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Social activity and network formation

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This paper develops a simple model in which a social hierarchy emerges endogenously when agents form a network for complementary interaction ("activity"). Specifically, we assume that agents are ex ante identical and their best response activity, as well as their value function, increases (strictly) concavely in the total activity of their neighbors in the network. There exists a unique and stable positive activity equilibrium on exogenous networks under mild conditions. When we endogenize network formation, equilibria become strongly structured: more active players have more neighbors, i.e., a higher degree, but tend to sponsor fewer links. Additionally, in strict equilibria, agents separate themselves into groups characterized by the symmetric activity of their members. The characteristic activity decreases in group size and the network is a complete multipartite graph.

KEYWORDS. Network formation, strategic complementarity, supermodularity, peer effects, social hierarchy.

JEL CLASSIFICATION. C72, D00, D85.

1. INTRODUCTION

"Hierarchy [...] appears to be one of the most fundamental features of social relations. Leaders of groups naturally emerge from interactions..." (Magee and Galinsky 2008, p. 352).

Optimal actions of individuals embedded in a network often depend on the actions of their direct neighbors or friends. In recent years, mutually reinforcing actions—that is, strategic complements—have received particular attention in the applied game theoretic literature. However, many papers following the seminal work of Ballester et al. (2006) maintain two assumptions: first, optimal actions increase *linearly* in the action of neighbors; second, the network of social interactions is fixed and *exogenously* given. This paper varies both assumptions: we study agents whose optimal action increases *concavely*; they are allowed to form costly links and thus create the network of social interactions *endogenously*. As a result, equilibrium networks become strongly structured;

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a *social hierarchy* emerges through the interaction of homogenous agents in a simple one-shot game.

Before we describe the model and our results in more detail, let us briefly consider some examples of strategic complementarity in networks. First, strategic complementarity has been documented in criminal networks, where delinquents learn how to become a criminal—that is, to identify opportunities and adopt behaviors—through their social ties.[1] Similarly, there is strong evidence for strategic complementarity in the learning effort of adolescent friends (see, e.g., Calvó-Armengol et al. 2009). In R&D networks, strategic complementarity may arise when the investment of firms not only generates innovation, but also develops the firm's ability to identify, assimilate, and exploit external knowledge (Cohen and Levinthal 1989, 1990). On a very different note, casual observation indicates that user activity of friends in online social networks often displays complementarity.[2] Last and on a broader perspective, strategic complementarity arises in any network in which an agent's incentive to perform economic activity increases in the activity of his neighbors. For instance, this can be the case in networks of bilateral collaborations in which a generalized investment benefits all partners, networks of firms selling technologically complementary goods, supply chain networks, and trading networks.

In this paper, we put forward a simple model that may be applied to the situations above. Agents are ex ante identical—except possibly for their position in the network—and simultaneously choose a one-dimensional action (their "activity"). In particular, their *best response function*, as well as their *value function*, increases (strictly) concavely in the total activity of their neighbors in the network.

We begin our analysis with a fixed and connected interaction network. Provided a condition on the slope of the best response function holds, there exists a unique and stable equilibrium with strictly positive activity. In particular, adding a link to the network strictly increases every player's equilibrium activity—and thus everyone's payoff—due to strategic complementarity.

Thereafter, we endogenize network formation. Each agent can now form links at a fixed marginal cost simultaneously with his activity choice. As a result, equilibria become strongly structured: more active players have more neighbors (i.e., a higher degree) but tend to sponsor fewer links.[3] In comparison to interactions on exogenous networks as in Ballester et al. (2006), a much simpler measure of network centrality thus suffices to rank individual activity.[4]

[1] For empirical evidence, see Case and Katz (1991), Glaeser et al. (1996), Ludwig et al. (2001), Bayer et al. (2009), Damm and Dustmann (2014). For further theoretical work directly tailored to delinquent networks, see Ballester et al. (2010), Calvó-Armengol and Zenou (2004), Patacchini and Zenou (2008).

[2] As a starting point, Ugander et al. (2011) provide cross-sectional evidence for a positive correlation between the site engagement of Facebook friends.

[3] More precisely, the marginal value of any given link is smaller for the more active adjacent player. Thus there always exists a corresponding equilibrium in which all links are sponsored by the less active adjacent players and that is more robust to changes in the linking costs. See Section 4.2 for a detailed discussion.

[4] Studying linear best responses, Ballester et al. (2006) show that each player's equilibrium activity is proportional to his Bonacich centrality—the total number of direct and indirect paths in the network originating from him.

A *social hierarchy* emerges endogenously in equilibrium and our model replicates several key stylized facts of real-world hierarchies.[5] The resemblance becomes especially clear in strict equilibria, which are robust to the introduction of small heterogeneity in the players' linking costs and their best response functions:

(a) Agents separate themselves into groups characterized by the symmetric activity of their members. Specifically, all groups with more than one member have a different size and the characteristic activity decreases in group size. Similarly, many real-world social hierarchies have "a stratified structure, a pyramid shape with fewer people at the top than at the bottom" (Magee and Galinsky 2008, p. 352).

(b) Agents try to "climb up the social ladder": each player forms links to all more active players, receives links from all less active ones, and has no connection to others with similar activity (unless possibly when he is one of the most active players).

(c) Highly active agents choose high activity *because* many agents link to them. But many agents link to them *because* they choose high activity. Real-world hierarchies have a similar self-enforcing nature: "Social interaction can shape individuals' behaviour in a hierarchy-reinforcing manner by guiding behaviour so that it conforms to and becomes consistent with status-based expectations. Once expectations are formed, people often treat targets in an expectancy-consistent manner and, as a result, elicit expectancy-consistent responses from these targets, leading to the unwitting fulfillment of those expectations" (Magee and Galinsky 2008, p. 373).

We now discuss how the paper fits into the literature. Recently, games with *linear best responses* on *exogenous networks* have received considerable attention. The influential paper of Ballester et al. (2006) studies local strategic complements and global substitutes for small network effects. Other important contributions include Bramoullé and Kranton (2007), Belhaj et al. (2012), Bramoullé et al. (2014). We extend this line of research to *increasing concave best responses* but (mainly) focus on *endogenous network formation*.

The joint analysis of action choices and endogenous network formation in *one-shot simultaneous-move games* was initiated by Galeotti and Goyal (2010), who study local public goods games (i.e., *strategic substitutes*). Our paper essentially adapts Galeotti and Goyal (2010) to strategic complements. The star network constitutes the most prominent robust equilibrium under strategic substitutes (Propositions 2 and 3 in Galeotti and Goyal 2010) and is also a possible outcome under strategic complements.[6]

[5] A social hierarchy is commonly defined as "an implicit or explicit rank order of individuals or groups with respect to a valued social dimension" (Magee and Galinsky 2008, p. 354). In our model, individual equilibrium activity constitutes this valued social dimension.

[6] In other words, a similar outcome may emerge in equilibrium under opposite interactions. In both cases, ex ante identical agents separate themselves into two different groups, namely a central node (the "center") and $n-1$ peripheral nodes ("spokes"). Under *strategic substitutes*, the spokes free ride on the center and choose zero investment; the center chooses the optimal investment otherwise made by an isolated player. In contrast, under *strategic complements*, spokes choose positive activity and the center benefits from every additional spoke that is included.

Zhang et al. (2011) extend this line of research to *imperfect substitutes* and Hiller (2013), independently, extends it also to *strategic complements*. The latter differs from our paper by assuming a *convex value function*, which leads to a type of "bang–bang" solution: the unique equilibrium network is complete for low linking costs, while it is empty for high linking costs. For costs in between, any other possible equilibrium network has a core–periphery architecture. In our model, much richer social structures and activity patterns exist.[7]

Finally, there is a related literature studying action choices and network formation in *dynamic games* with *myopic agents*. In contrast to the setting here, the complementarity effect from any two neighbors is *reinforcing* in Lagerås and Seim (2012).[8] Furthermore, the network structure converges to so-called nested split graphs, whereas many equilibrium networks are non-nested graphs in our model.[9] König et al. (2014) analyze a related dynamic model and test it empirically. Several other papers consider learning dynamics in coordination games with a *finite* action space (see, e.g., Jackson and Watts 2002, Goyal and Vega-Redondo 2005). These papers pursue a different objective from ours: the authors try to determine whether (under different timings of action and link revisions) the risk-dominant or the efficient action prevails, and are less interested in the equilibrium network.

The paper proceeds as follows: In the next section, we introduce the model and our solution concept. In Section 3, we study the pure activity game on exogenous networks. Section 4 considers the extended game with endogenous network formation and is split into three subsections: first, we provide five simple equilibrium conditions that are individually necessary and jointly sufficient. Thereafter, we determine general properties that hold in any equilibrium and discuss the connection between an agent's activity and his position in the network. Finally, we apply an equilibrium refinement and study strict equilibria. Section 5 discusses the following additional important issues: linear best responses and network formation; welfare considerations; a two-stage game; two-sided link formation. The last section concludes. All proofs are presented in the Appendix.

2. Model

Let $N = \{1, 2, \dots, n\}$ be the set of agents with $n \geq 3$. All agents $i \in N$ choose a level of (social) activity $x_i \in X = [0, \infty)$ simultaneously and the vector $\mathbf{x} = (x_1, x_2, \dots, x_n)$ collects their choices. The social interaction structure is either exogenously given or is formed endogenously and simultaneously with the activity choices.

Social interaction structure

(a) *Social interactions.* Social interactions are bilateral. The corresponding interaction network is represented by an undirected graph $\bar{g} \in \bar{G}$, where $\bar{G}$ is the set of

[7] In fact, optimal deviations are quite different in both models, which has a significant effect on equilibrium analysis (see footnote 16). For further comparisons of both models, see also Section 5.

[8] That is, the cross-partial derivative of the best response function is weakly positive in their model, whereas it is weakly negative in our model. For instance, Assumption 4 from Lagerås and Seim (2012) is satisfied neither in our Example 1 nor Example 2.

[9] In nested split graphs, the neighborhood of any lower degree player is contained in the neighborhood of all higher degree players.

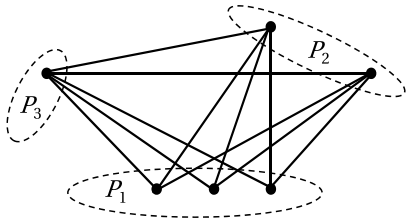

FIGURE 1. A complete three-partite graph.

symmetric Boolean $n \times n$ matrices with zeros on the main diagonal.[10] Let $\bar{g}_{ij}$ denote the (i, j)th entry of $\bar{g}$; players i and j interact if and only if they share a link in network $\bar{g}$, that is, $\bar{g}_{ij} = \bar{g}_{ji} = 1$. The operation $\bar{g} \oplus ij$ adds a link to network $\bar{g}$, i.e., the (i, j)th and (j, i)th entry of $\bar{g}$ are set to 1, keeping the remaining network fixed.

The set $N_{i,\bar{g}} = \{j \in N : \bar{g}_{ij} = 1\}$ defines player i's *neighborhood* in $\bar{g}$ and its cardinality $n_{i,\bar{g}} = |N_{i,\bar{g}}|$ is called i's degree. Two players i and j are *connected* if either $\bar{g}_{ij} = 1$ or there are players $i_1, \ldots, i_l$ with $\bar{g}_{ii_1} = \bar{g}_{i_1 i_2} = \cdots = \bar{g}_{i_l j} = 1$. Network $\bar{g}$ is *connected* if every pair of players is connected. An *independent set* is a non-empty subset of players who do not share any direct links among themselves.

In a *complete multipartite graph*, the set of agents N can be partitioned into an arbitrary number of independent sets (so-called *partite sets* or *parts*) such that every agent shares a link with all agents outside of his own part (see, for instance, Figure 1). There are several special cases: an *empty network* without any links is a complete one-partite graph. A *complete* network—in which every pair of nodes shares a link—is a complete n-partite graph. A *complete biregular bipartite graph* consists of two different-sized partite sets and is called a *star* if one of these sets contains a single player, the *center*, and the other one contains $n - 1$ *spokes*. Finally, a *complete core–periphery graph* consists of an arbitrary number of singleton partite sets, the *core players*, and a single partite set that contains the remaining *periphery players*.

(b) *Link sponsorship.* If the interaction network is formed endogenously, link sponsorship is represented by a directed graph $g \in G$, where G denotes the set of Boolean $n \times n$ matrices with zeros on the main diagonal. Player i sponsors, forms, or supports a link to player j if and only if entry $g_{ij} = 1$. Most importantly, we assume one-sided link formation. That is, g induces the interaction network $\bar{g}$ with $\bar{g}_{ij} = \max\{g_{ij}, g_{ji}\}$.

Let $\eta_{i,g} = |\{j \in N : g_{ij} = 1\}|$ count i's self-sponsored links in g. Sponsorship vector g_i, the ith row of g, summarizes i's link sponsorship and $\mathfrak{g}_i$ denotes the set of i's possible sponsorship vectors. The operation

$$g_{-i} \odot g'_i \equiv (g^t_1, \ldots, g^t_{i-1}, g'^t_i, g^t_{i+1}, \ldots, g^t_n)^t$$

replaces i's sponsorship vector in g by $g'_i \in \mathfrak{g}_i$.

[10] For simplicity, we identify the network or graph with its adjacency matrix.

Payoffs and modeling assumptions

Let $\pi_i(\mathbf{x}, \bar{g})$ denote player i's gross payoff (i.e., without linking costs) under activity $\mathbf{x}$ on interaction network $\bar{g}$. If the interaction network is formed endogenously, players additionally incur a fixed cost $k > 0$ for each self-sponsored link and net payoffs read as

$$\Pi_i(\mathbf{x}, g) = \pi_i(\mathbf{x}, \bar{g}) - \eta_{i,g}k, \quad \forall i \in N. \tag{1}$$

Our model differs from the existing literature by combining two crucial assumptions. First, personal activity and the total activity of neighbors are *strategic complements* such that a player desires to be active if one of his neighbors is active and then increases his activity at a diminishing rate.[11]

Assumption 1 (Best response function). *Player i's unique best response to activity $\mathbf{x}_{-i}$ on interaction network $\bar{g}$ satisfies*

$$x_i^*(\mathbf{x}_{-i}, \bar{g}) = f\Bigg(\sum_{j \in N_{i,\bar{g}}} x_j\Bigg), \quad \forall i \in N,$$

where $f(0) = 0$, $f'(0) > 1$, $0 \leq \lim_{x\to\infty} f'(x) < 1/(n-1)$, and $f'' < 0$.

Second, the benefits from social activity are limited. Specifically, agents who optimize their personal activity receive *diminishing marginal (indirect) utility* from total neighbor activity.[12]

Assumption 2 (Value function). *Player i's maximized gross payoff under activity $\mathbf{x}_{-i}$ on interaction network $\bar{g}$ satisfies*

$$\pi_i^*(x_i^*, \mathbf{x}_{-i}, \bar{g}) = h\Bigg(\sum_{j \in N_{i,\bar{g}}} x_j\Bigg), \quad \forall i \in N,$$

where $h(0) = 0$, $h' > 0$ and $h'' < 0$.

Below, we introduce two specific additively separable gross payoff functions that satisfy Assumptions 1 and 2. To fix ideas for the first of these payoff functions, assume that the interaction network $\bar{g}$ represents a network of bilateral collaborations and activity x_i is a generalized investment that benefits all partners. Specifically, there are quadratic costs for personal investment $\frac{1}{2}cx_i^2$ and benefits from investment are proportional to a concave function $v(\cdot)$ of total partner investment.

[11] The assumptions on f' ensure that a positive activity equilibrium exists; see Section 3. In particular, *sufficient concavity* bounds total activity and prevents an infinite solution. Under (increasing) *linear* best responses, a restriction on the slope of the best response function (Ballester et al. 2006) or a bounded action space (Belhaj et al. 2012) are usually imposed for the same reason.

[12] The "opposite" case of a convex value function is analyzed by Hiller (2013).

EXAMPLE 1. We have

$$\pi_i(\mathbf{x}, \bar{g}) = v\Big(\sum_{j\in N_{i,\bar{g}}} x_j\Big)x_i - \tfrac{1}{2}cx_i^2,$$

where $c > 0$, $v(0) = 0$, $v'(0) > c$, $0 \le \lim_{x\to\infty} v'(x) < c/(n-1)$, and v^2 is strictly concave. ◊

The payoff function from Example 1 implies the best response and value function

$$x_i^* = \frac{1}{c}v\Big(\sum_{j\in N_{i,\bar{g}}} x_j\Big) \quad \text{and} \quad \pi_i^* = \frac{1}{2c}v\Big(\sum_{j\in N_{i,\bar{g}}} x_j\Big)^2.$$

In Section 4.3, we employ a different example, the "baseline model," to obtain additional insights. Here, the gross payoff is additively separable into a Cobb–Douglas utility/production function with diminishing returns to scale and linear activity costs.

EXAMPLE 2 (The baseline model). We have

$$\pi_i(\mathbf{x}, \bar{g}) = 2\Big(\sum_{j\in N_{i,\bar{g}}} x_j\Big)^{q/2} x_i^{1/2} - cx_i,$$

where $q \in (0, 1)$ and $c > 0$. ◊

The payoff function of the baseline model implies the best response and value function

$$x_i^* = \frac{1}{c^2}\Big(\sum_{j\in N_{i,\bar{g}}} x_j\Big)^q \quad \text{and} \quad \pi_i^* = \frac{1}{c}\Big(\sum_{j\in N_{i,\bar{g}}} x_j\Big)^q.$$

Equilibrium conditions

(a) *Exogenous networks.* In Section 3, we assume that a (connected) interaction network $\bar{g}$ is exogenously given and known to all players. A player's strategy is merely his level of activity $x_i \ge 0$, which is chosen simultaneously by all players. Thus an activity vector $\mathbf{x}^*$ constitutes a (strict) Nash equilibrium if and only if it solves the best response functions of all players simultaneously:

$$x_i^* = f\Big(\sum_{j\in N_{i,\bar{g}}} x_j^*\Big), \quad \forall i \in N. \tag{2}$$

In particular, every player chooses strictly positive activity in a *positive activity equilibrium* $\mathbf{x}^* > \mathbf{0}$.

(b) *Endogenous network formation.* In Section 4, the network is formed endogenously: all players simultaneously choose their activity $x_i \ge 0$ and a vector of self-sponsored links $g_i \in \mathfrak{g}_i$. Thus each player's strategy is described by a pair (x_i, g_i)

and their net payoff is given by (1). A pair $(\mathbf{x}^*, g^*)$ constitutes a Nash equilibrium if and only if no player gains from choosing some alternative strategy (x_i', g_i'), that is,

$$\Pi_i(x_i^*, \mathbf{x}_{-i}^*, g^*) \geq \Pi_i(x_i', \mathbf{x}_{-i}^*, g_{-i}^* \odot g_i'), \quad \forall i \in N, x_i' \geq 0, g_i' \in \mathfrak{g}_i. \tag{3}$$

Throughout the paper, we apply two equilibrium refinements. *Generic* equilibria are robust to small changes of linking costs.[13] Moreover, in *strict* equilibria, the condition in (3) holds as a strict inequality for all players.

3. Exogenous networks

This section studies the pure activity game on exogenous (connected) networks, which is closely related to supermodular games. In the latter games, π_i has to be supermodular in $(x_i, \mathbf{x}_{-i})$, i.e.,

$$\frac{\partial^2 \pi_i}{\partial x_i \, \partial x_j} \geq 0, \quad \text{for all } j \neq i \in N,$$

and X is a closed interval in $\mathbb{R}$. Indeed, the pure activity game on exogenous networks has the following related property.

Remark 1. The function π_i has positive cross-partial derivatives at best response activity by Assumption 1, i.e.,

$$\left.\frac{\partial^2 \pi_i}{\partial x_i \, \partial x_j}\right|_{x_i = x_i^*(\mathbf{x}_{-i}, \bar{g})} \geq 0, \quad \text{for all } j \neq i \in N,$$

and $X = [0, \infty)$ is a left-closed interval in $\mathbb{R}$.

As a consequence, we can adapt concepts from the literature on supermodular games by restricting the strategy space conveniently.[14] An equilibrium with strictly positive activity exists due to the shape of the best response function (Assumption 1): (i) a sufficiently steep incline at the origin prevents ever-decreasing activity while (ii) a sufficiently small slope at high levels of neighbor activity prevents ever-increasing play. Furthermore, strict concavity makes the positive activity equilibrium unique and stable. It also bounds individual equilibrium activity below total neighbor activity. Finally, adding a link to the network increases every player's equilibrium activity (and thus everyone's gross payoff) due to strategic complementarity and connectedness. In summary, the following results hold on any *connected* network $\bar{g}$.[15]

[13] More precisely, we say that a pair $(\mathbf{x}^*, g^*)$ constitutes a *generic* equilibrium for linking costs k if and only if $(\mathbf{x}^*, g^*)$ constitutes an equilibrium for all $k' \in (k - \epsilon, k + \epsilon)$ for some $\epsilon > 0$.

[14] The main results were developed in the classic papers Topkis (1979), Vives (1990), and Milgrom and Roberts (1990). For a more recent overview, see Vives (2005).

[15] In a disconnected network, the proposition holds true for any component of at least three players. For a connected pair of players, Proposition 1(ii) is not valid as $x_i^* = x_j^* > 0$, whereas an isolated player optimally chooses zero activity.

PROPOSITION 1. *A unique positive activity equilibrium* $\mathbf{x}^* > \mathbf{0}$ *exists and*

(i) $\mathbf{x}^*$ *is globally asymptotically stable for* $\mathbf{x} > \mathbf{0}$ *under best response dynamics*

(ii) $x_i^* < \sum_{j \in N_i} x_j^*$

(iii) $\mathbf{x}^*_{\bar{g} \oplus jj'} > \mathbf{x}^*_{\bar{g}}$ *and* $\pi_i^*(\bar{g} \oplus jj') > \pi_i^*(\bar{g})$, $\forall i \in N$ *if* $\bar{g}_{jj'} = 0$.

4. ENDOGENOUS NETWORK FORMATION

This section treats the extended game with endogenous network formation, in which players simultaneously choose a level of activity and a vector of self-sponsored links. As a result, a Nash equilibrium now consists of a pair $(\mathbf{x}^*, g^*)$.

We split the subsequent analysis into three subsections: first, we use Assumptions 1 and 2 to find five simple equilibrium conditions that are individually necessary and jointly sufficient. On this basis, we then derive general properties that hold in any equilibrium and show a strong connection between a player's activity, degree, and total linking costs. To obtain further insights, we finally apply an equilibrium refinement and study how players separate themselves into groups of symmetric activity in strict equilibria.

4.1 *Five simple equilibrium conditions*

Recall that (3) provides a (general) equilibrium condition for the extended game with endogenous networks formation. We build intuition for the simplified version in Lemma 1 below through three observations. First, each player's *equilibrium* activity has to be optimal given the activity of his neighbors in the induced interaction network $\bar{g}^*$, which leads to condition (i) of the lemma. Assume for the moment that a *candidate equilibrium* meets this condition and that no link is sponsored by both adjacent players (condition (ii)), which would be wasteful.

The second observation examines an "optimized" deviation from link sponsorship in the candidate equilibrium. Consider a deviating player who alters his total neighbor activity. As he has a unique best response activity $x_i^*(\mathbf{x}_{-i}, \bar{g})$, his "optimized" deviation from link sponsorship involves simultaneous activity adjustment to his new neighbor activity. Consequently, only his *value* from any induced post-deviation network $\bar{g}'$ needs to be considered.

Finally, we exploit the concavity of the value function and the linearity of linking costs. If an arbitrary "optimized" deviation from link sponsorship is profitable, then so is one of the following simple (activity-adjusted) deviations: either to add a *single* link to the most active non-neighbor (tested by condition (iii)), or to delete the *single* least valuable self-sponsored link (tested by condition (iv)), or to do both simultaneously (tested by condition (v)).[16] Thus we arrive at five simple equilibrium conditions:

[16]This result nicely illustrates a core difference to Hiller (2013). Assume a player desires to deviate from a candidate equilibrium with a complete network. Under a *concave* value function, the player maximizes his average deviation benefit by deleting a *single* self-sponsored link. Conversely, under a *convex* value function, he gains his highest average deviation benefit by deleting *all* self-sponsored links.

LEMMA 1. *A pair* $(\mathbf{x}^*, g^*)$ *constitutes a Nash equilibrium under endogenous network formation for linking costs* $k > 0$ *if and only if for all* $i, i' \in N$,

(i) $x_i^* = f(\sum_{j \in N_{i,\bar{g}^*}} x_j^*)$

(ii) $g_{ii'} + g_{i'i} \in \{0, 1\}$

(iii) $h(\sum_{j \in N_{i,\bar{g}^*}} x_j^* + \bar{x}_i) - h(\sum_{j \in N_{i,\bar{g}^*}} x_j^*) \leq k$

(iv) $h(\sum_{j \in N_{i,\bar{g}^*}} x_j^*) - h(\sum_{j \in N_{i,\bar{g}^*}} x_j^* - \underline{x}_i) \geq k$

(v) $\underline{x}_i \geq \bar{x}_i$,

where $\bar{x}_i = \max_{j:\, \bar{g}_{ij}^* = 0}\{x_j^*, 0\}$, $\underline{x}_i = \min_{j:\, g_{ij}^* = 1}\{x_j^*, \infty\}$, *and* $h(-\infty) = -\infty$.

4.2 *General properties of Nash equilibria*

It is one of the main objectives of this section to investigate structural properties of equilibria that hold independently from the direction of link sponsorship—that is, finding the set of equilibrium interaction networks and the corresponding activity levels. This exercise would be greatly simplified by making a particular assumption on the direction of link sponsorship—namely that each link is sponsored by the less active adjacent player ("upward linking"). The next paragraph discusses the context in which this assumption is valid.

For certain specifications of the model, there exist equilibria *without* upward linking. However, any link has a higher marginal value for the less active adjacent player. Consequently, any equilibrium without upward linking corresponds to one *with* upward linking. The corresponding equilibrium with upward linking has the same structure (interaction network and activity levels) but is sustainable under a (weakly) greater interval of linking costs. Lemma 2 below summarizes this finding. As a result, we may assume without loss of generality that upward linking holds so as to derive structural properties of equilibria—as long as we are not concerned with the direction of link sponsorship.

LEMMA 2. *Let* $(\mathbf{x}^*, g^*)$ *be an equilibrium and consider* g' *with (i)* $g'_{ij} = 1$ *and* $g'_{ji} = 0$ *if* $\bar{g}_{ij}^* = 1$ *and* $x_i^* < x_j^*$, *and (ii)* $g'_{ij} = g_{ij}^*$ *otherwise. Then* $(\mathbf{x}^*, g')$ *is an equilibrium for a weakly greater interval of linking costs* k *and* $\bar{g}' = \bar{g}^*$.

Assuming upward linking, we can show a strong connection between an agent's activity and his position in the network. Since the value function increases in total neighbor activity, more active players are more "popular" friends. As a result, they have more neighbors in any equilibrium.

However, despite their larger neighborhood, more active players tend to sponsor fewer links themselves: their marginal value of any particular link—and thus their willingness to pay for that link—is smaller. Thereby, highly active players benefit in two ways from their attractiveness: they not only have many friends, but also tend to pay little for link formation.

There are two further regularities in generic equilibria with upward linking: first, an agent who sponsors a link to another agent with some particular level of activity also forms links to all agents with higher activity. Second, agents who access the same neighbor activity through incoming links behave similarly: they sponsor the same number of links and choose the same level of activity. The following proposition describes these properties formally.

PROPOSITION 2. *Consider any generic equilibrium* $(\mathbf{x}^*, g^*)$ *with upward linking.*

(i) If $x_i^* > x_j^*$, *then* $\eta_i \leq \eta_j$ *and* $n_i > n_j$.

(ii) If $g_{ij} = 1$ *and* $x_{j'}^* > x_j^*$, *then* $g_{ij'} = 1$.

(iii) If and only if $\sum_{l:\, g_{li}=1} x_l^* = \sum_{l:\, g_{lj}=1} x_l^*$, *then* $\eta_i = \eta_j$ *and* $x_i^* = x_j^*$.

4.3 *Equilibrium refinement: Strict Nash equilibria*

This section applies an equilibrium refinement and studies *strict* Nash equilibria. To begin with, we briefly motivate this choice of refinement by comparing weak (generic) equilibria with strict equilibria. As discussed in Section 2, generic equilibria are, by definition, robust to small changes of (individual) linking costs, that is, Lemma 1(iii) and (iv) hold as strict inequalities. However, when Lemma 1(v) holds with equality, generic equilibria are not strict. Then some player i sponsors a link to player j with activity $\hat{x}$ but only shares links with a strict subset of *all* players with that activity. In particular, player i can relocate his link from j to a non-neighbor from this group without affecting his payoff. As a result, these *weak* equilibria are not robust to small changes of the best response activity and are only meaningful in a perfectly symmetric world.

In contrast, *strict* equilibria are robust to the introduction of small heterogeneity in the best response functions. More precisely, consider the homogeneous model and a heterogeneous variation in which each player's best response function is *slightly* modified. For any strict equilibrium $(\mathbf{x}^*, g^*)$ of the homogeneous model, there exists an equilibrium $(\mathbf{x}', g^*)$ in the heterogeneous model such that the equilibrium activity vector is slightly altered but link sponsorship is unchanged. Definitions 1 and 2 in the Appendix formalize these ideas, and the following proposition summarizes our findings.

PROPOSITION 3. *An equilibrium* $(\mathbf{x}^*, g^*)$ *is robust to the introduction of small heterogeneity in the best response functions if and only if the equilibrium is strict.*

As strict equilibria are robust to small departures from symmetry, their properties deserve special attention when thinking about applications. In fact, in strict equilibria, players separate themselves into groups (partite sets) characterized by the symmetric activity of their members. Moreover, every pair of players from different partite sets shares a link while no pair of players from the same group is connected. Formally, the equilibrium network is a complete multipartite graph with different-sized non-singleton parts. Figure 2 shows a schematic illustration of such a graph. Note that the characterization allows, among other things, for the empty network, the complete

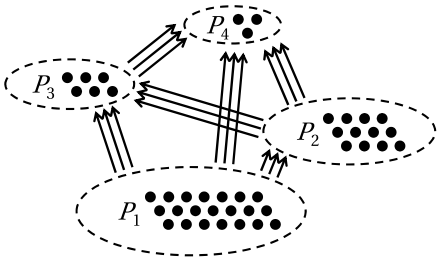

FIGURE 2. Schematic illustration of a directed complete four-partite graph with different-sized partite sets and upward linking.

network, complete biregular bipartite networks, and complete core–periphery networks (see the definitions in Section 2). The next proposition summarizes our findings.

PROPOSITION 4. *Consider any strict equilibrium* $(\mathbf{x}^*, g^*)$.

(i) The equilibrium network $\bar{g}^*$ *is a complete multipartite graph.*

(ii) All non-singleton partite sets are of different size.

(iii) An agent's activity depends on, and decreases in, the size of his part.

A *social hierarchy* emerges endogenously in equilibrium: if there are singleton partite sets, the corresponding players display the highest level of activity. Apart from that, the symmetric activity within a partite set decreases in its size. The endogenous hierarchy becomes even more apparent under upward linking (as introduced in Section 4.2): each player forms links to *all* more active players, receives links from *all* less active players, and has no connection to others with similar activity (unless possibly when he is one of the most active players and constitutes a singleton partite set).

Although Proposition 4 cleanly characterizes strict equilibria, it does not deal with equilibrium existence. Our next result shows that strict equilibria with up to two levels of activity exist—independently of the details of the model—for proper intervals of linking costs k.

PROPOSITION 5. *Consider any complete multipartite graph* $\hat{g}$ *with different-sized non-singleton parts. A strict equilibrium* $(\mathbf{x}^*, g^*)$ *with* $\bar{g}^* = \hat{g}$ *exists if*

(i) $\hat{g}$ *is the empty network for* $k \in (0, \infty)$ *with* $x_i^* = 0$

(ii) $\hat{g}$ *is the complete network for* $k \in (0, k_c)$ *with* $x_i^* = x_c > 0$

(iii) $\hat{g}$ *is a complete biregular bipartite network for* $k \in (\underline{k}, \bar{k})$ *with* $x_i^* > 0 \in \{x_1, x_2\}$.

Unfortunately, we are unable to provide any further general theoretical results about equilibria with more levels of activity or other networks. Instead, we made an extensive numerical analysis of the baseline model (Example 2), leading to two major insights: first, there exist strict equilibria with many levels of activity for certain specifications of

the model. For instance, a complete four-partite graph as illustrated in Figure 2 constitutes an equilibrium with four levels of activity for some specifications. Second, it depends critically on the details of the model—i.e., in the baseline model on exponent q—whether there exist strict equilibria with particular graphs (allowed by Proposition 4 but different from those in Proposition 5) for a non-empty interval of linking costs. In conclusion, we make the following important observation:

REMARK 2. Depending on the details of the model—i.e., the functional forms of $f(\cdot)$ and $h(\cdot)$—there may exist strict equilibria with many activity levels.

5. DISCUSSION

In this section, we discuss further aspects and variations of the model, which may be used as a basis for future research.

(a) *Linear best responses and network formation.* Our results from Sections 4.1–4.3 readily extend to linear best responses as long as the value function remains *concave*.[17] However, the classic linear-quadratic payoff function with local complements as introduced in Ballester et al. (2006), that is,

$$\pi_i(\mathbf{x}, \bar{g}) = \alpha x_i - \tfrac{1}{2}x_i^2 + \delta x_i \sum_{j \in N_{i,\bar{g}}} x_j,$$

implies the *convex* value function

$$h = \tfrac{1}{2}\left(\alpha + \delta \sum_{j \in N_{i,\bar{g}}} x_j\right)^2.$$

As a result, this particular payoff function is covered by the analysis in Hiller (2013): the equilibrium network is either empty, complete, or a core–periphery graph.

(b) *Welfare analysis.* A welfare analysis of our model would be desirable, yet turns out to be complicated. When payoffs are weakly convex in the effort levels of neighbors, convexity leads to a type of bang–bang solution and the efficient network is either empty or complete (Hiller 2013). Conversely, when payoffs are concave in neighbor effort, as in our model, we conjecture that the set of efficient networks is much larger.

(c) *Two-stage game.* In our model, players simultaneously choose their link sponsorship and level of activity. However, in practice, activity adjustments are made much more frequently than link alternations. Thus a two-period model seems to be a reasonable modeling alternative: agents decide about their link sponsorship

[17] So as to guarantee a positive activity equilibrium on any (exogenous) network, agents must now have strictly positive activity in isolation and the best response function must be sufficiently flat; see Bramoullé et al. (2014).

in the first period, anticipating its effects on everyone's activity choice in the second period. Unfortunately, such a model is a lot less tractable: to solve it via backward induction, one would need to characterize the equilibrium activity on *any* exogenous network—an as yet unsolved problem for an arbitrary concave best response function.

(d) *Two-sided link formation*. We have focused our analysis on one-sided link formation. For future research, it would also be interesting to consider two-sided link formation so as to compare the equilibrium networks and activity patterns (see Jackson and Wolinsky 1996 for a formal concept). As both adjacent players now have to sponsor the link, Lemma 2 (that is, upward linking) is no longer applicable, so our main results may not carry over.

6. Conclusion

Social hierarchies and behavioral complementarities (peer effects) are well documented empirical phenomena. This paper brings both of them together in a tractable theoretical model: a social hierarchy is the emergent property of a game in which ex ante identical agents form a network for complementary interaction. The model may be applied to understand hierarchies in very different circumstances, for instance, in criminal networks or among school friends. Our results show that there exists a unique and stable positive activity equilibrium on exogenous networks. Furthermore, endogenous equilibrium networks are strongly structured; a simple measure of network centrality—the number of neighbors—suffices to rank individual activity. Finally, in strict equilibria, the network is a complete multipartite graph with higher individual activity in smaller partite sets.

Apart from the points raised in the last section, there are two particularly promising directions for future research: first, we only consider one-shot interactions and, second, our results are only robust to small levels of heterogeneity. We believe that extending our research to a dynamic model allowing for arbitrary heterogeneity would be strongly appealing to social scientists from other disciplines. For instance, there is a well established sociological literature on how status differences may amplify over time—known as the Matthew effect (Merton 1968)—which such a model may elaborate and substantiate.

Appendix: Proofs from the main body

Proof of Remark 1. Assumption 1 states that there is a unique best response $x_i^*(\mathbf{x}_{-i}, \bar{g})$ that is an increasing, concave function of i's total neighbor activity. Thus x_i^* maximizes π_i and $\partial x_i^*/\partial x_j \geq 0$.

The necessary condition for a maximum implies

$$\left.\frac{\partial \pi_i}{\partial x_i}\right|_{x_i = x_i^*(\mathbf{x}_{-i}, \bar{g})} = 0.$$

Using the implicit function theorem and rewriting, we get

$$\left.\frac{\partial^2 \pi_i}{\partial x_i \, \partial x_j}\right|_{x_i = x_i^*(\mathbf{x}_{-i}, \bar{g})} = -\left.\frac{\partial^2 \pi_i}{\partial x_i \, \partial x_i}\right|_{x_i = x_i^*(\mathbf{x}_{-i}, \bar{g})} \frac{\partial x_i^*}{\partial x_j}.$$

As x_i^* is a maximum,

$$\left.\frac{\partial^2 \pi_i}{\partial x_i \, \partial x_i}\right|_{x_i = x_i^*(\mathbf{x}_{-i}, \bar{g})} < 0$$

and the claim follows. □

Proof of Proposition 1. First, we prove existence and uniqueness of the positive activity equilibrium $\mathbf{x}^* > \mathbf{0}$ and subsequently show its properties. Let us define the function $f_{\bar{g}} : X^n \to X^n$ as

$$f_{\bar{g}}(\mathbf{x}) = \begin{pmatrix} f(\sum_{j \in N_{1,\bar{g}}} x_j) \\ \vdots \\ f(\sum_{j \in N_{n,\bar{g}}} x_j) \end{pmatrix};$$

$f_{\bar{g},i}(\mathbf{x})$ denotes the ith entry of $f_{\bar{g}}(\mathbf{x})$. From the equilibrium condition in (2), it is clear that $\mathbf{x}^*$ is an equilibrium of the pure activity game on exogenous network $\bar{g}$ if and only if $\mathbf{x}^*$ is a fixed point of $f_{\bar{g}}$, that is, $f_{\bar{g}}(\mathbf{x}^*) = \mathbf{x}^*$.

First, consider the Taylor expansion of f around 0 evaluated at $\epsilon > 0$:

$$f(\epsilon) = f(0) + f'(0)\epsilon + \frac{f''(0)}{2!}\epsilon^2 + \cdots.$$

Together with Assumption 1, it follows that $f(\epsilon) - \epsilon > 0$ for sufficiently small ϵ. Furthermore, as $\bar{g}$ is connected and f is strictly increasing, $f(n_{i,\bar{g}}\epsilon) \geq f(\epsilon) > \epsilon > 0$, i.e., there exists a vector $\mathbf{a}$ with $f_{\bar{g}}(\mathbf{a}) > \mathbf{a} > \mathbf{0}$.

Second, consider the function $g(x) = f((n-1)x) - x$. Note that $g(x)$ is decreasing for sufficiently high x by Assumption 1. As $g(x)$ is also concave, there exists some x_0 with $g(x) < 0$ for all $x \geq x_0$ or, equivalently, b with $f((n-1)b) < b$. Furthermore, since f strictly increases and each player has at most $n-1$ neighbors, there exists a vector $\mathbf{b}$ with $f_{\bar{g}}(\mathbf{b}) < \mathbf{b}$.

Finally, as $f_{\bar{g}}$ is non-decreasing, $\mathbf{a} < f_{\bar{g}}(\mathbf{x}) < \mathbf{b}$ for $\mathbf{x} \in [\mathbf{a}, \mathbf{b}]$. Therefore, $f_{\bar{g}} : [\mathbf{a}, \mathbf{b}] \to [\mathbf{a}, \mathbf{b}]$, Tarski's fixed point theorem (Tarski 1955) is applicable, and $f_{\bar{g}}$ has at least one fixed point $\mathbf{x}^* \in [\mathbf{a}, \mathbf{b}]$.

To show uniqueness of $\mathbf{x}^* > \mathbf{0}$, we use three important properties of $f_{\bar{g}}$. First, $f_{\bar{g}}$ is non-decreasing, that is, $\mathbf{x} \geq \mathbf{y}$ implies $f_{\bar{g}}(\mathbf{x}) \geq f_{\bar{g}}(\mathbf{y})$. This holds true because f is strictly increasing and so $f(\sum_{j \in N_{i,\bar{g}}} x_j) \geq f(\sum_{j \in N_{i,\bar{g}}} y_j)$, $\forall i$. Second, $\mathbf{x} > \mathbf{0}$ implies $f_{\bar{g},i}(\mathbf{x}) > 0$ as $\bar{g}$ is connected so that each player i has at least one neighbor. Third, $f_{\bar{g}}(\lambda \mathbf{x}) > \lambda f_{\bar{g}}(\mathbf{x})$ for $\lambda \in (0,1)$ and $\mathbf{x} > \mathbf{0}$ as f is strictly concave and $\bar{g}$ connected, i.e., $f(\sum_{j \in N_{i,\bar{g}}} \lambda x_j) > \lambda f(\sum_{j \in N_{i,\bar{g}}} x_j)$, $\forall i$.

Let $u(\mathbf{x}) = f_{\bar{g}}(\mathbf{x}) - \mathbf{x}$. Using notation from Kennan (2001), u is strictly R-quasiconcave: $u(\mathbf{x}) = \mathbf{0}$, $\mathbf{x} > \mathbf{0}$, and $\lambda \in (0,1)$ implies $u(\lambda \mathbf{x}) = f_{\bar{g}}(\lambda \mathbf{x}) - \lambda \mathbf{x} > \lambda(f_{\bar{g}}(\mathbf{x}) - \mathbf{x}) = \mathbf{0}$.

Also, u is quasi-increasing: $x_i = y_i$ and $x_j \geq y_j$ for all $j \in N$ implies $u_i(\mathbf{x}) = f_{\bar{g},i}(\mathbf{x}) - x_i \geq f_{\bar{g},i}(\mathbf{y}) - y_i = u_i(\mathbf{y})$. Thus the conditions of Theorem 3.1 in Kennan (2001) are satisfied and $f_{\bar{g}}$ has at most one fixed point $\mathbf{x}^* > \mathbf{0}$.

We now turn to the properties of $\mathbf{x}^* > \mathbf{0}$.

Part (i). The proof follows similar reasoning as applied in Theorem 8 of Milgrom and Roberts (1990). Fix some $\tilde{\mathbf{x}} > \mathbf{0}$. Then $\mathbf{x}^*$ is globally asymptotically stable under best response dynamics if iterative application of $f_{\bar{g}}$ on $\tilde{\mathbf{x}}$ converges to $\mathbf{x}^*$. Using the same notation as above, we can find $\mathbf{a}, \mathbf{b}$ such that $\mathbf{x}^*, \tilde{\mathbf{x}} \in (\mathbf{a}, \mathbf{b})$ and restricted $f_{\bar{g}} : [\mathbf{a}, \mathbf{b}] \to (\mathbf{a}, \mathbf{b})$, as $\mathbf{a}$ can be arbitrarily close to $\mathbf{0}$ and $\mathbf{b}$ can be arbitrarily large. Consider three sequences starting at $\alpha^1 \in \{\mathbf{a}, \mathbf{b}, \tilde{\mathbf{x}}\}$ and $\alpha^s = f_{\bar{g}}(\alpha^{s-1})$. As $f_{\bar{g}}$ is non-decreasing and $f_{\bar{g}}(\mathbf{a}) > \mathbf{a}$, the sequence $\mathbf{a}^s$ is non-decreasing. As it is also bounded above by $\mathbf{b}$, it converges to some $\mathbf{x}^0 = \sup(\mathbf{a}^s)$. As both sides of $\mathbf{a}^s = f_{\bar{g}}(\mathbf{a}^{s-1})$ converge to $\mathbf{x}^0$, $\mathbf{x}^0$ is a fixed point of $f_{\bar{g}}$ in $(\mathbf{a}, \mathbf{b})$ and, thus, $\mathbf{x}^0 = \mathbf{x}^*$. For similar reasons, $\mathbf{b}^s$ is a nonincreasing sequence converging to $\mathbf{x}^*$. Finally, $\tilde{\mathbf{x}}^s \in [\mathbf{a}^s, \mathbf{b}^s]$ at the sth elements of the three series, as $f_{\bar{g}}$ is non-decreasing so that the sequence $\tilde{\mathbf{x}}^s$ converges to $\mathbf{x}^*$ as well.

For the proofs of the other two parts, we need to provide a stronger finding to facilitate *strict* comparative statics:

Claim 1. *Consider some $\mathbf{x}^1 > \mathbf{0}$ with $f_{\bar{g}}(\mathbf{x}^1) \geq \mathbf{x}^1$ and at least one strict entry $f_{\bar{g},i}(\mathbf{x}^1) > x_i^1$. Then the sequence $\mathbf{x}^s$ with $f_{\bar{g}}(\mathbf{x}^s) = \mathbf{x}^{s-1}$ converges to the unique positive fixed point $\mathbf{x}^* > \mathbf{x}^1$.*

Proof. From the reasoning above, it is clear that the sequence $\mathbf{x}^s$ converges to the unique positive fixed point $\mathbf{x}^* = f_{\bar{g}}(\mathbf{x}^*)$. As $f_{\bar{g}}(\mathbf{x}^1) \geq \mathbf{x}^1$ and $f_{\bar{g}}$ is non-decreasing, the sequence $\mathbf{x}^s$ is also non-decreasing and, thus, $\mathbf{x}^* \geq \mathbf{x}^1$. Assume, by contradiction, that there exists $i \in N$ with $x_i^* = x_i^1$. As $\mathbf{x}^* \geq \mathbf{x}^1$ and, by the fixed point property, $x_i^* = f_{\bar{g},i}(\mathbf{x}^*)$, it follows immediately that $x_j^* = x_j^1$ for all neighbors $j \in N_{i,\bar{g}}$. Iterating the argument implies $\mathbf{x}^* = \mathbf{x}^1$ since $\bar{g}$ is connected. As $\mathbf{x}^*$ is a fixed point, $f_{\bar{g}}(\mathbf{x}^1) = \mathbf{x}^1$, a contradiction to the assumptions of the claim. ◁

Part (ii). Note that there is a unique positive $\hat{x}$ for which $\hat{x} = f(\hat{x})$ by Assumption 1. Furthermore, $y > \hat{x}$ implies $f(y) < y$. Consider any connected graph $\bar{g}$ with $n > 2$ and assume the hypothetical starting effort level $\mathbf{x}^1 = \hat{\mathbf{x}}$. There is at least one player k who has more than one neighbor in $\bar{g}$ and for whom

$$f_{\bar{g},k}(\mathbf{x}^1) \geq f(2\hat{x}) > f(\hat{x}) = \hat{x} = x_k^1.$$

All other players have at least one neighbor, i.e., $f_{\bar{g},i}(\mathbf{x}^1) \geq x_i^1$. By Claim 1, this implies $\mathbf{x}^* > \mathbf{x}^1 = \hat{\mathbf{x}}$. As $\sum_{j \in N_i} x_j^* > \hat{x}$ for any player i, it follows from the reasoning above that

$$x_i^* = f\left(\sum_{j \in N_i} x_j^*\right) < \sum_{j \in N_i} x_j^*, \quad \forall i \in N.$$

Part (iii). Consider some network with $\bar{g}_{ij} = 0$. Since $f_{\bar{g}}(\mathbf{x}_{\bar{g}}^*) = \mathbf{x}_{\bar{g}}^* > \mathbf{0}$, it follows that $f_{\overline{g \oplus ij}}(\mathbf{x}_{\bar{g}}^*) \geq f_{\bar{g}}(\mathbf{x}_{\bar{g}}^*) = \mathbf{x}_{\bar{g}}^*$ and $f_{\overline{g \oplus ij},i}(\mathbf{x}_{\bar{g}}^*) > f_{\bar{g},i}(\mathbf{x}_{\bar{g}}^*)$. By Claim 1, the sequence $\mathbf{x}^s = f_{\overline{g \oplus ij}}(\mathbf{x}^{s-1})$ with $\mathbf{x}^1 = \mathbf{x}_{\bar{g}}^*$ then converges to $\mathbf{x}_{\overline{g \oplus ij}}^* > \mathbf{x}^1 = \mathbf{x}_{\bar{g}}^*$. □

PROOF OF LEMMA 1. The proof simplifies the general equilibrium condition in (3). First, this general condition implies that every player chooses his unique best response activity $x_i^*(\mathbf{x}_{-i}^*, \bar{g}^*)$ in equilibrium (see Assumption 1): otherwise, a player improves his payoff by merely changing his activity. That is, condition (i) of the lemma is a necessary equilibrium condition.

By Assumption 2, a player then receives gross payoff $\pi_i^* = h(\sum_{j\in N_{i,\bar{g}^*}} x_j^*)$ from sticking to his (candidate) equilibrium strategy. Since there is a unique best response activity $x_i^*(\mathbf{x}_{-i}, \bar{g})$ for any network $\bar{g}$, an optimal deviation from (equilibrium) link sponsorship involves an adjustment of activity,

$$\pi_i(x_i', \mathbf{x}_{-i}^*, \overline{g_{-i}^* \odot g_i'}) \leq \pi_i(x_i^*(\mathbf{x}_{-i}^*, \overline{g_{-i}^* \odot g_i'}), \mathbf{x}_{-i}^*, \overline{g_{-i}^* \odot g_i'}) = h\Bigg(\sum_{j\in N_{i,\overline{g_{-i}^* \odot g_i'}}} x_j^*\Bigg)$$

for all $i \in N$, $x_i' \geq 0$, $g_i' \in \mathfrak{g}_i$. Thus, a pair $(\mathbf{x}^*, g^*)$ constitutes an equilibrium if and only if each agent chooses optimal activity (condition (i) of the lemma) and no player gains from changing his link sponsorship while simultaneously adjusting his activity to the new network. Mathematically, the second condition reads as

$$h\Bigg(\sum_{j\in N_{i,\bar{g}^*}} x_j^*\Bigg) - \eta_{i,g^*}k \geq h\Bigg(\sum_{j\in N_{i,\overline{g_{-i}^* \odot g_i'}}} x_j^*\Bigg) - \eta_{i,g_{-i}^* \odot g_i'}k, \quad \forall i \in N, g_i' \in \mathfrak{g}_i. \tag{4}$$

It is easy to check that (4) implies conditions (ii)–(v) of the lemma. Conversely, we are going to show that a violation of (4) implies a violation of at least one of the conditions—i.e., conditions (ii)–(v) jointly imply (4). Assume some vector $g_i' \in \mathfrak{g}_i$ violates (4) in a candidate equilibrium $(\mathbf{x}^*, g^*)$ in which each link is only sponsored by one of the adjacent players (i.e., condition (ii) holds true).

If deviation g_i' strictly increases the set of players to whom i supports a link, then adding a link to the most active non-neighbor, $\arg\max_{j:\, \bar{g}_{ij}^*=0}\{x_j^*\}$, must also be a profitable deviation (a violation of condition (iii)). If deviation g_i' instead strictly decreases the set of players to whom i supports a link, deleting only the least valuable self-sponsored link, $\arg\min_{j:\, g_{ij}^*=1}\{x_j^*\}$, must also be a profitable deviation (a violation of condition (iv)).

Finally, consider a profitable deviation g_i' that demands deleting some self-sponsored links and forming some new ones instead. Then either condition (v) is violated or the activity of all players to whom i forms a new link is weakly lower than the activity of all players to whom i deletes his links. Consequently, there is a weakly better deviation g_i'': if deviation g_i' increases i's number of self-sponsored links, g_i'' demands forming some new links without deleting any old links. If i's number of self-sponsored links decreases instead, g_i'' demands deleting some links without adding any. Both these cases are discussed above. □

PROOF OF LEMMA 2. First note that g' differs from g^* only insofar as links between two players with unequal activity are now sponsored by the less active player; in particular, $\bar{g}' = \bar{g}^*$ and $N_{i,\bar{g}'} = N_{i,\bar{g}^*}$. Therefore, $\mathbf{x}^*$ is an equilibrium activity vector on $\bar{g}'$—that is,

$(\mathbf{x}^*, g')$ meets the first equilibrium condition of Lemma 1—and the total neighbor activity accessed by any player is unaltered, that is, $\sum_{j\in N_{i,\bar{g}'}} x_j^* = \sum_{j\in N_{i,\bar{g}^*}} x_j^*$. Likewise, every link is only sponsored by one adjacent player (i.e., the second condition holds true).

As $\bar{g}' = \bar{g}^*$ and every player has unaltered activity x_i^* in g', no player wants to form any additional link(s) in g' under any linking costs k under which he refrains from adding a link in g^*; hence, in particular, $(\mathbf{x}^*, g')$ meets the third equilibrium condition from Lemma 1.

Consider any link that is sponsored by the same player in g' as in g^*, i.e., $g'_{ij} = g^*_{ij} = 1$. For the same reason as above, no player wants to delete (or switch) such a link for any linking costs k under which he refrains from deleting (or switching) it in g^*. Thus equilibrium conditions (iv) and (v) from Lemma 1 are met for these links.

Instead, consider some players m and l who share a link with altered sponsorship, i.e., $g'_{lm} = g^*_{ml} = 1$. As the link between l and m is sponsored by l in g' but by m in g^*, then $x_l^* < x_m^*$ by the assumptions of the lemma. To finalize the proof, we are going to show (i) that l refrains from deleting his self-sponsored link to m in g' for a greater interval of linking costs k than the interval under which m refrains from deleting his link to l in g^* (equilibrium condition (iv)). Subsequently, we are going to show (ii) that l does not want to switch his link (equilibrium condition (v)).

(i) As $(\mathbf{x}^*, g^*)$ is an equilibrium and $g^*_{ml} = 1$, equilibrium condition (iv) of Lemma 1 implies

$$h\Big(\sum_{j\in N_m} x_j^*\Big) - h\Big(\sum_{j\in N_m\setminus\{l\}} x_j^*\Big) \geq k.$$

Thus l refrains from deleting his self-sponsored link to m in g' for a greater interval of linking costs k if

$$h\Big(\sum_{j\in N_l} x_j^*\Big) - h\Big(\sum_{j\in N_l\setminus\{m\}} x_j^*\Big) > h\Big(\sum_{j\in N_m} x_j^*\Big) - h\Big(\sum_{j\in N_m\setminus\{l\}} x_j^*\Big).$$

Since activity $x_l^* < x_m^*$ and as f is strictly increasing, condition (i) of Lemma 1 implies that total neighborhood activity of l is smaller than m's: $\sum_{j\in N_l} x_j^* < \sum_{j\in N_m} x_j^*$ under both g^* and g'. Thus $x_l^* < x_m^*$ implies $\sum_{j\in N_l\setminus\{m\}} x_j^* < \sum_{j\in N_m\setminus\{l\}} x_j^*$. As h is strictly increasing and concave, and from the findings above,

$$\begin{aligned} h\Big(\sum_{j\in N_l} x_j^*\Big) - h\Big(\sum_{j\in N_l\setminus\{m\}} x_j^*\Big) &> h\Big(\sum_{j\in N_l\setminus\{m\}} x_j^* + x_l^*\Big) - h\Big(\sum_{j\in N_l\setminus\{m\}} x_j^*\Big) \\ &> h\Big(\sum_{j\in N_m\setminus\{l\}} x_j^* + x_l^*\Big) - h\Big(\sum_{j\in N_m\setminus\{l\}} x_j^*\Big). \end{aligned}$$

(ii) Finally, let us turn to condition (v). If this condition does not hold for $(\mathbf{x}^*, g')$, then there must be some player $o \in N$ with $x_o^* > x_m^*$ and $\bar{g}'_{lo} = \bar{g}^*_{lo} = 0$ so that l prefers to form a link to o instead of m. As shown below, this leads to a contradiction as $(\mathbf{x}^*, g^*)$ would not be an equilibrium if such a player o exists.

Assume $(\mathbf{x}^*, g^*)$ is an equilibrium, but, by contradiction, player o as described above exists. On the one side, since player m supports a link to l in g^* and $x_o^* > x_l^*$, we infer

$$\begin{aligned} k &\leq h\Bigg(\sum_{j\in N_m\setminus\{l\}} x_j^* + x_l^*\Bigg) - h\Bigg(\sum_{j\in N_m\setminus\{l\}} x_j^*\Bigg) \\ &< h\Bigg(\sum_{j\in N_m\setminus\{l\}} x_j^* + x_o^*\Bigg) - h\Bigg(\sum_{j\in N_m\setminus\{l\}} x_j^*\Bigg). \end{aligned}$$

Since player l does not form a link to o in g^* (and o is currently no neighbor of l), we also know that

$$k \geq h\Bigg(\sum_{j\in N_l} x_j^* + x_o^*\Bigg) - h\Bigg(\sum_{j\in N_l} x_j^*\Bigg).$$

As h is strictly concave, these two inequalities imply $\sum_{j\in N_l} x_j^* > \sum_{j\in N_m\setminus\{l\}} x_j^*$ or

$$x_l^* > \sum_{j\in N_m} x_j^* - \sum_{j\in N_l} x_j^*. \tag{5}$$

On the other side, we also know that player l only supports links to players with activity $x_i^* \geq x_o^*$ in g^*, as he does not support a link to o. Conversely, as m supports a link to l in g^*, he must share links with all players of activity $x_i^* > x_l^*$—i.e., in particular, $x_i^* \geq x_o^*$. Thus every player to whom l forms a link is a neighbor of m. Furthermore, every player who forms a link to l must share links with all players of activity $x_i^* > x_l^*$—i.e., in particular, to m.

These two findings and the fact that o is a neighbor of m but not of l imply

$$\sum_{j\in N_m} x_j^* - \sum_{j\in N_l} x_j^* \geq x_o^* + x_l^* - x_m^* > x_l^*,$$

in contradiction to (5). □

Proof of Proposition 2. *Part (ii).* Assume $g_{ij} = 1$ in equilibrium and, by contradiction, that there exists some $j' \in N$ with $x_{j'}^* > x_j^*$ but $g_{ij'} = 0$. As i sponsors a link to j and there is upward linking, $x_j^* \geq x_i^*$. This implies $x_{j'}^* > x_i^*$ so that $g_{j'i} = 0$, i.e., j' does not sponsor a link to i. But then i can increase his payoff by deleting the link to j and forming one to j' instead, a contradiction.

Part (i). Assume, by contradiction, $x_i^* > x_j^*$ but $\eta_i > \eta_j$ in some generic equilibrium $(\mathbf{x}^*, g^*)$ with upward linking. Let $\underline{x} = \min\{x_l : g_{il} = 1\}$, i.e., the minimum level of activity accessed by i through a self-sponsored link, and let $\kappa_i = |\{l \in N : x_l = \underline{x} \wedge g_{il} = 1\}|$ be the number of players with activity $\underline{x}$ to whom i sponsors a link. Note that $\underline{x} \geq x_i^*$ as there is upward linking so that $\underline{x} > x_j^*$ and no player with activity $\underline{x}$ or higher sponsors a link to j. As each player prefers to link to more active players (see part (ii) above) and $\eta_j < \eta_i$, it follows $\kappa_j < \kappa_i$ so that j sponsors less links to players with activity $\underline{x}$.

On the one hand, as i sponsors a link to somebody with activity $\underline{x}$ in equilibrium and j refrains from adding a link to one more person with activity $\underline{x}$ (which would be possible as $\kappa_j < \kappa_i$ and players with activity $\underline{x}$ do not sponsor links to j), Lemma 1 tells us that

$$h\Big(\sum_{l\in N_i} x_l^*\Big) - h\Big(\sum_{l\in N_i} x_l^* - \underline{x}\Big) \geq k \geq h\Big(\sum_{l\in N_j} x_l^* + \underline{x}\Big) - h\Big(\sum_{l\in N_j} x_l^*\Big). \tag{6}$$

As h is strictly concave, this can only hold true if

$$\sum_{l\in N_i} x_l^* \leq \sum_{l\in N_j} x_l^* + \underline{x}. \tag{7}$$

On the other hand, as $x_i^* > x_j^*$, any player l who sponsors a link to j has to sponsor a link to i as well by the first statement of this lemma so that i receives more total neighbor activity than j through incoming links. And as $\kappa_j < \kappa_i$, i also receives at least $(\kappa_i - \kappa_j)\underline{x}$ more total neighbor activity than j through self-sponsored links. Together this implies

$$\sum_{l\in N_i} x_l^* \geq \sum_{l\in N_j} x_l^* + (\kappa_i - \kappa_j)\underline{x}. \tag{8}$$

Combining (7) and (8), we get

$$\sum_{l\in N_j} x_l^* + (\kappa_i - \kappa_j)\underline{x} \leq \sum_{l\in N_i} x_l^* \leq \sum_{l\in N_j} x_l^* + \underline{x},$$

which can only hold true if $\kappa_i - \kappa_j = 1$ and $\sum_{l\in N_i} x_l^* = \sum_{l\in N_j} x_l^* + \underline{x}$. The latter finding together with (6) implies

$$h\Big(\sum_{l\in N_i} x_l^*\Big) - h\Big(\sum_{l\in N_i} x_l^* - \underline{x}\Big) = k = h\Big(\sum_{l\in N_j} x_l^* + \underline{x}\Big) - h\Big(\sum_{l\in N_j} x_l^*\Big).$$

Then for any small $\epsilon > 0$, under linking costs $k' = k + \epsilon$, player i would prefer to delete a link, and for $k' = k - \epsilon$, player j would prefer to add a link. In other words, the equilibrium is non-generic, a contradiction. Thus $x_i^* > x_j^*$ implies $\eta_i \leq \eta_j$.

We now show $x_i^* > x_j^*$ also implies $n_i > n_j$. Let N_j^{in} be the set of players who sponsor links to j. By part (ii) and $x_i^* > x_j^*$, everybody who sponsors a link to j sponsors a link to i so that $N_j^{in} \subseteq N_i^{in}$ is also the set of players who sponsor links to i as well as to j. Let $A = N_j^{in} \cup \{i, j\}$. Then $N_j \setminus A$ is the set of players to whom j sponsors links apart from i with $|N_j \setminus A| = \eta_j - \bar{g}_{ij}$. Similarly, $N_i \setminus A$ is the set of i's neighbors apart from j who do not sponsor links to i as well as j. Since we delete in both cases the same *number* of neighbors, we have

$$|N_i \setminus A| = n_i - (n_j - |N_j \setminus A|) = n_i - n_j + \eta_j - \bar{g}_{ij}.$$

From optimal activity, $x_i^* > x_j^*$ implies

$$\sum_{l\in N_i\setminus A} x_l > \sum_{l\in N_j\setminus A} x_l. \tag{9}$$

Consider the set $N \setminus A$ and relabel players such that $\tilde{x}_1 \geq \cdots \geq \tilde{x}_{|N\setminus A|}$. By part (ii), j sponsors links to some set of most active players, that is,

$$\sum_{l\in N_j\setminus A} x_l = \sum_{l=1}^{\eta_j-\bar{g}_{ij}} \tilde{x}_l,$$

whereas the remaining neighbor activity that i accesses is bounded from above by

$$\sum_{l\in N_i\setminus A} x_l \leq \sum_{l=1}^{n_i-n_j+\eta_j-\bar{g}_{ij}} \tilde{x}_l.$$

Together with (9), $n_i > n_j$ follows.

"Only if" claim of part (iii). As there is upward linking and every player prefers to link to more active players, $\eta_i = \eta_j$ implies $\sum_{l:\, g_{il}=1} x_l^* = \sum_{l:\, g_{jl}=1} x_l^*$. In addition, $x_i^* = x_j^*$ implies $\sum_{l\in N_i} x_l^* = \sum_{l\in N_j} x_l^*$ from optimal activity. Thus the claim holds.

"If" claim of part (iii). Assume $\sum_{l:\, g_{li}=1} x_l^* = \sum_{l:\, g_{lj}=1} x_l^*$ and, by contradiction, $x_i^* > x_j^*$. Part (i) of the lemma then implies $\eta_i \leq \eta_j$. As i only forms links to players with $x_{j'}^* \geq x_i^* > x_j^*$ due to upward linking, players j' to whom i sponsors links do not sponsor a link to j. Thus j could copy i's linking decisions and strictly increase his payoff, a contradiction.

The equalities $\sum_{l:\, g_{li}=1} x_l^* = \sum_{l:\, g_{lj}=1} x_l^*$ and $x_i^* = x_j^*$ together imply $\sum_{l:\, g_{il}=1} x_l^* = \sum_{l:\, g_{jl}=1} x_l^*$. As there is upward linking, $\eta_i = \eta_j$ follows by part (ii). □

Definition 1. Consider the (homogeneous) game $\Gamma = [N, \{X \times \mathfrak{g}_i\}, h(\cdot), f(\cdot), k]$ with endogenous network formation, which consists of the set of players N, the action spaces for activity and link sponsorship $\{X \times \mathfrak{g}_i\}$, the value function h, the best response function f, and the linking costs k. Then $\Gamma_\delta = [N, \{X \times \mathfrak{g}_i\}, h, \{f_i\}, k]$ denotes its heterogeneous δ-variation where the individual best response functions $\{f_i\}$ satisfy Assumption 1 and $\delta \equiv \sup_{i\in N, x\geq 0}\{|f_i(x) - f(x)|\}$.

Definition 2. Let $(\mathbf{x}^*, g^*)$ be an equilibrium of the homogenous game and let $(\mathbf{x}_\delta^*, g_\delta^*)$ be an equilibrium of a δ-variation. The equilibrium $(\mathbf{x}^*, g^*)$ is robust to the introduction of small heterogeneity in the best response functions if and only if there exists $\bar{\delta} > 0$ such that for all δ-variations with $\delta < \bar{\delta}$, there exist equilibria $(\mathbf{x}_\delta^*, g_\delta^*)$ with $g_\delta^* = g^*$ and $\lim_{\delta\to 0} \mathbf{x}_\delta^* = \mathbf{x}^*$.

Proof of Proposition 3. First consider a weak equilibrium. By definition, one of the conditions (iii)–(v) from Lemma 1 holds as an equality for some player i. If condition (v) holds with equality, then there exist $j, j' \in N_i$ such that $x_j^* = x_{j'}^*$ and $g_{ij} = 1$, whereas $\bar{g}_{ij'} = 0$. The equilibrium network is not robust to a small increase in the best response activity of j', as player i would desire to switch his self-sponsored link from j to j'. Similar problems exist when conditions (iii)–(iv) hold with equality.

Conversely, we need to show that any strict equilibrium $(\mathbf{x}^*, g^*)$ satisfies Definition 2. First, note that the results from Proposition 1 readily extend to any heterogeneous

δ-variation (see Definition 1). In particular, there exists a unique equilibrium activity vector $\mathbf{x}^*_\delta > \mathbf{0}$ on any (connected) exogenous network $\bar{g}$.

Second, we show that $\lim_{\delta\to 0}\mathbf{x}^*_\delta = \mathbf{x}^* > \mathbf{0}$ on any (connected) exogenous network $\bar{g}$. Define $\bar{f}(\cdot) = f(\cdot) + \delta$ and $\underline{f} = f - \delta$, and let δ be sufficiently small so that $\underline{f}(x) > x$ for some $x > 0$. By an extension of Proposition 1, there exists equilibrium activity $\bar{\mathbf{x}}(\delta) > \mathbf{0}$ when every player has the best response function $\bar{f}$; equilibrium activity $\underline{\mathbf{x}}(\delta) > \mathbf{0}$ exists accordingly. By strict complementarity, $\bar{\mathbf{x}}(\delta)$ strictly increases in δ, whereas $\underline{\mathbf{x}}(\delta)$ strictly decreases and both converge to $\mathbf{x}^*$ for $\delta \to 0$. To see the latter, assume that the players have activity $\mathbf{x}^*$, their best responses shift from f to $\bar{f} = f + \delta$, and they can simultaneously update their activity k times, leading to activity $\bar{\mathbf{x}}^k(\delta)$. From Proposition 1, we know $\lim_{k\to\infty}\bar{\mathbf{x}}^k(\delta) = \bar{\mathbf{x}}(\delta)$ for any δ. For $k = 1$, we have $\lim_{\delta\to 0}\bar{\mathbf{x}}^1(\delta) = \mathbf{x}^*$ as

$$\bar{x}^1_i(\delta) = f\Bigg(\sum_{j\in N_{i,\bar{g}}} x^*_j\Bigg) + \delta, \quad \forall i \in N.$$

If $\lim_{\delta\to 0}\bar{\mathbf{x}}^{k'}(\delta) = \mathbf{x}^*$ for some k', then

$$\lim_{\delta\to 0}\bar{x}^{k'+1}_i(\delta) = \lim_{\delta\to 0}\Bigg(f\Bigg(\sum_{j\in N_{i,\bar{g}}}\bar{x}^{k'}_j(\delta)\Bigg) + \delta\Bigg) = x^*_i, \quad \forall i \in N.$$

Consequently, $\bar{\mathbf{x}}(\delta)$ converges to $\mathbf{x}^*$ for $\delta \to 0$ (and $\underline{\mathbf{x}}(\delta)$ accordingly). The claim $\lim_{\delta\to 0}\mathbf{x}^*_\delta = \mathbf{x}^* > \mathbf{0}$ follows as $\mathbf{x}^*_\delta \in [\underline{\mathbf{x}}(\delta), \bar{\mathbf{x}}(\delta)]$ by strict complementarity.

Finally, $(\mathbf{x}^*, g^*)$ strictly satisfies conditions (iii)–(v) of Lemma 1 by definition. Thus there exists $\hat{\delta} > 0$ such that $(\mathbf{x}^*_\delta, g^*)$ also satisfies conditions (iii)–(v) for all $\mathbf{x}^*_\delta$ with $\|\mathbf{x}^*_\delta - \mathbf{x}^*\| < \hat{\delta}$. Together with the finding above, we conclude that every strict equilibrium $(\mathbf{x}^*, g^*)$ satisfies Definition 2. □

Proof of Proposition 4. In any strict equilibrium $(\mathbf{x}^*, g^*)$, we can partition the set of players N into L strictly ordered sets of equally active players N^l, i.e., $\bigcup_{l=1}^L N^l = N$ and $x^*_i = x^l$, $\forall i \in N^l$ with $x^1 < x^2 < \cdots < x^L$. Furthermore, we assume without loss of generality that links are sponsored by an adjacent player with weakly smaller activity (Lemma 2).

For our proof, two observations are decisive: (i) If player i sponsors a link to some player $j \in N^r$ in a *strict* equilibrium, then he shares a link with all players $j' \in \bigcup_{l=r}^L N^l \setminus \{i\}$. Otherwise player i can weakly improve his payoff by relocating his self-sponsored link from j to some non-neighbor $j' \in \bigcup_{l=r}^L N^l \setminus \{i\}$, a contradiction.

(ii) If some player i shares a link with all other players $j \in N \setminus \{i\}$, then his activity is maximal—i.e., $i \in N^L$ and $x^*_i = x^L$—and any other player $i' \in N^L$ also shares a link with players $j \in N \setminus \{i'\}$. Again, we use a proof by contradiction. First, assume that there exists $j \in N$ with $x^*_j > x^*_i$. Then from optimal activity, we have

$$x^*_i = f\Bigg(\sum_{s\in N_i} x^*_s\Bigg) < f\Bigg(\sum_{s\in N_j} x^*_s\Bigg) = x^*_j \quad \Rightarrow \quad \sum_{s\in N} x^*_s - x^*_i < \sum_{s\in N_j} x^*_s \le \sum_{s\in N} x^*_s - x^*_j$$
$$\Rightarrow \quad x^*_j < x^*_i.$$

Consequently, $x_i^* = x^L$ and $i \in N^L$. Second, assuming that any other player $i' \in N^L$ does not share a link with players $j \in N \setminus \{i'\}$ leads to a similar contradiction.

We now turn to the main proof. Consider players in N^1. There are three possible configurations: (a) If players in N^1 do not sponsor any links, then they have maximal activity $x^1 = x^L$: due to complementarity, any hypothetical more active players would have to share some links among themselves, in contradiction to Proposition 2(i). Thus the network is empty, i.e., a complete one-partite graph.

(b) Some player $i \in N^1$ sponsors a link to another player in N^1. By the two observations above, $L = 1$ and all pairs of players share a link, that is, the network is a complete n-partite graph.

(c) Players in N^1 (only) sponsor links to more active players (and N^1 is an independent set). By Proposition 2, they sponsor the same number of links and prefer to sponsor links to more active players. Assume that they did not sponsor any links to players in N^2. Then Proposition 2 implies $x^1 = x^2$, a contradiction. As players in N^1 sponsor some links to players in N^2, from observation (i) above and by upward linking, players in N^1 sponsor links to all more active players $\bigcup_{l=2}^{L} N^l$.

Repeating a similar argument for N^2 to N^L shows that the network is a complete multipartite graph. In particular, each set of equally active players N^l is either a non-singleton partite set of the graph or $N^l = N^L$ and consists of $|N^L|$ singleton partite sets. Consider two non-singleton partite sets $N^{l'}$ and $N^{l''}$ with $l' < l''$, and assume, by contradiction, $|N^{l'}| \leq |N^{l''}|$. Then

$$\begin{aligned} x^{l'} < x^{l''} \quad &\Leftrightarrow \quad f\left(\sum_{i\in N} x_i - |N^{l'}|x^{l'}\right) < f\left(\sum_{i\in N} x_i - |N^{l''}|x^{l''}\right) \\ &\Leftrightarrow \quad |N^{l''}|x^{l''} < |N^{l'}|x^{l'} \\ &\Rightarrow \quad x^{l''} < x^{l'}. \end{aligned}$$

As a result, all non-singleton partite sets have a different size; all agents within a partite set have symmetric activity that is smaller in larger partite sets. □

Proof of Proposition 5. Part (i). The zero activity equilibrium with an empty network exists for any linking costs as all conditions in Lemma 1 are met: it is optimal for an isolated player to choose zero activity and forming a link to any non-active player is costly as $k > 0$.

Part (ii). Consider the complete network $\bar{g}_c$ and assume that each link is sponsored (arbitrarily) by one of the adjacent players. By Proposition 1, there is a unique positive activity equilibrium on $\bar{g}_c$; the equilibrium activity solves $x_c = f((n-1)x_c) > 0$ and the players' gross payoff reads as $\pi_i = h((n-1)x_c)$. As the interaction network is complete, no player gains from adding or relocating any self-sponsored link. Conversely, no player weakly increases his payoff by deleting one of his self-sponsored links if and only if

$$0 < k < h((n-1)x_c) - h((n-2)x_c) \equiv k_c.$$

In conclusion, all conditions from Lemma 1 are (strictly) satisfied for $k \in (0, k_c)$.

Part (iii). Consider an arbitrary complete bipartite graph $\bar{g}_{bp}$ with different-sized partite sets P_1, P_2 and assume w.l.o.g. $|P_1| > |P_2|$. By Proposition 1, there is a unique positive activity equilibrium on the (exogenous) network $\bar{g}_{bp}$; by Proposition 4, players in P_1 choose symmetric activity x_1, whereas players in P_2 choose symmetric activity $x_2 > x_1$ in this equilibrium. Applying Lemma 2, we assume that links are (only) sponsored by players in P_1. To prove the statement, we need to show that equilibrium conditions (iii)–(v) from Lemma 1 strictly hold for $k \in (\underline{k}, \bar{k}) \neq \varnothing$ given the equilibrium activity pattern described above.

It is obvious that condition (v) is met as $x_1 < x_2$; links are sponsored by players in P_1 and directed toward players in P_2. For the same reason, condition (iv) (preventing link deletion) is written as

$$h(|P_2|x_2) - h(|P_2|x_2 - x_2) \geq k. \tag{10}$$

Condition (iii) (preventing link creation) can be split into two conditions, namely that players in P_2 do not gain from adding a link to another player in P_2 and that players in P_1 do not gain from adding a link to a player in P_1. However, as h is strictly increasing and concave, and $|P_2|x_2 < |P_1|x_1$ by optimal activity, the following two inequalities hold:

$$h(|P_1|x_1 + x_2) - h(|P_1|x_1) < h(|P_2|x_2 + x_2) - h(|P_2|x_2)$$
$$h(|P_2|x_2 + x_1) - h(|P_2|x_2) < h(|P_2|x_2 + x_2) - h(|P_2|x_2).$$

Therefore, the condition below preventing players in P_1 from adding a link to a hypothetical player in P_2, is sufficient for equilibrium condition (iii):

$$h(|P_2|x_2 + x_2) - h(|P_2|x_2) \leq k. \tag{11}$$

As h is strictly concave, we know that

$$h(|P_2|x_2 + x_2) - h(|P_2|x_2) < h(|P_2|x_2) - h(|P_2|x_2 - x_2)$$

so that (10) and (11) simultaneously strictly hold for some interval $(\underline{k}, \bar{k}) \neq \varnothing$. □

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