Majority runoff elections: Strategic voting and Duverger's hypothesis

Laurent Bouton
Department of Economics, Georgetown University

Gabriele Gratton
School of Economics, UNSW Business School

The majority runoff system is widely used around the world, yet our understanding of its properties and of voters' behavior is limited. In this paper, we fully characterize the set of strictly perfect voting equilibria in large three-candidate majority runoff elections. Considering all possible distributions of preference orderings and intensities, we prove that only two types of equilibria can exist. First, there are always equilibria in which only two candidates receive votes. Second, there may exist an equilibrium in which three candidates receive votes. Its characteristics challenge common beliefs: (i) neither sincere voting by all voters nor pushover tactics (i.e., supporters of the front-runner voting for a less preferred candidate so as to influence who will face the front-runner in the second round) are supported in equilibrium, and (ii) the winner does not necessarily have democratic legitimacy since the Condorcet winner may not even participate in the second round.

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1. Introduction

In a majority runoff election, a candidate wins outright in the first round if she obtains an absolute majority of the votes. If no candidate wins in the first round, then a second round is held between the two candidates who have the most first-round votes. The winner of that round wins the election.

Over the past decades, most newly minted democracies have adopted the majority runoff system to elect their presidents as well as other important government officials.[1] The majority runoff system is also widely used in longstanding democracies (see, e.g., Blais et al. 1997 and Golder 2005).[2] Moreover, debates about whether it should be implemented even more widely are recurrent (e.g., *La Repubblica*, June 20, 2012). These debates and the widespread inclination in favor of the majority runoff system rely on both formal and informal arguments. On the one hand, the majority runoff system is commonly believed (i) to be more conducive to preference and information revelation than plurality, and (ii) to ensure a large mandate to the winner, thereby providing her with more democratic legitimacy. On the other hand, the majority runoff system is commonly believed (i) to suffer from a nonmonotonicity problem that may induce a harmful strategic behavior in the first round called pushover,[3] and (ii) to increase the risk that a left-wing and a right-wing candidate "squeeze" the median candidate and exclude her from the second round.[4]

The scant empirical literature on majority runoff elections is not widely supportive of these arguments. First, as reviewed in Bouton (2013), the evidence that the runoff systems are more conducive to preference and information revelation than plurality is mixed. Second, there are many examples of majority runoff elections in which the winner is not the candidate preferred by the majority and thus lacks democratic legitimacy. For instance, in Peru's presidential election in 2006, Lourdes Flores Nano (Unidad Nacional) did not make it to the second round, despite opinion polls indicating that she was the majority candidate. Indeed, polls showed that she would have won a second round against the two other serious candidates: Ollanta Humala Tasso (Union for Peru) and Alan García Pérez (Aprista Party).[5] Finally, as far as we know, evidence of pushover behavior in runoff elections has never been documented (see Blais 2004a, 2004b and Dolez and Laurent 2010 for evidence against pushover).[6]

Such discrepancies beg an explanation. Arguably, part of the problem is that beliefs about the majority runoff system either have not been formally proven or have not

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[1]Most new democracies in Eastern Europe and in Africa have adopted this system to elect their presidents (Golder 2005, Golder and Wantchekon 2004). This is also true for the new democracies in Latin America, although to a lesser extent.

[2]For instance, in the United States, runoff primaries are a trademark in southern states, and most large cities have a runoff provision (Bullock III and Johnson 1992, Engstrom and Engstrom 2008). In Italy, the majority runoff system is used for the elections of mayors in all major cities and provincial presidents.

[3]In runoff elections, an additional vote in favor of a candidate can reduce her likelihood of victory (Smith 1973). Hence, there might be situations in which supporters of the front-runner prefer to vote for a less preferred candidate so as to influence who will face the front-runner in the second round. This is called *pushover*. It is deemed harmful because it may allow a minority group to influence the outcome of the elections to its advantage (Cox 1997, Saari 2003).

[4]It has also been argued that runoff elections may make it easier for voters to communicate their policy preferences to candidates (see, e.g., Piketty 2000, Castanheira 2003, Blais 2004a, 2004b). Such a behavior is beyond the scope of this paper.

[5]See Schmidt (2007). The 2007 French presidential election is another striking example (Spoon 2008).

[6]Based on anecdotal evidence, it has been argued that pushover occurred in the 2002 French presidential election. Two studies explicitly search for strategic behavior in that election using survey data. Blais (2004a, 2004b) finds evidence of strategic voting in the 2002 French presidential election but *not* of pushover.

been proven robust. Despite recent theoretical advances (Martinelli 2002, Morton and Rietz 2007, and Bouton 2013), there is still no complete characterization of the set of voting equilibria in a general setup. Thus, the properties of the majority runoff system established in the literature might be inaccurate. The problem is twofold (Myerson 2002, 2013). First, the lack of generality of the structure of preferences implies that the equilibria characterized in the literature and/or their properties might be artifacts of restrictions on voters preferences. Second, without a full characterization of *all* voting equilibria, one cannot exclude the existence of other equilibria featuring different properties.

In this paper, we attempt to fill this gap by generalizing Bouton's (2013) approach of three-candidate runoff elections in three directions.[7] First, we generalize the set of voter preferences to a continuum of types that allows for all possible preference orderings and intensities.[8] Second, we consider any positive and constant risk of upset victory in the second round, as well as risk converging to zero when the electorate grows large.[9] Third, we fully characterize the set of equilibria. For reasons detailed below, we focus on the set of strictly perfect equilibria (Okada 1981).

We demonstrate that in majority runoff elections, when sufficiently many voters are strategic, the set of strictly perfect equilibria features three main properties. First, a strictly perfect equilibrium always exists. Our proof is constructive: we show the existence of three Duverger's law equilibria, in which only two candidates receive a positive fraction of the votes. In these equilibria, an outright victory in the first round always occurs. Second, we show that a Duverger's hypothesis equilibrium, in which three candidates receive a positive fraction of the votes, may exist. Third, we show that there are no other strictly perfect equilibria in majority runoff elections. The unique Duverger's hypothesis equilibrium (i) never supports pushover (see footnote 3), (ii) never supports sincere voting by all voters, i.e., all voters voting for their most preferred candidate, and (iii) can lead to the exclusion of the Condorcet winner from the second round.[10]

These results strongly qualify some of the aforementioned common beliefs about majority runoff elections. First, regarding the idea that majority runoff elections should ensure a large mandate to the winner, we show that even when there are more than two serious candidates in the first round, the Condorcet winner is not guaranteed to

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[7]There are also two other less important differences. First, we exclude the possibility of a positive measure of voters being indifferent between two (or more) candidates. Such indifferences might seem non-generic, yet indifference is a convenient way to capture the existence of partisan/nonstrategic voters in large elections. This is one reason why we discuss the robustness of our results to the presence of such voters in Section 5. Second, we focus on majority runoff elections (the most used of the runoff systems), whereas Bouton (2013) considers all possible threshold for first-round victory strictly below 100%.

[8]By allowing for all possible preference orderings and intensities, we can consider any situation of interest as a particular case. For instance, combinations of policy platforms such that the median candidate is "squeezed" between two moderate candidates (see, e.g., Solow 2013, and Van Der Straeten et al. 2013) or the majority is divided (see, e.g., Myerson and Weber 1993, Piketty 2000, and Myatt 2007) are easy to represent in our framework. Therefore, our results show how voters behave in these situations of particular interest in the literature, as well as any other that can come to mind (e.g., Osborne and Slivinski 1996, Callander 2005, and Bordignon et al. 2013).

[9]Bouton (2013) focuses on a specific version of the latter. The former structure might be rationalized if a change in the distribution of voters' preferences between the two rounds is possible. See Section 3.3.

[10]The Condorcet winner is a candidate who would win a one-to-one contest against any other candidate.

participate in the second. Therefore, the fact that the eventual winner of the election obtains more than 50% of the votes in the second round cannot be considered a strong proof of legitimacy. This only ensures that a potential Condorcet loser never wins.[11] We argue that this result is closely related to the belief that the median candidate may be "squeezed" by a left-wing and a right-wing candidate (see, e.g., Solow 2013 and Van Der Straeten et al. 2013) and, hence, excluded from the second round. We thus show that, perhaps surprisingly, such a squeezing can indeed happen when voters are strategic.

Second, we show that the idea that majority runoff elections are more conducive to preference and information revelation than plurality elections is quite overrated. The initial argument is the following: since voters can use the second round to coordinate against a minority candidate, in the first round they feel free to vote "sincerely" for their most preferred candidate (Duverger 1954, Riker 1982, Cox 1997, Piketty 2000, Martinelli 2002). Our results actually reinforce Bouton's (2013) argument against this perceived benefit of the majority runoff system. First, we prove that Duverger's law equilibria exist even if voters have heterogeneous preference intensities.[12] Second, we show that when the structure of preferences is sufficiently rich, the only type of Duverger's hypothesis equilibrium that may exist is such that some voters do not vote sincerely. Contrasting the latter result with Bouton (2013, Theorem 2), we can actually conclude that the sincere voting equilibrium only exists if one arbitrarily restricts the set of possible preference orderings.

Third, we show that harmful pushover tactics perceived to be induced by the non-monotonicity of the runoff system (Cox 1997, Saari 2003) do not happen in any (strictly perfect) equilibrium. The generality of our approach makes sure that this conclusion holds for any possible distribution of preferences in the electorate.[13] It thus appears that, in line with empirical evidence, pushover is not a robust phenomenon in runoff elections: it requires voters to have excessively precise information about the expected outcome of the elections. Therefore, the main concern with the nonmonotonicity of the runoff system is that it might prevent sincere voting in equilibrium, not that it can induce harmful voting behavior.

Typically, there are many equilibria in multicandidate elections. In an environment as rich as the one considered in this paper, the multiplicity is even greater than usual. This is not undesirable per se. Indeed, equilibrium multiplicity captures the risk of coordination failure that exists in multicandidate elections (see, e.g., Myerson and Weber 1993, Bouton and Castanheira 2012). However, it has been argued that some equilibria of voting games are neither robust nor reasonable (see, e.g., Fey 1997 for a discussion of equilibrium multiplicity in plurality elections). It is thus proper to refine the set of equilibria when studying multicandidate elections. In this paper, we focus on the set

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[11]The Condorcet loser is a candidate who would lose a one-to-one contest against any other candidate.

[12]We also show the robustness of this result to many different ways of modeling the second round.

[13]By contrast, the restrictions on the structure of preferences in Bouton (2013) prevent the analysis of pushover equilibria since they imply that voters' incentives to push over are brought to a minimum. Hence, he remains completely silent about the existence or nonexistence of those equilibria.

of strictly perfect equilibria (Okada 1981). As we show in Technical Appendix C (Technical Appendices C and D are available in a supplementary file on the journal website, http://econtheory.org/supp/1642/supplement.pdf), in our framework, the idea behind strict perfection boils down to equilibria being stable against arbitrary slight perturbations of the model (and not only strategic uncertainty).

There are several reasons for using strict perfection as an equilibrium concept in Poisson voting games (see Technical Appendix C for proofs and more details). First, less stringent concepts such as perfection and properness have very little bite in Poisson voting games (De Sinopoli and Pimienta 2009). For instance, they do not eliminate equilibria in plurality elections that have been deemed unstable and undesirable in terms of voters' information and expectational stability (Fey 1997).[14] Fey's argument is that some voting equilibria are unreasonable because they require (i) excessive coordination among voters and (ii) very precise information about the expected outcome of the election that public opinion polls are unlikely to provide. By contrast, strict perfection rules out exactly those equilibria.[15] Second, multiple strictly perfect equilibria always exist in our model, and in general in most voting games. This suggests that, though stringent, strict perfection is not too stringent a concept in voting games. Finally, we prove, for a general class of Poisson games that strict perfection can be defined in a way that is simple and easy to use. Using strict perfection actually makes the complete characterization of the set of equilibria significantly simpler.

2. The model

The majority runoff system works as follows. There are three candidates, $i \in \mathcal{C} \equiv \{A, B, C\}$, who all participate in a first round of an election. If, in the first round, a candidate receives more than half of the votes, then she is elected.[16] Otherwise, the two candidates with the largest shares of votes will face each other in a second ballot. To lighten notation, we assume that ties for the second place are resolved by alphabetical order: A wins over both B and C, and B wins over C.[17]

We conduct the analysis under the assumption that the size of the electorate ν is distributed according to a Poisson distribution of mean n: $\nu \sim \mathcal{P}(n)$ (Appendix A summarizes some properties of Poisson games and applies them to runoff elections).[18] Each

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[14]In our setup, any equilibrium in which all three candidates get a positive fraction of the votes is perfect and proper.

[15]See Technical Appendix C for a proof of the equivalence of expectational stability and strict perfection in our setup.

[16]It is straightforward to extend our results to runoff systems with a threshold for first-round victory above 50%. For runoff systems with a threshold below 50%, the situation becomes quite intractable because a ballot can be pivotal in many new and different ways.

[17]This particular tie-breaking rule will prove irrelevant for the proofs of our results.

[18]Myerson (2000, p. 24) shows that, for large n, in a setup with a fixed number of voters whose preferences follow a multinomial distribution, pivot probabilities are a monotonic transformation of their Poisson equivalent. In other words, the magnitude ratios for multinomial distributions behave similarly as under Poisson distributions. This implies that most of our results extend to a setup with a fixed number of voters whose preferences follow a multinomial distribution.

voter has preferences over the candidates defined by her type $t \in \mathcal{T}$.[19] Types are assigned by independent and identically distributed (i.i.d.) draws from a distribution F that admits a density and has full support over $\mathcal{T}$.[20,21] We label the set of such distributions as $\mathcal{F}$. Note that we do not allow for aggregate uncertainty in the preferences of voters. The utility of a voter of type t when candidate i is elected is given by $U(i|t)$. Voters of type t with $U(i|t) > U(j|t) \ \forall j \in \mathcal{C}, \ j \neq i$ prefer candidate i over any other candidate and we shall call them i's *supporters*. We denote by γ_{ij} the (expected) fraction of voters with preferences $i \succ j \succ k$.

From Bouton (2013), we know that what happens in the second round influences dramatically the behavior of voters in the first round. In particular, a crucial element is the risk of upset victory in that round. The way we model the second round is thus a sensitive matter. One obvious modeling choice would be to endogenize the second round as in Bouton (2013). Assuming that there is uncertainty about the realized distribution of preferences in the electorate after the first round, we would have that the risk of an upset victory converges to zero when the expected number of voters grows large. The speed of convergence to zero would depend on the particular assumptions on the distribution of preferences made to obtain such uncertainty. In Technical Appendix D, we show that all our results hold (at least qualitatively) for any speed of convergence to zero of the risk of upset victory.

While this modeling choice is appealing, it also makes the proofs and the exposition of the results much more cumbersome. For the sake of expositional clarity, in the core of the paper, we work under the assumption that at the time of the first round, the probabilities of victory are given and constant. We denote by $\Pr(i|ij)$, $i, j \in \mathcal{C}$, the probability that candidate i defeats candidate j in the second round opposing these two candidates. Hence, $\Pr(j|ij) = 1 - \Pr(i|ij)$. We assume that all these second round probabilities are strictly positive, i.e., $\Pr(i|ij) \in (0,1)$, and constant. This includes (but is not limited to) any "realistic" restriction (e.g., the front-runner or the candidate with the largest (expected) number of supporters being more likely to win in the second round). Hence, at the time of the first round, the result of any eventual second round ballot is not certain.[22] Given the probabilities of victory in the second round, we can determine the expected utility of a second round opposing i to j for a type t:

$$U(i, j|t) = \Pr(i|ij)U(i|t) + \Pr(j|ij)U(j|t).$$

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[19]Without loss of generality, we restrict attention to $\mathcal{T} = \mathbb{R}^2$.

[20]This ensures that the measure of voters who are indifferent between any pair of candidates is zero.

[21]None of our results depends on the fact that our model includes all possible preference orderings and intensities over the set of candidates (i.e., we can prove the existence of the two types of equilibria in a setup that only includes an adequately chosen subset of the preference orderings and intensities).

[22]We want to argue that the assumption that the chances of victory in the second round are interior, $\Pr(i|ij) \in (0,1)$, is not only for the sake of simplicity. We regard that assumption as capturing small shocks to preferences between the two rounds. In the period between the two rounds, electoral campaigns go on and new information regarding the candidates can arise. This can affect the voters' preference orderings and thus the expected outcome of the second round. There are plenty of anecdotal evidence of such a phenomenon. We could formally model this argument by allowing the distribution of voters' preferences in the second round to be a random variable with a sufficiently large support. Our results can thus be interpreted as follows: we identify equilibria that are robust to such an uncertainty about the events that may occur between the two rounds.

The action set for each voter is $\{A, B, C\} = \mathcal{C}$. A voting strategy is $\sigma : \mathcal{T} \to \Delta(\mathcal{C})$, where σ_t denotes the strategy of a voter of type t.[23] Call $\sigma \equiv ((\sigma_t(i))_{i \in \mathcal{C}})_{t \in \mathcal{T}} \in \Delta(\mathcal{C})^{\mathcal{T}}$ a profile of voting strategies. Define $\tau : \Delta(\mathcal{C})^{\mathcal{T}} \times \mathcal{F} \to \Delta(\mathcal{C})$,

$$\tau(\sigma, F) \equiv \left(\int_{\mathcal{T}} \sigma_t(i) \, dF(t) \right)_{i \in \mathcal{C}},$$

where an element of $\tau(\sigma, F)$, $\tau_i(\sigma, F) \geq 0$, is the measure of voters' types voting for candidate i in the first round. This is also the expected share of votes received by candidate i in the first round. For any distribution of preferences F, a profile of voting strategies σ identifies a unique profile of expected vote shares.

The number of players who choose action i (the day of the election) is denoted by x_i, where $i \in \mathcal{C}$. This number is random (voters do not observe it before going to the polls) and its distribution depends on the strategy through $\tau_i(\sigma, F)$. For the sake of readability, we henceforth often omit (σ, F) from the notation.

In any equilibrium, there is a candidate who is expected to receive a share of votes larger than or equal to that of any other candidate. This candidate is the *front-runner*. Also, one of the two remaining candidates has higher chances than the other to win a second round ballot against the front-runner. Without loss of generality, in what follows, the set of candidates $\mathcal{C}$ is defined by a *front-runner*, R, a *strong opponent*, S, and a *weak opponent*, W, with $\Pr(R|RW) > \Pr(R|RS)$. For example, suppose that, in equilibrium, candidate A is expected to receive 40% of the votes, B is expected to receive 35%, and C is expected to receive 25%. In what follows, we refer to candidate A as R. Between B and C, we refer to the candidate who is more likely to defeat A in the second round as S and the other candidate is referred to as W. The action set for each voter thus becomes $\{R, S, W\}$. This relabeling allows us to reduce the number of cases to consider without losing generality. For example, we prove that there exist equilibria where only two candidates get votes, i.e., R and S or R and W. Since the existence of such equilibria does not depend on the distribution of preferences, then all possible equilibria in which only two candidates receive votes (A and B, B and C, or A and C) always exist. Of course, it is always possible to reconstruct whether an outcome is possible for a given distribution of preferences. For example, in equilibrium, if R receives votes only from voters who rank her first and $\tau_R(\sigma, F) > \frac{1}{3}$, then there exists such an equilibrium with $A = R$ only if $\gamma_{AB} + \gamma_{AC} > \frac{1}{3}$.

3. Pivot probabilities, payoffs, and equilibrium concept

Since voters are instrumental, their behavior depends on the probability that a ballot affects the final outcome of the elections, i.e., its probability of being pivotal. This section identifies all the pivotal events. Then we compute voters' expected payoffs of the different actions and define the best response correspondence.[24]

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[23]More precisely, the strategy $\sigma_t(i)$ is the marginal distribution of Milgrom–Weber distributional strategies (Milgrom and Weber 1985).

[24]This paragraph and Section 3.1 are based on similar sections in Bouton (2013).

As explained in Appendix A (which summarizes the properties of Poisson games and applies them to runoff elections), the probability that a pivotal event E occurs is exponentially decreasing in n. The (absolute value of the) *magnitude* of event E, denoted $\mu(E) \leq 0$ and formally defined in Appendix A, represents the "speed" at which the probability decreases toward zero: the more negative the magnitude, the faster the probability goes to zero. Unless two events have the same magnitude, their likelihood ratio converges either to zero or infinity when the electorate grows large (see Lemma 2 in Appendix A). Proofs in this paper rely extensively on this property and, thus, on the comparison of magnitudes of pivotal events. Lemma 3 (in Appendix A) computes the magnitudes of the different pivotal events. It shows that the magnitude of a pivotal event *piv* is larger when the expected outcome of that round is close to the conditions necessary for event *piv* to occur. Generally, the smaller is the deviation with respect to the expected outcome required for the pivotal event to occur, the larger is the magnitude.

3.1 *Pivotal events*

The first round influences the final result either directly (if one candidate wins outright) or indirectly (through the identity of the candidates participating in the second round).

Due to the alphabetical order tie-breaking rule, the precise conditions for the pivotal events actually depend on the alphabetical order of the candidates. Yet we define the different pivotal events for any candidates $i, j, k \in \{R, S, W\}$ and $i \neq j \neq k$, abstracting from the candidates' alphabetical order. These conditions are thus necessarily loose.[25]

A ballot is *threshold pivotal i/ij*, denoted $piv_{i/ij}$, if candidate i lacks one vote (or less) to obtain a majority of the votes in the first round. Thus, without an additional vote in favor of i, a second round opposing i to j is held. The complementary event is the threshold pivotability ij/i, denoted $piv_{ij/i}$, that refers to an event in which any ballot against candidate i, i.e., in favor of either j or k, prevents an outright victory of i in the first round and ensures that a second round opposing i to j is held.

A ballot may also affect the final outcome if it changes the identity of the two candidates participating in the second round. This happens when a ballot changes the identity of the candidates who rank second and third in the first round. A ballot is *second-rank pivotal ki/kj*, denoted $piv_{ki/kj}$, when candidate k ranks first (but does not obtain an absolute majority of the votes), and candidates i and j tie for second place. An additional vote in favor of candidate i allows her, instead of j, to participate in the second round with k.

Table 1 summarizes the different first-round pivotal events that influence the first-round voting behavior.

3.2 *Payoffs and best responses*

Let $G_t(i, n\tau)$ denote the *expected gain* of playing action $i \in \mathcal{C}$ for a voter of type t when the expected share of votes is τ. As usual in the literature, this expected gain is defined

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[25]In the third column of Table 1, depending on the candidates alphabetical order, (i) the conditions might feature weak inequality signs instead of strict ones or conversely, and (ii) the minus 1 might not be there. As proved in Myerson (2000, Theorem 2), such small approximations in the definition of the pivotal events do not matter for the computation of magnitudes.

Event	Notation	Condition
Threshold pivotal i/ij	$piv_{i/ij}$	$x_i + \frac{1}{2} > \frac{1}{2}(x_i + x_j + x_k) \geq x_i \geq x_j \geq x_k$
Threshold pivotal ij/i	$piv_{ij/i}$	$\frac{1}{2}(x_i + x_j + x_k + 1) \geq x_i > \frac{1}{2}(x_i + x_j + x_k) > x_j$ $x_i \geq x_j \geq x_k$
Second-rank pivotal ki/kj	$piv_{ki/kj}$	$x_i = x_j - 1$ $\frac{1}{2}(x_i + x_j + x_k) \geq x_k > x_j$

TABLE 1. First-round pivotal events.

as the difference between the expected utility for a voter of type t if she votes for i and if she abstains. This gain depends on the voter's type and on the strategy function for all voters, σ. Strategies determine the expected number of votes received by each candidate in the first round and, thus, the pivot probabilities. It easy to show that for a type t, the expected gain of playing action i in the first round is

$$
\begin{aligned}
G_t(i, n\tau) = {} & \Pr(piv_{ki/kj})[U(k, i|t) - U(k, j|t)] + \Pr(piv_{ji/jk})[U(j, i|t) - U(j, k|t)] \\
& + \Pr(piv_{i/ik})[U(i|t) - U(i, k|t)] + \Pr(piv_{i/ij})[U(i|t) - U(i, j|t)] \\
& + \Pr(piv_{ki/k})[U(k, i|t) - U(k|t)] + \Pr(piv_{ji/j})[U(j, i|t) - U(j|t)] \\
& + \Pr(piv_{kj/k})[U(k, j|t) - U(k|t)] + \Pr(piv_{jk/j})[U(j, k|t) - U(j|t)],
\end{aligned}
\tag{1}
$$

where $i, j, k \in \mathcal{C}$ and $i \neq j \neq k$. The first line in (1) reads as follows: if a ballot in favor of i is second-rank pivotal ki/kj, then the second round opposes k to i instead of k to j; if a ballot in favor of i is second-rank pivotal ji/jk, then the second round opposes j to i instead of j to k. The three last lines refer to the gains when the ballot is threshold pivotal.

By Theorem 8 in Myerson (1998), when players behave according to a strategy profile σ, the number of voters voting for candidate i follows a Poisson distribution with mean $n\tau_i(\sigma, F)$. Thus, for any finite n, a strategy profile σ and a distribution F uniquely identify the probability of any event, including the probability that a single vote is pivotal between two electoral outcomes. That is, the vector of all pivot probabilities is a function of $\tau(\sigma, F)$. Hence, we can define the best response correspondence $\mathcal{B} : \mathcal{T} \times \Delta(\mathcal{C}) \rightrightarrows \Delta(\mathcal{C})$. For a voter of type t, a strategy profile σ, and a distribution of types F,

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\mathcal{B}_t(\tau) \equiv \arg \max_{\sigma_t \in \Delta(\mathcal{C})} \sum_{i \in \mathcal{C}} \sigma_t(i) G_t(i, n\tau).
$$

3.3 *Equilibrium concept*

In voting games, the object of the analysis is often the limit of the set of equilibria as $n \to \infty$. We refer to an element of this set as an asymptotic equilibrium.

DEFINITION 1. Let $\hat{\Gamma} \equiv \{\Gamma_n\}_{n \to \infty}$ be a sequence of games $\Gamma_n \equiv (n, \mathcal{T}, F, \mathcal{C}, u)$. A strategy profile σ_t^* for all $t \in \mathcal{T}$ is an *asymptotic equilibrium* of $\hat{\Gamma}$ if there exists a sequence of Nash equilibria $\{\sigma_n^*\}_{n \to \infty}$ of Γ_n such that $\sigma_{n,t}^* \to \sigma_t^*$ for almost all $t \in \mathcal{T}$.

Let us clarify the meaning of this definition by means of an example. Take a Duverger's law equilibrium in a plurality voting game. In such an equilibrium, only two serious candidates receive a positive expected vote share. For any finite n, there exists a positive measure of voters who vote for a third candidate—those who are almost indifferent between the two serious candidates, but like this third candidate very much. What we mean when we say that there is a Duverger's law equilibrium as $n \to \infty$ is that as n grows large, the measure of voters voting for a third candidate goes to zero.

In what follows, we fully characterize the set of strictly perfect equilibria (Okada 1981) as the size of the electorate n goes to infinity.[26] The original idea behind strict perfection is that a sensible equilibrium should be stable against arbitrary slight perturbations of the strategy set. In our setup, strict perfection encompasses more than "strategic uncertainty." As we show in Technical Appendix C, strict perfection is equivalent to requiring robustness to some perturbations of "the model," i.e., the expected distribution of preferences in the electorate. In terms of the model's predictive power, this implies that strictly perfect equilibria are not an artifact of the precise distribution of preferences the modeler assumes.

In the context of voting games, election polls are a natural device that permits voters to approximately predict the outcome of the election. We think of strictly perfect equilibria as equilibria robust to small errors in a hypothetical election poll.[27] For example, one can construct an equilibrium in which some of R's supporters vote for R while others "push over" W. Such an equilibrium requires the first round vote shares to be exactly equal to some values. In general, such values can be approximated numerically, e.g., 28% for W, 24.451% for S, and 47.549% for R. This equilibrium is not strictly perfect because even an arbitrarily small perturbation of the expected results would change the best response of a nontrivial fraction of the population. For example, any drop in the voting share of S would result in a large fraction of pushover votes to move from W to R.

In Technical Appendix C, we show that for all Poisson games with infinite type sets, a strictly perfect equilibrium can be defined as follows.

DEFINITION 2 (Strictly perfect equilibrium). A strategy profile σ^* is a strictly perfect equilibrium if and only if there exists $\epsilon > 0$ such that if $\tau \in \Delta(\mathcal{C})$: $|\tau - \tau(\sigma^*, F)| < \epsilon$, then $\sigma_t^* \in \mathcal{B}_t(\tau)$ for all $t \in \mathcal{T}$.

Given our setup, we have to interpret strict perfection as a limit condition. Indeed, for any finite n, there exists a positive measure of voters' types for which $\sigma_t^* \notin \mathcal{B}(\tau_n)$, where $|\tau_n - \tau(\sigma_n^*, F)| < \epsilon$. Therefore, we say that a series of equilibria is strictly perfect if this measure goes to zero as n grows large. On the contrary, we say that an equilibrium is not strictly perfect if this measure remains bounded away from zero even as n grows large (see Technical Appendix C).

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[26]Okada (1981) defines strictly perfect equilibria for finite games. In Technical Appendix C, we provide a straightforward extension to Poisson games.

[27]This is also the intuition behind expectationally stable equilibria, a common concept in the voting literature (Fey 1997).

DEFINITION 3. An asymptotic equilibrium σ^* is *asymptotically strictly perfect* if there exists a sequence of Nash equilibria $\{\sigma_n^*\} \to \sigma^*$ for almost all $t \in \mathcal{T}$ such that for any $\delta > 0$, there exist $N \in \mathbb{N}$ and $\epsilon > 0$ such that for any $n > N$, if $\tau_n \in \Delta(\mathcal{C})$: $|\tau_n - \tau(\sigma_n^*, F)| < \epsilon$, then $\Pr[t \in \mathcal{T} : \sigma_t^* \notin \mathcal{B}_t(\tau_n)] < \delta$.

Proposition S4 in Technical Appendix C greatly simplifies the equilibrium analysis since it dramatically reduces the number of sequences of strictly perfect equilibria to consider as $n \to \infty$. This proposition shows that if a strategy profile is a best response to itself only if two pivotal events have identical magnitudes, then it is not an asymptotically strictly perfect equilibrium. Importantly, this does *not* imply that a strategy profile that does not generate a unique largest magnitude (as n goes to infinity) cannot be an asymptotically strictly perfect equilibrium. For example, there exists an asymptotically strictly perfect equilibrium with $\tau_R > \tau_S > \tau_W = 0$ and two pivotal events with largest magnitude: $piv_{R/RS}$ and $piv_{S/RS}$. For ease of exposition, we reproduce Proposition S4 in Technical Appendix C here as Lemma 1.

LEMMA 1. *Let σ^* be an asymptotic equilibrium if and only if two pivotal events have equal magnitudes under $\tau(\sigma^*, F)$. Then σ^* is not asymptotically strictly perfect.*

For the proof, see Proposition S4 in Technical Appendix C.

In the rest of the paper, we focus on asymptotic and asymptotically strictly perfect equilibria. For the sake of readability, we will from now on refer to asymptotic equilibria as "equilibria," and to asymptotically strictly perfect equilibria as "strictly perfect equilibria."

4. EQUILIBRIUM ANALYSIS

This section analyzes the set of strictly perfect equilibria in majority runoff elections. We prove three main results. First, a strictly perfect equilibrium always exists. Our proof is constructive: we show that three *Duverger's law* equilibria exist for any distribution of preferences.

DEFINITION 4 (Duverger's law equilibrium). A Duverger's law equilibrium is an equilibrium in which only two candidates obtain a nonzero expected vote share.

Second, we prove that a *Duverger's hypothesis* equilibrium may exist and be strictly perfect.

DEFINITION 5 (Duverger's hypothesis equilibrium). A Duverger's hypothesis equilibrium is an equilibrium in which all three candidates obtain a nonzero expected vote share.

Third, we show that there is only one type of Duverger's hypothesis equilibrium that is strictly perfect. Interestingly, neither the *sincere voting* equilibrium nor *pushover* equilibria exist.

DEFINITION 6 (Sincere equilibrium). An equilibrium is sincere if and only if all voters vote for their most preferred candidate.

DEFINITION 7 (Pushover equilibrium). A pushover equilibrium is an equilibrium where some supporters of the front-runner R vote for the weak opponent W with nonzero probability.

4.1 *Existence: Duverger's law*

In this section, we prove that a strictly perfect equilibrium always exists. Our proof is constructive.

PROPOSITION 1. *There always exist three strictly perfect Duverger's law equilibria.*

See Appendix B for the proof.

The intuition behind this result is straightforward. If a voter expects only two candidates to receive a positive share of votes, as the expected number of votes grows large, his vote can only be decisive in determining which of these two candidates will be elected outright in the first round. That is because if only two candidates receive any vote, then one of them will receive a majority of the votes in all cases except when both candidates receive exactly a 50% share. There are three different Duverger's law equilibria because there are three different combinations of two candidates receiving all votes. It is easy to show that if there are N candidates, then there are $N!/((N-2)!2!)$ strictly perfect Duverger's law equilibria.

Proposition 1 can be illustrated through a numerical example. Suppose that (i) 10% of the voters are W supporters, and (ii) if all voters who prefer R to S vote for R and all voters who prefer S to R vote for S, then $\tau_R = 60\% > \tau_S = 40\% > \tau_W = 0\%$. In this case, all magnitudes are equal to -1 except for $\mu(piv_{R/RS})$ and $\mu(piv_{S/SR})$, which are equal to -0.0202. This means that, conditional on being pivotal, voters choose between an outright victory of either R or S in the first round, and a second round opposing R to S. Since both candidates have a positive chance of winning a second round, voters who prefer R to S vote for R to avoid the risk of S's victory in the second round. Similarly, voters who prefer S to R vote for S to avoid the risk of R's victory in the second round.

Importantly, these best responses would not change if $\mu(piv_{R/RS})$ and $\mu(piv_{S/SR})$ were different (but still the two largest magnitudes). Consider the case in which $\mu(piv_{R/RS}) > \mu(piv_{S/SR})$. All voters who prefer R to S will vote for R. Indeed, by ensuring an outright victory of R in the first round, they avoid the risk of a victory of S in the second round. For voters who prefer S to R, the choice is slightly more complex. If their decision is based only on this most likely scenario, then they would vote against R but be indifferent between voting for S or W. Indeed, any of these two actions would have the same result: decreasing the probability of an outright victory of R and increasing the probability of a victory of S (through a second round). Thus, their choice between S and W depends on the second most likely pivotal event, i.e., $piv_{S/SR}$ in the case under consideration. Thus, to avoid the risk of an upset victory of R in the second round, voters who prefer S to R vote for S.

Threshold magnitudes	Second-rank magnitudes
$\boldsymbol{\mu(piv_{R/RS}) = -0.005}$	$\mu(piv_{RS/RW}) = -0.0927$
$\mu(piv_{R/RW}) = -0.0927$	$\mu(piv_{SR/SW}) = -0.181$
$\boldsymbol{\mu(piv_{S/SR}) = -0.0461}$	$\mu(piv_{WR/WS}) = -0.196$
$\mu(piv_{S/SW}) = -0.1897$	
$\mu(piv_{W/WR}) = -0.4$	
$\mu(piv_{W/WS}) = -0.40755$	

TABLE 2. Magnitudes.

The strict perfection of Duverger's law equilibria ensue from the continuity of the magnitudes in the probability distribution over actions. Since small perturbations to the strategies generate small changes to the magnitudes, there is always a small enough deviation from σ^* such that the two largest magnitudes are $\mu(piv_{R/RS})$ and $\mu(piv_{S/SR})$.

The following example illustrates the robustness of the force underlying Duverger's law equilibria. Consider the distribution of voters of the previous example. Now let all W supporters (who represent 10% of the electorate) vote for W, whereas the other voters adopt the same strategy as above. Then, for instance, we have $\tau_R = 55\% > \tau_S = 35\% > \tau_W = 10\%$. As shown in Table 2, for this expected vote shares, the largest magnitude is $\mu(piv_{R/RS})$ and the second largest magnitude is $\mu(piv_{S/SR})$. This is thus not an equilibrium: W supporters prefer to vote for either R or S.

Contrasting Proposition 1 with Theorem 1 in Bouton (2013) highlights one specificity of the model in which the risk of upset victory in the second round is positive and constant. For the case of majority runoff, Bouton (2013) shows that Duverger's law equilibria exist if the expected vote share of the candidate expected to rank second is large enough. This condition arises because, in Bouton (2013), the risk of victory of the minority candidate in the second round converges to zero when n grows large. The rate of convergence depends on the expected vote share of the minority candidate. If the expected vote share is too small, this risk converges to zero too fast and then voters disregard it. In the model under consideration, when all candidates have a positive and constant probability of victory in the second round, the threat of the minority candidate in the second round is always large enough to trigger a coordination in the first round. In Technical Appendix D, we show that whenever the risk of upset victory in the second round converges to zero as n grows large, there always exist at least two Duverger's law equilibria. We also find that the Duverger's law equilibrium in which W is the runner-up might not exist if the support for W against the front runner is small enough. This is another illustration of what can be missed by not including all preference orderings and intensities in the model.

4.2 Duverger's hypothesis

The Duverger's hypothesis suggests that in runoff elections, voters have incentives to disperse their votes on more than two candidates. In this section, we show that these incentives exist and that they can lead to the existence of a Duverger's hypothesis equilibrium.

In a Duverger's hypothesis equilibrium, all three candidates receive a positive expected vote share. In general, there is a *front-runner* (the candidate with the largest expected vote share), a *runner-up* (the candidate with the second largest expected vote share), and a third candidate who is expected to receive less votes than any other candidate. Notice that the distinction between the runner-up and the third candidate is determined in equilibrium by the first-round expected voting shares of the two opponents. This is different from the distinction between a strong and a weak opponent, which is determined by the relative likelihood of defeating the front-runner in a second-round ballot.

As we show in Section 4.3 (Proposition 3), the only strictly perfect Duverger's hypothesis equilibria are those identified in the following proposition.

PROPOSITION 2. *For some distribution of preferences, there exist strictly perfect equilibria in which three candidates receive a positive share of the votes. In these equilibria, all voters who prefer the front-runner to the runner-up vote for the front-runner. Some, but not all, of the supporters of the weak opponent vote for the strong opponent, regardless of which opponent is expected to receive more votes.*

See Appendix B for the proof.

To understand the intuition of this result, we must first understand voters' reaction when they must choose between an outright victory of R and a second round where R opposes the runner-up (i.e., either $piv_{R/RS}$ or $piv_{R/RW}$ has the largest magnitude). All voters who prefer R to the runner-up will vote for R. Indeed, by ensuring an outright victory of R in the first round, they avoid the risk of a victory for the runner-up in the second round. For voters who prefer the runner-up to R, the choice is slightly more complex. If their decision is based only on this most likely scenario, then they would vote against R but remain indifferent between S or W. Indeed, either of these two actions would have the same result: decreasing the probability of an outright victory of R and increasing the probability of a victory for the runner-up (through a second round). Thus, their choice between S and W depends on the second most likely pivotal event. There are two cases to consider: $piv_{S/SR}$ (or $piv_{W/WR}$) and $piv_{RS/RW}$.

If the threshold pivotability S/SR (or W/WR) dominates (which happens when both R and the runner-up have a large advantage with respect to the third candidate), the incentives are the same as in a Duverger's law equilibrium: all voters who prefer the runner-up to R vote for the runner-up. Therefore, we cannot have a Duverger's hypothesis equilibrium. Suppose, on the contrary, that the second-rank pivotability RS/RW dominates (which happens when S and W are sufficiently close to each other). In this situation, voters voting against R realize that they determine whether S or W faces R in the second round. Consider the choice of a supporter of S who prefers the runner-up to the front-runner. He casts his ballot considering what to do if R does not pass the threshold (by voting against R, he actually maximizes this probability). He would surely prefer to vote for S, and for two good reasons: first, because he prefers S to W; second, because S has more chances than W to win against R in the second round. Consider the choice of a W supporter who prefers the runner-up to the front-runner. He prefers W to

Threshold magnitudes	Second-rank magnitudes
$\mu(piv_{R/RS}) = -0.02968$	$\boldsymbol{\mu(piv_{RS/RW}) = -0.02693}$
$\boldsymbol{\mu(piv_{R/RW}) = -0.005}$	$\mu(piv_{SR/SW}) = -0.05982$
$\mu(piv_{S/SR}) = -0.2178$	$\mu(piv_{WR/WS}) = -0.05519$
$\mu(piv_{S/SW}) = -0.2178$	
$\mu(piv_{W/WR}) = -0.04$	
$\mu(piv_{W/WS}) = -0.08233$	

TABLE 3. Magnitudes.

S, but he also knows that S has a better chance of winning against R. Since he prefers S to R, he faces a trade-off between the likelihood of a second-round victory against R and how much he prefers W to S. If he is sufficiently close to indifference between S and W, then he votes for the former; otherwise, he votes for W.

In an equilibrium such as those described in Proposition 2, the Condorcet winner might be the candidate who has the smallest expected vote share (she would thus be very unlikely to reach the second round if held). This happens when the Condorcet winner is the second best choice of a large fraction of the voters, but the first choice of only a minority. Hence, in the first round, a large fraction of the support she would receive in a pairwise ballot is lost in favor of a third candidate.[28]

REMARK 1. For some distribution of preferences, there exists a strictly perfect equilibrium in which three candidates receive a positive share of votes and the Condorcet winner receives the smallest share.

We can illustrate this result through a numerical example. Consider the following situation: supporters of S represent 35% of the voters ($\gamma_{SW} = 15\%$ and $\gamma_{SR} = 20\%$), while the share of R's and W's supporters is equal to 25% ($\gamma_{RS} = 16\%$ and $\gamma_{RW} = 9\%$) and 40% ($\gamma_{WR} = 10\%$ and $\gamma_{WS} = 30\%$), respectively. It is easy to verify that (i) S is the Condorcet winner, (ii) R is the Condorcet loser, and (iii) S is a stronger opponent of R than W. In this case, there exists an equilibrium in which (i) the Condorcet winner, S, is expected not to reach the second round and (ii) the weak opponent, W, is expected to defeat the front-runner in the second round. In particular, the expected vote shares in that equilibrium are $\tau_R = 45\%$, $\tau_W = 36\%$, and $\tau_S = 19\%$. For those expected vote shares, the magnitudes are given in Table 3. Since the largest magnitude is $\mu(piv_{R/RW}) = -0.005$, we have that all voters who prefer R to W vote for R ($\gamma_{RS} + \gamma_{RW} + \gamma_{SR} = 45\%$) and that all voters who prefer W to R ($\gamma_{WR} + \gamma_{WS} + \gamma_{SW} = 55\%$) vote against R, either for W or for S. Since the second largest magnitude is $\mu(piv_{RW/RS}) = -0.0263$, the choice between S and W is determined by the utility difference between a second round where R opposes S, and a second round where R opposes W. As detailed in the proof of Proposition 2, this

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[28]Thus, strict perfection does not exclude coordination failures among the voters who prefer the Condorcet winner to the ultimate winner of the election. This is in stark contrast with Messner and Polborn (2007), who consider coalition-proof equilibria and find that when a Condorcet winner exists, then it is the unique coalition-proof equilibrium outcome.

difference depends on (i) the intensity of the relative preference between W and S, and (ii) the probabilities of victory in the second round. Since $\Pr(R|RS) < \Pr(R|RW)$, some voters who prefer (only slightly) W to S vote for S because she is more likely than W to defeat R in the second round. There are many different combinations of distribution of preferences F, $\Pr(R|RS)$, and $\Pr(R|RW)$ such that 4% of the voters, all of whom prefer W to both R and S, vote for S.

As mentioned in the Introduction, a common critique of the majority runoff system is that the median candidate may be "squeezed" by a left-wing and a right-wing candidate, and hence excluded from the second round (see, e.g., Van Der Straeten et al. 2013 and Solow 2013). Arguably, such squeezing happened during the 2007 Presidential election in France (Spoon 2008). Indeed, even if most polls indicated that Francois Bayrou (UDF, centrist) would have defeated any other candidate in the second round (i.e., he was the Condorcet winner), he did not make it to that round. This was because, in the first round, he was squeezed between Nicolas Sarkozy (UMP, rightist) and Ségolène Royal (PS, leftist). Our result that the Condorcet winner may not qualify for the second round is closely related to that issue. Indeed, the Condorcet winner must be a median candidate. Therefore, we show that such squeezing of the median candidate is possible in runoff elections when voters are strategic.[29]

4.3 *No other equilibria*

In the previous sections, we have identified two types of strictly perfect equilibria: Duverger's law equilibria and Duverger's hypothesis equilibria as described in Proposition 2. The following proposition establishes that these are the only two types of strictly perfect equilibria.

PROPOSITION 3. *There is no strictly perfect equilibrium other than those characterized in Propositions 1 and 2.*

See Appendix B for the proof.

Proposition 3 shows that runoff elections produce only two types of equilibria: Duverger's law equilibria and only one type of Duverger's hypothesis equilibrium, the one described in Proposition 2. The proof follows these lines. First, we show (Lemma 4 in Appendix A) that the most likely pivotal event is either a threshold pivotability between the front-runner and the runner-up or it is the second-rank pivotability RS/RW. In the first case, we have either a Duverger's law equilibrium or Duverger's hypothesis equilibrium as in Proposition 2. The second case cannot happen in equilibrium. To see this, notice that all voters prefer to vote for one of the two opponents. Indeed, conditional on

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[29]We can formally represent such a case in our framework. To do so, we need to consider candidates located along a one-dimensional policy space over which voters have single-peaked preferences. This directly implies that the electorate is composed of "only" four groups of voters: (i) those who prefer Royal to Bayrou to Sarkozy, (ii) those who prefer Bayrou to Royal to Sarkozy, (iii) those who prefer Bayrou to Sarkozy to Royal, and (iv) those who prefer Sarkozy to Bayrou to Royal. It is then easy to find an equilibrium in which Bayrou (the Condorcet winner) is squeezed between Royal and Sarkozy, and hence excluded from the second round.

being pivotal, voters choose whom of S and W will oppose R in the second round. A vote for R is irrelevant to that choice and thus useless. It follows that nobody votes for R, and thus that R cannot be the front-runner, contradicting the hypothesis that RS/RW is the most likely pivotal event.

Arguably, the most interesting implication of Proposition 3 is that *sincere voting* and *pushover tactics*, two types of voting behavior that are commonly believed to arise in (three-candidate) runoff elections (Duverger 1954, Cox 1997, Martinelli 2002), are not supported in equilibrium. There are two main differences between our analysis and previous studies that explain why such behaviors do not arise in our model: the richness of the preference structure and the focus on strictly perfect equilibria. Our results thus show implicitly that both sincere voting and pushover tactics are not robust phenomena in runoff elections and are, therefore, unlikely to be observed empirically.

To see why sincere voting is not an equilibrium, notice that in all equilibria the most likely pivotal event is the threshold pivotability R/Ri, where i is the runner-up. Thus, conditional on being pivotal, voters choose between an outright victory of R and a second round where R opposes the runner-up. But then all voters who prefer R to the runner-up prefer to vote for R. This includes all the voters whose most preferred candidate is the third candidate ($j \succeq R \succ i$, $j \neq R, i$) whose vote is, therefore, not sincere.

Perhaps surprisingly, we do not need strict perfection to exclude the existence of the sincere voting equilibrium.[30] The crucial ingredient for this result is that the set of types includes all orders of preferences.

Pushover is the incentive to vote for an unpopular candidate in the first round with the sole purpose of helping the front-runner win in the second round.[31] It works as follows. Suppose that a voter ranks candidate R higher than both S and W. He expects R to gain enough votes to reach the second round, but not enough to win outright in the first round. In his expectations, S and W will receive a much lower share than R, but the difference between the expected shares of S and W is small. For which candidate should our voter vote? A vote for his most preferred candidate, R, is of no use: it is very unlikely that such a vote will push R above the threshold of 50% (neither is it likely that a vote will be needed to ensure R's participation in the second round). On the other hand, a vote for either S or W is likely to change the composition of the second round. Since R has higher chances of winning a second round against the weak opponent, W, our R supporter prefers to vote for W to ensure a higher chance of his most preferred candidate winning the election.

For a supporter of R to push over and vote for W in equilibrium, one cannot have that a unique pivotal event is more likely than all others. Indeed, all R's supporter vote for R when a threshold pivotability (R/RS or R/RW) dominates, and vote for either S or W (making it impossible that R is the front-runner) if a second-rank pivotability (RS/RW) dominates. We thus need two pivotal events to dominate: an impossibility in

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[30]Let RS/RW and R/Ri, i being the runner-up, have equal magnitude. This condition requires an expected tie between the top two contenders. Thus, for generic distributions of preferences, the condition is met only if some voters vote sincerely with probability strictly less than 1.

[31]Pushover is intrinsically related to the "nonmonotonicity" of runoff systems, i.e., the fact that increasing the vote share of a candidate may decrease her probability of victory (Smith 1973).

any strictly perfect equilibrium (Lemma 1). Thus, pushover is not a robust phenomenon in runoff elections.

Importantly, we do not show that pushover equilibria that are *not* strictly perfect never exist. Actually, there are situations in which such equilibria might exist: when voters expect a tie between the top two contenders in the first round (see the example in Section 3.3). We show that no such equilibrium is strictly perfect. In a sense, this means that pushover is not robust to voters being uncertain about the expected outcome of the first round. Arguably, such a precise information is possible in small committees, but unlikely in large elections.

Though not supported in (any strictly perfect) equilibrium, pushover incentives do affect the voting behavior of voters. For instance, as explained above, there are situations in which the desire to qualify a weak opponent for the second round induces R supporters to behave nonsincerely.

Together, Propositions 1, 2, and 3 allow us to draw a general conclusion about the nature of the support in the two rounds. The front-runner always receives the support of all the voters who prefer her to the runner-up. An implication is that the vote share of the front-runner should not increase between the first and the second round if the distribution of voters remains unchanged between the two rounds. Thus, unless the front-runner wins outright in the first round, he is expected to lose in the second round. This is not an appealing feature of our model. Indeed, such a scenario seems to happen very frequently in real life elections. For instance, Bullock III and Johnson (1992) report empirical evidence on U.S. data according to which the election winner corresponds to the first-round winner approximately 70% of the times. However, it appears that this feature of our model is an artifact of two assumptions: (i) the probabilities of victory in the second round are exogenous, positive, and constant (i.e., independent of the size of the electorate), and (ii) all voters are strategic. In Technical Appendix D, we show that when the first assumption is relaxed, the model accommodates easily for changes in the vote share of the front-runner between the two rounds. In the next section, we show that a model including nonstrategic voters also does so.

5. Nonstrategic voters

The empirical literature on strategic voting (see, e.g., Kawai and Watanabe 2013, Spenkuch 2014 and references therein) shows that the electorate is composed of both strategic and nonstrategic voters. Nonstrategic voters vote for their most preferred candidate no matter what other voters do, whereas strategic voters maximize their expected utility, taking the behavior of the other voters into account. Models including only one type of voters are thus at odds with empirical findings. In this section, we discuss the robustness of our results to the presence of nonstrategic voters.

There is no reason to believe that voters of some types, i.e., with some given preferences, are more likely to be strategic than others. Therefore, we adopt a neutral position: we assume that each voter, no matter his type, is strategic with probability λ and nonstrategic with probability $(1 - \lambda)$. This implies that among the supporters of, say, R, a fraction $(1 - \lambda)$ vote for R no matter what they expect others to do.

The best response of strategic voters is not affected by the presence of nonstrategic voters, yet the presence of nonstrategic voters may affect the equilibrium properties of majority runoff elections. We illustrate this influence through numerical examples. First, with nonstrategic voters, the vote share of the front-runner can increase in the second round.[32] This is in stark contrast with the predictions of the model without nonstrategic voters. Given the empirical evidence on U.S. data according to which the election winner corresponds to the first-round winner approximately 70% of the times, this example suggests that a model including both strategic and nonstrategic voters outperforms a model including only strategic voters. Second, in the presence of nonstrategic voters, pushover can be supported in equilibrium, yet, we identify a necessary condition for the existence of a pushover equilibrium: it requires an unreasonably large fraction of nonstrategic voters in the electorate.

5.1 *Increase of the front-runner's vote share in the second round*

In the presence of nonstrategic voters in the electorate, we can prove the existence of a strictly perfect Duverger's hypothesis equilibrium in which the vote share of the front-runner increases in the second round. To compute the vote shares in the second round, we assume that voters are sequentially rational. Therefore, all voters vote for their most preferred participating candidate.

Suppose that $\lambda = 30\%$, i.e., 70% of the voters are expected to be nonstrategic. Suppose also the following expected distribution of preferences in the electorate:

Preferences	Expected share
$R \succ S \succ W$	0.21
$R \succ W \succ S$	0.2
$S \succ R \succ W$	0.11
$S \succ W \succ R$	0.05
$W \succ S \succ R$	0.33
$W \succ R \succ S$	0.1

Therefore, R is the Condorcet winner and S is the Condorcet loser. However, S is a stronger opponent of R than W (49% of the votes in the second round for S and 48% for W when opposed to R). In this case, we can prove the existence of a Duverger's hypothesis equilibrium as the one identified in Proposition 2. When strategic voters who prefer R to W vote for R and those who prefer W to R vote for their most preferred candidate, the expected vote shares are $\tau_R = 0.21 + 0.2 + 0.3 * 0.11 = 0.443$, $\tau_W = 0.33 + 0.1 = 0.43$, and $\tau_S = 0.7 * 0.11 + 0.05 = 0.127$. For these expected vote shares, the largest magnitude is $\mu(piv_{R/RW}) = -0.00652$ and the second largest magnitude is $\mu(piv_{RW/RS}) = -0.08962$.

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[32]We assume that all voters are sequentially rational. Therefore, in the second round, they all vote for their most preferred participating candidate. See Bouton (2013) for a formal analysis of voting behavior in the second round of a runoff election.

Hence, the postulated strategy is indeed a best response for all strategic voters (see Section 4.2).[33] The (expected) vote share of R in the second round is 52% if opposed to W and 51% if opposed to S. This is substantially higher than the 44.3% of the votes that R is expected to receive in the first round.

5.2 *Pushover*

We prove two results in this subsection. First, we show that a strictly perfect pushover equilibrium may exist. Second, we prove that the fraction of strategic voters must be sufficiently small for a pushover equilibrium to exist.

Suppose that $\lambda = 0.11\%$, i.e., 89% of the voters are expected to be nonstrategic. Suppose also the following expected distribution of preferences in the electorate:

Preferences	Expected share
$R \succ S \succ W$	0.13
$R \succ W \succ S$	0.28
$S \succ R \succ W$	0.145
$S \succ W \succ R$	0.155
$W \succ S \succ R$	0.23
$W \succ R \succ S$	0.06

In this example, sincere voting would lead (in expectation) to a second round where R opposes S and then to an expected victory of S in the second round. However, there is a pushover equilibrium in which S is expected to rank third. In that equilibrium, all strategic voters (even those who rank R first) vote for W if they prefer W to S and vote for S if they prefer S to W. Nonstrategic voters vote for their most preferred candidate. We therefore have $\tau_R = 0.89 * (0.13 + 0.28) = 0.3649$, $\tau_W = 0.23 + 0.06 + 0.11 * 0.28 = 0.3208$, and $\tau_S = 0.145 + 0.155 + 0.11 * 0.13 = 0.3143$. For these expected vote shares, the largest magnitude is $\mu(piv_{RW/RS}) = -3.32633 \times 10^{-5}$. Hence, the postulated strategy is indeed a best response for all strategic voters.[34] This example highlights how, by pushing over, strategic R supporters can influence the outcome of the election to their advantage.

We now identify a necessary condition on the fraction of nonstrategic voters for the pushover equilibrium to exist. To be the front-runner, a candidate must receive strictly more than $\frac{1}{3}$ of the votes. In a pushover equilibrium, we know that all strategic R supporters vote for W. Therefore, the fraction of nonstrategic R supporters must be strictly

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[33]Here we implicitly assume that all W's supporters have sufficiently strong preferences in favor of W that they prefer a second round where R opposes W instead of S even if S is a stronger opponent of R.

[34]Here we implicitly assume that all W supporters have sufficiently strong preferences in favor of W that they prefer a second round where R opposes W instead of S, even if W is expected to be defeated by R in the second round. This is not necessary for the existence of a pushover equilibrium. We also assume that all strategic R supporters prefer a second round of R vs. W rather than R vs. S. This is satisfied if, for instance, all R supporters have sufficiently intense preferences in favor of R and against S and/or W. Note that we could relax this assumption.
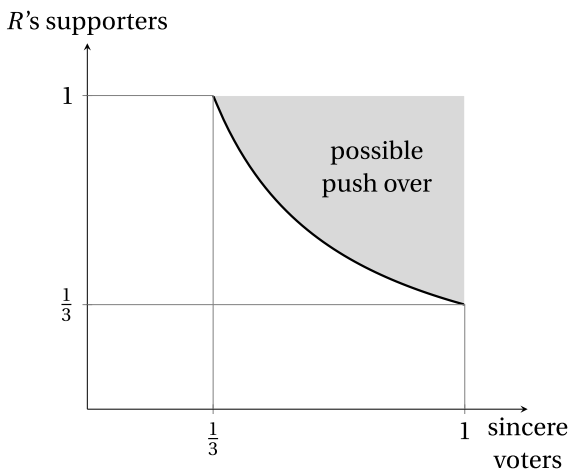
R's supporters

FIGURE 1. A necessary condition for pushover.

larger than $\frac{1}{3}$:

$$\underbrace{(1-\lambda)}_{\text{sincere voters}} \times \underbrace{(\gamma_{RS}+\gamma_{RW})}_{R\text{'s supporters}} > \tfrac{1}{3}.$$

Figure 1 shows the percentage of R's supporters necessary for a pushover equilibrium to exist as a function of the fraction of sincere voters. Ultimately, the existence of a pushover equilibrium is thus an empirical question. Considering the fraction of strategic voters found by Kawai and Watanabe (2013), i.e., between 63.4% and 84.9% of the electorate, our model predicts that a necessary (but far from sufficient) condition for a pushover equilibrium to exist is that at least 91.1% of the electorate prefer R to both S and W. Arguably, this is quite unlikely. The lower estimates in Spenkuch (2014), i.e., between 34.3% and 38.7%, imply the less demanding condition that at least 50.7% of the electorate prefer R to both S and W.[35]

Dolez and Laurent (2010) test directly for pushover behavior. They find that "the number of the 'ingenious' voters is zero, that is no respondent intended to desert temporarily his/her preferred party on the first round to favor it at the second" (p. 10) This supports our result that pushover is unlikely to arise in real-life majority runoff elections.

6. Conclusions

In this paper, we characterized the set of strictly perfect equilibria in three candidate runoff elections. In all equilibria, the front-runner receives the votes of all the voters who prefer her to the runner-up. An equilibrium where all the remaining voters coordinate on the runner-up candidate always exists. That is, as in plurality elections, there always

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[35]Spenkuch (forthcoming) sets a lower bound for the fraction of strategic voters at around 10%. Thus a necessary condition for a pushover equilibrium to exist is that at least 37% of the electorate prefer R to both S and W.

exists a Duverger's law equilibrium in which only two candidates receive a positive vote share. We also showed that there is, at most, one Duverger's hypothesis equilibrium in which all three candidates receive a positive fraction of the votes. The characteristics of that unique Duverger's hypothesis equilibrium challenge common beliefs about runoff elections: (i) some voters do not vote for their most preferred candidate (i.e., the sincere voting equilibrium does not exist), (ii) supporters of the front-runner do not vote for a less preferred candidate so as to "choose" who will face the front-runner in the second round (i.e., there is no pushover equilibrium), and (iii) the Condorcet winner might not qualify for the second round.

Our theoretical analysis delivers interesting testable predictions. First, the theory delivers a clear prediction about the set of equilibria: Duverger's law equilibria always exist, but a Duverger's hypothesis equilibrium does not always exist. One could easily test the emergence of different types of equilibria in runoff elections using controlled laboratory experiments. Second, the theory delivers interesting comparative statics about the behavior of one type of voter while playing the Duverger's hypothesis equilibrium. In particular, the behavior of the supporters of the weak runner-up varies with the intensity of their preferences and the probabilities of victory in the second round: the more intense supporters vote for their champion, whereas the others abandon her. Again, this is easily testable in the laboratory.

Our results also suggest a novel way to estimate the fraction of strategic voters in the electorate. As mentioned in Section 4.3, in a setup including only strategic voters, if the distribution of voters remains unchanged between the two rounds, the vote share of the front-runner should not increase between the two rounds. By contrast, we show in Section 5.1 that such an increase, which is frequent in practice, can be explained by the presence of nonstrategic voters. Together, these results suggest that one could use the increase in the vote share of the front-runner between the two rounds to estimate the fraction of strategic voters in the electorate. Let us consider the following simple example: an election in which (i) the vote share of the front-runner (R) is 44.3% in the first round and 52% in the second round, and (ii) the vote share of the runner-up (W) is 43% in the first round. We work under the assumptions that (i) each voter in the electorate is strategic with probability λ (as in Section 5) and (ii) the distribution of preferences remains unchanged between the two rounds. We notice that (i) the condition $\gamma_{SR} \leq 52\%$ must be satisfied, and (ii) the difference between the first- and second-round vote shares of candidate R must come from sincere voters who prefer S over R over W, i.e., $(1 - \lambda)\gamma_{SR} = 7.7\%$. Thus, we obtain a lower bound on the fraction of nonstrategic voters in the electorate of 14.81%. With information about the distribution of preferences in the electorate, one can get a tighter lower bound, an upper bound, or even a point estimate.

Appendix A: Large Poisson games in runoff elections

A Poisson game $\Gamma \equiv (n, \mathcal{T}, F, \mathcal{C}, u)$ is defined by the expected number of voters $n \in \mathbb{N}$, the set of types $\mathcal{T}$, a probability measure F defined over $\mathcal{T}$, a set of actions $\mathcal{C}$, and a vector of payoffs $u_t : \mathcal{C} \times Z(\mathcal{C}) \to \mathbb{R}$, each $t \in \mathcal{T}$, where $Z(\mathcal{C})$ is the set of all action profiles for the

players. The probability of the action profile x depends on τ, which itself depends on σ and F. In particular, this probability is

$$\Pr(x|\tau) = \prod_{i \in C} \left(\frac{\exp(-n\tau_i)(n\tau_i)^{x_i}}{x_i!} \right).$$

To lighten notation, we will omit the τ from the notation of the probability of any action profile or set of action profiles.

An event E is a set of action profiles that satisfy given constraints, i.e., a subset of $Z(\mathcal{C})$. As shown in Myerson (2000, Theorem 1), for a large population of size n, the probability of an event E is such that

$$\mu(E) \equiv \lim_{n \to \infty} \frac{\log[\Pr(E)]}{n} = \max_{x \in E} \sum_i \frac{x_i}{n} \left(1 - \log\left(\frac{x_i}{n\tau_i} \right) \right) - 1.$$

That is, the probability that event E occurs is exponentially decreasing in n; $\mu(E) \in [-1, 0]$ is called the magnitude of event E. Its absolute value represents the "speed" at which the probability decreases toward 0: the more negative is the magnitude, the faster the probability goes to 0.

Furthermore, Myerson (2000, Corollary 1) shows the following lemma.

LEMMA 2. *Compare two events E and E' with different magnitudes: $\mu(E) < \mu(E')$. Then the probability ratio of the former over the latter event goes to zero as n increases:*

$$\mu(E) < \mu(E') \quad \Rightarrow \quad \frac{\Pr(E)}{\Pr(E')} \underset{n \to \infty}{\to} 0.$$

The intuition is that the probabilities of different events do not converge toward zero at the same speed. Hence, unless two events have the same magnitude, their likelihood ratio converges either to zero or to infinity when the electorate grows large. Myerson calls this result the *magnitude theorem*. Proofs in this paper rely extensively on this property of large Poisson games.

We make use of the magnitude theorem to identify the properties of the set of strictly perfect equilibria as $n \to \infty$. As explained in Section 3, there are two types of pivotal events in a majority runoff election: the threshold pivotabilities and the second-rank pivotabilities. As proven in Bouton (2013), the magnitude of a pivotal event piv is larger when the expected outcome of the first round, τ, is close to the conditions necessary for event piv to occur. For instance, the pivotal event $piv_{i/ij}$ is more likely to occur when $\frac{1}{2} = \tau_i > \tau_j > \tau_k$ than when $\frac{1}{2} > \tau_k > \tau_j > \tau_i$. Indeed, the occurrence of the pivotal event in the latter case requires a "larger deviation with respect to the expected outcome."

LEMMA 3. *The magnitudes of the pivot probabilities are the following:*

(a) Threshold pivot probability i/ij and ij/i:

$$\mu(piv_{i/ij}) = \mu(piv_{ij/i}) = \begin{cases} 2\sqrt{(\tau_j + \tau_k)\tau_i} - 1 & \text{if } \frac{\tau_j}{\tau_j + \tau_k} \geq \frac{1}{2} \\ 2\sqrt{2\tau_i}\sqrt{\tau_j\tau_k} - 1 & \text{otherwise.} \end{cases}$$

(b) Second-rank pivot probability ki/kj and kj/ki:

$$\mu(piv_{ki/kj}) = \mu(piv_{kj/ki})$$

$$= \begin{cases} -(\sqrt{\tau_i} - \sqrt{\tau_j})^2 & \text{if } 2\sqrt{\tau_i\tau_j} > \tau_k > \sqrt{\tau_i\tau_j} \\ 2\sqrt{2\tau_k\sqrt{\tau_i\tau_j}} - 1 & \text{if } \tau_k > 2\sqrt{\tau_i\tau_j} \\ 3(\tau_i\tau_j\tau_k)^{1/3} - 1 & \text{if } \sqrt{\tau_i\tau_j} > \tau_k. \end{cases}$$

We are now in position to establish some preliminary results on the equilibrium be-havior of the magnitudes of different pivot probabilities. In particular, Lemma 4 says that the magnitude of $piv_{R/Ri}$—the event that a single vote is decisive between the front-runner winning outright and a second round between the front-runner and the runner-up—is never less than the magnitude of any other first-round pivot probability. Also, the magnitude of $piv_{RS/RW}$—the event that a vote is pivotal in determining which can-didate will face the front-runner in a second round—is never less than the magnitude of any other second-round pivot probability and is strictly larger unless the front-runner and the runner-up have the same expected share of votes.

LEMMA 4. *Let $i, j \neq R$ be two candidates such that $\tau_i \geq \tau_j$. There are three possible rank-ings of the two largest magnitudes:*

(i) $\mu(piv_{R/Ri}) \geq \mu(piv_{i/Ri}) \geq$ *any other magnitude*

(ii) $\mu(piv_{R/Ri}) \geq \mu(piv_{RS/RW}) \geq$ *any other magnitude*

(iii) $\mu(piv_{RS/RW}) > \mu(piv_{R/Ri}) \geq$ *any other magnitude.*

PROOF. We first compare $\mu(piv_{R/Ri})$ with other threshold magnitudes and show that this is the largest threshold magnitude,

$$\mu(piv_{R/Ri}) = 2\sqrt{(\tau_j + \tau_i)\tau_R} - 1 \geq 2\sqrt{(\tau_j + \tau_R)\tau_i} - 1 = \mu(piv_{i/Ri}),$$

and the expression holds with equality only if $\tau_j = 0$ or $\tau_i = \tau_R$. Also, trivially

$$\mu(piv_{R/Ri}) = 2\sqrt{(\tau_j + \tau_i)\tau_R} - 1 > 2\sqrt{(\tau_i + \tau_R)\tau_j} - 1 = \mu(piv_{j/Rj})$$

unless $\tau_R = \tau_i$ and

$$\mu(piv_{i/Ri}) = 2\sqrt{(\tau_j + \tau_R)\tau_i} - 1 > 2\sqrt{(\tau_i + \tau_R)\tau_j} - 1 = \mu(piv_{j/Rj})$$

unless $\tau_j = \tau_i$. We can also show that

$$\mu(piv_{R/Ri}) = 2\sqrt{(\tau_j + \tau_i)\tau_R} - 1 > 2\sqrt{2\tau_R\sqrt{\tau_i\tau_j}} - 1 = \mu(piv_{R/Rj})$$

unless $\tau_i = \tau_j$. Indeed, $2\sqrt{(\tau_j + \tau_i)\tau_R} - 1 > 2\sqrt{2\tau_R\sqrt{\tau_i\tau_j}} - 1 \Leftrightarrow \tau_R(\tau_i + \tau_j) - 2\tau_R\sqrt{\tau_i\tau_j} > 0$. The left-hand side of the last inequality can be rewritten as $\tau_R[\tau_j + \tau_i - 2\sqrt{\tau_i\tau_j}]$ and

$$\tau_R[\tau_j + \tau_i - 2\sqrt{\tau_i\tau_j}] = \tau_R(\sqrt{\tau_i} - \sqrt{\tau_j})^2 > 0$$

if $\tau_i > \tau_j$.

It remains to show that $\mu(piv_{R/Ri})$ is larger than $\mu(piv_{i/ij})$ and $\mu(piv_{j/ij})$. The first condition is satisfied if $\tau_R > \tau_i$ or $\tau_i > \tau_j$ since

$$\mu(piv_{R/Ri}) > \mu(piv_{i/ij})$$

$$2\sqrt{(\tau_j + \tau_i)\tau_R} - 1 > 2\sqrt{2\tau_i\sqrt{\tau_j\tau_R}} - 1$$

$$\frac{(\tau_j + \tau_i)}{2}\tau_R > \sqrt{\tau_j\tau_i}\sqrt{\tau_i\tau_R}.$$

Notice that the first element of the left-hand side is greater than or equal to (only if $\tau_R = \tau_i = \tau_j = \frac{1}{3}$) than the first element of the right-hand side since a geometric mean of $x, y, \ldots$ is always less than or equal to the arithmetic mean of $x, y, \ldots$, with the equality holding only if $x = y = \cdots$. Also, since $\tau_R \geq \tau_i$, the second element is also greater than or equal to the second element of the right-hand side. The last case, i.e., $\mu(piv_{R/Ri}) > \mu(piv_{j/ij})$, follows a very similar argument.

Furthermore,

$$\mu(piv_{i/Ri}) = 2\sqrt{(\tau_j + \tau_R)\tau_i} - 1 > 2\sqrt{2\tau_i\sqrt{\tau_j\tau_R}} - 1 = \mu(piv_{i/ij})$$

unless $\tau_j = \tau_R$; similarly, $\mu(piv_{i/Ri}) > \mu(piv_{j/ij})$.

We now compare $\mu(piv_{RS/RW})$ with other second-rank magnitudes and we show that this is the largest second-rank magnitude. First, notice that for $\tau_R \geq \tau_i \geq \tau_j$, $\tau_R \geq \sqrt{\tau_i\tau_j}$, $\tau_j \leq \sqrt{\tau_R\tau_i}$, and $\tau_i > 2\sqrt{\tau_R\tau_j} \Rightarrow \tau_R > 2\sqrt{\tau_i\tau_j}$. Hence, to prove that $\mu(piv_{RS/RW}) > \mu(piv_{jR/ji})$, it is sufficient to show two conditions:

(i) If $\tau_R < 2\sqrt{\tau_i\tau_j}$, then it is sufficient to show that $-(\sqrt{\tau_i} - \sqrt{\tau_j})^2 > 3(\tau_i\tau_j\tau_R)^{1/3} - 1$. The inequality can be rewritten as (using $\tau_R + \tau_i + \tau_j = 1$)

$$\tau_i + \tau_j - 2\sqrt{\tau_i\tau_j} < \tau_i + \tau_j + \tau_R - 3(\tau_i\tau_j\tau_R)^{1/3}$$

and, therefore, as

$$\frac{\tau_R + 2\sqrt{\tau_i\tau_j}}{3} > (\sqrt{\tau_i\tau_j}^2\tau_R)^{1/3}.$$

The right-hand side and the left-hand side are, respectively, the weighted geometric and arithmetic means of $\sqrt{\tau_i\tau_j}$ and τ_R with weights 2 and 1. It follows that they are equal if and only if $\tau_R = \tau_i = \tau_j = \frac{1}{3}$; otherwise, the inequality holds.

(ii) If $\tau_R \geq 2\sqrt{\tau_i\tau_j}$, then it is sufficient to show that $2\sqrt{2\tau_R\sqrt{\tau_i\tau_j}} - 1 \geq 3(\tau_R\tau_i\tau_j)^{1/3} - 1$. Taking logs and simplifying, we get

$$\frac{3}{2}\ln 2 - \ln 3 \geq \frac{\ln \tau_i\tau_j - 2\ln \tau_R}{12}$$

$$\ln\left[\left(\frac{\sqrt{2}^3}{3}\right)^{12}\right] \geq \ln\left(\frac{\tau_i\tau_j}{\tau_R^2}\right),$$

which simplifies to $\tau_R \geq (\frac{\sqrt{2}^3}{3})^6 \sqrt{\tau_i \tau_j}$. Notice that $(\frac{\sqrt{2}^3}{3})^6 \approx 0.7023 < 2$. Hence, since $\tau_R > 2\sqrt{\tau_i \tau_j}$, we have shown that $\mu(piv_{RS/RW}) > \mu(piv_{jR/ji})$.

To show that $\mu(piv_{RS/RW}) \geq \mu(piv_{iR/ij})$, we divide the analysis into three cases.

Case 1. If $\tau_i < \sqrt{\tau_R \tau_j}$, then $\mu(piv_{iR/ij}) = \mu(piv_{jR/ji})$ and we have just shown that $\mu(piv_{RS/RW}) \geq \mu(piv_{jR/ji})$ with equality holding only if $\tau_R = \tau_i = \tau_j = \frac{1}{3}$.

Case 2. If $\tau_i > 2\sqrt{\tau_R \tau_j}$, then $\mu(piv_{RS/RW}) \geq \mu(piv_{iR/ij})$ if and only if

$$2\sqrt{2\tau_R \sqrt{\tau_i \tau_j}} - 1 \geq 2\sqrt{2\tau_i \sqrt{\tau_R \tau_j}} - 1$$
$$\tau_R \sqrt{\tau_i \tau_j} \geq \tau_i \sqrt{\tau_R \tau_j}$$
$$\sqrt{\tau_R} \geq \sqrt{\tau_i}$$

which is trivially true for all $\tau_R \geq \tau_i$, with equality only if $\tau_R = \tau_i$.

Case 3. If $\sqrt{\tau_R \tau_j} < \tau_i < 2\sqrt{\tau_R \tau_j}$ and $\tau_R < 2\sqrt{\tau_i \tau_j}$, then $\mu(piv_{RS/RW}) \geq \mu(piv_{iR/ij})$ if and only if

$$-(\sqrt{\tau_i} - \sqrt{\tau_j})^2 \geq -(\sqrt{\tau_R} - \sqrt{\tau_j})^2$$
$$\sqrt{\tau_i} \leq \sqrt{\tau_R},$$

which is trivially true for all $\tau_R \geq \tau_i$, with equality only if $\tau_R = \tau_i$.

Last, we compare $\mu(piv_{R/Ri})$ and $\mu(piv_{i/Ri})$ with $\mu(piv_{RS/RW})$. Notice that if $\tau_R \geq 2\sqrt{\tau_i \tau_j}$, then $\mu(piv_{RS/RW}) = \mu(piv_{R/Rj}) \leq \mu(piv_{R/Ri})$ with the last inequality holding with strict sign unless $\tau_i = \tau_j$. Otherwise, if $\tau_R \leq 2\sqrt{\tau_i \tau_j}$, then there exist two regions of $\Delta(\mathcal{C})$ such that $\mu(piv_{R/Ri}) > \mu(piv_{RS/RW})$ and $\mu(piv_{R/Ri}) < \mu(piv_{RS/RW})$, respectively. Furthermore, if $\tau_R \geq 2\sqrt{\tau_i \tau_j}$, $\mu(piv_{i/Ri}) \geq \mu(piv_{RS/RW}) \Rightarrow$

$$2\sqrt{(\tau_j + \tau_R)\tau_i} - 1 \geq 2\sqrt{2\tau_R \sqrt{\tau_i \tau_j}} - 1$$
$$(\tau_j + \tau_R)\tau_i \geq 2\tau_R \sqrt{\tau_i \tau_j}.$$

Using $\tau_j + \tau_R = 1 - \tau_i$ and $\tau_R = 1 - \tau_i - \tau_j$, we rewrite the last inequality as

$$(1 - \tau_i)\tau_i \geq 2(1 - \tau_i - \tau_j)\sqrt{\tau_i \tau_j},$$

and we can easily notice that as $\tau_j \to 0$, the inequality holds with strict sign, while as $\tau_j \to \tau_i$, the opposite is true. Hence, there exist two regions of $\Delta(\mathcal{C})$ such that $\mu(piv_{i/Ri}) > \mu(piv_{RS/RW})$ and $\mu(piv_{i/Ri}) > \mu(piv_{RS/RW})$, respectively. Similarly, if $\tau_R < 2\sqrt{\tau_i \tau_j}$, there exist two regions of $\Delta(\mathcal{C})$ such that $\mu(piv_{i/Ri}) > \mu(piv_{RS/RW})$ and $\mu(piv_{i/Ri}) > \mu(piv_{RS/RW})$, respectively. □

APPENDIX B: PROOFS FOR SECTION 4

PROOF OF PROPOSITION 1. We want to show that for any $j \in \{W, S\}$, as n grow large, there exists an equilibrium where j receives a share zero of votes. Let i be the candidate who is neither R nor j. In equilibrium, all voters who strictly prefer R to i vote for R and all those who strictly prefer i to R vote for i. Note that the measure of voters who are indifferent between R and i is zero by assumption.

Note that under these voting strategies, $\mu(piv_{R/Ri}) = \mu(piv_{i/Ri}) >$ any other magnitude. Thus,

$$\lim_{n \to \infty} \frac{G_t(j, n\tau)}{\Pr(piv_{R/Ri})} = U(R, i|t) - U(R|t) + \phi[U(R, i|t) - U(i|t)]$$

$$= (1 + \phi)U(R, i|t) - U(R|t) - \phi U(i|t)$$

$$\lim_{n \to \infty} \frac{G_t(i, n\tau)}{\Pr(piv_{R/Ri})} = U(R, i|t) - U(R|t) + \phi[U(i|t) - U(R, i|t)]$$

$$= (1 - \phi)U(R, i|t) - U(R|t) + \phi U(i|t)$$

$$\lim_{n \to \infty} \frac{G_t(R, n\tau)}{\Pr(piv_{R/Ri})} = U(R|t) - U(R, i|t) + \phi[U(R, i|t) - U(i|t)]$$

$$= (\phi - 1)U(R, i|t) + U(R|t) - \phi U(i|t),$$

$$(2)$$

$$(3)$$

where

$$\phi \equiv \lim \frac{\Pr(piv_{S/Ri})}{\Pr(piv_{R/Ri})} > 0.$$

Recall that $U(R, i|t)$ is a strictly convex combination of $U(R|t)$ and $U(i|t)$. This implies that (i) $\lim_{n \to \infty} G_t(R, n\tau) > \lim_{n \to \infty} G_t(j, n\tau)$ if $U(R|t) > U(i|t)$ and (ii) $\lim_{n \to \infty} G_t(i, n\tau) > \lim_{n \to \infty} G_t(j, n\tau)$ if $U(i|t) > U(R|t)$. Thus, as n grows large, only voters who are indifferent between R and i prefer to vote for j. By inspection of (2) and (3), all voters who prefer R to i strictly prefer to vote for R and those who prefer i to R strictly prefer to vote for i.

To show that this equilibrium is strictly perfect, consider a small deviation from the share of votes expected in equilibrium. Then candidate j is expected to receive a share $\epsilon > 0$. Notice that all magnitude formulae are continuous. Indeed, for ϵ sufficiently small, the order of the magnitudes is $\mu(piv_{R/Ri}) > \mu(piv_{i/Ri}) >$ any other magnitude. Thus,

$$\lim_{n \to \infty} \frac{G_t(j, n\tau)}{\Pr(piv_{R/Ri})} = U(R, i|t) - U(R|t)$$

$$\lim_{n \to \infty} \frac{G_t(i, n\tau)}{\Pr(piv_{R/Ri})} = U(R, i|t) - U(R|t)$$

$$\lim_{n \to \infty} \frac{G_t(R, n\tau)}{\Pr(piv_{R/Ri})} = U(R|t) - U(R, i|t),$$

and

$$\lim_{n\to\infty} \frac{G_t(i, n\tau) - G_t(j, n\tau)}{\Pr(piv_{i/Ri})} = U(i|t) - U(R, i|t).$$

It follows that as n grows large, all voters who prefer R to i vote for R and all those who prefer i to R vote for i.   □

PROOF OF PROPOSITION 2. The starting point of the proof is to consider a tuple (σ^*, F) such that $\mu(piv_{R/Ri}) > \mu(piv_{RS/RW}) \geq \mu(piv_{R/Rj})$ and $\mu(piv_{RS/RW}) >$ any other magnitude (that is, i is the runner-up candidate). We then divide the rest of the proof is into three parts. First, we show that all voters with $U(R|t) > U(i|t)$ vote for candidate R. Second, we analyze the behavior of voters with $U(R|t) < U(i|t)$. We show that those with $U(R, S|t) > U(R, W|t)$ vote for S, whereas the others vote for W. Finally, we prove that the equilibrium is strictly perfect.

   Given the order of magnitudes above,

$$\lim_{n\to\infty} \frac{G_t(j, n\tau)}{\Pr(piv_{R/Ri})} = U(R, i|t) - U(R|t)$$

$$\lim_{n\to\infty} \frac{G_t(i, n\tau)}{\Pr(piv_{R/Ri})} = U(R, i|t) - U(R|t)$$

$$\lim_{n\to\infty} \frac{G_t(R, n\tau)}{\Pr(piv_{R/Ri})} = U(R|t) - U(R, i|t).$$

Thus, all voters with $U(R|t) > U(i|t)$ vote for candidate R.

   We divide the analysis of the behavior of voters with $U(R|t) < U(i|t)$ into two cases.

   Case 1: $\mu(piv_{RS/RW}) > \mu(piv_{R/Rj})$. Divide the expected gains by $\Pr(piv_{RS/RW})$:

$$\lim_{n\to\infty} \frac{G_t(S, n\tau) - G_t(W, n\tau)}{\Pr(piv_{RS/RW})} = U(R, S|t) - U(R, W|t) - U(R, W|t) + U(S, W|t)$$

$$= 2[U(R, S|t) - U(R, W|t)].$$

   Thus, a voter of type t (with $U(R|t) < U(i|t)$) votes for S only if

$$U(R, S|t) > U(R, W|t)$$

$$\Leftrightarrow \quad \Pr(R|RS)U(R|t) + (1 - \Pr(R|RS))U(S|t) \tag{4}$$

$$> \Pr(R|RW)U(R|t) + (1 - \Pr(R|RW))U(W|t).$$

Otherwise, she votes for W. Notice that the condition in (4) is independent of which candidate, S or W, is expected to receive more votes. Furthermore, since $\Pr(R|RS) < \Pr(R|RW)$, a type t who is indifferent between S and W must be a W supporter.

Case 2: $\mu(piv_{RS/RW}) = \mu(piv_{R/Rj})$. Divide the expected gains by $\Pr(piv_{RS/RW})$:

$$\lim_{n\to\infty} \frac{G_t(S, n\tau) - G_t(W, n\tau)}{\Pr(piv_{RS/RW})} = U(R, S|t) - U(R, W|t) - U(R, W|t) + U(S, W|t)$$

$$+ \phi'[U(R, j|t) - U(R|t) - U(R, j|t) + U(R|t)]$$

$$= 2[U(R, S|t) - U(R, W|t)]$$

with

$$\phi' = \frac{\Pr(piv_{R/Rj})}{\Pr(piv_{RS/RW})} > 0.$$

Hence, the condition for voting for S or W is not changed (see Case 1).

To show that this equilibrium is strictly perfect, consider any $\tau \in \Delta(\mathcal{C})$: $|\tau - \tau(\sigma^*, F)| < \epsilon$ for some $\epsilon > 0$ and $\tau_j(\sigma^*, F)$: $\mu(piv_{R/Ri}) > \mu(piv_{RS/RW}) > \mu(piv_{R/Rj})$. Notice that all magnitude formulae are continuous in $\Delta(\mathcal{C})$. For ϵ sufficiently small, the order of the magnitudes is unchanged, in which case, σ^* is a best response. □

PROOF OF PROPOSITION 3. Propositions 1 and 2 characterize the set of equilibria when the order of magnitudes is as in points (i) and (ii) in Lemma 4. Together, Lemmata 1 and 4 imply that no other strictly perfect equilibrium σ^* can exist unless $\tau(\sigma^*, F)$ implies point (iii) in Lemma 4, i.e., when $\mu(piv_{RS/RW})$ is the (strictly) largest magnitude. Notice that this implies $\tau_R(\sigma^*, F) > 0$ (by definition, R is the front-runner).

When $\mu(piv_{RS/RW})$ is the (strictly) largest magnitude,

$$\lim_{n\to\infty} \frac{G_t(S, n\tau)}{\Pr(piv_{RS/RW})} = U(R, S|t) - U(R, W|t)$$

$$\lim_{n\to\infty} \frac{G_t(W, n\tau)}{\Pr(piv_{RS/RW})} = U(R, W|t) - U(R, S|t)$$

$$\lim_{n\to\infty} \frac{G_t(R, n\tau)}{\Pr(piv_{RS/RW})} = 0.$$

Hence, as $n \to \infty$, voting for R is a best response for a measure zero of voter types, those with $U(R, S|t) = U(R, W|t)$. Hence, $\tau_R(\sigma^*, F) = 0$, contradicting the assumption that $\mu(piv_{RS/RW})$ is the largest magnitude. □

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