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Strategic uncertainty and the ex post Nash property in large games

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This paper elucidates the conceptual role that independent randomization plays in non-cooperative game theory. In the context of large (atomless) games in normal form, we present precise formalizations of the notions of a mixed strategy equilibrium (MSE) and of a randomized strategy equilibrium in distributional form (RSED). We offer a resolution of two longstanding open problems and show that (i) any MSE *induces* a RSED and any RSED can be *lifted* to a MSE, and (ii) a mixed strategy profile is a MSE if and only if it has the ex post Nash property. Our substantive results are a direct consequence of an *exact* law of large numbers that can be formalized in the analytic framework of a Fubini extension. We discuss how the "measurability" problem associated with a MSE of a large game is automatically resolved in such a framework. We also present an approximate result pertaining to a sequence of large but finite games.

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1. Introduction

In this paper, we present two theorems regarding atomless games in normal form, which is to say one-shot games of complete information played by a continuum of players. First, we show an equivalence between randomized strategy equilibrium in distributional form and mixed strategy Nash equilibrium in that the first can be consolidated and *lifted up* into the second, and that the second can be personalized to *induce* the first. Without going into details in this introductory paragraph, this is to answer in the affirmative the question as to whether individually subjective and independent randomizations can be consolidated into an objective, grand randomization for the collective, and if the latter is given a priori, whether it can be seen as having been obtained from individual randomizing devices used in isolation from each other. Our second result builds on the first and again answers in the affirmative the question as to whether a mixed strategy equilibrium, after the resolution of uncertainty, leads to pure strategy profiles that are individually regret-free—that an ex ante Nash equilibrium is also an ex post Nash equilibrium. This is to assert that in an equilibrium, there is no incentive for any nonnegligible set of agents to deviate from their pure strategies that result from the realization of information that formed the basis of their individual randomized strategies in distributional form in the first place. This is the thrust of the consolidation of the individual randomizing devices into the resulting measure of societal responses. As such, the first question presupposes the second: if a randomized strategy profile in distributional form is not successfully consolidated as one in mixed strategies, the second question can hardly even be posed.

Both questions are relatively longstanding, even if one confines oneself to large atomless games.[1] The first question concerning equivalence was already posed in the early fifties, and in the transfer of the pioneering analysis for a finite game to a large game, a conceptual connection was explicitly made in the early nineties to the classical law of large numbers and an invocation of Kolomogorov's extension theorem for the construction of the relevant continuum product sample space for the consolidated strategic uncertainty. However, as has been well known since the late thirties, the irregularity of the sample functions of a stochastic process defies resolution in a noncooperative context for which the independence of shocks is an essential *desideratum*. As a result of this so-called *measurability problem*, only an approximate, rather unsatisfactory, answer could be obtained. It is only in the late nineties that an exact law of large numbers (henceforth ELLN) for a continuum of independent random variables was proved through the framework of Fubini extension. However, the first question being considered here was not explicitly taken up. As regards the second question concerning robustness of Nash equilibria, following up on the distinction and classification

[1] We shall be scrupulous in relating our work to the relevant literature in the sequel.

of decisions along ex ante, ex post, and interim lines in the mid-eighties, the notion of an *ex post Nash equilibrium* was precisely formulated and successfully applied in auction theory as a problem of auction design.[2] To be sure, this notion of an ex post Nash equilibrium was already considered a decade ago for a large game with complete information rather than a specific finite game of incomplete information, as an auction, and again connected to the ELLN to deal with games with independent shocks. A result pertaining exclusively to strategic uncertainty was to be had a decade ago, but again, as for the first question, only an approximate, rather unsatisfactory, answer could be obtained by an appeal to the classical law of large numbers.[3]

It is perhaps worthwhile to be a little more explicit about the measurability problem and the relevance of the ELLN to the theory of large games and to the two questions considered in this paper. The key game-theoretic notion involved is the concept of a mixed strategy Nash equilibrium. A natural mathematical representation of a mixed strategy profile is that each player's strategy is a random variable taking values in her action space while the random variables across players are independent.[4] There is no measurability issue associated with a finite number of independent random variables. However, when one substitutes a continuum of players for a finite number of players, one needs to work with a process with a continuum of independent random variables. And this leads to subtle measurability issues. First, the archetype Lebesgue unit interval is not suitable for modeling a continuum of players acting independently simply because it has too few measurable sets in the sense that countably many measurable subsets of players determine the Lebesgue σ-algebra. Second, events in the usual continuum product sample space depend only on the actions of countably many players, and therefore such a sample probability space is inadequate for the study of a continuum of independent players, especially when one needs to consider the aggregate behavior of all the players. Third, and this may possibly be the most decisive consideration, no matter what probability spaces are used to model the player and sample spaces, a process with a continuum of independent random variables can *never* be jointly measurable with respect to the usual product σ-algebra. Therefore, to ensure that there are enough measurable sets, we work with saturated probability spaces such as a saturated extension of the Lebesgue unit interval, to model the space of players, and rely on a rich Fubini extension as the relevant joint agent–sample space.[5]

[2]The point was that an auctioneer could design a regret-free mechanism under which bidders would not have an incentive to revise their bids made in the absence of the knowledge of the other bids even after all uncertainty was resolved and this complete knowledge was to be had.

[3]These claims relating to an exact ex post property of Nash equilibrium for *general* uncertainty and an approximate version for *strategic* uncertainty will be documented in Sections 4 and 5.

[4]For more detailed discussions on this point, see Section 2 below.

[5]The framework of a Fubini extension resolves the joint measurability issue automatically. The first and second measurability issues can also be viewed in this framework. As noted in Footnote 25 below, it is shown, respectively, in Sun (2006, Corollary 4.3 and Propositions 6.1) that almost all sample functions of a nontrivial independent process *must* be non-Lebesgue measurable under a Fubini extension, and that the usual continuum product sample space cannot be a marginal probability space of a Fubini extension. The second measurability issue also indicates that even though the usual continuum product sample space is rich with measurable sets, it is still problematic regardless of whether the space of players is modeled by the Lebesgue unit interval or not; see Sun (2006, Remark 6.3).

With this prelude, two further questions can be asked and answered in a preliminary way, even at this introductory stage. First, how is this measurability problem, once directly faced, circumvented to obtain satisfyingly exact answers to the question posed above? And second, given the concern of this work with atomless games in normal form,[6] what is the interest in an ex post notion that involves uncertainty? As regards the first question, what one needs is an extension of the joint space of players and samples that is hospitable to integration relevant to a process with a continuum of independent random variables. It turns out that if an extension retains the Fubini property that allows the change of order for an iterated integral, an ELLN can then be proved for a measurable independent process in such a Fubini extension. Since the payoffs as considered in a large game usually depend not on the individual plays of each of the other players, but on a summary statistic of the plays as a whole, the ELLN will play a key role in working with a mixed strategy Nash equilibrium. In particular, it allows us to relate a mixed strategy Nash equilibrium to a randomized strategy profile in distributional form as well as to a pure strategy Nash equilibrium, and thereby establish the no-regret property of the randomized Nash equilibrium of a large game. And it is in this way that the ELLN enters the picture and has to be relied upon in answering the questions being considered here. And so, in the context of finite-player games, our first result reduces to triviality, and the second one reduces to an impossibility: the point is that in a finite static game with complete information, pure strategy Nash equilibrium does not exist in general; an ex post realization of uncertainty can never exactly equal an ex ante starting point in equilibrium.[7]

In keeping with this introduction, the plan of this paper is as follows. Section 2 presents the model and the antecedent results on the previous use of saturated spaces for the existence of a Nash equilibrium in pure strategies in a large game and the so-called measurability problem concerning a continuum of independent random variables for the noncooperative context. With all this, rather essential, background at hand, Section 3 introduces the key notion of a Fubini extension, and uses it to define a meaningful mixed strategy Nash equilibrium and to show an equivalence result in the context of large games. Sections 4 and 5 present results on ex post Nash equilibria. Section 6 concludes the paper. Other than a short Appendix devoted to an argument based on nonstandard analysis, the proofs are included in the relevant sections themselves: they are routine manipulations for anyone with a first course in the subject.[8]

2. Theoretical antecedents

A finite game in normal form consists of a set of players I, each of whom has a finite action set A_i and a real-valued payoff function u_i depending on the Cartesian product

[6]From now on, we shall bow to convention, and use the phrase *normal form* to refer to games of complete information with simultaneous (one-shot) plays by a continuum of players (in our case).

[7]We shall return to this statement when we relate our results to the literature. As will be seen, this is clearly understood by the best past writing on the subject.

[8]The reader is referred to Dudley (1989) for details as to the mathematical prerequisites of this paper.

$\Pi_{i\in I} A_i$. It is by now conventional to see a large (atomless) game in normal form as being constituted by three basic objects: an abstract atomless probability space[9] $(I, \mathcal{I}, \lambda)$ representing the space of players' names, a compact metric space A representing a common action space of each player, and a space of payoffs $\mathcal{U}_A$ built on the set A. A large game is a measurable function from I to $\mathcal{U}_A$, and its Nash equilibrium is another measurable function from I to A. We now spell out the notational details for a formal development of these notions.

Endowed with its Borel σ-algebra, the action set leads to the Borel measurable space $(A, \mathcal{B}(A))$ and, through it, to the space $\mathcal{M}(A)$ of all probability measures on A endowed with its weak topology.[10] The space $\mathcal{M}(A)$ is then also a compact metric space and it represents the distributions of possible plays in the game, and, through them, "society's plays." In terms of the vernacular of economic theory, these distributions represent "externalities" to an individual player.[11] The space of players' payoffs $\mathcal{U}_A$ is then given by the space of all continuous functions on the product space $A \times \mathcal{M}(A)$, which, when endowed with its sup-norm topology and the resulting Borel σ-algebra, can also be conceived as a measurable space $(\mathcal{U}_A, \mathcal{B}(\mathcal{U}_A))$. This space can be taken to represent the space of players' characteristics, one independent of the space of players' names $(I, \mathcal{I}, \lambda)$. A large game, as conventionally defined, is then a measurable function on $(I, \mathcal{I}, \lambda)$ taking values in $\mathcal{U}_A$, and its *pure strategy profile* is again a measurable function with the same domain taking values in A. A *pure strategy Nash equilibrium* of a game is a pure strategy profile that satisfies Nash conditions. Formally, we can state the following definition.

Definition 1. A large game is a measurable function $\mathcal{G}$ from I to $\mathcal{U}_A$. A pure strategy profile of $\mathcal{G}$ is a measurable function $f : I \longrightarrow A$. A pure strategy Nash equilibrium f^* of $\mathcal{G}$ is a pure strategy profile of the game such that for λ-almost all $i \in I$,

$$u_i(f^*(i), \lambda \circ (f^*)^{-1}) \geq u_i(a, \lambda \circ (f^*)^{-1}) \quad \text{for all } a \in A,$$

with u_i abbreviated for $\mathcal{G}(i)$.

All this is now standard.[12]

Next, we turn to an antecedent existence result that renders the results of this paper to be nonvacuous. For this, we need the notion of a *saturated* probability space,[13] a concept that has emerged to be crucial for the theory. It has been shown that in general, for the existence of a pure strategy Nash equilibrium, the space of players' names must be a *saturated* probability space and that such a requirement is not only sufficient but also *necessary.* We begin with the basic definition.

[9]Throughout the paper, we use the convention that a probability space is always a complete, countably additive measure space.

[10]We conform to standard usage and forgo referring to this as the weak*-topology, the formally correct designation.

[11]This terminology is merely meant to be suggestive: it is the distributions that delineate the dependence of the players on each other's actions, and in giving the formulation its game-theoretic content, distinguish it from a collection of individual optimization problems.

[12]See the survey chapter in Khan and Sun (2002) and its references.

[13]See Khan et al. (2013) for a detailed discussion and bibliographic details.

Definition 2. A probability space is said to be (essentially) countably generated if its σ-algebra can be generated by a countable number of subsets together with the null sets; otherwise, it is not countably generated. A probability space $(I, \mathcal{I}, \lambda)$ is saturated if it is nowhere countably generated, in the sense that, for any subset $S \in \mathcal{I}$ with $\lambda(S) > 0$, the restricted probability space $(S, \mathcal{I}^S, \lambda^S)$ is not countably generated, where $\mathcal{I}^S := \{S \cap S' : S' \in \mathcal{I}\}$ and λ^S is the probability measure rescaled from the restriction of λ to $\mathcal{I}^S$.

The following result is summarized from Keisler and Sun (2009) on the existence of pure strategy Nash equilibria.

Proposition 1. *Let $(I, \mathcal{I}, \lambda)$ be an atomless probability space. Then every large game $\mathcal{G}: I \longrightarrow \mathcal{U}_A$ has a pure strategy Nash equilibrium if and only if $(I, \mathcal{I}, \lambda)$ is a saturated probability space.*

The final result of this section, also an antecedent result to this work, addresses this need for a saturated probability space from another angle. Consider, to begin with, a strategic form game G with a finite player set $I = \{1, \dots, n\}$ in which player i has action set A_i and $(\Omega, \mathcal{F}, P)$ some probability space. A correlated strategy profile in G is a vector $\{g_1, \dots, g_n\}$, where each $g_i: \Omega \longrightarrow A_i$ is measurable. A mixed strategy profile in G is a correlated strategy profile $\{g_1, \dots, g_n\}$ for which $\{g_i : i \in I\}$ is a collection of independent random variables and whose independence captures the non-cooperative behavior of players—what von Neumann referred to in 1928 as the "free decisions" of the players.[14] One can also consider another strategy profile with randomization—a randomized strategy profile in distributional form—a measure-valued mapping from the player space I to the space of distributions over actions $\mathcal{M}(A)$, which is to say, a function $h: I \longrightarrow \mathcal{M}(A)$. When I and A are finite, we obtain the tuple considered by Nash, and when I is itself nondenumerable and of a single type, we obtain the transition probabilities considered by Schmeidler (1973) and Khan and Sun (2002). In a strategic form game with a finite player set, a mixed strategy profile is trivially equivalent to a randomized strategy profile in distribution. Indeed, it is by now well understood that if each A_i is finite and Ω is "big enough," then every randomized strategy profile in distributional form $(h_1, \dots, h_n)$ with $h_i \in \mathcal{M}(A_i)$ can be induced as the distribution of some mixed strategy profile $\{g_1, \dots, g_n\}$. This is a simple consequence of the standard form of Lyapunov's theorem. However, in a large game, it is not clear if such an equivalence holds. And in fact, it is no longer even clear how to define a meaningful mixed strategy profile in such a game.[15]

[14]In Aumann (1964), a more vivid language is used for these ideas: "Mathematically, the random device—the set of sides of the coin or of points on the edge of a roulette wheel—constitutes a probability measure space, sometimes called a *sample space*; a mixed strategy is a function from this sample space to the set of all pure strategies. In other words *what we have here is precisely a random variable whose values are pure strategies.*" Also see Aumann (1963).

[15]See Pascoa (1993, 1998) for a more extended discussion of this. One can also draw the analogy to *behavior* strategies in finite player games of incomplete information, as considered by Aumann, Radner and Rosenthal, Milgrom and Weber, and Khan and Sun; see Khan and Sun (2002, Section 4.1) for references and details. In particular, one could pursue the point in games in extensive form, as considered by Kuhn (1953)

The "measurability problem" arises naturally in the context of the formalization of a mixed strategy profile of a large game where an atomless space is used to model the space of players. Such a strategy profile is a *process* from the product of the player space and a sample space to a common action set such that all the independent randomizations have been consolidated into one "large" sample space Ω. In short, a mixed strategy profile should be a measurable function $F: I \times \Omega \longrightarrow A$ such that for any realization $\omega \in \Omega$, we obtain a pure strategy $F(\cdot, \omega): I \longrightarrow A$, and for any two players' i and j in I, the random variables $F(i, \cdot)$ and $F(j, \cdot)$ are independent. It is this independence that goes to the heart of non-cooperative game theory—the players are not coordinating their randomizations and acting "out of concert" so to speak. To be sure, there is no difficulty in defining an independent collection of random variables $F(i, \cdot)_{i \in I}$: the delicate issue arises when one wants this independence along with the process $F(\cdot, \cdot)$, a function of two variables, also to be jointly measurable in both variables, which is to say $\mathcal{I} \otimes \mathcal{F}$-measurable.[16] This is the measurability problem, a difficulty whose sharpest articulation is furnished by a result that we develop next.

For any two probability spaces $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$, we write $\mathcal{I} \otimes \mathcal{F}$ as the usual product σ-algebra (including all the null subsets) generated by $\{S \times T : S \in \mathcal{I}, T \in \mathcal{F}\}$, and write $\lambda \otimes P$ as the product probability measure on $\mathcal{I} \otimes \mathcal{F}$. Given any mapping F from $I \times \Omega$ to a Polish space X, for any $i \in I$ and $\omega \in \Omega$, let F_i denote the marginal mapping $F(i, \cdot)$ on Ω, and let F_ω denote the marginal mapping $F(\cdot, \omega)$ on I. We begin with the relevant formalization of independence in Sun (1998, 2006).

Definition 3. A process F is said to be *essentially pairwise independent*[17] if for λ-almost all $i \in I$, F_i and $F_{i'}$ are independent for λ-almost all $i' \in I$.

We can construct an essentially pairwise independent process as follows. Let $[0, 1]$ be the unit interval endowed with the Borel σ-algebra $\mathcal{B}_{[0,1]}$ and the uniform distribution. For an atomless probability space $(I, \mathcal{I}, \lambda)$, let $\Omega = [0, 1]^I$ represent the space of all functions from I to the unit interval $[0, 1]$. By the Kolmogorov's extension theorem, we can consider the continuum product probability space $(\Omega, \mathcal{F}', P')$, where $\mathcal{F}'$ is the σ-algebra generated by cylinders of the form $\{\omega \in \Omega : \omega(i) \in B\}$ for all $B \in \mathcal{B}_{[0,1]}$, and P' is the continuum product probability measure on $(\Omega, \mathcal{F}')$. Next define π to be a process from $I \times \Omega$ to $[0, 1]$ by letting $\pi(i, \omega) := \omega(i)$ for all $(i, \omega) \in I \times \Omega$. Here the marginal function π_i is the ith coordinate function on $(\Omega, \mathcal{F}', P')$. It is clear that π_i induces the uniform distribution on $[0, 1]$ for any $i \in [0, 1]$, and π_i, π_j are independent for $i \neq j$. Accordingly, the process π is an essentially pairwise independent process. However, it is

and Aumann (1964), and use the Fubini extension framework to overcome the measurability problem in defining a behavior strategy by incorporating independence across types when types are nondenumerable. We avoid any extended discussion of this analogy in this paper, and to prevent a confounding of ideas, do not use the terminology of a *behavior strategy* in the formal development of our theorems.

[16] In terms of the analogy to extensive form games with a finite number of players but with infinite information sets, as alluded to in Footnote 15, a conceptually similar issue arises with the notion of a behavior strategy.

[17] Given that $(I, \mathcal{I}, \lambda)$ is an atomless (complete) probability space, a single point (and thus up to countably many points) has measure zero, and thus essential pairwise independence is more general than the usual pairwise and mutual independence.

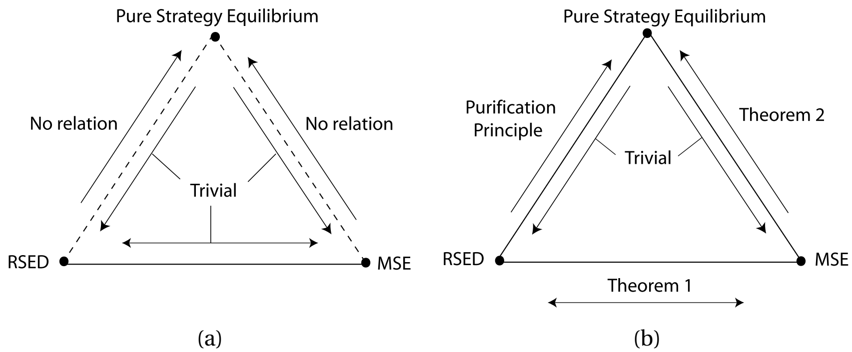

Figure 1. (a) Classification of equilibrium strategies in finite games, (b) classification of equilibrium strategies in infinite games.

well known that this process π is not $\mathcal{I} \otimes \mathcal{F}'$-measurable. Indeed, the essentially pairwise independence and the joint measurability of a process with respect to the usual product σ-algebra are never compatible with each other except for the trivial case that almost all random variables are essentially constant.

Proposition 2. *Let F be a function from $I \times \Omega$ to a Polish space X. If F is jointly measurable on the product probability space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ and if F is essentially pairwise independent, then, for λ-almost all $i \in I$, F_i is a constant random variable.*[18]

With the terminological clarity and specificity that has been hereby obtained, we are now in a position to delineate the relationships among equilibria in the strategies with randomization (mixed strategy equilibrium (MSE) and randomized strategy equilibrium in distributional form (RSED)), and those without randomization (pure strategy Nash equilibria).[19] This delineation is pictured in Figures 1(a) and 1(b). As brought out in Figure 1(a) pertaining to normal form games with a finite set of players, a MSE and a RSED, which exist in general, are trivially equivalent; but there is no relation between them and a pure strategy Nash equilibrium that may not even exist, other than the trivial statement that a pure strategy Nash equilibrium is automatically a MSE and a RSED. In the setting of games with an atomless player space, one can obtain a pure strategy Nash equilibrium from a RSED via a purification procedure.[20] Theorem 1 below shows

[18] See Sun (2006, Proposition 2.1) and Doob (1953, p. 67) for the special case that the process is independent and identically distributed (iid) with the Lebesgue interval as its parameter space, and note that this result is valid in the presence of atoms.

[19] Note also that a pure strategy Nash equilibrium is also a degenerate MSE or RSED. As such, the randomization is nondegenerate.

[20] For atomless games, be they one with an atomless space of players as in Schmeidler (1973) or of incomplete diffused information as in Radner and Rosenthal (1982) and Milgrom and Weber (1985), it is well understood that the Dvoretzky–Wald–Wolfowitz (DWW) purification principle (Dvoretzky et al. 1951) guarantees the existence of a pure strategy Nash equilibrium in the setting of a finite action set. For more details on this, see Khan et al. (2006). For an atomless game with countable actions, the existence of pure strategy Nash equilibria can be done though the Halmos–Vaughan marriage lemma by the purification of its RSED; see the survey in Khan and Sun (2002) and its references. And in the recent development of atomless

that one can obtain a MSE from a RSED and vice versa, while Theorem 2 below demonstrates that the ex post realization of a MSE is a pure strategy Nash equilibrium with probability 1.

Since it is not clear how to incorporate the notion of independence of a continuum of random variables in a measurable process, the literature on finite games with a non-denumerable type space or that of large games with a continuum of players has relinquished the use of independent random variables altogether and, thereby, the important concept of MSE. The whole point of this paper and the *raison d'etre* of this section is precisely to establish the point that with the right mathematical tools, this need not be so for a large game. In the next section, we provide *the* framework to give a meaningful and proper definition of a mixed strategy profile, and once this is done, we provide further results to show an exact equivalence between ex ante and ex post Nash notions—a complete characterization among three equilibria for large games concretized in Theorems 1 and 2.[21]

3. MSE and RSED: A relationship

With this background, we can now turn to our results. Our first concept is based on the use of a single probability space formalizing the set of players' names. From the point of view of game-theoretic substance, this notion of a *randomized strategy equilibrium in distributional form* complements Definition 1 above for the *pure strategy Nash equilibrium* of a large game.

Definition 4. A randomized strategy profile in distributional form of a large game $\mathcal{G}: I \longrightarrow \mathcal{U}_A$ is a measurable function $h: I \longrightarrow \mathcal{M}(A)$ with the latter being endowed Borel σ-algebra generated by the weak topology. A randomized strategy equilibrium in distributional form (RSED) $h^*: I \longrightarrow \mathcal{M}(A)$ of $\mathcal{G}$ is a randomized strategy profile in distributional form such that for λ-almost all $i \in I$,

$$\int_A u_i\left(a, \int_I h^*(j)\,\mathrm{d}\lambda\right)\mathrm{d}h_i^* \geq \int_A u_i\left(a, \int_I h^*(j)\,\mathrm{d}\lambda\right)\mathrm{d}\nu \tag{1}$$

for all $\nu \in \mathcal{M}(A)$.[22]

In keeping with the discussion in Section 2 of the measurability problem pertaining to simultaneous independence and measurability, it is unclear whether and how

games with arbitrary actions, saturated spaces are used to characterize the existence of pure strategy Nash equilibria; see Keisler and Sun (2009) and Khan et al. (2013).

[21] Since the classical law of large numbers in the sequential case can be restated to the exact setting as an integral with respect to a purely finitely additive measure on a countable player space, a natural question arises as to whether one can work *meaningfully* with a MSE on such a countable player space. The answer is definitely no; see Sun (2006, Proposition 6.5). This point will be elucidated in future work.

[22] Note that the measurability of the mapping h^* is equivalent to the measurability of $h^*(\cdot)(B): I \longrightarrow [0,1]$ for any given $B \in \mathcal{A}$. This is a standard result; see, for example, Khan and Sun (2002, Section 7) and their references.

the individual randomizing devices that underlie the notion of a randomized strategy profile in distributional form can be consolidated into one grand objective randomization device for society as a whole underlying a mixed strategy profile, a consolidation that respects the fact that these individual decisions are independent explicitly. But a mathematical solution to the measurability problem has been available since the late nineties.[23] In other words, to overcome the above noncompatibility problem of measurability and independence, we need to work with the framework of a *Fubini extension*, an enrichment of the usual product probability space on which the Fubini property is retained. We turn to this in the first part of this section. Toward this end, the following definition is taken from Sun (2006, Definitions 2.2 and 5.1).

Definition 5. A probability space $(I \times \Omega, \mathcal{W}, Q)$ is said to be a Fubini extension of the usual product probability space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ if for any real-valued Q-integrable function F on $(I \times \Omega, \mathcal{W})$, the following statements hold:

(i) The function F_i is P-integrable on $(\Omega, \mathcal{F}, P)$ for λ-almost all $i \in I$, and F_ω is λ-integrable on $(I, \mathcal{I}, \lambda)$ for P-almost all $\omega \in \Omega$;

(ii) The integrals $\int_\Omega F_i \,\mathrm{d}P$ and $\int_I F_\omega \,\mathrm{d}\lambda$ are integrable on $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$, respectively. In addition, $\int_{I\times\Omega} F \,\mathrm{d}Q = \int_I (\int_\Omega F_i \,\mathrm{d}P)\,\mathrm{d}\lambda = \int_\Omega (\int_I F_\omega \,\mathrm{d}\lambda)\,\mathrm{d}P$.

A Fubini extension $(I \times \Omega, \mathcal{W}, Q)$ is said to be rich if there is a $\mathcal{W}$-measurable process G from $I \times \Omega$ to the interval $[0, 1]$ such that G is essentially pairwise independent, and G_i induces the uniform distribution on $[0, 1]$ for λ-almost all $i \in I$. We say that such a rich Fubini extension is based on $(I, \mathcal{I}, \lambda)$, and the process G witnesses the richness of the Fubini extension.[24]

In a Fubini extension $(I \times \Omega, \mathcal{W}, Q)$, note that the marginal probability measures of Q on $(I, \mathcal{I})$ and $(\Omega, \mathcal{F})$ are λ and P, respectively. To reflect this property, we follow the attendant literature and denote the Fubini extension $(I \times \Omega, \mathcal{W}, Q)$ by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$.

Next, we connect the existence of a rich Fubini extension to the saturation property of a probability space that is formalized in Definition 2 and with which we have been working so far. The following result is from Sun (2006, Proposition 4.2) and Podczeck (2010, Theorem 1), and is summarized in Wang and Zhang (2012, Corollary 1).

Proposition 3. *The probability space $(I, \mathcal{I}, \lambda)$ is saturated if and only if there is a rich Fubini extension based on it.*

Note that this result is phrased in terms of the single probability space $(I, \mathcal{I}, \lambda)$, and whereas this is no impediment for the sufficiency part of the result, the requisite sample space has to be constructed for the necessity part. And so the necessity claim, when elaborated, comes down to asserting the existence of a probability space $(\Omega, \mathcal{F}, P)$

[23] See Sun (1998), which forcefully argues for the necessity of extending the usual product space considered in Proposition 2.

[24] For the existence of a rich Fubini extension, see Sun (1998, Theorem 6.2), Sun (2006, Theorem 5.6), Sun and Zhang (2009, Theorem 1), and Podczeck (2010, Theorem 1).

extending $(\Omega, \mathcal{F}', P')$, as defined after Definition 3 above, such that there exists a rich Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ on which the process of coordinate functions π is $\mathcal{I} \boxtimes \mathcal{F}$-measurable and witnesses the richness of the Fubini extension.[25] Finally, we also record a convenient universality property of a rich Fubini extension based on a saturated probability space. A rich Fubini extension satisfies the universality property in the sense that one can construct processes on it with essentially pairwise independent random variables that have any given variety of distributions on a general Polish space. The following result is available in Sun (2006, Proposition 5.3).

Proposition 4. *Let $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ be a rich Fubini extension, let X be a Polish space, and let h be a measurable mapping from $(I, \mathcal{I}, \lambda)$ to $\mathcal{M}(X)$. Then there exists an $\mathcal{I} \boxtimes \mathcal{F}$-measurable process $F: I \times \Omega \longrightarrow X$ such that the process F is essentially pairwise independent and $h(i)$ is the induced distribution by F_i for λ-almost all $i \in I$.*

This now formalizes the fact that, unlike the Lebesgue unit interval, saturated probability spaces are hospitable to independence and measurability, and that also in a strong sense they admit processes whose random variables have a full and arbitrarily given variety of distributions.[26]

We now turn to the definition of a *mixed strategy profile* of a large game. For any such game, $\mathcal{G}: I \longrightarrow \mathcal{U}_A$ as in Definition 1, we use the Fubini extension to overcome the measurability problem to ensure that almost any two agents play independent mixed strategies in a non-cooperative setting. Toward this end, from now on, let $(I, \mathcal{I}, \lambda)$ be a saturated probability space and let $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ be a rich Fubini extension of the product space $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$.

Definition 6. A mixed strategy profile of a large game $\mathcal{G}: I \longrightarrow \mathcal{U}_A$ is an $\mathcal{I} \boxtimes \mathcal{F}$-measurable function $g: I \times \Omega \longrightarrow A$, where the process g is assumed to be essentially pairwise independent.[27] A mixed strategy Nash equilibrium (MSE) of $\mathcal{G}$ is a mixed strategy profile g^*, such that for λ-almost all $i \in I$,

$$\int_\Omega u_i(g_i^*(\omega), \lambda \circ (g_\omega^*)^{-1})\,\mathrm{d}P \geq \int_\Omega u_i(\eta(\omega), \lambda \circ (g_\omega^*)^{-1})\,\mathrm{d}P \tag{2}$$

for any $\eta \in \mathrm{Meas}(\Omega, A)$, where $\mathrm{Meas}(\Omega, A)$ is the set of all random variables from $(\Omega, \mathcal{F}, P)$ to A.

[25] Three observations make explicit the relationship of a Fubini extension and the measurability issues discussed in the Introduction. First, recall that Doob (1937, Theorem 2.2) points out that when a continuum of independent random variables is constructed based on the Lebesgue unit interval and the usual continuum product, the sample functions *may* not be Lebesgue measurable; see also Judd (1985) and Feldman and Gilles (1985). It is shown in Sun (2006, Corollary 4.3) that under a Fubini extension, almost all sample functions of a nontrivial independent process *must* be non-Lebesgue measurable. It means that the Lebesgue unit interval cannot be used as a player space associated with mixed strategies in a large game. Second, Propositions 6.1 and 6.2 of Sun (2006) show that the usual continuum product sample space is inadequate for the study of a continuum of independent players no matter what atomless measures are imposed on any σ-algebra on the unit interval. Finally, Proposition 2 above addresses the issue arising from the use of the usual product σ-algebra; see Sun (2006, p. 54) for more details.

[26] This arbitrary nature is only modulated by the fact that the distributions are stitched together by a measurable function—the function F in Proposition 4.

[27] See the saturated Lebesgue extension established in Sun and Zhang (2009).

Now that we have developed the necessary background to define a MSE of a large game, we turn to inquire into the relationship between a MSE and a RSED of the game. We need the next result, which is taken from Corollary 2.9 of Sun (2006), on a version of the ELLN in the framework of Fubini extension.

Proposition 5. *Assume that $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a Fubini extension. If F is an essentially pairwise independent and $\mathcal{I} \boxtimes \mathcal{F}$-measurable process, then the sample distribution $\lambda \circ (F_\omega)^{-1}$ is the same as the distribution $(\lambda \boxtimes P) \circ F^{-1}$ for P-almost all $\omega \in \Omega$.*

We are now ready to show the relationship between a randomized strategy profile in distributional form—a function from the space of players' names to a distribution on their alternatives—and a mixed strategy profile—a random process from a product of the space of players' names and a sample space to the space of alternatives. Note that there is no presumption that these two kinds of profiles are equilibrium profiles.[28]

Lemma 1. *In a large game $\mathcal{G} : I \longrightarrow \mathcal{U}_A$, every mixed strategy profile induces a randomized strategy profile in distributional form, and every randomized strategy profile in distributional form can be lifted to some mixed strategy profile.*

Proof. Fix any mixed strategy profile g of $\mathcal{G}$. Let h be $h(i) = P \circ g_i^{-1}$ for all $i \in I$. As $h(i) \in \mathcal{M}(A)$ for all $i \in I$ and h is measurable, it is easy to see that h is a randomized strategy profile in distributional form of $\mathcal{G}$.

Fix any randomized strategy profile in distributional form h of $\mathcal{G}$. It is clear that h is a measurable function from I to $\mathcal{M}(A)$. Thus, given that $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a rich Fubini extension, by Proposition 4, there is an $\mathcal{I} \boxtimes \mathcal{F}$-measurable process g from $I \times \Omega$ to A such that g is an essentially pairwise independent process, and the distribution $P \circ g_i^{-1}$ is the given distribution $h(i)$ for λ-almost all $i \in I$. By Definition 5, g is indeed a mixed strategy profile of $\mathcal{G}$. □

We now can present the following equivalence theorem of MSE and RSED in a large game.

Theorem 1. *The following equivalence holds for a large game $\mathcal{G} : I \longrightarrow \mathcal{U}_A$: (i) Every MSE induces a RSED and (ii) every RSED can be lifted to a MSE.*

Proof. (i) Suppose g^* is a MSE of $\mathcal{G}$. Let $h^*(i) = P \circ (g_i^*)^{-1}$ for all $i \in I$. By the ELLN in Proposition 5, it is clear that for P-almost all $\omega \in \Omega$,

$$\int_I h^*(i)\, \mathrm{d}\lambda = \int_I P \circ (g_i^*)^{-1}\, \mathrm{d}\lambda = \lambda \circ (g_\omega^*)^{-1}. \tag{3}$$

Because g^* is a MSE, (2) holds for g^*. Now note that for any random variable $\eta : \Omega \longrightarrow A$, $P \circ \eta^{-1} \in \mathcal{M}(A)$. Moreover, as $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is rich, Ω is atomless, and, therefore, for any $\nu \in \mathcal{M}(A)$, there exists a random variable $\eta : \Omega \longrightarrow A$ such that $\nu = P \circ \eta^{-1}$.

[28] In the statement of the results to follow, we have not formally defined the words "induce" and "lift": we feel it would be pedantic to do so given that their meaning is clear from the context and especially from the rather straightforward proofs.

Therefore, we can again apply the change of variables theorem and (3) to assert that (2) is equivalent to (1) for h^*, where $h^*(i) = P \circ (g_i^*)^{-1}$ for all $i \in I$. By Definition 4, this measurable function h^* is a RSED of the game $\mathcal{G}$.

(ii) Now suppose h^* is a RSED of a game $\mathcal{G}$. By Lemma 1, there exists a mixed strategy profile g^* of $\mathcal{G}$ such that $P \circ (g_i^*)^{-1} = h^*(i)$ for λ-almost all $i \in I$. Furthermore, we can apply the ELLN to assert that (3) holds for such g^* and h^*. Therefore, we can appeal to (1) and the change of variables theorem to guarantee that this mixed strategy profile g^* from $I \times \Omega$ to A also satisfies (2). Hence, g^* is a MSE of the game $\mathcal{G}$. □

The basic intuition of this proof can be simply expressed. In one direction, any given MSE (g^*) induces a measure-valued mapping h^* whose distribution is guaranteed by ELLN to equal almost surely the empirical distribution. Thus, the social responses used in both (1) and (2) are almost surely the same and thereby guarantee the optimality of h^*. In the other direction, a rich Fubini framework guarantees that any given RSED h^* can be decomposed as an essentially pairwise independent process g^*, and ELLN can be applied again to yield the conclusion that is desired.

We now conclude this section by an analogy already alluded to in Footnotes 15 and 16: a conceptually similar issue arises with the notion of a behavior strategy as a jointly measurable process in an extensive form game with finitely many players, each of whom faces nondenumerably infinite information states. Within the Fubini framework, a behavior strategy in such a game can be well defined (in the Kuhn–Aumann sense) as a collection of random processes such that in each stage, the collection of random variables indexed by states of information are essentially pairwise independent.

4. Mixed and pure strategies: An ex post relationship

In a section titled "Large games with independent idiosyncratic shocks," Khan and Sun (2002) observed that the notion of externalities in the form of a distribution of the actions of all players—a distinguishing characteristic of the theory of large games—allows one to make a rather novel claim: this is the assertion that in a setting of idiosyncratic shocks, "in equilibrium, societal responses do not depend on a particular sample realization, and each player is *justified* in ignoring other players' risks."[29] We begin this section by transcribing Theorem 7 in Khan and Sun (2002) into the vocabulary of a rich Fubini extension. We begin with the following definition.

Definition 7. A large game with idiosyncratic uncertainty is a measurable function $\mathcal{G}^U$ from $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to $\mathcal{U}_A$ such that $\mathcal{G}^U$ is essentially pairwise independent.

We can now present[30] the following proposition.

[29]See Section 11 in Khan and Sun (2002), which provides an analog of the ex post Walrasian equilibria considered in Section 3 of Sun (1999). The quote is taken from Section 11.3 on page 1792, where we substitute "player" for "agent." In this connection, see Assumption C and its discussion in Crémer and McLean (1985, p. 346); also see Footnote 32 below.

[30]Though we allow ω to enter the payoffs, what we consider here is not a large Bayesian game with independent types, since we do not require the ex post equilibrium f to be measurable with respect to some exogenously given independent type spaces. The consideration of large Bayesian games is beyond the scope of this paper; see Footnote 45 below.

Proposition 6. *Let $\mathcal{G}^U$ be a large game with idiosyncratic uncertainty. Then there is a process $f: I \times \Omega \longrightarrow A$ such that f is a Nash equilibrium of the game $\mathcal{G}^U$, the random strategies $f(i, \cdot)$ are essentially pairwise independent, and for P-almost all $\omega \in \Omega$, $f(\cdot, \omega)$ is an equilibrium of the large game $\mathcal{G}^U(\cdot, \omega)$ with constant societal distribution $(\lambda \boxtimes P) \circ f^{-1}$.*

A two-line proof of this proposition is furnished in Khan and Sun (2002). The basic idea is straightforward. One can regard the game $\mathcal{G}^U$ as a large game modeled on the space of players' names to be the joint space $I \times \Omega$, and given joint measurability on such a space, one can deduce the existence of a Nash equilibrium $g: I \times \Omega \longrightarrow A$ from the standard result. This is to assert that there exists a measurable function such that

$$\mathcal{G}^U_{(i,\omega)}(g(i, \omega), (\lambda \boxtimes P) \circ g^{-1}) \geq \mathcal{G}^U_{(i,\omega)}(a, (\lambda \boxtimes P) \circ g^{-1}) \quad \text{for all } a \in A.$$

The point is that this measurable function is a selection from the set-valued process

$$(i, \omega) \longrightarrow F(i, \omega) = \arg\max_{a \in A} \mathcal{G}^U_{(i,\omega)}(a, (\lambda \boxtimes P) \circ g^{-1}).$$

We can now finish the proof by appealing to the following proposition (which is Theorem 2 of Sun 1999) and to the ELLN as stated in Proposition 5.

Proposition 7. *Let F be a set-valued process from $I \times \Omega$ to a complete separable metric space A. Assume that $F(i, \cdot)$ are essentially pairwise independent. Let g be a selection of F with distribution μ. Then there is another selection f of F such that the random variables $f(i, \cdot)$ are essentially pairwise independent and the distribution of f is viewed as a random variable on $I \times \Omega$ is μ.*

So rather than the proof, it is the interpretation of Proposition 6 that is of interest. The context is one of exogenous uncertainty whereby the individual payoffs, as well as the individual randomized strategies, are independent, and the proposition rigorously develops the intuition that once uncertainty is resolved, a player has no incentive to depart ex post from her optimal strategy taken in the ex ante game when she finds herself in the realized ex post game.[31] What was missed of course was that Aumann (1974, Section 8) is devoted to *a posteriori equilibria* and explicitly considers the issue:[32]

[31]As emphasized by an anonymous referee, the game $\mathcal{G}^U$ can be viewed as an extensive form game where after Nature chooses a public signal $\omega \in \Omega$, all players observe ω and then choose actions in A. This is to say, the elements of Ω also label the common information sets of all players. Then any equilibrium $g: I \times \Omega \longrightarrow A$ of $\mathcal{G}^U$ in Proposition 6 is a *pure* behavior strategy equilibrium as defined in the Kuhn–Aumann sense. The existence of the process f in Proposition 6 shows that $f(\cdot, \omega)$, the restriction of f to the subgame $\mathcal{G}^U(\cdot, \omega): I \longrightarrow \mathcal{U}_A$ indexed by ω, is a pure strategy Nash equilibrium of $\mathcal{G}^U(\cdot, \omega)$. Therefore, Proposition 6 also asserts the existence of a pure strategy subgame perfect equilibrium in a large game with idiosyncratic uncertainty.

[32]This 1974 request of Aumann's is the basis of our reference to "longstanding open problems" in the abstract and elsewhere. Aumann (1964) also works with a process and with mixed and behavior strategies, but bypasses measurability considerations arising out of the independence issue.

> To legitimize the view of an equilibrium point as a self-enforcing agreement, one can either (a) make assumptions under which the possibility of some player wanting to renege is assigned probability zero; or (b) construct a model in which it is possible to define equilibrium points at which no player *ever* wants to renege.

Since these lines were written, a rich literature has developed on equilibrium notions involving the ex post concept; see, for example, McLean and Postlewaite (2002) and Bergemann and Morris (2008) and their references. Moving on to the context of a large but finite game, Kalai (2004) writes:

> A particular modeling difficulty of noncooperative game theory is the sensitivity of Nash equilibrium to the rules of the game, e.g., the order of the players' moves and the information structure. Since such details are often not available to the modeler or even to the players of the game, equilibrium prediction may be unreliable. For this purpose, we define a Nash equilibrium of a game to be extensively robust[33] if it remains a Nash equilibrium in *all extensive versions* of the simultaneous-move game. Extensive robustness means in particular that an equilibrium must be *ex post Nash.* Even with perfect hindsight knowledge of the types and selected actions of all of his opponents, no player regrets, or has an incentive to revise, his own selected action.

One can look on the dependence of the payoffs on the space Ω in Proposition 6 as the proxy for the variety of phenomena emphasized by Kalai and not explicitly modeled.[34]

To be sure, what makes Proposition 6 work is the existence of a rich Fubini extension and the ELLN. But the point can be sharpened still if rather than work with a large game with idiosyncratic uncertainty $\mathcal{G}^U$, one works instead with a deterministic large game as in Definition 1. In this case, the uncertainty underlying a mixed strategy arises only from the uncertainty regarding the moves, randomized or otherwise, of everyone else's plays. To put the matter another way, a natural question concerns the possibility of such a claim in situations when there is no exogenous parametric uncertainty, but one introduced as a result of players' playing mixed strategies based on independent randomizations, as befits a non-cooperative game setting. If all these independent randomizations can be consolidated in one large space Ω with the mixed strategy profile again being a process from $I \times \Omega$ to the space of actions, we are back in the situation considered in Khan and Sun (2002), but with the underlying space of uncertainty being generated *only* from the independent randomized strategies of the players.

Since a mixed strategy profile $g: I \times \Omega \longrightarrow A$ is defined in this paper as an $\mathcal{I} \boxtimes \mathcal{F}$-measurable function where the process g is assumed to be essentially pairwise independent, it is easy to see that for any realized sample ω, g_ω is a pure strategy profile in any large game. One can then again ask whether each player is *justified* in ignoring other players' risks and has no incentive to depart ex post from her optimal strategy

[33]Kalai (2004, p. 1632) emphasizes, "This is a new notion of robustness, different from other robustness notions used in economics or game theory." While of course accepting the validity of this statement, one can usefully connect it to Crémer and McLean (1985, p. 347), who write, "Then we utilize an equilibrium concept, called ex post Nash equilibrium, which states that, after seeing the bids of others, buyers will not want to revise their bids."

[34]Referring to *extensive versions* of the simultaneous-move game, Kalai refers to "wide flexibility in the order of players' moves, as well as information leakage, commitment and revision possibilities, cheap talk, and more."

taken in the ex ante game when all the randomizations of each player have been individually realized. In other words, this is to ask, in the terminology adopted by Kalai (2004), whether a mixed strategy equilibrium has an ex post purification. Since the question is being posed in a deterministic large game, an affirmative answer is even easier to obtain than in the situation considered in Proposition 6 above. The definition of the ex post property of a mixed strategy profile in a large game is provided as follows.

Definition 8. A mixed strategy profile g^* of a large game $\mathcal{G}: I \longrightarrow \mathcal{U}_A$ is said to have the *ex post Nash property* if for P-almost all $\omega \in \Omega$, g^*_ω is a pure strategy Nash equilibrium for the same game with the empirical action distribution $\lambda \circ (g^*_\omega)^{-1}$.

Since a mixed strategy profile can now be rigorously defined in a framework with strategic uncertainty, the definition simply means that it is ex post Nash if no player has any incentive to unilaterally change her selected action after the realized state, the realized action distribution being induced by the selected actions of all other players in the given state. This observation leads us to the following result for a large game.

Theorem 2. *A mixed strategy profile of a large game $\mathcal{G}: I \longrightarrow \mathcal{U}_A$ is a MSE if and only if it has the ex post Nash property.*

Proof. Suppose g^* is a MSE. We shall show that g^* has the ex post Nash property. Toward this end, first note that by the ELLN as stated in Proposition 5, for any mixed strategy profile g, we have, for P-almost all ω,

$$\lambda \circ g_\omega^{-1}(\cdot) = \int_I (P \circ g_i^{-1})(\cdot)\, \mathrm{d}\lambda. \tag{4}$$

Let $\xi = \int_I (P \circ (g_i^*)^{-1})(\cdot)\, \mathrm{d}\lambda$. Because g^* is a MSE, (2) holds for such a g^*. By (4), (2) can be rewritten, for λ-almost all $i \in I$, as

$$\int_\Omega u_i(g_i^*(\omega), \xi)\, \mathrm{d}P \geq \int_\Omega u_i(\eta(\omega), \xi)\, \mathrm{d}P \quad \text{for all random variables } \eta: \Omega \longrightarrow A, \tag{5}$$

which implies, for λ-almost all $i \in I$, for P-almost all $\omega \in \Omega$,

$$u_i(g_i^*(\omega), \xi) = \max_{a \in A} u_i(a, \xi).$$

Then, by the Fubini property of a Fubini extension, we have, for P-almost all $\omega \in \Omega$, for λ-almost all $i \in I$,

$$u_i(g^*_\omega(i), \xi) = \max_{a \in A} u_i(a, \xi). \tag{6}$$

By the ELLN again, hence, for P-almost all $\omega \in \Omega$, λ-almost all $i \in I$,

$$u_i(g^*_\omega(i), \lambda \circ (g^*_\omega)^{-1}) = \max_{a \in A} u_i(a, \lambda \circ (g^*_\omega)^{-1}). \tag{7}$$

This means, for P-almost all $\omega \in \Omega$, g^*_ω is a pure strategy Nash equilibrium and, therefore, g^* has the ex post Nash property.

Now suppose that a mixed strategy profile g has the ex post Nash property. This is to say, for P-almost all $\omega \in \Omega$, for λ-almost all $i \in I$,

$$u_i(g_\omega(i), \lambda \circ g_\omega^{-1}) = \max_{a \in A} u_i(a, \lambda \circ g_\omega^{-1}).$$

By (4) and the Fubini property of a Fubini extension, we have, for λ-almost all $i \in I$, P-almost all $\omega \in \Omega$,

$$u_i(g_i(\omega), \lambda \circ g_\omega^{-1}) = \max_{a \in A} u_i(a, \lambda \circ g_\omega^{-1}).$$

Thus, for any random variable $\eta : \Omega \longrightarrow A$, we have, for λ-almost all $i \in I$, for P-almost all $\omega \in \Omega$,

$$u_i(g_i(\omega), \lambda \circ g_\omega^{-1}) \geq u_i(\eta(\omega), \lambda \circ g_\omega^{-1}),$$

which implies that for λ-almost all $i \in I$,

$$\int_\Omega u_i(g_i(\omega), \lambda \circ g_\omega^{-1})\, \mathrm{d}P \geq \int_\Omega u_i(\eta(\omega), \lambda \circ g_\omega^{-1})\, \mathrm{d}P \quad \text{for any random variable } \eta : \Omega \longrightarrow A.$$

This verifies that g is a MSE by Definition 6. $\square$

The intuition behind the proof above is simple. The ELLN implies (4), which tells us that with a MSE (g^*) being played, the empirical distribution of actions after uncertainty is realized is almost surely the same as the ex ante average of players' action distributions (ξ). This leads to the equilibrium condition (2) implying (5), a statement that asserts that for almost all i, playing the equilibrium strategy g_i^* yields the highest expected payoff when societal responses are ξ. This then ensures that for almost all i, all her ex post plays with g_i^* must almost surely be i's best responses to ξ. The Fubini property now implies that once uncertainty is resolved, it happens almost surely that for almost all players i, i's selected action is i's best response given ξ. Finally, since ξ is almost surely the same as any empirical distribution of actions, we obtain that almost surely, once the state ω is realized, for almost all i, the selected action is the best response of the empirical distribution of actions that is induced by the entire selected strategy profile. This proves that a MSE must possess the ex post Nash property. And because the argument can be carried out in reverse, we can also establish that a mixed strategy profile with the ex post Nash property must be an equilibrium itself.

Without belaboring the point, the observation needs to be made that the above result is a complete resolution to the dilemma, identified in Section 2, pertaining to the simultaneous requirement of independence and joint measurability in the modeling of a mixed strategy profile. It is precisely to bypass this dilemma that Kalai works with an increasing sequence of large but finite games, and emphasizes an approximate ex post Nash notion and an equicontinuity property of payoffs that plays no role in the result presented here. However, his discussion of the property itself is illuminating:[35]

[35]See, for example, Kalai (2004, Lemma 6.1) and related discussion.

> An immediate consequence of the ex post Nash property is a *purification* property in large games. First, for normal-form games the ex post Nash property provides stronger conclusions than Schmeidler's (1973) on the role of pure strategy equilibria in large anonymous games. Working in the limit with a continuum of players, Schmeidler shows that every "mixed strategy" equilibrium[36] may be "purified." This means that for any mixed strategy equilibrium one can construct a pure strategy equilibrium with the same individual payoffs. The ex post Nash theorem…shows (asymptotically) that in large semi-anonymous games there is no need to purify since it is done for us automatically by the laws of large numbers. So every mixed strategy may be thought of as a "self-purifying device."

The point is that these words go with, and underscore, Proposition 6 and Theorem 2. The technical vocabulary that constitutes Theorem 2 presented above and that delivers what needs to be substantively delivered, was simply not available to Pascoa (1998) and to Kalai (2004). This is the reason for the turn to approximations. In a slightly different setting of large but finite games, Kalai (2004) establishes an approximate ex post Nash property of equilibrium involving uncertainty where the independence condition across players holds exactly. One may also mention here that in the other direction, as shown in Cartwright and Wooders (2009, Example 2), a strategy profile that has the approximate ex post property may not necessarily be an equilibrium, even an approximate one.[37]

We conclude this section with an example to illustrate the applicability of the two theorems of the paper. Consider a one-shot game of complete information modified from a game of *regime change* by Angeletos et al. (2007, Section 2.1). We shall work with a saturated extension $(I, \mathcal{I}, \lambda)$ of the Lebesgue unit interval $(I, \mathcal{L}, \ell)$. The former supports a rich Fubini extension, whereas the Lebesgue unit interval does not.

Example. Let $(I, \mathcal{I}, \lambda)$ be the space of players' names. All players move simultaneously, playing one of two actions: to attack the status quo $(\bar{a})$ or to refrain from doing so $(\bar{n})$. Let $A = \{\bar{a}, \bar{n}\}$. For each $i \in I$, let characteristics u_i be such that $u_i(\bar{a}, \tau) = (1 - c)(\tau(\bar{a}) - \theta)$ and $u_i(\bar{n}, \tau) = 0$ for all $\tau \in \mathcal{M}(A)$, where exogenously given $c \in (0, 1)$ and $\theta \in (0, 1]$ parameterize the relative cost of an attack and the strength of the status quo, respectively. The status quo is abandoned only if $\tau(\bar{a}) \geq \theta$. Assume that θ is commonly known by all players. Let G be such that $G(i) = u_i$ for any $i \in I$. It is straightforward to see that G is a game that fits Definition 1.

For any pure strategy profile $f: I \longrightarrow A$ in G, it is clear that the payoff to play $\bar{n}$ is always zero, while the payoff from attacking is at least as good as playing $\bar{n}$ only if $(\lambda \circ f^{-1})(\bar{a}) \geq \theta$. Now let $h: I \longrightarrow \mathcal{M}(A)$ be the randomized strategy profile in

[36] We would like to point out that the concept of a *mixed strategy equilibrium* (MSE) in this quotation is what we are referring to as a *randomized strategy equilibrium in distributional form* (RSED). In this connection, Kalai (2004, Footnote 11) is confusing in its statement, "As Schmeidler points out in his paper, it is difficult to define a 'real mixed strategy' equilibrium due to failings of the law of large numbers in the case of continuously many random variables." Schmeidler has no reference to the law of large numbers or to the independence condition, and his phrase that "in many real gamelike situations a mixed strategy has no meaning" refers to difficulties in reality rather than to those of the model.

[37] We return to this issue of asymptotic implementation in the next section.

distributional form such that $h(i)(\{\bar{a}\}) = \theta$ for all $i \in I$. In this strategy profile, all players just play $\bar{a}$ with probability θ, and it is easy to check that h is a RSED. By Theorem 1, h can be lifted to a MSE $g: I \times \Omega \longrightarrow A$. By Theorem 2, g also has the ex post Nash property so that with probability 1, the realized pure strategy profile is a pure strategy Nash equilibrium, where θ proportion of players choose $\bar{a}$ and the rest $(1-\theta)$ choose $\bar{n}$. ◊

5. Ex post relationship: An asymptotic implementation

Theorems 1 and 2 pertain to an idealized complete-information game in normal (strategic) form with a continuum of agents. The continuum of agents are formalized as a saturated probability space. Since hyperfinite Loeb counting probability spaces are saturated, these exact results hold trivially for such spaces, and it is now well understood that it is a basic property of the nonstandard model that an exact result on hyperfinite Loeb counting probability spaces can be translated to an approximate result for a sequence of large but finite probability spaces; see Loeb and Wolff (2000) and their references for details. Thus, one can implement this methodology to show that the exact limiting results presented as Theorems 1 and 2, as well as the result in Section 11 of Khan and Sun (2002) on a large game, have asymptotic analogs for games with a large but finite number of players. In this section, we present an asymptotic version of Theorem 2, one in which the independence condition across players is only satisfied in an approximate sense and thereby allows the presentation of a more general asymptotic result.[38]

However, we preface this result by returning to the proof of the sufficiency claim in Theorem 2 for the idealized continuum. In some sense, its simplicity and analytical sophistication is also its expository failing. One wants to proceed more slowly so as to unravel the proof and delineate how and why the ELLN and the Fubini property, as well as the continuity assumption on payoffs, are intimately involved in it. Such a discussion,[39] heuristic and intuitive to be sure, would serve as a natural bridge between it and the large but finite result presented as Proposition 8 below. This will enable us not only to unravel the result presented as Theorem 2, but also to bring out why the naive argument needs supplementation by a rigorous one.

We begin by reminding the reader that a player's payoff is represented by a continuous function of two variables: actions (a compact metric space A) and probability distributions on actions ($\mathcal{M}(A)$), and as such, it is abstract and difficult to visualize. Think instead of a finite world: the set of players $I_s = \{1, \ldots, s\}$, $A = \{a_1, \ldots, a_t\}$, and a finite sample space $(\Omega, \mathcal{F}, P)$ with $P(\omega) > 0$ for all $\omega \in \Omega$. Since the action set is finite, the payoff of an arbitrary player i can now be explicitly written out in terms of relative frequencies with which each action is being played; i.e., the abstract function of two variables is replaced by one of $(t+1)$ variables. More explicitly, let player i's payoff be a continuous function $u_i: A \times E \longrightarrow \mathbb{R}$, where $E = \{(e_1, \ldots, e_t): \text{all } e_k \geq 0 \text{ and } \sum_{k=1}^t e_k = 1\}$.

[38] See Sun (1998, Proposition 9.4) for such a condition. The reader may compare the corresponding one with Kalai (2004).

[39] We were prodded to do this by an anonymous referee. We also note here that in this attempt, we take our lead and inspiration from Aumann (1975, Section 8).

Now a mixed strategy profile $(g_1, \ldots, g_s)$ is a collection of independent random variables $g_i : \Omega \longrightarrow A$. If $(g_1, \ldots, g_s)$ is a Nash equilibrium in mixed strategies, then for any mixed strategy h_i of player $i \in I_s$,

$$\sum_{\omega\in\Omega} u_i\biggl(g_i(\omega), \biggl(\frac{1}{s}|\phi(a_1|\omega)|, \ldots, \frac{1}{s}|\phi(a_t|\omega)|\biggr)\biggr)P(\omega) \geq \sum_{\omega\in\Omega} u_i\biggl(h_i(\omega), \biggl(\frac{1}{s}|\phi^h(a_1|\omega)|, \ldots, \frac{1}{s}|\phi^h(a_t|\omega)|\biggr)\biggr)P(\omega),$$

where $\phi(a_k|\omega) = \{j \in I_s : g_j(\omega) = a_k\}$ and $\phi^h(a_k|\omega) = \{j \in I_s : (g'_j)(\omega) = a_k\}$ with $g'_j = g_j$ if $j \neq i$ and $g'_j = h_i$ otherwise. Clearly, $|\phi(a_k|\omega)|$ is the number of players playing the action a_k if everyone plays the equilibrium profile under the assumed state $\omega \in \Omega$, while $|\phi^h(a_k|\omega)|$ is the number of players playing a_k when player i plays $h_i(\omega)$ while others play their equilibrium strategy under ω. As we are working with a counting measure $\lambda_s(\{j\}) = 1/s$ for all j, the above Nash inequality can be translated to the notation analogous to that of the Nash inequality in a large game, i.e., (2) that we used in Theorem 2, and rewritten as

$$\sum_{\omega\in\Omega} u_i\bigl(g_i(\omega), (\lambda_s \circ g_\omega^{-1}(a_1), \ldots, \lambda_s \circ g_\omega^{-1}(a_t))\bigr)P(\omega) \geq \sum_{\omega\in\Omega} u_i\bigl(h_i(\omega), (\lambda_s \circ (g'_\omega)^{-1}(a_1), \ldots, \lambda_s \circ (g'_\omega)^{-1}(a_t))\bigr)P(\omega). \tag{8}$$

Now turn to the implication of the law of large numbers and allow s to vary. For any a_k in A, the empirical distribution of a_k,

$$(\lambda_s \circ g_\omega^{-1})(a_k) = \frac{1}{s}\bigl|\{j \in I : g_j(\omega) = a_k\}\bigr| = \frac{1}{s}\sum_{j=1}^{s} 1_{\{a_k\}}(g_j(\omega)),$$

has the same limit of $(1/s)\sum_{j=1}^{s} P(\{\omega \in \Omega : g_j(\omega) = a_k\})$ by the strong law of large numbers. The latter is equal to the expected distribution of a_k, $\sum_{j=1}^{s}(P \circ g_j^{-1})(a_k)\lambda_s(j)$, which is an analog for $\int_I P(\{\omega \in \Omega : g_j(\omega) = a_k\})\, \mathrm{d}\lambda(j)$ defined as (4) in Theorem 2. That is to say, with a mixed strategy profile being played, the proportion of players who plays a_k at any realized state ω converges to the expected proportion of players who play a_k before uncertainty is revealed. Moreover, as A is finite, for any given ϵ_1, we can find a large s such that

$$\sup_{a_k\in A}\left|(\lambda_s \circ g_\omega^{-1})(a_k) - \sum_{j=1}^{s}(P \circ g_j^{-1})(a_k)\lambda_s(j)\right| \leq \epsilon_1. \tag{9}$$

This is exactly the finite analogy of (4) as shown in the proof of Theorem 2.

Furthermore, note that $\sup_{a_k\in A}|\lambda_s \circ (g_\omega)^{-1} - \lambda_s \circ (g'_\omega)^{-1}(a_k)| \to 0$. Then, by (9) and the enough equicontinuity assumption of u_i, together with (8), the following inequality holds for any given ϵ' when s is large enough:

$$\begin{aligned}\sum_{\omega\in\Omega} u_i&\left(g_i(\omega), \left(\sum_{j=1}^{s}(P\circ g_j^{-1})(a_1)\lambda_s(j), \ldots, \sum_{j=1}^{s}(P\circ g_j^{-1})(a_t)\lambda_s(j)\right)\right)P(\omega) \\ &\geq \sum_{\omega\in\Omega} u_i\left(h_i(\omega), \left(\sum_{j=1}^{s}(P\circ g_j^{-1})(a_1)\lambda_s(j), \ldots, \sum_{j=1}^{s}(P\circ g_j^{-1})(a_t)\lambda_s(j)\right)\right)P(\omega) - \epsilon'.\end{aligned} \tag{10}$$

Note that the equation above is the finite, approximate analog of (5). We need now to transfer the inequality (10) in terms of expected payoff to the inequality in terms of payoff at the realized state ω just as is done from (5) to (6). An intuitive way of doing this would be as follows: choose any $\omega \in \Omega$ and $a \in A$, and then choose a strategy h_i defined as $h_i(\omega) = a$ and $h_i(\omega') = g_i(\omega')$ if $\omega' \neq \omega$, to obtain,

$$\begin{aligned}u_i&\left(g_i(\omega), \left(\sum_{j=1}^{s}(P\circ g_j^{-1})(a_1)\lambda_s(j), \ldots, \sum_{j=1}^{s}(P\circ g_j^{-1})(a_t)\lambda_s(j)\right)\right) \\ &\geq u_i\left(a, \left(\sum_{j=1}^{s}(P\circ g_j^{-1})(a_1)\lambda_s(j), \ldots, \sum_{j=1}^{s}(P\circ g_j^{-1})(a_t)\lambda_s(j)\right)\right) - (\epsilon'/P(\omega)).\end{aligned} \tag{11}$$

If $(\epsilon'/P(\omega))$ can be made arbitrarily small, then one can conclude from (11) that $g_i(\omega)$ is an approximate best response against the (nonrandom) expected frequency profile, and then, by another appeal to (9) and to the continuity of payoffs, that it is also an approximate best response against the (ω-dependent) empirical frequency profile $(\lambda \circ g_\omega^{-1}(a_1) \cdots \lambda \circ g_\omega^{-1}(a_t))$. A further appeal to continuity gives, for any $\epsilon > 0$,

$$\begin{aligned}u_i&\big(g_i(\omega), (\lambda_s \circ g_\omega^{-1}(a_1), \ldots, \lambda_s \circ g_\omega^{-1}(a_t))\big) \\ &\geq u_i\big(a, (\lambda_s \circ (g'_\omega)^{-1}(a_1), \ldots, \lambda_s \circ (g'_\omega)^{-1}(a_t))\big) - \epsilon\end{aligned} \tag{12}$$

for all $a \in A$. This allows the conclusion that the original mixed strategy equilibrium satisfies the ex post Nash property approximately, (7) being the approximate version of (12). But the point is that $(\epsilon'/P(\omega))$ *cannot* be made arbitrarily small.

It is important for the reader to appreciate why such a conclusion *cannot* be had and where the failure rests. It rests on the fact that in the move from (10) to (11), the size of Ω guarantees that one can always find a small enough $P(\omega)$ such that $\epsilon'/P(\omega)$ goes out of control. The main reason for such a failure is that one can never expect to derive the approximate best response for each $\omega \in \Omega$, but to be able to obtain (12) from (10) only with a large probability, as will be shown below.[40] Note that in Theorem 2, it is

[40] In terms of a schematic sketch, we have the following steps: (i) expression (8) as an analog of (2); (ii) (9) as an analog of (4); (iii) (10) as an analog of (5); (iv) (11) and (12) as analogs of (6) and (7). However, the argument fails from (iii) and (iv): the key construction from (10) to (11) does not (and cannot) deliver (12) for all $\omega \in \Omega$. This is because the requirement that the mixed strategies of different player are independent when the number of players (s) is "large" can only be fulfilled when the sample space is also "sufficiently large." As it happens, the move from (10) to (11) is illegitimate. If we only consider a subset Ω_s of Ω, one could possibly obtain (12) for all $\omega \in \Omega_s$ from (10) with a high probability. But this requires elaboration: a rather long and tedious management of different epsilons.

the Fubini property that allows us to switch between the two "almost all" quantifiers.[41] Thus, a more careful, and carefully rigorous and correct, formulation is needed.

In turning to this, we need the basic notion of a *tight* sequence of mappings. Corresponding to a sequence of large but finite games of players, we obtain a sequence of measures on the space of characteristics, and can then appeal to the conventional[42] formulation that a sequence of mappings from a measure space to a topological space is *tight* if for any ϵ, there exists a compact set of characteristics such that most of the mass, in the sense measured by ϵ, of all of the measures is in the compact set. We need such an idea of a *tight* sequence of mappings so as to put some control on the extent to which the characteristics are allowed to vary.

For each $n \geq 1$, let $I^n = \{1, 2, \ldots, n\}$ and let λ^n be the counting probability measure on the power set $\mathcal{I}^n$ of I^n. Let $(\Omega, \mathcal{F}, P)$ be a sample space. Let $\mathcal{G}^n$ be a game, which is to say, a measurable function from I^n to $\mathcal{U}_A$. We now develop the notion of asymptotic Nash equilibria in mixed strategies.

Definition 9. Let $\mathcal{G}^n$ be a game and let g^n be a process from $(I^n \times \Omega, \mathcal{I}^n \otimes \mathcal{F}, \lambda^n \otimes P)$ to A. The sequence $\{g^n\}_{n=1}^{\infty}$ is said to be a sequence of asymptotic Nash equilibria in mixed strategies if the following two conditions hold:

(i) Asymptotic independence. For any $\delta > 0$, $\lim_{n\to\infty} \lambda^n \otimes \lambda^n(\{(i_1, i_2) \in I^n \times I^n : \rho_2(P \circ (g^n_{i_1}, g^n_{i_2})^{-1}, P \circ (g^n_{i_1})^{-1} \otimes P \circ (g^n_{i_2})^{-1}) \leq \delta\}) = 1$, where ρ_2 is the Prohorov metric on $\mathcal{M}(A \times A)$.

(ii) Asymptotic Nash property. For any $\delta > 0$, $\lim_{n\to\infty} \lambda^n(I^n_\delta) = 1$, where

$$I^n_\delta = \left\{ i \in I^n : \int_\Omega \mathcal{G}^n_i(g^n_i(\omega), \lambda^n \circ (g^n_\omega)^{-1})\, \mathrm{d}P \geq \int_\Omega \mathcal{G}^n_i(h(\omega), \lambda^n \circ (g^n_\omega)^{-1})\, \mathrm{d}P - \delta \right.$$
$$\left. \text{for any } h \in \mathrm{Meas}(\Omega, A) \right\}.$$

We can now present the following proposition.

Proposition 8. *Consider a tight sequence of finite games $\{\mathcal{G}^n\}_{n=1}^{\infty}$. If $\{g^n\}_{n=1}^{\infty}$ is a sequence of asymptotic Nash equilibria in mixed strategies, then it is a sequence of asymptotic ex post Nash equilibria in the sense that for any $\epsilon > 0$, there is a positive integer N such that for any $n > N$, there exists $\Omega_n \in \mathcal{F}$ with $P(\Omega_n) > 1 - \epsilon$ and with the property that for any $\omega \in \Omega_n$,*

$$\lambda^n\big(\big\{i \in I^n : \mathcal{G}^n_i(g^n_\omega(i), \lambda^n \circ (g^n_\omega)^{-1}) \geq \mathcal{G}^n_i(a, \lambda^n \circ (g^n_\omega)^{-1}) - \epsilon \textit{ for all } a \in A\big\}\big) \geq 1 - \epsilon.$$

[41] As a perceptive reader may have already realized, in the proof of Theorem 2, we can only claim the optimality condition for λ-almost all agents $i \in I$ and for P-almost all states $\omega \in \Omega$. In our setting, f_i differs from g_i by one point and is thus essentially the same measurable function. We can "generalize" the result to work for all $i \in I$ but this difficulty of the exchange of quantifiers would remain, and little would be accomplished.

[42] See Khan and Sun (2002, Section 10.2, Footnote 86) and its references.

A fully rigorous proof is provided in the Appendix for interested readers. Here we simply note that with this result, we obtain the asymptotic ex post Nash property for large finite games with asymptotically approximate versions of independence and of a Nash equilibrium. The result is considerably more general than the case for large finite games with exact versions of independence and of a Nash equilibrium.

6. Conclusion

In this paper, we offer a precise definition of a mixed strategy profile for a large game by relying on the notion of a Fubini extension to characterize independent risks in a non-cooperative context. After a much needed and overdue clarification of the measurability problem, we furnish a complete resolution of two longstanding[43] open problems: Theorem 1 shows the equivalence between a MSE and a RSED, and Theorem 2 relies on this equivalence to exhibit the ex post Nash property of randomized strategy equilibria. More generally, our application of the ELLN brings out the ease with which one can perform operations on a continuum of independent random variables and provides a rigorous measure-theoretic microfoundation that can be used to model other macroeconomic and microeconomic scenarios.

In terms of future work and direction, note that the notions of a MSE and a RSED admit direct translation to a more elaborate model of a large game, one in which agent names and agent traits are disentangled, as considered in Khan et al. (2013). We leave it to the interested reader to check that Theorems 1 and 2 are still valid for such a model, and, indeed, virtually the exact proofs carry over to this more elaborate setting.[44] But substantively more to the point, it is of interest to ask how the framework offered in this paper, with or without traits, can be transferred into a context that is both Bayesian and dynamic. This is to ask, for example, whether the equivalence theorem and the ex post property of randomized strategies still hold in situations where there are many stages and in each stage, only a player's name is known with certainty, but her individual traits are random. This is thereby a movement from a large game of complete information to one with a multistage incomplete information game; see Morris and Shin (2003), Angeletos et al. (2007), and their references for the rich variety of applications, and Aumann and Dreze (2008) for a discussion of Bayesian rationality in a finite-player setting.[45]

Appendix

For the interested reader, we present a proof of Proposition 8 based on a rigorous Loeb-space approximation argument.[46]

[43] See Footnote 32 and the text that it footnotes.

[44] These theorems and their proofs are available online and also from the authors on request.

[45] In "Cancellation of individual risks and Fubini extension: A complete characterization," presented at the First Singapore Economic Theory Workshop, 16–17 August 2007 and available as a report from the National University of Singapore, Sun considers large Bayesian games with stochastic types in a static setting.

[46] The reader is referred to Loeb and Wolff (2000) for a working knowledge of nonstandard analysis.

Proof of Proposition 8. Fix any real number $\epsilon > 0$. We transfer the given sequences $\{\mathcal{G}^n\}_{n=1}^{\infty}$ and $\{g^n\}_{n=1}^{\infty}$ to the nonstandard universe to obtain internal sequences $\{\mathcal{G}^n\}_{n\in {}^*\mathbb{N}}$ and $\{g^n\}_{n\in {}^*\mathbb{N}}$. Fix $n \in {}^*\mathbb{N}_\infty$. Let $(I, \mathcal{I}, \lambda)$ be the corresponding Loeb space of $(I^n, \mathcal{I}^n, \lambda^n)$. The tightness assumption on the original sequence $\{\mathcal{G}^n\}_{n\in\mathbb{N}}$ implies that for each $n \in {}^*\mathbb{N}_\infty$, $\mathcal{G}^n$ is near standard in the sense that for λ-almost all $i \in I$, $\mathcal{G}^n(i)$ has a standard part ${}^o(\mathcal{G}^n(i))$. For $i \in I$, let $\mathcal{G}(i) = {}^o(\mathcal{G}^n(i))$. Let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ be the Loeb space of $({}^*\Omega, {}^*\mathcal{F}, {}^*P)$. For $(i, \omega) \in (I \times \bar{\Omega})$, let $g(i, \omega) \in {}^o(g^n(i, \omega))$. It is clear that g is $\mathcal{I} \boxtimes \bar{\mathcal{F}}$-measurable. By asymptotic independence, as specified in Definition 9(i), and the spillover principle, we can find some positive infinitesimal θ such that

$$\lambda^n \otimes \lambda^n\big(\big\{(i_1, i_2) \in I^n \times I^n : {}^*\rho_2({}^*P \circ (g^n_{i_1}, g^n_{i_2})^{-1}, {}^*P \circ (g^n_{i_1})^{-1} \otimes {}^*P \circ (g^n_{i_2})^{-1}) \le \theta\big\}\big) \ge 1 - \theta.$$

It is easy to see that

$${}^*P \circ (g^n_{i_1}, g^n_{i_2})^{-1} \simeq \bar{P} \circ (g_{i_1}, g_{i_2})^{-1}$$

and that

$${}^*P \circ (g^n_{i_1})^{-1} \otimes {}^*P \circ (g^n_{i_2})^{-1} \simeq \bar{P} \circ (g_{i_1})^{-1} \otimes \bar{P} \circ (g_{i_2})^{-1}.$$

Hence we can claim that for $(\lambda \boxtimes \lambda)$-almost all $(i_1, i_2) \in I \times I$,

$$\bar{P} \circ (g_{i_1}, g_{i_2})^{-1} = \bar{P} \circ (g_{i_1})^{-1} \otimes \bar{P} \circ (g_{i_2})^{-1}.$$

This means that the process g is essentially pairwise independent. By the asymptotic Nash property, as specified in Definition 9(ii), we can find some positive infinitesimal θ' such that

$$\lambda^n\bigg(\bigg\{i \in I^n : \int_{{}^*\Omega} \mathcal{G}^n_i(g^n_i(\omega), \lambda^n \circ (g^n_\omega)^{-1})\, \mathrm{d}^*P \ge \int_{{}^*\Omega} \mathcal{G}^n_i(h(\omega), \lambda^n \circ (g^n_\omega)^{-1})\, \mathrm{d}^*P - \theta' \\ \text{for all } h \in {}^*(\mathrm{Meas}(\Omega, A))\bigg\}\bigg) \ge 1 - \theta'.$$

It is easy to see that $\lambda^n \circ (g^n_\omega)^{-1} \simeq \lambda \circ g_\omega^{-1}$. Furthermore, note that

(i) $\mathcal{G}^n_i \simeq \mathcal{G}_i$

(ii) $\mathcal{G}^n_i(g^n_i(\omega), \lambda^n \circ (g^n_\omega)^{-1}) \simeq \mathcal{G}_i(g_i(\omega), \lambda \circ g_\omega^{-1})$

(iii) $\int_{{}^*\Omega} \mathcal{G}^n_i(g^n_i(\omega), \lambda^n \circ (g^n_\omega)^{-1})\, \mathrm{d}^*P \simeq \int_{\bar{\Omega}} \mathcal{G}_i(g_i(\omega), \lambda \circ g_\omega^{-1})\, \mathrm{d}\bar{P}$

(iv) $\int_{{}^*\Omega} \mathcal{G}^n_i(h(\omega), \lambda^n \circ (g^n_\omega)^{-1})\, \mathrm{d}^*P \simeq \int_{\bar{\Omega}} \mathcal{G}^n_i({}^o h(\omega), \lambda \circ g_\omega^{-1})\, \mathrm{d}\bar{P}$.

Hence for λ-almost all $i \in I$,

$$\int_{\bar{\Omega}} \mathcal{G}_i(g_i(\omega), \lambda \circ g_\omega^{-1})\, \mathrm{d}\bar{P} \ge \int_{\bar{\Omega}} \mathcal{G}_i(\bar{h}(\omega), \lambda \circ g_\omega^{-1})\, \mathrm{d}\bar{P}$$

for any measurable function $\bar{h}$ from $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{P})$ to A. By Theorem 2, we obtain that for $\bar{P}$-almost all $\omega \in \bar{\Omega}$, λ-almost all $i \in I$,

$$\mathcal{G}_i(g_\omega(i), \lambda \circ g_\omega^{-1}) \ge \mathcal{G}_i(a, \lambda \circ g_\omega^{-1}) \quad \text{for all } a \in A,$$

which implies that

$$\mathcal{G}_i^n(g_\omega^n(i), \lambda^n \circ (g_\omega^n)^{-1}) \geq \mathcal{G}_i^n(a, \lambda^n \circ (g_\omega^n)^{-1}) - \epsilon \quad \text{for all } a \in {}^*A.$$

Therefore, there is an internal set $\Omega_n \in {}^*\mathcal{F}$ with ${}^*P(\Omega_n) > 1 - \epsilon$ such that

$$\lambda^n\big(\big\{i \in I^n : \mathcal{G}_i^n(g_i^n(\omega), \lambda^n \circ (g_\omega^n)^{-1}) \geq \mathcal{G}_i^n(a, \lambda^n \circ (g_\omega^n)^{-1}) - \epsilon \text{ for all } a \in {}^*A\big\}\big) \geq 1 - \epsilon.$$

Since n is chosen as an arbitrary hyperinteger in ${}^*N_\infty$, we can use the spillover principle to claim the existence of a positive integer N such that for any $n > N$, there exists $\Omega_n \in \mathcal{F}$ with $P(\Omega_n) > 1 - \epsilon$ and with the property that for any $\omega \in \Omega_n$,

$$\lambda^n\big(\big\{i \in I^n : \mathcal{G}_i^n(g_i^n(\omega), \lambda^n \circ (g_\omega^n)^{-1}) \geq \mathcal{G}_i^n(a, \lambda^n \circ (g_\omega^n)^{-1}) - \epsilon \text{ for all } a \in A\big\}\big) \geq 1 - \epsilon. \quad \square$$

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