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The Foster–Hart measure of riskiness for general gambles

Frank Riedel
Center for Mathematical Economics, Bielefeld University

Tobias Hellmann
Center for Mathematical Economics, Bielefeld University

Foster and Hart propose a measure of riskiness for discrete random variables. Their defining equation has no solution for many common continuous distributions. We show how to extend consistently the definition of riskiness to continuous random variables. For many continuous random variables, the risk measure is equal to the worst-case risk measure, i.e., the maximal possible loss incurred by that gamble. For many discrete gambles with a large number of values, the Foster–Hart riskiness is close to the maximal loss. We give a simple characterization of gambles whose riskiness is or is close to the maximal loss.

Keywords. Risk measures, bankruptcy, continuous random variable.

JEL classification. D81, G11.

Foster and Hart (2009) introduce a measure of riskiness for finite lotteries. If the lottery is described by a random variable X, its riskiness is the unique positive solution $\rho > 0$ of the equation

$$E \log(1 + X/\rho) = 0, \tag{1}$$

where ρ is a critical wealth level. An investor who rejects gambles if his current wealth is below the riskiness avoids bankruptcy almost surely; agents accepting gambles at lower wealth levels can lose their wealth against a malevolent nature with positive probability.

Until now, the Foster–Hart measure of riskiness has only been studied for gambles with finitely many outcomes; even the finite examples were mostly confined to gambles with few values. Many financial applications involve distributions with a large number or a continuum of outcomes; it seems natural and important to generalize the concept of critical wealth level to such cases. Distributions with densities being the limit of finite gambles with many outcomes, we will learn something about finite lotteries as well.

Frank Riedel: frank.riedel@uni-bielefeld.de
Tobias Hellmann: tobias.hellmann@uni-bielefeld.de
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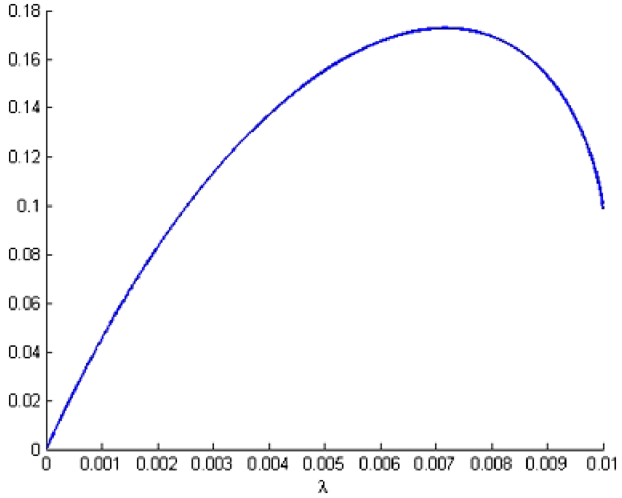


FIGURE 1. The function $\lambda \mapsto E\log(1+\lambda X)$ for the uniform distribution over $[-100, 200]$ has no zero.

A straightforward generalization to general gambles via (1) is not possible. Even for the simple case of a uniform distribution, the defining equation of Foster and Hart does not always have a finite solution, as the following example demonstrates.

EXAMPLE 1. Let X be uniformly distributed over $[-100, 200]$. The random variable X has the positive expectation 50 and losses occur with positive probability. It thus qualifies as a gamble in the sense of Foster and Hart. Replacing ρ by $\lambda = 1/\rho$, we study the equation

$$\phi(\lambda) := E\log(1+\lambda X) = 0. \tag{2}$$

The function ϕ is well defined for positive values $\lambda \le 1/L$, where $L = 100$ is the gamble's maximal loss. We plot the function $\phi(\lambda)$ in Figure 1. No solution for $\lambda > 0$ to (2) exists. For a formal proof, note that ϕ is continuous and concave on $[0, 1/L]$, with positive slope in 0 as $EX > 0$ (see the argument in Foster and Hart (2009)). Thus, there exists a root for the defining equation if and only if $\phi(1/L) < 0$. For the maximal possible value $\lambda^*(X) = 1/L = \frac{1}{100}$, we have

$$\begin{aligned} E\log(1+\lambda^*(X)X) &= \int_{-100}^{200} \tfrac{1}{300}\log(1+x/100)\,dx \\ &= \Big[\tfrac{1}{3}((1+x/100)\log(1+x/100) - (1+x/100))\Big]_{-100}^{200} \\ &= \log 3 - 1 \simeq 0.0986 > 0. \end{aligned}$$

We conclude that $\phi(\lambda) > 0$ for all $\lambda \in (0, \lambda^*(X)]$. ◊

n	Grid size	Riskiness
2	300	200
3	150	145.74
5	75	119.46
11	30	104.997
21	15	101.197
31	10	100.3651
41	7.5	100.1195
61	5	100.0137
101	3	100.0002

TABLE 1.

As a reaction to the example above, one might point out that the value $\lambda = 0$ solves the defining equation. The value $\lambda = 0$ corresponds to a riskiness of infinity; while rejecting the favorable gamble at all wealth levels certainly avoids bankruptcy, such a rule would seem economically implausible.

Let us study next how the riskiness of discrete distributions that approximate the uniform one looks. We call a random variable $X : (\Omega, \mathcal{F}) \to \mathbb{R}$ defined on a probability space $(\Omega, \mathcal{F}, P)$ a gamble if its expectation is positive ($EX > 0$), losses occur with positive probability ($P[X < 0] > 0$), and its maximal loss is bounded ($L(X) := \operatorname{ess\,sup}(-X) < \infty$).[1] We call a gamble finite if its support is finite.

Now we approximate the uniform distribution over $[-100, 200]$ by finite gambles. We consider discrete and uniformly distributed gambles on the grid $-100, -100 + 300/(n-1), \dots, -100 + 300k/(n-1), \dots, 200$. The riskiness is the root of

$$f_n(\lambda) = \frac{1}{n} \sum_{k=0}^{n-1} \log\left(1 + \lambda\left(-100 + \frac{300k}{n-1}\right)\right).$$

For the simplest case, $n = 2$, one can easily verify that

$$\tfrac{1}{2}(\log(1 - 100\lambda) + \log(1 + 200\lambda)) = 0$$

yields a riskiness $\rho = 1/\lambda = 200$.

In Table 1, the riskiness numbers for different grid sizes are shown. We observe that the riskiness decreases and converges to the maximal loss as the grid becomes finer and finer. As the single weights on specific losses vanish, the investor might accept the gambles at ever lower wealth levels. In the limit, he is able to gamble as long as his wealth suffices to cover the maximal loss without taking any risk of bankruptcy. Note that the riskiness is very close to the maximal loss 100 even for quite large grid sizes ($n = 21$, e.g.).

Let us now go beyond specific examples and clarify for which gambles the Foster–Hart index is equal or close to the maximal loss. We will characterize such distributions by a simple condition.

[1]We define as usual $\operatorname{ess\,sup}(-X) := \inf\{x \in \mathbb{R} \mid P(-X > x) = 0\}$.

There are two classes of gambles. For one class, the defining equation of Foster and Hart has a finite solution, and one can use this number as its riskiness. For the other class, including the uniform one described above, the defining equation has no solution. For all finite gambles that approximate these distributions in a suitable way, the Foster–Hart riskiness converges to the maximal loss.

Theorem 1. *Let X be a gamble with maximal loss $L > 0$. Let X_n be a sequence of finite gambles with $X_n \uparrow X$ a.s., where each X_n has the same maximal loss L. Denote by $\rho_n := \rho(X_n) > L$ their Foster–Hart riskiness. Then the following statements hold true:*

(i) The sequence (ρ_n) is decreasing. We write $\rho_\infty = \lim \rho_n \geq L$ for its limit.

(ii) If $E \log(1 + X/L) < 0$, then $\rho_\infty > L$ and ρ_∞ is the unique positive solution of the Foster–Hart equation (1).

(iii) If $E \log(1 + X/L) \geq 0$, then the Foster–Hart equation has no solution and $\rho_\infty = L(X)$.

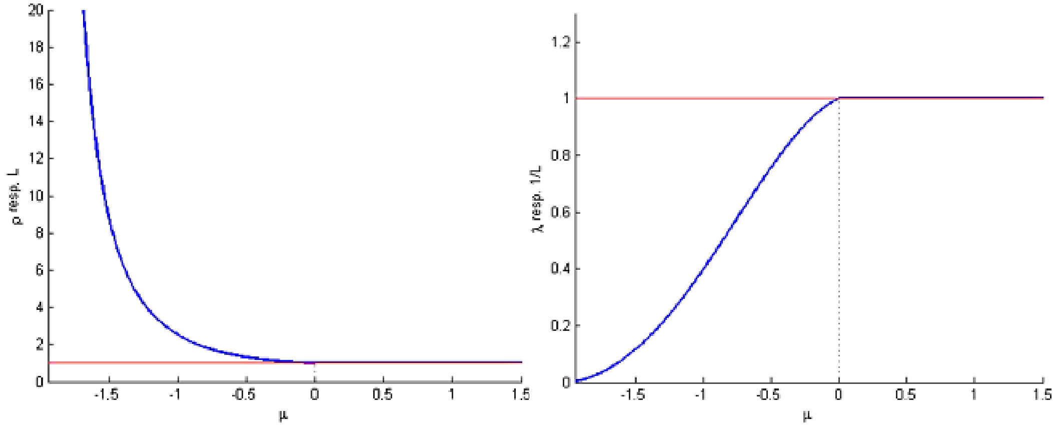
The proof is postponed to the end of the text. For gambles where the defining equation does not have a solution, our theorem suggests the use of the maximal loss as their riskiness.

The previous theorem also gives a simple test to see whether the Foster–Hart riskiness is equal to (or close to) the maximal loss of a distribution. Indeed, the sign of the expectation $E \log(1 + X/L)$ determines whether the riskiness is equal or close to the maximal loss.

The maximal loss is indeed obtained for a large number of gambles. For example, for the uniform distribution on $[-100, 200]$ and for a uniform distribution on, say, $[-100, 10^{12}]$, the riskiness is the same, namely 100 (and similarly for finite gambles with such a support on a dense grid; compare Example 3 below for more details). The Foster–Hart riskiness index then boils down to the so-called worst-case risk measure.

This property appears to be undesirable. Why would uniform gambles on $[-100, 200]$ and the much more favorable uniform gambles on $[-100, 10^{12}]$ have the same riskiness? Let us look at the operational interpretation of the riskiness that Foster and Hart had in mind. The aim is to find a critical wealth level that ensures solvency with probability 1 if it is used as a decision rule for acceptance and rejection of gambles. For solvency, losses clearly play a much more important role than potential gains, and our analysis shows that frequently the *maximal loss* only determines whether one should accept or reject a gamble.

In Hellmann and Riedel (2014), we extend the Foster–Hart result on solvency to our gambles. One does avoid bankruptcy with probability 1 if one uses our extended riskiness as a decision rule. In particular, if one faces a sequence of independent uniformly distributed gambles with sufficiently high maximal gains, one stays solvent with probability 1 if one accepts every gamble whose maximal loss is below one's wealth. As the operational interpretation of Foster and Hart (2009) carries over, this provides another justification for using the maximal loss as an extension of the Foster–Hart riskiness.



(a) Generalized Foster–Hart riskiness ρ

(b) $\lambda = 1/\rho$

Figure 2. Variables ρ and λ for log-normally distributed gambles over $(-1, \infty)$ with $\sigma = 2$.

A riskiness equal to the maximal loss is quite a contrast to the simple Bernoulli example discussed by Foster and Hart where you need a wealth of at least \$600 to accept a gamble with outcomes of $+\$120$ and $-\$100$ occurring with equal probability.

Our theorem also shows that, for certain gambles, the Foster–Hart index does not care about the way gains are distributed. Whether you have specific gains with a certain density or point masses on some numbers does not matter. Further examples illustrate this point.

Example 2. The log-normal distribution is used in many financial applications, for instance, in the widely used Black–Scholes options pricing model. Therefore, it seems to be important to be able to apply the measure of riskiness for this distribution.

A random variable X is said to be log-normally distributed if its density φ is (see Johnson et al. 1995)

$$\varphi(x; \mu, \sigma, L) = \frac{1}{(x+L)\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2}\frac{(\log(x+L)-\mu)^2}{\sigma^2}\right), \quad x > -L,$$

where μ and σ are, respectively, the expected value and the standard deviation of the normally distributed random variable $X^N = \log(X+L)$, and L is the maximal loss of X.

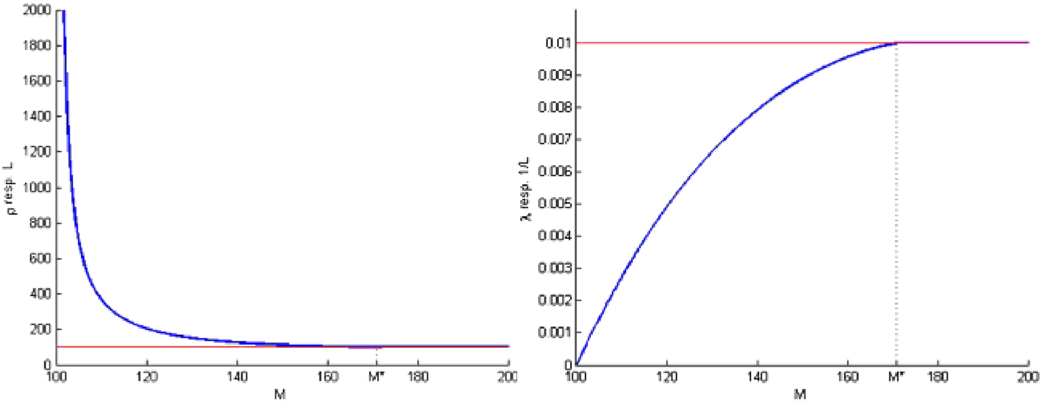
For the special case of log-normal distributed gambles with $L = 1$, we can obtain an interesting result. $\diamond$

Proposition 1. *For the log-normal distributed random variable* $X = \exp(X^N) - 1$ *with* $EX > 0$, *there exists a solution for the defining (1) if and only if* $EX^N < 0$.

Proof. We can easily check that

$$E\log\left(1 + \frac{\exp(X^N) - 1}{1}\right) = E\log(\exp(X^N)) = EX^N$$

and, therefore, $E\log(1 + X/L(X)) < 0$ if and only if $EX^N < 0$. $\square$



(a) Generalized Foster–Hart riskiness ρ (b) $\lambda = 1/\rho$

FIGURE 3. Variables ρ and λ for uniform distributed gambles over $[-100, M]$.

Now, if we also fix $\sigma = 2$, we can numerically compute the riskiness as a function of μ. The result is drawn in Figure 2. As Proposition 1 already says, we observe that the critical value for which there exists no zero for the defining equation is $\mu^* = 0$.

EXAMPLE 3. Let us consider the motivating example again. We fix $L = 100$ and check for which value M^* of the maximal gain the defining equation (1) has a solution for the uniformly distributed gamble over $[-100, M^*]$, i.e., we need to find M^* such that $E\log(1 + X/L) = 0$. Therefore,

$$\begin{aligned} E\log\left(1 + \frac{X}{100}\right) &= \int_{-100}^{M^*} \frac{1}{100 + M^*} \log\left(1 + \frac{x}{100}\right) dx \\ &= \left[\frac{100}{100 + M^*}\left(\left(1 + \frac{x}{100}\right)\log\left(1 + \frac{x}{100}\right) - \left(1 + \frac{x}{100}\right)\right)\right]_{-100}^{M^*} \\ &= \frac{100}{100 + M^*}\left(\left(1 + \frac{M^*}{100}\right)\log\left(1 + \frac{M^*}{100}\right) - \left(1 + \frac{M^*}{100}\right)\right). \end{aligned}$$

Setting this equal to zero yields

$$\log\left(1 + \frac{M^*}{100}\right) = 1,$$

which implies

$$M^* = L(e - 1) \simeq 171.8.$$

Hence, for all values $M < M^*$, there exists a solution to the defining equation and we take this solution as the riskiness. For all $M \geq M^*$ there does not exist a finite solution and, therefore, we take the maximal loss $L = 100$ as the riskiness.

In Figure 3, the graph of the riskiness ρ as well as the solution λ of (2) is plotted against the maximal gain M of the gambles. To get a positive expectation, we consider only values of M with $M > 100$.

The graph of the riskiness is a continuous function; the riskiness tends to the maximal loss $L = 100$ as we approach the critical value M^* and converges to infinity as the expectation of the gamble goes to 0 (i.e., $M \downarrow 100$). ♢

Example 4. Consider mixed gambles that have a discrete part as well as a continuous part. For instance, we take a gamble X that is uniformly distributed over the interval $[-100, 0]$ and that places a probability of 50% on the event $\{X = M\}$, where $M > 50$ to ensure a positive expectation. For $M > M^* := 100(e-1) \simeq 171.8$, no solution to the defining equation exists as a calculation similar to the previous example shows:

$$\begin{aligned}
E\log\left(1+\frac{X}{100}\right) &= \tfrac{1}{2}\log\left(1+\frac{M}{100}\right)+\tfrac{1}{2}\int_{-100}^{0}\tfrac{1}{100}\log\left(1+\frac{x}{100}\right)dx \\
&= \tfrac{1}{2}\left(\log\left(1+\frac{M}{100}\right)+\left(1+\tfrac{0}{100}\right)\log\left(1+\tfrac{0}{100}\right)-\left(1+\tfrac{0}{100}\right)\right) \\
&= \tfrac{1}{2}\left(\log\left(1+\frac{M}{100}\right)-1\right).
\end{aligned}$$

For $M > M^*$, this expression is positive.

We observe that the critical value for this mixed distribution is the same as for the uniform distribution over $[-100, 171.8]$ (see Example 3). This seems to be surprising at first sight, as we replaced a uniform distribution over an interval by a positive mass on the maximal gain. But notice that the Foster–Hart measure of riskiness is more sensitive on the loss side than on the gain side. The decrease of the probability of the event $\{X < 0\}$ from $\frac{1}{2}$ to $\simeq 0.37$ outweighs the higher gains of the mixed gamble and the critical value is exactly the same.

On the other hand, if we take a mixed distribution that has a point mass on its maximal loss L, the defining equation always has a solution. Indeed, due to the fact that the event $\{X = -L\}$ has a positive probability, we have $\lim_{\lambda\to 1/L} E\log(1+\lambda X) = -\infty$ and hence a solution to (1) exists. ♢

Example 5. Let us consider beta distributed gambles. The density of a random variable X that is beta distributed over the compact interval $[-L, M]$ is, for instance, given in Johnson et al. (1995) as

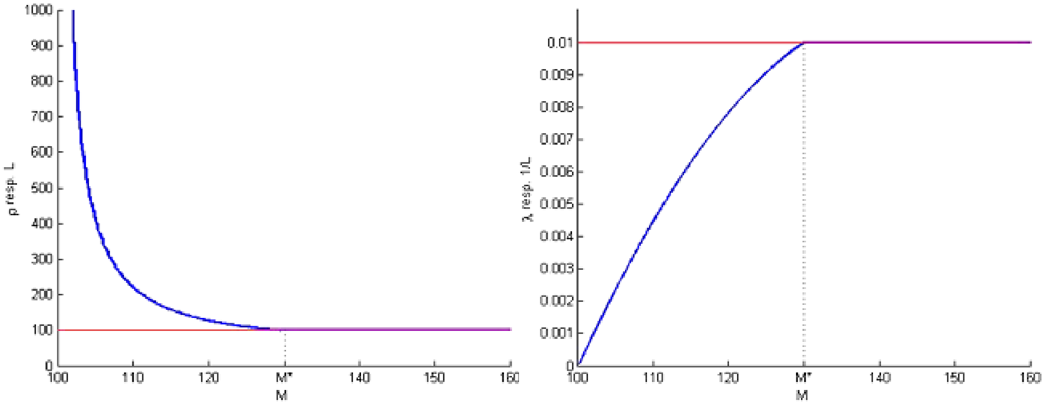
$$\varphi(x;\alpha,\beta,L,M) = \frac{1}{B(\alpha,\beta)}\frac{(x+L)^{\alpha-1}(M-x)^{\beta-1}}{(M+L)^{\alpha+\beta-1}}, \quad x\in[-L,M],\ \alpha,\beta>0,$$

where $B(\alpha, \beta)$ denotes the beta function defined as

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\,dt.$$

The mean of X is given by

$$EX = \frac{\alpha M-\beta L}{\alpha+\beta}.$$



(a) Generalized Foster–Hart riskiness ρ

(b) $\lambda = 1/\rho$

FIGURE 4. Variables ρ and λ for beta distributed gambles over $[-100, 200]$ with $\alpha = 2$.

We can parameterize our beta distributed gamble X by

$$X = cZ - L,$$

where Z is a beta distributed random variable over $[0, 1]$ and $c = M + L$. Using this parameterization, we can now explicitly compute for which value of M (or c) no solution to the defining equation exists. Let us fix $L = 100$, $\alpha = 2$, and $\beta = 2$. We have

$$E \log\left(1 + \frac{X}{L}\right) = E \log\left(1 + \frac{cZ - L}{L}\right) = E \log\left(\frac{cZ}{L}\right) = \log(c) - \log(L) + E \log(Z).$$

Thus, we are searching for c^* that solves

$$\log(c^*) = \log(L) - E \log(Z).$$

Now,

$$\begin{aligned} E \log(Z) &= \int_0^1 \frac{\log(x)}{B(2, 2)} x(1 - x)\, dx \\ &= \frac{1}{B(2, 2)} \left[\log(x)\left(\tfrac{1}{2}x^2 - \tfrac{1}{3}x^3\right) - \left(\tfrac{1}{4}x^2 - \tfrac{1}{9}x^3\right)\right]_0^1 \\ &= -\tfrac{5}{6}. \end{aligned}$$

Hence,

$$c^* = 100 \exp\left(\tfrac{5}{6}\right) \simeq 230.09,$$

which means

$$M^* \simeq 130.09.$$

Figure 4 shows the graph of the riskiness and of $\lambda = 1/\rho$ against M. For $M > M^*$, where no positive solution exists, the maximal loss $L = 100$ is used to determine the riskiness. The figure demonstrates that the maximal loss is a continuous extension of the Foster–Hart riskiness. ◊

Proof of Theorem 1. The first statement follows directly by the monotonicity of the Foster–Hart measure of riskiness; see Proposition 2 in Foster and Hart (2009). It is easier to prove the converse of the latter two statements. We define $1/\rho_n = \lambda_n$, and $1/\rho_\infty = \lambda_\infty$. Without loss of generality, we take $L = 1$ (else replace X by X/L). Let us start by assuming $\lambda_\infty < 1$. In that case, the sequence

$$Z_n = \log(1 + \lambda_n X_n)$$

is uniformly bounded. Indeed,

$$-\infty < \log(1 - \lambda_\infty) \le Z_n \le \log(1 + |X|) \le |X| \in L^1.$$

As we have $Z_n \to \log(1 + \lambda_\infty X)$ a.s., we can then invoke Lebesgue's dominated convergence theorem to conclude

$$0 = \lim E Z_n = E \lim Z_n = E \log(1 + \lambda_\infty X).$$

In particular, (2) has a positive solution $\lambda_\infty < 1$. As $\phi(\lambda) = E \log(1 + \lambda X)$ is strictly concave and strictly positive on $(0, \lambda_\infty)$, we conclude that we must have $\phi(1) = E \log(1 + X) < 0$. This proves the second claim.

Now let us assume $\lambda_\infty = 1$. In that case, we cannot use Lebesgue's theorem. However, the sequence

$$Z'_n = -\log(1 + \lambda_n X_n)$$

is bounded from below by $-\log(1 + |X|) \ge -|X| \in L^1$. We can then apply Fatou's lemma to conclude

$$-E \log(1 + X) = E \lim Z'_n \le \liminf -E \log(1 + \lambda_n X_n) = 0$$

or

$$E \log(1 + X) \ge 0.$$

This proves the first claim. □

References

Foster, Dean P. and Sergiu Hart (2009), "An operational measure of riskiness." *Journal of Political Economy*, 117, 785–814. [1, 2, 4, 9]

Hellmann, Tobias and Frank Riedel (2014), "A dynamic extension of the Foster–Hart measure of riskiness." Working Paper 528, Center for Mathematical Economics, Bielefeld University. [4]

Johnson, Norman Lloyd, Samuel Kotz, and Narayanaswamy Balakrishnan (1995), *Continuous Univariate Distributions*. Wiley-Interscience, New York. [5, 7]

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