) and Svensson (1991), where the "tightness" of the market is demonstrated to have a significant impact on manipulation possibilities.

Definition 12. Let $(a, x) \in \mathcal{A}$.

(i) Agent $i \in N$ is *linked* to agent $k \in N$ at allocation (a, x) if there exists an indifference chain $(i_0, \dots, i_t)$ at allocation (a, x) with $i = i_0$ and $i_t = k$.

(ii) Allocation (a, x) is *agent k-linked* if each agent $i \in N$ is linked to agent $k \in N$.

Thus, at an agent k-linked allocation, each agent is linked to agent $k \in N$ through some indifference chain. The following is a slightly stronger result than Theorem 2 whereby we also show that a fair and budget-balanced allocation is agent k-preferred if and only if it is agent k-linked.

Theorem 6. *Let $k \in N$.*

(i) For each profile $u \in \mathcal{U}$, there exists an agent k-preferred allocation in $F(u)$, i.e., $\psi^k(u) \neq \varnothing$. Moreover, for any allocation $(a, x) \in F(u)$, $(a, x) \in \psi^k(u) \Leftrightarrow (a, x)$ is agent k-linked.

(ii) ψ^k is an allocation rule.

Proof. Let $k \in N$ and $u \in \mathcal{U}$. First, we show (i). Note that agent k-preferred allocations exist in $F(u)$ since $F(u)$ is compact. Thus, $\psi^k(u) \neq \varnothing$.

Let $(a, x) \in \psi^k(u)$. We show that (a, x) is agent k-linked. By contradiction, suppose that (a, x) is not k-linked, i.e., that there is an agent $l \in N$ that is not linked to agent k. Let

$$G = \{i \in N : i \text{ is linked to } k \text{ at } (a, x)\} \cup \{k\}.$$

Because $k \in G$ and $l \in N - G$, both G and $N - G$ are nonempty. Moreover, by construction, $u_{ia_i}(x) > u_{ia_j}(x)$ if $i \in N - G$ and $j \in G$. From the perturbation lemma in Alkan et al. (1991), it then follows that there exists another allocation $(b, y) \in F(u)$ such that $y_{a_i} > x_{a_i}$ for all $i \in G$.[16] Then by fairness and monotonicity in money, we have

$$u_{ib_i}(y) \geq u_{ia_i}(y) = v_{ia_i} + y_{a_i} > v_{ia_i} + x_{a_i} = u_{ia_i}(x) \quad \text{for all } i \in G.$$

Because $k \in G$, it follows that $u_{kb_k}(y) > u_{ka_k}(x)$, which contradicts the fact that (a, x) maximizes k's utility in $F(u)$. Hence, any agent k-preferred fair allocation is agent k-linked.

Next we show (ii) and that any agent k-linked allocation maximizes agent k's utility in $F(u)$.[17] It suffices to show that if $(a, x), (b, y) \in F(u)$ are agent k-linked, then $u_{ia_i}(x) = u_{ib_i}(y)$ for all $i \in N$. By the first part of the proof for (i), any agent k-preferred allocation is agent k-linked. Thus, we may suppose without loss of generality that $(b, y) \in \psi^k(u)$.

[16]Because preferences are quasi-linear, this can be simply done by infinitesimally increasing equally the compensations of $\{a_i : i \in G\}$ and infinitesimally decreasing equally the compensations of $\{a_i : i \in N - G\}$ (while preserving budget balance).

[17]Again, to make the presentation self-contained, we include the proof (which follows Theorem 6 in Alkan et al. 1991).

We first demonstrate the analogue of Lemma 3 for agent k-linked fair allocations: if $(a, x), (b, y) \in F(u)$ are agent k-linked, then $x = y$. To see this, note that (a, y) is also fair by Lemma 2. First, we show that (a, y) is agent k-linked if (b, y) is agent k-linked. Fairness implies

$$u_{ia_i}(y) = u_{ib_i}(y) \quad \text{for all } i \in N. \tag{2}$$

Since (b, y) maximizes the utility of agent k in $F(u)$, (2) implies that (a, y) also maximizes the utility of agent k in $F(u)$. Thus, by the first part of the proof for (i), (a, y) is agent k-linked. Hence, without loss of generality we may assume $a = b$.

Suppose that the fair allocations (a, x) and (a, y) are agent k-linked but $x \neq y$. Then by budget-balance and $x \neq y$, there must be two nonempty groups of agents:

$$A = \{i \in N : x_{a_i} > y_{a_i}\}$$
$$B = \{i \in N : x_{a_i} \leq y_{a_i}\}.$$

Note that for all $i \in A$ and all $j \in B$, $u_{ia_i}(x) > u_{ia_i}(y) \geq u_{ia_j}(y) \geq u_{ia_j}(x)$. Hence, no agent in A can be linked to any agent in B at allocation (a, x). Because (a, x) is agent k-linked, we must have $k \in A$. Let $j \in B$ and $i \in A$. By fairness and monotonicity in money,

$$u_{ja_j}(y) \geq u_{ja_j}(x) \geq u_{ja_i}(x) > u_{ja_i}(y).$$

Thus, at allocation (a, y), no agent in B can be linked to any agent in A. Hence, by $k \in A$, allocation (a, y) cannot be agent k-linked, which contradicts our assumption.

Let $(a, x), (b, y)$ be agent k-linked and $i \in N$. By the above, we have $x = y$. Obviously, if $a_i = b_i$, then $u_{ia_i}(x) = u_{ib_i}(y)$. If $a_i \neq b_i$, then by fairness both $u_{ia_i}(x) \geq u_{ib_i}(x)$ and $u_{ia_i}(y) \leq u_{ib_i}(y)$. Hence, by $x = y$, we have $u_{ia_i}(x) = u_{ib_i}(y)$, the desired conclusion. □

Theorem 6 implies that (i) the set of agent k-linked fair allocations and the set of agent k-preferred fair allocations coincide, and (ii) all agents are indifferent between all fair allocations that maximize agent k's utility in $F(u)$.

We next introduce a more demanding notion of indifference structures, namely indifference components. In each indifference component, any two agents are linked through an indifference chain in this component and there is no superset of this component where any two agents are linked.

Definition 13. Let $(a, x) \in \mathcal{A}$. An *indifference component* at allocation (a, x) is a nonempty set $G \subseteq N$ such that for all $i, k \in G$, there exists an indifference chain at (a, x) in G, say $g = (i_0, \ldots, i_k)$ with $\{i_0, \ldots, i_k\} \subseteq G$, such that $i = i_0$ and $i_k = k$, and there exists no $G' \supsetneq G$ satisfying the previous property at allocation (a, x).

The next result states an important characteristic of indifference components, namely that if there are two allocations that are fair and budget-balanced at some profile $u \in \mathcal{U}$ and if there is an indifference component at one of these allocations, then the very same indifference component must be present at the other allocation. In other words, indifference components at fair and budget-balanced allocations only depend on the preference profile $u \in \mathcal{U}$ because they are invariant with respect to the selected fair and budget-balanced allocation.

Lemma 4. *Suppose that allocations* (a, x) *and* (b, y) *are fair and budget-balanced at profile* $u \in \mathcal{U}$. *If* G *is an indifference component at allocation* (a, x), *then* G *is an indifference component at allocation* (b, y).

Proof. By Lemma 2, we know that (a, y) is fair. First we show that the indifference component G is present at (a, y).

Because G is an indifference component at (a, x), G consists of indifference chains $g = (i_0, i_1, \dots, i_k)$ such that $i_k \rightarrow_{(a,x)} i_0$. Thus, we have $i_0 \rightarrow_{(a,x)} i_1 \rightarrow_{(a,x)} \cdots \rightarrow_{(a,x)} i_k \rightarrow_{(a,x)} i_0$. We show $i_0 \rightarrow_{(a,y)} i_1 \rightarrow_{(a,y)} \cdots \rightarrow_{(a,y)} i_k \rightarrow_{(a,y)} i_0$.

For any $i \in N$, let $\Delta_{a_i} = y_{a_i} - x_{a_i}$. To obtain a contradiction, suppose that we do not have $i_0 \rightarrow_{(a,y)} i_1 \rightarrow_{(a,y)} \cdots \rightarrow_{(a,y)} i_k \rightarrow_{(a,y)} i_0$, say $u_{i_0 a_{i_0}}(x) = u_{i_0 a_{i_1}}(x)$ but $u_{i_0 a_{i_0}}(y) > u_{i_0 a_{i_1}}(y)$. Thus, $\Delta_{a_{i_0}} > \Delta_{a_{i_1}}$. Now, fairness is respected among the agents in G at allocation (a, y) only if

$$\Delta_{a_{i_j}} \geq \Delta_{a_{i_{j+1}}} \quad \text{for all } j \in \{0, \dots, k-1\} \tag{3}$$

$$\Delta_{a_{i_k}} \geq \Delta_{a_{i_0}}. \tag{4}$$

From (3) and $\Delta_{a_{i_0}} > \Delta_{a_{i_1}}$, we obtain $\Delta_{a_{i_0}} > \Delta_{a_{i_k}}$. Hence, (4) is not satisfied. Thus, allocation (a, y) cannot be fair, which contradicts our assumption. Hence, $i_0 \rightarrow_{(a,y)} i_1 \rightarrow_{(a,y)} \cdots \rightarrow_{(a,y)} i_k \rightarrow_{(a,y)} i_0$. Note that there exists no $G' \supsetneq G$ such that G' is an indifference component at (a, y) because otherwise, using the previous arguments, any two agents in G' are connected through some indifference chain at (a, x) in G', which contradicts the definition of G being an indifference component at (a, x). Thus, the indifference component G is present at (a, y).

Next, we show that G must be also an indifference component at (b, y). Fairness implies

$$u_{ia_i}(y) = u_{ib_i}(y) \quad \text{for all } i \in N. \tag{5}$$

Let $j, k \in G$ and suppose that $j \rightarrow_{(a,y)} k$. If $a_k = b_k$, then by (5), $j \rightarrow_{(b,y)} k$. Let $a_k \neq b_k$ and $l_1 \in N$ be such that $a_{l_1} = b_k$. Obviously, (5) implies $k \rightarrow_{(a,y)} l_1$. More generally, let $l_1, \dots, l_t$ be such that $a_{l_r} = b_{l_{r-1}}$ with $r = 2, \dots, t$ and $a_k = b_{l_t}$. Note that such a "cycle" exists because $|N| = |M|$. Now obviously we have $k \rightarrow_{(a,y)} l_1$, $l_r \rightarrow_{(a,y)} l_{r+1}$ for all $r = 1, \dots, t-1$, and $l_t \rightarrow_{(a,y)} k$. Since $k \in G$ and G is an indifference component at (a, y), we must have $\{l_1, \dots, l_t\} \subseteq G$.

Now by (5), we have $u_{jb_j}(y) = u_{ja_j}(y) = u_{ja_k}(y) = u_{jb_{l_t}}(y)$, which implies $j \rightarrow_{(b,y)} l_t$. Note that by construction, we also have $l_1 \rightarrow_{(b,y)} k$ and $l_r \rightarrow_{(b,y)} l_{r-1}$ for all $r = 2, \dots, t$. This means that j and k are connected through the indifference chain $j \rightarrow_{(b,y)} l_t \rightarrow_{(b,y)} l_{t-1} \rightarrow_{(b,y)} \cdots \rightarrow_{(b,y)} l_1 \rightarrow_{(b,y)} k$ in G under (b, y) (if $a_k \neq b_k$). If $a_k = b_k$, then $j \rightarrow_{(b,y)} k$. Because this is true for any $j, k \in G$ such that $j \rightarrow_{(a,y)} k$, it also follows that any two agents belonging to G must be connected through an indifference chain in G at (b, y). Furthermore, there can be no $G' \supsetneq G$ satisfying this property under (b, y) because by the same argument G' would also satisfy this property under (a, x), which would contradict the definition of an indifference component. □

Lemma 4 implies that the set of components and the set of indifference components are identical, i.e.,

$$\mathcal{G}(u) = \{G \subseteq N : G \text{ is an indifference component at all } (a, x) \in F(u)\}.$$

Furthermore, Lemma 1 follows directly from Lemma 4 and Theorem 6: Given $u \in U$ and $k \in N$, by Lemma 4, there is a unique indifference component G such that $k \in G$. Then for all $i \in G$, by definition of an indifference component, any allocation $(a, x) \in \psi^k(u)$ is agent i-linked. Thus, by Theorem 6, $(a, x) \in \psi^i(u)$. Interchanging the roles of i and k, we now obtain $\psi^k(u) = \psi^i(u)$ for all $i \in G$ and $\psi^k(u) = \psi^G(u)$.

The existence of indifference components is closely related to the presence of isolated groups (or coalitions): a group of agents $C \subsetneq N$ is isolated if no agent outside this group can be linked to any agent in C.

Definition 14. A group of agents $C \subsetneq N$ is *isolated* at allocation (a, x) if $i \nrightarrow_{(a,x)} j$ for all $i \in N - C$ and all $j \in C$.

The following lemma relates isolated groups and indifference components.

Lemma 5. *Let φ be a fair and budget-balanced allocation rule, $u \in \mathcal{U}$ and $(a, x) \in \varphi(u)$. If $N - G$ is the (possibly empty) isolated group with maximal cardinality at allocation (a, x), then G is an indifference component at allocation (a, x).*

Proof. We first show that all $i, j \in G$ can be linked via an indifference chain in G. Suppose not, i.e., there exist $i, j \in G$ such that i cannot be linked to j via some indifference chain G. Let

$$H = \{k \in G : k \text{ can be linked to } j \text{ via some indifference chain in } G\}.$$

Since $i \in G - H$, we have $G - H \neq \varnothing$. Because no agent in $G - H$ can be linked to any agent in H, by construction, it follows that $(N - G) \cup H \subsetneq N$ (by $i \in G - H$), the set $(N - G) \cup H$ is isolated, and $|(N - G) \cup H| > |N - G|$, which contradicts the assumption that $N - G$ is the isolated group with maximal cardinality at allocation $(a, x) \in \varphi(u)$.

Now, the proof follows directly because the group $N - G$ is isolated at allocation (a, x), i.e., $i \nrightarrow_{(a,x)} j$ for all $i \in G$ and all $j \in N - G$. Consequently, there is no $G' \supsetneq G$ such that G' is an indifference component by Definition 13. □

A.2 *Manipulability and nonmanipulability*

Below we determine the (non-)manipulation possibilities of fair allocation rules. The first result describes the relation between isolated groups and the possibility to manipulate φ at a specific profile. We show that any coalition contained in an isolated group can manipulate the fair and budget-balanced allocation rule.[18]

[18]Note that Beviá's (2010) results do not allow for single-valued allocation rules whereas all our results hold any single-valued allocation rule (and Beviá's Theorem 2.1 does not have any implication for Lemma 6).

Lemma 6. *Let φ be a fair and budget-balanced allocation rule, $u \in \mathcal{U}$, and $(a, x) \in \varphi(u)$. If the nonempty group $G \subsetneq N$ is isolated at allocation (a, x), then each coalition $C \subseteq G$ can manipulate φ at profile $u \in \mathcal{U}$.*

Proof. Let $(a, x) \in \varphi(u)$ and suppose that $G \subsetneq N$ is a nonempty isolated coalition, i.e., that both $i \nrightarrow_{(a,x)} j$ and $u_{ia_i}(x) > u_{ia_j}(x)$ for all $i \in N - G$ and all $j \in G$. Now simultaneously all compensations for objects a_i $(i \in G)$ can be increased by the same amount and all compensations for objects a_j $(j \in N - G)$ can be decreased by the same amount without losing budget balance and fairness. Hence, there is a number $\tau > 0$ and $(a, y) \in F(u)$ such that $u_{ia_i}(y) > u_{ia_i}(x) + \tau$ for all $i \in G$ (and $y_{a_i} > x_{a_i} + \tau$ for all $i \in G$). Fix $0 < \varepsilon < \tau$ and define for any $i \in G$, the function $\hat{u}_i$ as follows: for all $j \in M$ and all $x' \in \mathbb{R}^M$, let

$$\hat{u}_{ij}(x') = (-y_j + \varepsilon_{ij}) + x'_j,$$

where $\varepsilon_{ij} = 0$ if $j \neq a_i$ and $\varepsilon_{ia_i} = \varepsilon > 0$. Note that $\hat{v}_{ij} = -y_j + \varepsilon_{ij}$. Let $C \subseteq G$ and $\hat{u}_C = (\hat{u}_i)_{i\in C}$. By construction of $\hat{u}_C$, we have $(a, y) \in F(\hat{u}_C, u_{-C})$.[19]

Let $(b, z) \in \varphi(\hat{u}_C, u_{-C})$. We first show $b_i = a_i$ for all $i \in C$. Let $\delta_j = z_j - y_j$ for all $j \in M$. Without loss of generality, order M such that $\delta_j \geq \delta_{j+1}$ for all $j = 1, \dots, |M| - 1$.

If $z = y$, then by fairness, $\hat{u}_{ib_i}(y) = \hat{u}_{ia_i}(y)$ for all $i \in C$. Since for all $i \in C$, $\hat{u}_{ia_i}(y) = \varepsilon$ and $\hat{u}_{ij}(y) = 0$ for $j \neq a_i$, we obtain $b_i = a_i$ for all $i \in C$.

If $z \neq y$, then by budget balance of both (b, z) and (a, y), $\delta_1 > 0$ and $\delta_n < 0$. Let $(j_l)_l$ be a subsequence of $(1, \dots, n)$ such that $j_l < j_{l+1}$, $\delta_{j_l} > \delta_{j_{l+1}}$ and $\delta_j = \delta_{j_l}$ if $j_l \leq j < j_{l+1}$. Let $S_l = \{i \in N : j_l \leq a_i < j_{l+1}\}$. Then for $i \in S_l$,

$$u_{ia_i}(z) = u_{ia_i}(y) + \delta_{a_i} \geq u_{ib_i}(y) + \delta_{a_i} > u_{ib_i}(y) + \delta_{b_i} = u_{ib_i}(z) \quad \text{if } b_i \geq j_{l+1} \text{ and } i \in N - C$$
$$\hat{u}_{ia_i}(z) = z_{a_i} - y_{a_i} + \varepsilon = \delta_{a_i} + \varepsilon > \delta_{b_i} = \hat{u}_{ib_i}(z) \quad \text{if } b_i \geq j_{l+1} \text{ and } i \in C.$$

Thus, by fairness, for all l, $i \in S_l$ implies $j_l \leq b_i < j_{l+1}$. Moreover, for $i \in C$, $\hat{u}_{ia_i}(z) = \delta_{a_i} + \varepsilon > \delta_{b_i} = \hat{u}_{ib_i}(z)$ if $b_i \neq a_i$ and $b_i \geq j_l$. Hence, by fairness, $b_i = a_i$ for all $i \in C$.

It remains to prove that $u_{ib_i}(z) > u_{ia_i}(x)$ for all $i \in C$, i.e., φ is manipulable at u by coalition C. From the above, we have $a_i = b_i$ for all $i \in C$. Since φ is fair, we have $(b, z) \in F(\hat{u}_C, u_{-C})$. Now we have for all $i \in C$ with $b_i \neq 1$,

$$\hat{u}_{ib_i}(z) = \hat{u}_{ia_i}(z) = z_{ia_i} - y_{ia_i} + \varepsilon \geq z_{i1} - y_{i1} = \hat{u}_{i1}(z).$$

Because $\delta_j = z_j - y_j$, it follows from the above condition that $\delta_{b_i} \geq \delta_1 - \varepsilon$ for $i \in C$ with $b_i \neq 1$. Note that this inequality holds trivially if $b_i = 1$ because $\varepsilon > 0$. Now this fact, the definition of δ_j, and our choice of $0 < \varepsilon < \tau$, $\delta_1 \geq 0$ and $a_i = b_i$ for all $i \in C$, yield for all $i \in C$,

$$\begin{aligned} u_{ia_i}(x) &< u_{ia_i}(y) - \tau \\ &= u_{ib_i}(y) - \tau \\ &= v_{ib_i} + z_{b_i} - (z_{b_i} - y_{b_i}) - \tau \\ &= u_{ib_i}(z) - \delta_{b_i} - \tau \end{aligned}$$

[19]Note that for all $i \in C$, $\hat{u}_{ia_i}(y) = \varepsilon$ and $\hat{u}_{ij}(y) = 0$ for $j \neq a_i$.

$$\leq u_{ib_i}(z) - \delta_1 - (\tau - \varepsilon)$$
$$< u_{ib_i}(z),$$

where the first inequality follows from $u_{ia_i}(y) > u_{ia_i}(x) + \tau$, the first equality follows from $a_i = b_i$ for $i \in C$, the second inequality follows from $-\delta_{b_i} \leq -(\delta_1 - \varepsilon)$, and the last inequality follows from $\delta_1 \geq 0$ and $\tau > \varepsilon$. Hence, $u_{ia_i}(x) < u_{ib_i}(z)$ for all $i \in C$, which is the desired conclusion. □

The second result shows that the agent k-preferred fair allocation rule cannot be manipulated by any coalition containing agent k. The intuition is as follows. If agent k can successfully manipulate the allocation rule, then by fairness, agent k must be assigned a consumption bundle where the monetary compensation increases. Since each agent is linked to agent k, then each agent must be assigned a consumption bundle where the monetary compensation increases, because if this is not the case, then fairness is violated at the new allocation. But then the budget must be exceeded. Hence, agent k cannot manipulate. The same intuition holds for any fair allocation rule choosing agent k-preferred fair allocations for some profile.

Lemma 7. *Let φ be a fair and budget-balanced allocation rule, $k \in N$, and $u \in \mathcal{U}$. If $\varphi(u) \subseteq \psi^k(u)$, then no coalition $C \subseteq N$ containing agent k can manipulate φ at profile $u \in \mathcal{U}$.*

Proof. Let $C \subseteq N$ be such that $k \in C$. Suppose that φ is manipulable at profile $u \in \mathcal{U}$ by coalition C. Then there is a profile $(\hat{u}_C, u_{-C}) \in \mathcal{U}$ and two allocations $(a, x) \in \varphi(u)$ and $(b, y) \in \varphi(\hat{u}_C, u_{-C})$ such that $u_{ib_i}(y) > u_{ia_i}(x)$ for all $i \in C$. Note that $\varphi(u) \subseteq \psi^k(u)$ and $(a, x) \in \psi^k(u)$. Thus, by Theorem 6, (a, x) is agent k-linked.

By fairness, $u_{ia_i}(x) \geq u_{ib_i}(x)$ for all $i \in C$. Hence, for all $i \in C$, $u_{ib_i}(y) > u_{ib_i}(x)$ and $y_{b_i} > x_{b_i}$. Because (b, y) satisfies budget balance, we must have $C \subsetneq N$. We distinguish two cases.

First, suppose $\{b_i : i \in C\} = \{a_i : i \in C\}$. Since $k \in C$ and (a, x) is an agent k-linked fair allocation, there exist $i \in N - C$ and $j \in C$ such that $i \to_{(a,x)} j$. By $j \in C$ and $\{b_i : i \in C\} = \{a_i : i \in C\}$, we have $y_{a_j} > x_{a_j}$. Now $u_{ia_i}(x) = u_{ia_j}(x)$, fairness, and monotonicity in money imply

$$u_{ib_i}(y) \geq u_{ia_j}(y) > u_{ia_j}(x) = u_{ia_i}(x) \geq u_{ib_i}(x).$$

Hence, $y_{b_i} > x_{b_i}$. Let $C^1 = C \cup \{i \in N : i \to_{(a,x)} j \text{ for some } j \in C\}$. Thus, we have $y_{b_i} > x_{b_i}$ for all $i \in C^1$ (and $C \subsetneq C^1$).

Second, suppose $\{b_i : i \in C\} \neq \{a_i : i \in C\}$. Let $i \in N - C$ be such that $a_i \in \{b_i : i \in C\}$. Then $y_{a_i} > x_{a_i}$, fairness, and monotonicity in money imply

$$u_{ib_i}(y) \geq u_{ia_i}(y) > u_{ia_i}(x) \geq u_{ib_i}(x).$$

Hence, $y_{b_i} > x_{b_i}$. Let $C^1 = C \cup \{i \in N - C : a_i \in \{b_i : i \in C\}\}$. Thus, we have $y_{b_i} > x_{b_i}$ for all $i \in C^1$ (and $C \subsetneq C^1$).

Using the same arguments, it follows for any l that (i) if $\{b_i : i \in C^l\} = \{a_i : i \in C^l\}$, then for each $i \in N$ such that $i \rightarrow_{(a,x)} j$ for some $j \in C^l$, we have $y_{b_i} > x_{b_i}$. Let $C^{l+1} = C^l \cup \{i \in N : i \rightarrow_{(a,x)} j \text{ for some } j \in C^l\}$; and (ii) if $\{b_i : i \in C^l\} \neq \{a_i : i \in C^l\}$, then for each $i \in N - C^l$ such that $a_i \in \{b_i : i \in C^l\}$, we have $y_{b_i} > x_{b_i}$. Let $C^{l+1} = C^l \cup \{i \in N - C^l : a_i \in \{b_i : i \in C^l\}\}$.

Because N is finite and (a, x) is agent k-linked, for some t, we obtain $C^t = N$ and $y_{b_i} > x_{b_i}$ for all $i \in C^t$, which is a contradiction to budget balance of (b, y). Hence, C cannot manipulate φ at profile $u \in \mathcal{U}$. □

The following theorem identifies all preference profiles $u \in \mathcal{U}$ at which any fair and budget-balanced allocation rule is (coalitionally) nonmanipulable.

Theorem 7. *A fair and budget-balanced allocation rule φ is (coalitionally) nonmanipulable at profile $u \in \mathcal{U}$ if and only if N is the unique indifference component at profile $u \in \mathcal{U}$ (i.e., $\mathcal{G}(u) = \{N\}$).*

Proof. The "only if" part follows directly from Lemma 6 since there always is an isolated group unless N is the unique indifference component by Lemma 5. To prove the "if" part, note that if N is the unique indifference component, any $(a, x) \in F(u)$ is agent i-linked for any $i \in N$ by Lemma 4. Since $\varphi(u) \subseteq F(u)$, Lemma 7 implies that no coalition containing $i \in N$ can manipulate φ at profile $u \in \mathcal{U}$. Hence, φ is both nonmanipulable at profile $u \in \mathcal{U}$ and coalitionally nonmanipulable at profile $u \in \mathcal{U}$. □

Lemma 4 and Theorem 7 imply our first main result, Theorem 1: a fair and budget-balanced allocation rule is nonmanipulable at a profile if and only if all fair and budget-balanced allocation rules are nonmanipulable at this profile. Furthermore, the same equivalence holds when considering coalitional nonmanipulability instead of individual nonmanipulability.

Theorem 1. *Let φ and ψ be two arbitrary fair and budget-balanced allocation rules. Then φ is (coalitionally) nonmanipulable at profile $u \in \mathcal{U}$ if and only if ψ is (coalitionally) nonmanipulable at profile $u \in \mathcal{U}$.*

The proof follows directly from Lemma 4 and Theorem 7.

Note that for any $i \in N$, there is a unique (indifference) component $G \in \mathcal{G}(u)$ such that $i \in G$ (where $G = \{i\}$ is possible), i.e., any agent is included in exactly one indifference component at any profile $u \in \mathcal{U}$. Given this observation, we determine for any profile the precise number of agents and coalitions who *can* manipulate the agent k-linked fair allocation rule. Specifically, we demonstrate that ψ^k can be manipulated by less than 50% of all coalitions at any profile.

Theorem 8. *Let $k \in N$ and $u \in \mathcal{U}$.*

(i) If $k \in S \in \mathcal{G}(u)$, then ψ^k is manipulable by exactly $|N| - |S|$ agents and exactly $2^{|N|-|S|} - 1$ coalitions at profile $u \in \mathcal{U}$.

(ii) ψ^k is manipulable by less than 50% of all coalitions at any profile $u \in \mathcal{U}$.

Proof. To prove (i), note that since S is an indifference component, for all $i \in S$ and all $(a, x) \in \psi^k(u)$, allocation (a, x) is agent i-linked. Thus, by Theorem 6, $(a, x) \in \psi^i(u)$ and $\psi^k(u) = \psi^i(u)$. From Lemma 7, it then follows that no coalition containing agent $i \in S$ can manipulate ψ^k at profile $u \in \mathcal{U}$. Thus, at most $2^{|N|-|S|} - 1$ coalitions can manipulate ψ^k at profile $u \in \mathcal{U}$. Lemma 6 guarantees that this bound is tight, i.e., that exactly $2^{|N|-|S|} - 1$ coalitions can manipulate ψ^k at profile $u \in \mathcal{U}$. Because there are exactly $|N| - |S|$ nonempty singleton coalitions in the class of coalitions that can gain by manipulation, it follows that exactly $|N| - |S|$ agents can manipulate ψ^k at profile $u \in \mathcal{U}$.

To prove (ii), note that $|S| \geq 1$. Because $2^{|N|-|S|} \leq 2^{|N|-1}$ for any $|S| \geq 1$, it follows from (i) that ψ^k can be manipulated at profile $u \in \mathcal{U}$ by at most $2^{|N|-1} - 1$ coalitions. Since there are $2^{|N|} - 1$ nonempty coalitions of N and $2^{|N|} - 1 = 2(2^{|N|-1} - 1) + 1$, less than 50% of all coalitions can manipulate ψ^k at profile $u \in \mathcal{U}$. □

Therefore, if the agent k-preferred fair allocation rule is adopted, then to calculate the exact number of manipulating agents and coalitions at a given profile, one only needs to know the number of agents that are included in the (indifference) component containing agent k. Because indifference components are invariant with respect to the chosen fair allocation (Lemma 4), it is sufficient to find an arbitrary agent k-linked fair allocation at the given preference profile to find the exact number of manipulating agents and coalitions. This task can be achieved, for example, by using the algorithm in Klijn (2000). Because this algorithm is polynomially bounded, this is not even computationally hard. An algorithm (inspired by Klijn 2000) for calculating agent k-linked fair allocations is provided in Andersson et al. (2010a).

The corollary below follows from the above results.

Corollary 1. *(i) ψ^k cannot be manipulated by agent k at any profile $u \in \mathcal{U}$.*

(ii) For any two distinct agents $i, j \in N$, there exists no fair and budget-balanced allocation rule φ such that neither i nor j can manipulate φ at any profile $u \in \mathcal{U}$.

Note that Lemma 7 implies that the agent k-linked fair allocation rule cannot be manipulated by any coalition containing k at any profile. In particular, the agent k-linked fair allocation rule is not manipulable by agent k at any profile $u \in \mathcal{U}$, which is the first part of Corollary 1. The second part of Corollary 1 is easy to verify and is left to the reader.

Remark 1. In a paper subsequent to this, Fujinaka and Wakayama (2011) report similar results as ours regarding individual manipulation (possibilities): (a) Proposition 1 and Proposition 2 in Fujinaka and Wakayama (2011) are identical with Theorem 6, (b) Theorem 2 of Fujinaka and Wakayama (2011) is equivalent to (restricting attention to individual manipulation) Theorem 7 (using Theorem 6), and (c) Corollary 2 of Fujinaka and Wakayama (2011) is equivalent to Corollary 1 (using Lemma 6). There are two important differences between our paper and Fujinaka and Wakayama (2011): on the one hand, we allow for multivalued allocation rules whereas they only consider single-valued allocation rules; on the other hand, we consider only quasi-linear utility functions whereas they consider general utility functions satisfying (i) *monotonicity in money*, i.e., for any $x, y \in \mathbb{R}^M$, if $x_j > y_j$, then $u_{ij}(x) > u_{ij}(y)$, and (ii) *no infinite desirability in terms of money*, i.e., for any $j, k \in M$ and any $x \in \mathbb{R}^M$, there exists $y \in \mathbb{R}^M$ such that $u_{ij}(x) = u_{ik}(y)$.

A.3 *Minimal manipulability*

Theorem 3 follows from the result below.

Theorem 9. *Let φ be an arbitrary fair and budget-balanced allocation rule and let ϕ^κ be a maximally preferred fair allocation rule. Then the following hold:*

(i) φ is agents-counting-more manipulable than ϕ^κ.

(ii) If ϕ^κ is agents-counting-more manipulable than φ, then φ is a maximally preferred fair allocation rule.

Proof. First, we show (i). Let $u \in \mathcal{U}$. Suppose that $(a, x) \in \varphi(u)$ and $(b, y) \in \phi^\kappa(u)$, and let $N - G$ be a (possibly empty) isolated group with maximal cardinality at allocation $(a, x) \in \varphi(u)$. Then G is an indifference component at allocations (a, x) and (b, y) by Lemmas 4 and 5.

Note first that all agents in the isolated coalition $N - G$ can manipulate φ by Lemma 6. Consequently, at least $|N - G|$ agents can manipulate φ. Hence, to conclude the proof for (i), we need to show that at most $|N - G|$ agents can manipulate ϕ^κ.

Suppose that $\kappa(u)$ belongs to the indifference component $\hat{G} \subseteq \bar{G}(u)$ and note that $|\hat{G}| \geq |G|$ by construction of ϕ^κ. Since $\phi^\kappa(u) = \psi^k(u)$ for all $k \in \hat{G}$, it now follows from Lemma 7 that no agent $k \in \hat{G}$ can manipulate ϕ^κ at profile $u \in \mathcal{U}$. Thus, at most $|N - \hat{G}|$ agents can manipulate ϕ^κ. The conclusion of (i) then follows directly from the observation that $|\hat{G}| \geq |G|$ implies $|N - \hat{G}| \leq |N - G|$.

For (ii), note that then we have to have $|N - G| \leq |N - \hat{G}|$ and $|G| \geq |\hat{G}|$. Then $G \subseteq \bar{G}(u)$. Since $u \in \mathcal{U}$ was arbitrary, now φ is a maximally preferred fair allocation rule. □

Theorem 4 follows from the result below.

Theorem 10. *Let φ be a fair and budget-balanced allocation rule and let ϕ^κ be a maximally preferred fair allocation rule. Then the following hold:*

(i) φ is coalitions-counting-more manipulable than ϕ^κ.

(ii) If ϕ^κ is coalitions-counting-more manipulable than φ, then φ is a maximally preferred fair allocation rule.

Proof. First, we show (i). Let $u \in \mathcal{U}$. Suppose that $(a, x) \in \varphi(u)$ and $(b, y) \in \phi^\kappa(u)$, and let $N - G$ be the (possibly empty) isolated group with maximal cardinality at allocation $(a, x) \in \varphi(u)$. Then G is an indifference component at allocations (a, x) and (b, y) by Lemmas 4 and 5.

Note first that all coalitions in the isolated group $N - G$ can manipulate φ by Lemma 6. Consequently, at least $2^{|N-G|} - 1$ coalitions can manipulate φ. Hence, to conclude the proof, we need to show that at most $2^{|N-G|} - 1$ coalitions can manipulate ϕ^κ. Suppose now that $\kappa(u)$ belongs to the indifference component $\hat{G} \subseteq \bar{G}(u)$ and note that $|\hat{G}| \geq |G|$ by construction of ϕ^κ. It now follows from Lemma 7 and the construction of ϕ^κ that at most $2^{|N-\hat{G}|} - 1$ coalitions can manipulate ϕ^κ. The conclusion of (i) then follows directly from the observation that $|\hat{G}| \geq |G|$ implies $2^{|N-\hat{G}|} - 1 \leq 2^{|N-G|} - 1$.

For (ii), note that then we have to have $2^{|N-\hat{G}|} - 1 \geq 2^{|N-G|} - 1$ and $|G| \geq |\hat{G}|$. Then $G \subseteq \bar{G}(u)$. Since $u \in \mathcal{U}$ was arbitrary, now φ is a maximally preferred fair allocation rule. □

Theorem 5 follows from the result below.

Theorem 11. *Let φ be an arbitrary fair and budget-balanced allocation rule. Then the following hold:*

(i) There exists a selection $\kappa : \mathcal{U} \to N$ such that φ is agents-inclusion-more manipulable than ϕ^κ.

(ii) If ϕ^κ is agents-inclusion-more manipulable than φ, then $\varphi = \phi^\kappa$.

Proof. We construct $\kappa : \mathcal{U} \to N$ as follows: for all $u \in \mathcal{U}$, if for some $k \in N$, $\varphi(u) = \psi^k(u)$, then we set $\kappa(u) = k$; otherwise $\kappa(u)$ can be arbitrary.

First we show (i). Let $u \in \mathcal{U}$. If for all $k \in N$, $\varphi(u) \nsubseteq \psi^k(u)$, then any agent $i \in N$ belongs to an isolated group. Now by Lemma 6, $P^\varphi(u) = N$. Since $\phi^\kappa(u) \subseteq \psi^{\kappa(u)}(u)$, now by Lemma 7, $P^{\phi^\kappa}(u) \subseteq N - \{\kappa(u)\}$. Hence, $P^\varphi(u) \supseteq P^{\phi^\kappa}(u)$.

If for some $k \in N$, $\varphi(u) = \psi^k(u)$, then by construction of κ, we also have $\phi^\kappa(u) = \psi^k(u)$. But now we have $P^\varphi(u) \supseteq P^{\phi^\kappa}(u)$.

Hence, for all $u \in \mathcal{U}$, $P^\varphi(u) \supseteq P^{\phi^\kappa}(u)$ and φ is agents-inclusion-more manipulable than ϕ^κ, the desired conclusion for (i).

For (ii), note that then we have to have $P^{\phi^\kappa}(u) \supseteq P^\varphi(u)$. Then for $k = \kappa(u)$, we have $\phi^\kappa(u) = \psi^k(u)$ and, by $k \notin P^{\phi^\kappa}(u)$, $\varphi(u) = \psi^k(u)$, the desired conclusion. □

References

Abdulkadiroğlu, Atila, Tayfun Sönmez, and Utku Ünver (2004), "Room assignment-rent division: A market approach." *Social Choice and Welfare*, 22, 515–538. [755]

Aleskerov, Fuad and Eldeniz Kurbanov (1999), "Degree of manipulability of social choice procedures." In *Proceedings of the Third International Meeting of the Society for the Advancement of Economic Theory*, Springer, Berlin. [754, 758]

Alkan, Ahmet, Gabrielle Demange, and David Gale (1991), "Fair allocation of indivisible objects and criteria of justice." *Econometrica*, 59, 1023–1039. [754, 764, 766]

Andersson, Tommy, Lars Ehlers, and Lars-Gunnar Svensson (2010a), "Budget-balance, fairness and minimal manipulability." Cahier 18-2010, CIREQ. [757, 763, 773]

Andersson, Tommy, Lars Ehlers, and Lars-Gunnar Svensson (2012), "(Minimally) ϵ-incentive compatible competitive equilibria in economies with indivisibilities." Cahier 04-2012, CIREQ. [754]

Andersson, Tommy and Lars-Gunnar Svensson (2008), "Non-manipulable assignment of individuals to positions revisited." *Mathematical Social Sciences*, 56, 350–354. [755, 757, 758, 765]

Andersson, Tommy, Lars-Gunnar Svensson, and Zaifu Yang (2010b), "Constrainedly fair job assignment under minimum wages." *Games and Economic Behavior*, 68, 428–442. [765]

Aragones, E. (1995), "A derivation of the money Rawlsian solution." *Social Choice and Welfare*, 12, 267–276. [755]

Azacis, Helmuts (2008), "Double implementation in a market for indivisible goods with a price constraint." *Games and Economic Behavior*, 62, 140–154. [755]

Beviá, Carmen (2010), "Manipulation games in economies with indivisible goods." *International Journal of Game Theory*, 39, 209–222. [769]

Dubey, Pradeep (1982), "Price-quantity strategic markets." *Econometrica*, 50, 111–126. [765]

Foley, Duncan (1967), "Resource allocation and the public sector." *Yale Economic Essays*, 7, 43–98. [757]

Fujinaka, Yuji and Takuma Wakayama (2011), "Maximal manipulation in fair allocation." Working paper. [753, 754, 773]

Green, Jerry and Jean-Jacques Laffont (1979), *Incentives in Public Decision Making*. North-Holland, Amsterdam. [753, 754, 758]

Haake, Claus-Jochen, Matthias G. Raith, and Francis E. Su (2002), "Bidding for envy-freeness: A procedural approach to n-player fair division problems." *Social Choice and Welfare*, 19, 723–749. [755]

Kelly, Jerry S. (1988), "Minimal manipulability and local strategy-proofness." *Social Choice and Welfare*, 5, 81–85. [754, 758]

Kelly, Jerry S. (1993), "Almost all social choice rules are highly manipulable, but a few aren't." *Social Choice and Welfare*, 10, 161–175. [754, 758]

Klijn, Flip (2000), "An algorithm for envy-free allocations in an economy with indivisible objects and money." *Social Choice and Welfare*, 17, 201–215. [755, 773]

Ma, Jinpeng (1995), "Stable matchings and rematching-proof equilibria in two-sided matching markets." *Journal of Economic Theory*, 66, 352–369. [761]

Maus, Stefan, Hans Peters, and Ton Storcken (2007a), "Minimal manipulability: Anonymity and unanimity." *Social Choice and Welfare*, 29, 247–269. [754, 758]

Maus, Stefan, Hans Peters, and Ton Storcken (2007b), "Anonymous voting and minimal manipulability." *Journal of Economic Theory*, 135, 533–544. [754, 758]

Mishra, Debasis and David C. Parkes (2009), "Multi-item Vickrey–Dutch auctions." *Games and Economic Behavior*, 66, 326–347. [765]

Mishra, Debasis and Dolf Talman (2010), "Characterization of the Walrasian equilibria of the assignment model." *Journal of Mathematical Economics*, 46, 6–20. [765]

Moulin, Hervé (1980), "On strategy-proofness and single peakedness." *Public Choice*, 35, 437–455. [754]

Pathak, Parag A. and Tayfun Sönmez (2013), "School admissions reform in Chicago and England: Comparing mechanisms by their vulnerability to manipulation." *American Economic Review*, 103, 80–106. [754, 755, 758, 759, 760]

Sankaran, Jayaram K. (1994), "On a dynamic auction mechanism for a bilateral assignment problem." *Mathematical Social Sciences*, 28, 143–150. [765]

Sun, Ning and Zaifu Yang (2003), "A general strategy proof fair allocation mechanism." *Economics Letters*, 81, 73–79. [755, 757, 758]

Svensson, Lars-Gunnar (1983), "Large indivisibilities: An analysis with respect to price equilibrium and fairness." *Econometrica*, 51, 939–954. [754]

Svensson, Lars-Gunnar (1991), "Nash implementation of competitive equilibria in a model with indivisible goods." *Econometrica*, 59, 869–877. [765]

Svensson, Lars-Gunnar (2009), "Coalitional strategy-proofness and fairness." *Economic Theory*, 40, 227–245. [755, 757, 758, 764]

Tadenuma, Koichi and William Thomson (1991), "No-envy and consistency in economies with indivisibilities." *Econometrica*, 59, 1755–1767. [754]

Tadenuma, Koichi and William Thomson (1993), "The fair allocation of an indivisible good when monetary compensations are possible." *Mathematical Social Sciences*, 25, 117–132. [755]

Velez, Rodrigo A. (2011), "Are incentives against economic justice?" *Journal of Economic Theory*, 146, 326–345. [755]

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