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Strategy-proofness and efficiency with non-quasi-linear preferences: A characterization of minimum price Walrasian rule

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We consider the problem of allocating objects to a group of agents and how much agents should pay. Each agent receives at most one object and has non-quasi-linear preferences. Non-quasi-linear preferences describe environments where payments influence agents' abilities to utilize objects or derive benefits from them. The *minimum price Walrasian* (MPW) *rule* is the rule that assigns a minimum price Walrasian equilibrium allocation to each preference profile. We establish that the MPW rule is the unique rule satisfying *strategy-proofness*, *efficiency*, *individual rationality*, and *no subsidy for losers*. Since the outcome of the MPW rule coincides with that of the *simultaneous ascending* (SA) *auction*, our result supports SA auctions adopted by many governments.

Keywords. Minimum price Walrasian equilibrium, simultaneous ascending auction, strategy-proofness, efficiency, heterogeneous objects, non-quasi-linear preferences.

JEL classification. D44, D47, D71, D82.

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1. Introduction

1.1 *Purpose*

Since the 1990s, governments in numerous countries have conducted auctions to allocate a variety of objects or assets including spectrum rights, vehicle ownership licenses, and land. Although auctions sometimes make a large amount of government revenue, the announced goals of many government auctions are rather to allocate objects "efficiently," i.e., to agents who benefit most from them.[1] Agents who benefit more are willing to pay higher prices and thus, have a better chance to win the auctions. However, as mentioned below, large-scale auction payments would influence agents' abilities to utilize objects or benefit from them, thereby complicating efficient allocations. This article analyzes rules that allocate auctioned objects efficiently even when payments are so large that they impair agents' abilities to utilize them or realize their benefits. We investigate what types of allocation rules can allocate objects efficiently in such environments.

1.2 *Main result*

An *allocation rule*, or simply a *rule*, is a function that assigns to each preference profile an *allocation*, which consists of an assignment of objects and agents' payments. Each agent receives one object at most, and has a preference over objects and payments.[2] The *domain* of rules is the class of preference profiles. We assume that preferences satisfy monotonicity,[3] continuity, and *finiteness*, which means that, given an assignment, any change of assigned object is compensated by a finite amount of money. We call such preferences *classical*. It is well known that in this model, there is a minimum price Walrasian equilibrium (MPWE),[4] and that the allocation associated with the MPWE coincides with the outcome of a certain type of auction called the *simultaneous ascending* (SA) *auction*.[5] Under SA auctions, bids on all objects start simultaneously, and the sale of any object is not settled as long as new bids are made on some objects. We focus on the rule that assigns an MPWE allocation to each preference profile. We refer to this rule as the *minimum price Walrasian* (MPW) *rule*.

The MPW rule satisfies four desirable properties. The first is (*Pareto*) *efficiency*. An allocation is *efficient* if no agent can be made better off without either some other agents being made worse off or the government's revenue being reduced.[6] The second is *strategy-proofness*. Note that efficiency is evaluated based on agents' preferences. Thus, an efficient allocation cannot be chosen without information about preferences. Since

[1] For example, frequency auctions in the United States were introduced to promote "efficient and intensive use of the electromagnetic spectrum." See McAfee and McMillan (1996, p. 160).

[2] Each agent knows his own preference. In this sense, our model is one of private value models.

[3] More precisely, in this article, we introduce two types of monotonicity assumptions, which we call money monotonicity and desirability of objects. See Section 2 for the formal definitions.

[4] See Demange and Gale (1985).

[5] For example, see Demange et al. (1986).

[6] In our auction model, efficiency is defined by taking government revenue into account.

preferences are private information, agents may have an incentive to behave strategically to influence the final outcome in their favor. *Strategy-proofness* is an incentive-compatibility property, which gives a strong incentive for each agent to reveal his true preference. It says that for each preference profile, in the normal form game induced by the rule, it is a (weakly) dominant strategy for each agent to reveal his true preference. The MPW rule satisfies strategy-proofness[7] and chooses an efficient allocation corresponding to the revealed preferences.

The third property of the MPW rule is *individual rationality*, which requires that no agent should be made worse off than if he had received no object and paid nothing. This property induces voluntary participation. The fourth property is *no subsidy for losers*. Under the MPW rule, the governments never subsidize losers. This property prevents agents who do not need objects from flocking to auctions only to sponge subsidies.

The primary conclusion of this article is that only the minimum price Walrasian rule satisfies strategy-proofness, efficiency, individual rationality, and no subsidy for losers (Theorem 2). Since the outcome of the MPW rule coincides with that of the SA auction (Proposition 1), the result supports SA auctions adopted by many governments.

1.3 *Related literature*

Holmström (1979) establishes a fundamental result relating to our question that applies when agents' benefits from auctioned objects are not influenced by their payments, i.e., agents have "quasi-linear" preferences. He assumes that preferences are quasi-linear, and shows that only the Vickrey–Clarke–Groves (VCG)[8] type allocation rules satisfy strategy-proofness and efficiency.[9] His result implies that on the quasi-linear domain, only the Vickrey rule[10] satisfies strategy-proofness, efficiency, individual rationality, and no subsidy for losers.[11] As Marshall (1920) demonstrates, preferences are approximately quasi-linear if payments for goods we analyze are sufficiently low.[12] However, quasi-linearity is not an appropriate assumption for large-scale auctions. Excessive payments for the auctioned objects may damage bidders' budgets to purchase complements for effective uses of the objects and thus, may influence the benefits from the objects. Alternatively, bidders may need to obtain loans to bid high amounts, and typically financial costs are nonlinear in borrowings, which makes bidders' preferences on objects and payments non-quasi-linear.[13] In spectrum license auctions and vehicle ownership

[7]In addition, the MPW rule is *group strategy-proof*, i.e., by jointly misrepresenting their preferences, no group of agents should obtain assignments that they prefer.

[8]See Vickrey (1961), Clarke (1971), and Groves (1973).

[9]More precisely, Holmström (1979) studies public goods models. When agents have quasi-linear preferences, his result can be applied to the auction model.

[10]See Section 6 for the formal definition.

[11]Recall that the payment of an agent under the VCG rule is decomposed into two parts. The first part is what is called *Vickrey price*, the social opportunity cost to allocate him an object; the second part is the term that is independent of his preference. Individual rationality and no subsidy for losers imply that the second part is zero. See also Chew and Serizawa (2007).

[12]See also Vives (1987) and Hayashi (2013) for mathematical arguments.

[13]See Saitoh and Serizawa (2008) for numerical examples.

license auctions, license prices often equal or exceed bidders' annual revenues. Thus, bidders' preferences are non-quasi-linear for such important auctions.[14] As contrasted with Holmström (1979), our result applies to such environments.

Saitoh and Serizawa (2008) investigate a problem similar to ours in the case where the domain includes non-quasi-linear preferences and there are multiple copies of the same object. They generalize Vickrey rules by employing compensated valuations from no object and no payment, and characterize the generalized Vickrey rule by strategy-proofness, efficiency, individual rationality, and no subsidy.[15] We stress that when preferences are not quasi-linear, the heterogeneity of objects makes the MPW rule different from the generalized Vickrey rule.[16]

Although the assumption of quasi-linearity neglects the serious effects of large-scale auction payments in actual practice, it is difficult to investigate the above question without this assumption. Quasi-linearity simplifies the description of efficient allocations. More precisely, under quasi-linear preferences, an efficient allocation of objects can be achieved simply by maximizing the sum of realized benefits from objects (agents' net benefits), and hence, is independent of how much agents pay. In this sense, Holmström (1979) characterizes only the payment part of strategy-proof and efficient rules. On the other hand, without quasi-linearity, efficient allocations of objects do depend on payments and thus, cannot be simply identified in the same way as in the quasi-linear case. In this article, we overcome that difficulty. Furthermore, as mentioned earlier, on non-quasi-linear domains, the MPW rule is different from the generalized Vickrey rule, and the former outperforms the latter in terms of our desirable properties, i.e., strategy-proofness and efficiency are satisfied by the MPW rule, but not by the generalized Vickrey rule. Needless to say, Holmström's (1979) results cannot be applied to prove our results on the non-quasi-linear domain. It is worthwhile to mention that most standard results of auction theory, such as the revenue equivalence theorem, also depend on assuming quasi-linearity. Recently, Baisa (2013) studies an auction model where probabilistic allocations are accommodated and he demonstrates that the effect of non-quasi-linearity makes optimal mechanisms qualitatively different.

Since Hurwicz's (1972) seminal work, many authors have investigated efficient and strategy-proof rules in pure exchange economies.[17] In pure exchange economies, classical[18] preferences are standard, but no rule is strategy-proof, efficient, and individually rational on the classical domain. On the other hand, Demange and Gale (1985) show that, in the model studied in this article, the MPW rule is strategy-proof, efficient,

[14] Ausubel and Milgrom (2002) also discuss the importance of the analysis under non-quasi-linear preferences. See Baisa (2013) for more examples of non-quasi-linear preferences.

[15] Sakai (2008) also obtains a result similar to theirs.

[16] In Section 6, we give a detailed discussion on this point by contrasting the MPW rule with the generalized Vickrey rule.

[17] For example, see Zhou (1991), Barberà and Jackson (1995), Schummer (1997), Serizawa (2002), and Serizawa and Weymark (2003).

[18] In pure exchange economies, where consumption spaces are some multidimensional Euclidean space, classical preferences are assumed to satisfy convexity in addition to continuity and monotonicity. Clearly, the class of such preferences contains non-quasi-linear preferences.

and individually rational on the classical domain.[19] Generalizing the MPW rule to the situations where price ranges are bounded, Andersson and Svensson (2014) introduce the *minimum rationing price equilibrium rule*, and demonstrate that it satisfies (group) strategy-proofness and a weak variant of efficiency. Miyake (1998) shows that only the MPW rule satisfies strategy-proofness among *Walrasian rules*.[20] Note that the Walrasian rules are a small part of the class of allocation rules satisfying efficiency, individual rationality, and no subsidy for losers. By developing analytical tools different from Miyake's (1998),[21] we extend his characterization in that we establish the uniqueness of the rules satisfying the desirable properties without confinement to Walrasian rules.

Many authors have analyzed SA auctions in quasi-linear settings (e.g., Gul and Stacchetti 2000, Ausubel and Milgrom 2002, Ausubel 2004, 2006, de Vries et al. 2007, Mishra and Parkes 2007, Andersson et al. 2013). In non-quasi-linear settings, the MPW rules differ from the generalized Vickrey rules, and it is the MPWE allocation that coincides with the outcome of the SA auction. Alaei et al. (2013) construct an alternative algorithm computing MPWE in non-quasi-linear settings. Our result demonstrates that the SA auction and alternative algorithms analyzed by those authors are more important in non-quasi-linear settings.

The problems of allocating objects and money have been studied by many authors. One of the extensively studied problems not referenced above is the one of fair (*envy-free*) allocation (Svensson 1983, Maskin 1987, Alkan et al. 1991, Tadenuma and Thomson 1991).[22] In the context of strategy-proofness, fair allocation rules are investigated by Tadenuma and Thomson (1995), Sun and Yang (2003), Ohseto (2006), and Svensson (2004, 2009).[23]

When Svensson (2004, 2009) characterizes the class of strategy-proof and envy-free rules, he does not impose no subsidy for losers on rules, but imposes only the *nonnegativity of the sum of payments*—the requirement that the sum of the agents' payments be nonnegative.[24] This alternative requirement is mild and natural. However, we emphasize that envy-freeness is a strong requirement in his model and in ours. When each object is assigned to some agent, envy-freeness implies efficiency (Svensson 1983) and is almost equivalent to Walrasian equilibrium conditions. Given an allocation such that each object is assigned to some agent, take the price vector such that the price of each object is the payment of the agent who receives it. Envy-freeness implies that for this

[19] More precisely, Demange and Gale (1985) study two-sided matching markets that contain our model as a special case and show that the rules selecting an optimal stable assignment for one side of the market are group strategy-proof for the agents on that side.

[20] A *Walrasian rule* is the rule that assigns a Walrasian equilibrium allocation to each preference profile.

[21] In Appendix B, we discuss why different analytical tools are necessary.

[22] *Envy-freeness* (Foley 1967) is the requirement that no agent should prefer anyone else's assignment to his own.

[23] Some authors also investigate the problem by other fairness axioms. See, for example, Ashlagi and Serizawa (2012) and Mukherjee (2014) for the axiom of *anonymity in welfare*, and see Sakai (2013) and Adachi (2014) for the axiom of *weak envy-freeness for equals*.

[24] To be precise, he requires that the sum of the agents' payments have a lower bound. This requirement implies that the total subsidy is limited by a prespecified level, but not that the subsidy to an individual agent is limited.

price vector, each agent demands the object he receives in the given allocation. Since we do not impose envy-freeness on rules, our results and the results of Svensson (2004, 2009) are logically independent.

Other authors have investigated the existence of strategy-proof and *nonbossy* rules.[25] Miyagawa (2001) characterizes the class of strategy-proof, nonbossy, individually rational, and onto rules. Svensson and Larsson (2002) characterize the classes of strategy-proof and nonbossy rules with several additional desirable properties.[26] It is well known that nonbossiness together with strategy-proofness makes the analysis tractable. Since the MPW rules violate nonbossiness, we do not impose this demanding property and thus, cannot apply their proof techniques in our proof.

1.4 *Organization*

The article is organized as follows. Section 2 sets up the model and introduces basic concepts. Section 3 defines the MPWE and discusses its properties. Section 4 provides our main result. Section 5 defines the SA auction, and shows that its outcome coincides with the MPWE. Section 6 introduces the generalized Vickrey rules and contrasts them with the MPW rules. Section 7 concludes. Most proofs appear in the Appendix. Proofs omitted from the main paper are given in a supplementary file on the journal website, http://econtheory.org/supp/1470/supplement.pdf.

2. The model and definitions

There are n agents and m objects, where $2 \leq n < \infty$ and $1 \leq m < \infty$. We denote the set of agents by $N \equiv \{1, \dots, n\}$ and the set of objects by $M \equiv \{1, \dots, m\}$. Let $L \equiv \{0\} \cup M$. Each agent consumes one object at most. We denote the object that agent $i \in N$ receives by $x_i \in L$. Object 0 is referred to as the null object, and $x_i = 0$ means that agent i receives no "real" object. We denote the amount that agent i pays by $t_i \in \mathbb{R}$. For each $i \in N$, agent i's *consumption set* is $L \times \mathbb{R}$, and a (*consumption*) *bundle* for agent i is a pair $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$. Let $\mathbf{0} \equiv (0, 0)$.

Each agent i has a complete and transitive preference relation R_i on $L \times \mathbb{R}$. Let P_i and I_i, respectively, be the strict relation and the indifference relation associated with R_i. Given a preference R_i and a bundle z_i, let the upper contour set and lower contour set of R_i at z_i be $UC(R_i, z_i) \equiv \{z_i' \in L \times \mathbb{R} : z_i' \, R_i \, z_i\}$ and $LC(R_i, z_i) \equiv \{z_i' \in L \times \mathbb{R} : z_i \, R_i \, z_i'\}$, respectively. For each $i \in N$, agent i's preference R_i satisfies the following properties.

Money monotonicity. For each $x_i \in L$ and each $t_i, t_i' \in \mathbb{R}$, if $t_i' < t_i$, then $(x_i, t_i') \, P_i \, (x_i, t_i)$.

Finiteness. For each $t_i \in \mathbb{R}$ and each $x_i, x_i' \in L$, there exist $t_i', t_i'' \in \mathbb{R}$ such that $(x_i', t_i') \, R_i \, (x_i, t_i)$ and $(x_i, t_i) \, R_i \, (x_i', t_i'')$.

[25] *Nonbossiness* (Satterthwaite and Sonnenschein 1981) is the requirement that when an agent's preferences change, if his assignment remains the same, then the chosen allocation should remain the same.

[26] See also Schummer (2000) for the other analysis of strategy-proof and nonbossy rules.

CONTINUITY. For each $z_i \in L \times \mathbb{R}$, $UC(R_i, z_i)$ and $LC(R_i, z_i)$ both are closed.

Let $\mathcal{R}^E$ denote the class of money monotonic, finite, and continuous preferences—the *extended domain*. Given $R_i \in \mathcal{R}^E$, $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$, and $y_i \in L$, we define the *compensating valuation* $cv_i(y_i; z_i)$ of y_i from z_i for R_i by $(y_i, t_i + cv_i(y_i; z_i))\ I_i\ z_i$, and we let $CV_i(y_i; z_i) \equiv t_i + cv_i(y_i; z_i)$. We refer to $CV_i(y_i; z_i)$ as the *compensated valuation* of y_i from z_i for R_i. Note that by continuity and finiteness, $CV_i(y_i; z_i)$ exists, and by money monotonicity, $CV_i(y_i; z_i)$ is unique. The compensated valuation for R_i' is denoted by CV_i'.

We introduce another property of preferences.

DESIRABILITY OF OBJECTS. For each $x_i \in M$ and each $t_i \in \mathbb{R}$, $(x_i, t_i)\ P_i\ (0, t_i)$.[27]

DEFINITION 1. A preference R_i is *classical* if it satisfies money monotonicity, finiteness, continuity, and desirability of objects.

Let $\mathcal{R}^C$ denote the class of classical preferences—the *classical domain*. Note that $\mathcal{R}^C \subsetneq \mathcal{R}^E$.

DEFINITION 2. A preference R_i is *quasi-linear* if there is a "valuation function" $v_i : L \to \mathbb{R}_+$ such that (i) $v_i(0) = 0$, (ii) for each $x \in M$, $v_i(x) > 0$, and (iii) for each $z_i \equiv (x_i, t_i) \in L \times \mathbb{R}$ and each $z_i' \equiv (x_i', t_i') \in L \times \mathbb{R}$, $z_i\ R_i\ z_i'$ if and only if $v_i(x_i) - t_i \geq v_i(x_i') - t_i'$.

Let $\mathcal{R}^Q$ denote the class of quasi-linear preferences—the *quasi-linear domain*. Note that $\mathcal{R}^Q \subsetneq \mathcal{R}^C$.

An *object allocation* is an n-tuple $(x_1, \ldots, x_n) \in L^n$ such that for each $i, j \in N$, if $x_i \neq 0$ and $i \neq j$, then $x_i \neq x_j$, that is, no two agents receive the same object except when both receive the null object. Let X be the set of object allocations. A *(feasible) allocation* is an n-tuple $z \equiv (z_1, \ldots, z_n) \equiv ((x_1, t_1), \ldots, (x_n, t_n)) \in [L \times \mathbb{R}]^n$ of bundles such that $(x_1, \ldots, x_n) \in X$. Let Z be the set of feasible allocations. We denote the object allocation and the agents' payments at $z' \in Z$ by $x' \equiv (x_1', \ldots, x_n')$ and $t' \equiv (t_1', \ldots, t_n')$, respectively.

Let $\mathcal{R}$ be a class of preferences such that $\mathcal{R} \subseteq \mathcal{R}^E$. A *preference profile* is an n-tuple $R \equiv (R_1, \ldots, R_n) \in \mathcal{R}^n$. Given $R \equiv (R_1, \ldots, R_n) \in \mathcal{R}^n$ and $N' \subseteq N$, let $R_{N'} \equiv (R_i)_{i \in N'}$ and $R_{-N'} \equiv (R_i)_{i \in N \setminus N'}$.

An *allocation rule*, or simply a *rule*, on $\mathcal{R}^n$ is a function f from $\mathcal{R}^n$ to Z. Given a rule f and a preference profile $R \in \mathcal{R}^n$, we denote agent i's assigned object under f at R by $f_i^x(R)$ and denote his payment by $f_i^t(R)$, and we write

$$f_i(R) \equiv (f_i^x(R), f_i^t(R)), \qquad f(R) \equiv (f_1(R), \ldots, f_n(R)), \quad \text{and} \quad f^x(R) \equiv (f_j^x(R))_{j \in N}.$$

We introduce basic properties of rules. The efficiency condition defined below takes the auctioneer's preference into account and assumes that he is only interested in his revenue. An allocation $z' \in Z$ *(Pareto-) dominates* $z \in Z$ for $R \in \mathcal{R}^n$ if

[27]A preference R_i satisfies *weak desirability of objects* if for each $x_i \in M$, $(x_i, 0)\ P_i\ \mathbf{0}$. All the results in this article still hold if desirability of objects is replaced by weak desirability of objects.

(i) $\sum_{i\in N} t'_i \geq \sum_{i\in N} t_i$

(ii) for each $i \in N$, $z'_i \, R_i \, z_i$, and

(iii) for some $j \in N$, $z'_j \, P_j \, z_j$.

An allocation $z \in Z$ is *(Pareto) efficient* for $R \in \mathcal{R}^n$ if there is no feasible allocation that dominates z for R.

Efficiency. For each $R \in \mathcal{R}^n$, $f(R)$ is efficient for R.

Individual rationality says that a rule should never select an allocation at which some agent is worse off than if he had received the null object and paid nothing. *No subsidy* says that the payments should always be nonnegative. *No subsidy for losers* says that the payments of agents who obtain the null object should always be nonnegative. No subsidy implies no subsidy for losers.

Individual rationality. For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i(R) \, R_i \, \mathbf{0}$.

No subsidy. For each $R \in \mathcal{R}^n$ and each $i \in N$, $f_i^t(R) \geq 0$.

No subsidy for losers. For each $R \in \mathcal{R}^n$ and each $i \in N$, if $f_i^x(R) = 0$, then $f_i^t(R) \geq 0$.

The two properties below have to do with incentives. First, by misrepresenting his preferences, no agent should obtain an assignment that he prefers.

Strategy-proofness. For each $R \in \mathcal{R}^n$, each $i \in N$, and each $R'_i \in \mathcal{R}$, $f_i(R) \, R_i \, f_i(R'_i, R_{-i})$.

The second property is stronger: by jointly misrepresenting their preferences, no group of agents should obtain assignments that they prefer.

Group strategy-proofness. For each $R \in \mathcal{R}^n$ and each $N' \subseteq N$, there is no $R'_{N'} \in \mathcal{R}^{|N'|}$ such that for each $i \in N'$, $f_i(R'_{N'}, R_{-N'}) \, P_i \, f_i(R)$.[28]

3. Minimum price Walrasian equilibrium

3.1 *Definition of Walrasian equilibria*

We define Walrasian equilibrium and minimum price Walrasian equilibrium. Let $\mathcal{R} \subseteq \mathcal{R}^E$ in this section. All results in this section also hold on the classical domain $\mathcal{R}^C$.

Let $p \equiv (p^1, \ldots, p^m) \in \mathbb{R}_+^m$ be a price vector. The *budget set at prices* p is defined as $B(p) \equiv \{(x, p^x) : x \in L\}$, where $p^x = 0$ if $x = 0$. Given $i \in N$, $R_i \in \mathcal{R}$, and $p \in \mathbb{R}_+^m$, agent i's *demand set* is defined as $D(R_i, p) \equiv \{x \in L : \text{for each } y \in L, (x, p^x) \, R_i \, (y, p^y)\}$.

[28]Let $|A|$ denote the cardinality of set A.

Definition 3. Let $R \in \mathcal{R}^n$. A pair $((x,t),p) \in Z \times \mathbb{R}^m_+$ is a *Walrasian equilibrium for* R if

(WE-i) for each $i \in N$, $x_i \in D(R_i, p)$ and $t_i = p^{x_i}$, and

(WE-ii) for each $y \in M$, if for each $i \in N$, $x_i \neq y$, then $p^y = 0$.

Condition (WE-i) says that each agent receives an object he demands and pays its price. Condition (WE-ii) says that an object's price is zero if it is not assigned.

Fact 1. *For each $R \in \mathcal{R}^n$, there is a Walrasian equilibrium for R.*

Fact 1 is already known.[29] Given $R \in \mathcal{R}^n$, let $W(R)$ be the set of Walrasian equilibria for R, and let $Z(R)$ and $P(R)$ be the sets of Walrasian equilibrium allocations and prices for R, respectively, i.e.,

$$Z(R) \equiv \{z \in Z : \text{for some } p \in \mathbb{R}^m_+, (z,p) \in W(R)\} \quad \text{and}$$
$$P(R) \equiv \{p \in \mathbb{R}^m_+ : \text{for some } z \in Z, (z,p) \in W(R)\}.$$

Next is a first welfare theorem for our model.[30]

Fact 2. *Let $R \in \mathcal{R}^n$ and $z \in Z(R)$. Then z is efficient for R.*[31]

Fact 3 says that for each preference profile, there is a unique minimum Walrasian equilibrium price vector. The *minimum price Walrasian equilibrium* (hereafter MPWE) is the Walrasian equilibria associated with the minimum price.

Fact 3 (Demange and Gale 1985). *For each $R \in \mathcal{R}^n$, there is a unique $p' \in P(R)$ such that for each $p \in P(R)$, $p' \leq p$.*

Let $p_{\min}(R)$ denote this price vector for R.

Given $R \in \mathcal{R}^n$, let $W_{\min}(R)$ be the set of minimum price Walrasian equilibria for R and let

$$Z_{\min}(R) \equiv \big\{z \in Z : (z, p_{\min}(R)) \in W_{\min}(R)\big\}.$$

[29] For example, see Alkan and Gale (1990). Our model is a special case of theirs.

[30] See also Svensson (1983).

[31] To see this, suppose that $z \equiv (z_1, \dots, z_n)$ is not efficient for R. Then there is $z' \equiv (z'_1, \dots, z'_n)$ such that

(i) $\sum_{i \in N} t'_i \geq \sum_{i \in N} t_i$

(ii) for each $i \in N$, $z'_i \, R_i \, z_i$

(iii) for some $j \in N$, $z'_j \, P_j \, z_j$.

Since $z \in Z(R)$, there is a price vector $p \in \mathbb{R}^m_+$ such that $(z,p) \in W(R)$. Then, by (ii) and (WE-i), for each $i \in N$, $t'_i \leq p^{x'_i}$. By (iii) and (WE-i), $t'_j < p^{x'_j}$. Thus, $\sum_{i \in N} t'_i < \sum_{i \in N} p^{x'_i} = \sum_{i \in N} t_i$. This contradicts (i).

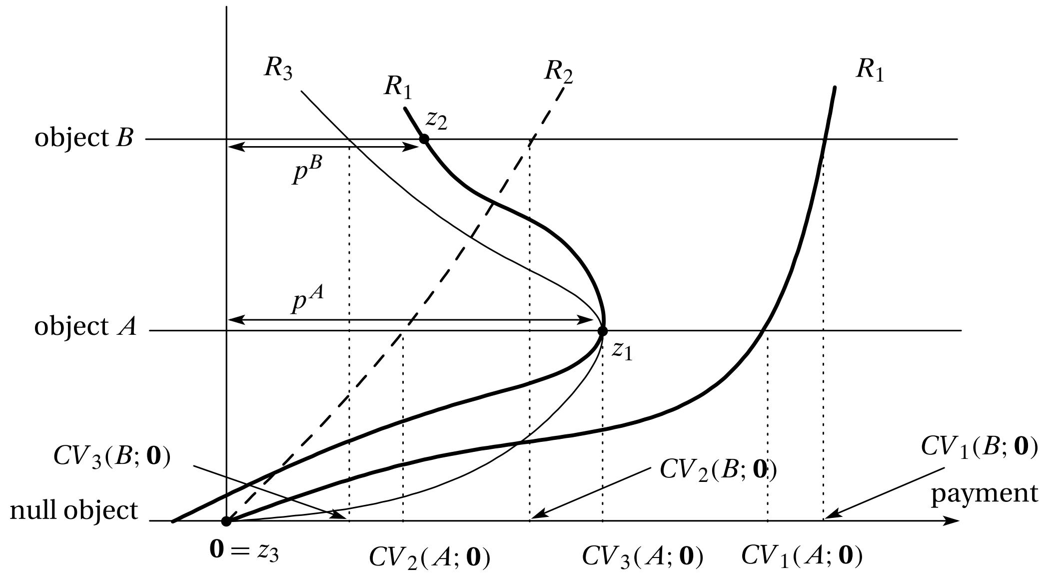

FIGURE 1. Illustration of non-quasi-linear preferences and the minimum price Walrasian equilibrium.

By Facts 1 and 3, for each $R \in \mathcal{R}^n$, the set $Z_{\min}(R)$ is nonempty. Although the correspondence $Z_{\min}$ is set-valued, it is *essentially single-valued*, i.e., for each $R \in \mathcal{R}^n$, each pair $z, z' \in Z_{\min}(R)$, and each $i \in N$, $z_i \, I_i \, z'_i$.[32]

As Demange et al. (1986), e.g., show for the quasi-linear domain, and as shown for our domain (Section 5), the SA auctions achieve the MPWE.

3.2 *Illustration of minimum price Walrasian equilibrium*

Figure 1 illustrates an MPWE for three agents, and two objects, say A and B. There are three horizontal lines. The lowest one corresponds to the null object. The middle and highest lines correspond to the real objects A and B, respectively. The intersection of the vertical line and each horizontal line denotes the bundle consisting of the corresponding object and no payment. For example, the origin $\mathbf{0}$ denotes the bundle consisting of the null object and no payment. For each point z_i on one of the three horizontal lines, the distance from z_i to the vertical line denotes payment. For example, z_1 denotes the bundle consisting of object A and payment p^A. Indifference between bundles is shown by a curvy line connecting them. Welfare increases with decreasing payments. Thus, in Figure 1, agent 1 prefers z_1 to $\mathbf{0}$.

Assume that preferences are as depicted in Figure 1. The compensated valuations from the origin are ranked as $CV_1(A; \mathbf{0}) > CV_3(A; \mathbf{0}) > CV_2(A; \mathbf{0})$ and $CV_1(B; \mathbf{0}) >$

[32]An allocation $z' \in Z$ *is obtained by an indifferent permutation from* $z \in Z$ if there is a permutation π on N such that for each $i \in N$, $z'_i = z_{\pi(i)}$ and $z'_i \, I_i \, z_i$ (Tadenuma and Thomson 1991). Note that for each pair $z, z' \in Z_{\min}(R)$, z' is obtained by an indifferent permutation from z.

$CV_2(B; \mathbf{0}) > CV_3(B; \mathbf{0})$. In Figure 1, agent 1's preference is not quasi-linear, but classical.[33] Thus, Figure 1 also illustrates that $\mathcal{R}^Q \subsetneq \mathcal{R}^C$.

The MPWE for the preference profile $R = (R_1, R_2, R_3)$ is as follows: Agent 1 receives object A and pays $CV_3(A; \mathbf{0})$, i.e., the price p^A of object A is $CV_3(A; \mathbf{0})$. His consumption is z_1. Agent 2 receives object B and pays $CV_1(B; z_1)$, i.e., the price p^B of object B is $CV_1(B; z_1)$. His consumption is z_2. Agent 3's consumption is $\mathbf{0}$ and is depicted as z_3.

Let us see why the allocation $z \equiv (z_1, z_2, z_3)$ is an MPWE for R. First, note that for each agent $i = 1, 2, 3$, z_i is maximal for R_i in the budget set $\{\mathbf{0}, (A, p^A), (B, p^B)\}$. Thus, z is a Walrasian equilibrium.

Next, let (p'^A, p'^B) be a Walrasian equilibrium price vector. We show $p'^A \geq p^A$ and $p'^B \geq p^B$. If $p'^A < p^A$ and $p'^B < p^B$, then all agents prefer (A, p'^A) or (B, p'^B) to $\mathbf{0}$, that is, all three agents demand A or B or both. In that case, one agent cannot receive an object he demands, contradicting (WE-i) in Definition 3. Thus, $p'^A \geq p^A$ or $p'^B \geq p^B$. If $p'^A < p^A$, then $p'^B \geq p^B$, and so both agents 1 and 3 prefer (A, p'^A) to $\mathbf{0}$ and (B, p'^B), that is, both demand only A. In that case, agents 1 or 3 cannot receive the object they demand, contradicting Walrasian equilibrium. Therefore, $p'^A \geq p^A$. If $p'^B < p^B$, both agents 1 and 2 prefer (B, p'^B) to $\mathbf{0}$ and (A, p'^A), and so agents 1 or 2 cannot receive the object they demand, contradicting Walrasian equilibrium. Therefore, $p'^B \geq p^B$. Hence, (z, p) is the MPWE.

3.3 *Overdemanded and underdemanded sets*

Next, we introduce the concepts of overdemanded set and underdemanded set (Mishra and Talman 2010, e.g.), and relate these concepts to Walrasian equilibria.

Definition 4. (i) A set $M' \subseteq M$ of objects is *(weakly) overdemanded at p for R* if

$$\left|\{i \in N : D(R_i, p) \subseteq M'\}\right| (\geq) > |M'|.$$

(ii) A set $M' \subseteq M$ of objects is *(weakly) underdemanded at p for R* if

$$[\forall x \in M', p^x > 0] \Longrightarrow \left|\{i \in N : D(R_i, p) \cap M' \neq \varnothing\}\right| (\leq) < |M'|.$$

In Figure 1, note that $\{i \in N : D(R_i, p) \subseteq \{A\}\} = \varnothing$, $\{i \in N : D(R_i, p) \subseteq \{B\}\} = \{2\}$, $\{i \in N : D(R_i, p) \subseteq \{A, B\}\} = \{1, 2\}$, $\{i \in N : D(R_i, p) \cap \{A\} \neq \varnothing\} = \{1, 3\}$, $\{i \in N : D(R_i, p) \cap \{B\} \neq \varnothing\} = \{1, 2\}$, and $\{i \in N : D(R_i, p) \cap \{A, B\} \neq \varnothing\} = \{1, 2, 3\}$. Thus, no set is overdemanded or weakly underdemanded.

Fact 4 and Theorem 1 below are established by Mishra and Talman (2010) for quasi-linear preferences. Fact 4 is a characterization of Walrasian equilibria by means of the concepts of overdemanded and underdemanded sets. Their proof also works for Fact 4 in the extended domain.

[33] Suppose that agent 1's preference is quasi-linear. Then since $CV_1(B, \mathbf{0}) > CV_1(A, \mathbf{0})$, agent 1's compensated valuation $CV_1(B, z_1)$ of object B from the point z_1 in Figure 1 must be greater than $CV_2(B, \mathbf{0})$. However, in Figure 1, agent 1 prefers z_1 to the point $(B, CV_2(B, \mathbf{0}))$. This is a contradiction.

FACT 4 (Mishra and Talman 2010). *Let $R \in \mathcal{R}^n$. A price vector p is a Walrasian equilibrium price vector for R if and only if no set is overdemanded and no set is underdemanded at p for R.*

Theorem 1 is a characterization of the *minimum* price Walrasian equilibrium by means of the concepts of overdemanded and *weakly* underdemanded sets. We emphasize, in contrast to Fact 4, that Mishra and Talman's (2010) proof crucially depends on quasi-linearity. It relies on the simple fact that when preferences are quasi-linear, if a set M' is weakly underdemanded at a Walrasian equilibrium price vector p, then all the prices of M' can be slightly lowered by the same amount while maintaining the Walrasian equilibrium conditions (WE-i) and (WE-ii).[34] However, this is not true when preferences are not quasi-linear. Theorem 1 is a novel result, and is the key to obtaining Theorem 2 and Proposition 1.

THEOREM 1. [35] *Let $R \in \mathcal{R}^n$. A price vector p is a minimum Walrasian equilibrium price vector for R if and only if no set is overdemanded and no set is weakly underdemanded at p for R.*

Corollary 1 says that if the number of objects is greater than or equal to the number of agents, the price of some objects is 0. It is used to prove Fact 6. Corollary 2 says that each object whose price is positive is "connected" by agents' demands to the null object or to an object with a price of 0. This corollary is used to prove Theorem 2.[36] For example, in Figure 1, object B has a positive equilibrium price, agent 1's demand connects objects A and B, and agent 3's demand connects object A and the null object.

COROLLARY 1 (Existence of free object). *Let $m \geq n$, $R \in \mathcal{R}^n$, and $z \in Z_{\min}(R)$. Then there is $i \in N$ such that $p_{\min}^{x_i}(R) = 0$.*

COROLLARY 2 (Demand connectedness). [37] *Let $R \in \mathcal{R}^n$ and $(z, p) \in W_{\min}(R)$. For each $x \in M$ with $p^x > 0$, there is a sequence $\{i_k\}_{k=1}^K$ of K distinct agents such that (i) $x_{i_1} = 0$ or $p^{x_{i_1}} = 0$, (ii) for each $k \in \{2, \dots, K-1\}$, $x_{i_k} \neq 0$ and $p^{x_{i_k}} > 0$, (iii) $x_{i_K} = x$, and (iv) for each $k \in \{1, \dots, K-1\}$, $\{x_{i_k}, x_{i_{k+1}}\} \subseteq D(R_{i_k}, p)$.*

Here, we also introduce a concept of d_i-truncation of a preference. This concept is important to prove Theorem 1. It says that the welfare position of each bundle $z_i \in M \times \mathbb{R}$ is lowered as much as d_i in terms of money, but their relative positions are kept.

Given $R_i \in \mathcal{R}$ and $d_i \in \mathbb{R}$, the *d_i-truncation of R_i* is the preference R_i' such that for each $z_i \in M \times \mathbb{R}$, $CV_i'(0; z_i) = CV_i(0; z_i) + d_i$. Given $R \in \mathcal{R}^n$ and $d \in \mathbb{R}^n$, the *d-truncation of R* is the preference profile R' such that for each $i \in N$, R_i' is the d_i-truncation of R_i.

The following remark and fact pertain to truncations. Remark 1(i) and Fact 5 are used to prove Theorem 1.

[34] For details, refer to the proof of Lemma 3 in Mishra and Talman (2010).
[35] Alaei et al. (2013) also establish this result independently by using different proof methods.
[36] See Lemma 12 for details.
[37] This structure is discussed by Demange et al. (1986) and Miyake (1998).

Remark 1. Let $R_i \in \mathcal{R}$, $d_i \in \mathbb{R}$, and R'_i be the d_i-truncation of R_i. Then the following statements hold:

(i) For each $z_i, \hat{z}_i \in M \times \mathbb{R}$, $z_i \; R_i \; \hat{z}_i$ if and only if $z_i \; R'_i \; \hat{z}_i$.

(ii) R'_i satisfies money monotonicity, finiteness, and continuity, and so $R'_i \in \mathcal{R}^E$.

(iii) For large d_i, R'_i violates desirability of objects.[38]

Fact 5 (Roth and Sotomayor 1990). *Let $R \in \mathcal{R}^n$ and let R' be a d-truncation of R such that for each $i \in N$, $d_i \geq 0$. Then $p_{\min}(R') \leq p_{\min}(R)$.*

4. Main results

In this section, we provide a characterization of the MPWE by means of properties of rules. Let $\mathcal{R} \subseteq \mathcal{R}^E$.

Definition 5. A rule f on $\mathcal{R}^n$ is a *minimum price Walrasian* (MPW) *rule* if for each $R \in \mathcal{R}^n$, $f(R) \in Z_{\min}(R)$.

4.1 *Properties of the minimum price Walrasian rule*

Let g be an MPW rule on $\mathcal{R}^n$. First, by Fact 2, for each $R \in \mathcal{R}^n$, $g(R)$ is efficient for R. Let $R \in \mathcal{R}^n$. Then there is a price vector $p \equiv (p^1, \ldots, p^m) \in \mathbb{R}^m_+$ such that for each $i \in N$, (a) $g_i(R) \in B(p)$, and (b) for each $z'_i \in B(p)$, $g_i(R) \; R_i \; z'_i$. Let $i \in N$. Note that, for each $x \in M$, $p^x \geq 0$ and $B(p) = \{(0, 0), (1, p^1), (2, p^2), \ldots, (m, p^m)\}$. Thus, by (a), $g^t_i(R) \geq 0$, and by (b), $g_i(R) \; R_i \; \mathbf{0}$. Therefore, the MPW rule satisfies efficiency, individual rationality, and no subsidy.

Fact 6 (Demange and Gale 1985). *The minimum price Walrasian rule is group strategy-proof.*

Theorem 1 allows a direct proof (see Appendix B).

4.2 *Characterizations*

In this subsection, we assume that each agent has a classical preference and the number of agents exceeds the number of objects. Recall that all results established in Section 3 also hold in this case. Theorem 2 is our main result of this article, a characterization of the MPW rule.

Theorem 2. *Let $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$. A rule f on $\mathcal{R}^n$ satisfies strategy-proofness, efficiency, individual rationality, and no subsidy for losers if and only if it is a minimum price Walrasian rule: for each $R \in \mathcal{R}^n$, $f(R) \in Z_{\min}(R)$.*

[38] Because of Remark 1(iii), a d_i-truncation of a classical preference may not be classical. However, this does not create any problems in the proofs of this article.

The proof is given in Appendix B. Since the MPW rules are group strategy-proof, Theorem 2 implies that only the MPW rules satisfy group strategy-proofness, efficiency, individual rationality, and no subsidy for losers. Since no subsidy implies no subsidy for losers, Theorem 2 also implies that only the MPW rules satisfy strategy-proofness, efficiency, individual rationality, and no subsidy.

4.3 *Indispensability of the axioms and assumptions*

The "only if" part of Theorem 2 fails if we drop any of the four axioms, as shown by the following examples.

Example 1 (Dropping strategy-proofness). Let f be the rule that chooses a "maximum" price Walrasian equilibrium allocation for each preference profile. Then f satisfies the axioms of Theorem 2 except for strategy-proofness.[39] $\Diamond$

Example 2 (Dropping efficiency). Let f be the rule such that for each preference profile, each agent receives the null object and pays nothing. Then f satisfies the axioms of Theorem 2 except for efficiency. $\Diamond$

Next, we introduce variants of Walrasian equilibria—those with "entry fees." Given an entry fee $e_i \in \mathbb{R}$, let $D(R_i, p, e_i) \equiv \{x \in L : \text{for each } y \in L, (x, p^x + e_i)\ R_i\ (y, p^y + e_i)\}$, where $p^x = 0$ if $x = 0$. A pair $((x, t), p) \in Z \times \mathbb{R}^m$ is a *Walrasian equilibrium with entry fees for* $R \in \mathcal{R}^n$ if there is an entry fee vector $e = (e_1, \dots, e_n) \in \mathbb{R}^n$ such that

(WE-i*) for each $i \in N$, $x_i \in D(R_i, p, e_i)$, and $t_i = p^{x_i} + e_i$, and

(WE-ii) for each $y \in M$, if for each $i \in N$, $x_i \neq y$, then $p^y = 0$.

Note that, similarly to Facts 1, 2, and 3, for each preference profile $R \in \mathcal{R}^n$ and each $e = (e_1, \dots, e_n) \in \mathbb{R}^n$, there is an MPWE with entry fees e, and it is efficient. A rule f is a *minimum price Walrasian rule with entry fees* if there is an entry fee vector $e \in \mathbb{R}^n$ and for each R, $f(R)$ is an MPWE with entry fees e. Then, MPW rules with entry fees are efficient. Similarly to Fact 6, we can show that they are also group strategy-proof.

Example 3 (Dropping individual rationality). Let $e = (e_1, \dots, e_n) \in \mathbb{R}^n$ be an entry fee vector such that for each $i \in N$, $e_i > 0$. Then the associated minimum price Walrasian rule with entry fees satisfies the axioms of Theorem 2 except for individual rationality. $\Diamond$

Example 4 (Dropping no subsidy for losers). Let $e = (e_1, \dots, e_n) \in \mathbb{R}^n$ be an entry fee vector such that for each $i \in N$, $e_i < 0$. Then the associated minimum price Walrasian rule with entry fees satisfies the axioms of Theorem 2 except for no subsidy for losers. $\Diamond$

[39] Demange and Gale (1985) also show that for each preference profile, there is a maximum price Walrasian equilibrium. When there is only one object, the maximum price Walrasian equilibrium corresponds to the first price auction. It is well known that the first price auction is not strategy-proof.

We further generalize MPW rules with entry fees as follows: a rule f is a minimum price Walrasian rule with *variable entry fees* if there is a list $\{e_i(\cdot)\}_{i\in N}$ of entry fee functions defined on $\mathcal{R}^n$, and for each R, $f(R)$ is an MPWE with entry fees $\{e_i(R)\}_{i\in N}$.

MPW rules with variable entry fees are also efficient. Note that if for each $i \in N$, the entry fee function $e_i(\cdot)$ depends only on the other agents' preferences R_{-i}, then the associated MPW rule with variable entry fees $\{e_i(\cdot)\}_{i\in N}$ on the quasi-linear domain is strategy-proof and so, by Holmström (1979), it is a rule, called Groves rule.[40] However, as illustrated in Example 5, an MPW rule with variable entry fees $\{e_i(\cdot)\}_{i\in N}$ is not strategy-proof on the classical domain even if for each $i \in N$, the entry fee function $e_i(\cdot)$ depends only on the other agents' preferences R_{-i}. This fact demonstrates the complexity of analysis on the classical domain.

Example 5 (A violation of strategy-proofness of an MPW rule with variable entry fees). Let $N \equiv \{1, 2\}$ and $M \equiv \{1\}$. Let f be the MPW rule with variable entry fees $\{e_i(\cdot)\}_{i\in N}$ such that for each R_2, $e_1(R_2) = 0$, and for each R_1, $e_2(R_1) = CV_1(1; \mathbf{0})$. Let R be a preference profile such that $CV_1(1; \mathbf{0}) \equiv 4$, $cv_2(1; (0, 4)) \equiv 2$, and $cv_2(1; (0, 7)) \equiv 1$. Then $f_1(R) = (1, 2)$. Let R'_1 be such that $CV'_1(1; \mathbf{0}) \equiv 7$. Then $f_1(R'_1, R_{-1}) = (1, 1)$. Thus, $f_1(R'_1, R_{-1})\, P_1\, f_1(R)$. ♢

One might wonder if the MPW rules with entry fees can be characterized by only strategy-proofness and efficiency. Our proof of Theorem 2 fails if individual rationality and no subsidy for losers are dropped. However, we have not found an example of a rule that satisfies strategy-proofness and efficiency, but is not an MPW rule with entry fees. Therefore, it is an open question whether the class of MPW rules with entry fees can be characterized by only strategy-proofness and efficiency.

One might also wonder if the assumption that $n > m$ can be dropped in Theorem 2. Our proof of Theorem 2 also fails if $n \le m$. However, we have not found an example of a rule that satisfies the four axioms of Theorem 2, but is not an MPW rule even if $n > m$ is dropped. Therefore, this question is also open.

5. Simultaneous ascending auction

We define a class of simultaneous ascending auctions and show that they achieve the MPWE. Let $\mathcal{R} \subseteq \mathcal{R}^E$, $R \in \mathcal{R}^n$, and $p \in \mathbb{R}^m_+$.

Definition 6. A set $M' \subseteq M$ is a *minimal* overdemanded set at p for R if M' is overdemanded at p for R and there is no $M'' \subsetneq M'$ such that M'' is overdemanded at p.

Under a (continuous time) simultaneous ascending auction, there is a constant $d > 0$, and at each time, each bidder submits his demand at the current price vector and the prices of the objects in a minimal overdemanded set are raised at a speed at least d. When there is no overdemanded set, the auction stops. Given a preference profile, a simultaneous ascending auction generates a "price path."

[40] See Section 6 for the definition of Groves rule.

Definition 7. A *simultaneous ascending* (SA) *auction* is a function τ from $\mathbb{R}_+ \times \mathbb{R}^m_+ \times \mathcal{R}^n$ to $\mathbb{R}^m_+$ such that the following statements hold:

(i) Given $R \in \mathcal{R}^n$, τ is integrable with respect to time $t \in \mathbb{R}_+$ and price $p \in \mathbb{R}^m_+$.

(ii) There is $d > 0$ such that for each $t \in \mathbb{R}_+$, each $p \in \mathbb{R}^m_+$, each $R \in \mathcal{R}^n$, and each $x \in M$,

  (ii-a) if x is in a minimal overdemanded set at p, then $\tau^x(t, p, R) \geq d$

  (ii-b) $\tau^x(t, p, R) = 0$ otherwise.

For each $R \in \mathcal{R}^n$, the *price path* generated by an SA auction τ is a function p from $\mathbb{R}_+$ to $\mathbb{R}^m_+$ such that the following statements hold:

(i) For each $x \in M$, $p^x(0) = 0$.

(ii) For each $x \in M$ and each $t \in \mathbb{R}_+$,

$$p^x(t) = \int_0^t \tau^x(s, p(s), R)\, ds.$$

Proposition 1 says that the outcome of an SA auction coincides with the MPWE.

Proposition 1. *For each $R \in \mathcal{R}^n$, the price path generated by any simultaneous ascending auction converges to the minimum Walrasian equilibrium price in a finite time.*

The proof is given in Appendix C. Proposition 1 implies that for each $R \in \mathcal{R}^n$, the price path $p(\cdot)$ generated by an SA auction has a *termination time* T such that for each $t \geq T$, $p(t) = p(T) = p_{\min}(R)$, and at the final prices $p(T)$, each agent receives an object from his demand and pays the final price of the object that he receives.

6. Generalized Vickrey rule

In this section, we introduce the generalized Vickrey rules and contrast them with the MPW rules.

6.1 *Generalized Vickrey rule*

Each quasi-linear preference R_i can be defined by means of a valuation function $v_i : L \to \mathbb{R}_+$, and a preference profile R in the quasi-linear domain corresponds to a valuation profile $v(R) \equiv (v_1(R_1), \ldots, v_n(R_n))$. Given a valuation profile $v = (v_1, \ldots, v_n)$, let $(x^*_1(v), \ldots, x^*_n(v)) \in \arg\max_{(x_1,\ldots,x_n)\in X} \sum_i v_i(x_i)$, $\sigma_{-i}(v) \equiv \sum_{j\neq i} v_j(x^*_j(v))$ and $\sigma'_{-i}(v) \equiv \max_{(x_1,\ldots,x_n)\in X} \sum_{j\neq i} v_j(x_j)$.[41] On the quasi-linear domain, the Vickrey rules are defined as follows.

[41]Note that effectively $\sigma'_{-i}(\cdot)$ is a function of v_{-i}.

Definition 8. A rule f on the quasi-linear domain is a *Groves rule* if (i) for each valuation profile v, $f^x(v) \in \arg\max_{(x_1,\dots,x_n)\in X} \sum_i v_i(x_i)$, and (ii) for each $i \in N$, there is a function h_i of the other agents' valuation profile v_{-i} such that for each valuation profile v, $f_i^t(v) = h_i(v_{-i}) - \sigma_{-i}(v)$. A Groves rule f is a *Vickrey rule* if for each $i \in N$, $h_i = \sigma'_{-i}$.

To generalize Vickrey rules to the classical domain, we need to use some valuation function v_i for each classical preference R_i. The compensated valuation $CV_i(\cdot; \mathbf{0})$ from the origin is defined for each classical preference R_i and a generalization of valuation function, and so is a natural candidate. Given a classical preference R_i, let $v_i(\cdot; R_i)$ be a function defined as, for each $x \in L$, $v_i(x; R_i) \equiv CV_i(x; \mathbf{0})$. Given a classical preference profile R, let $v'(R) \equiv (v_1(\cdot; R_1), \dots, v_n(\cdot; R_n))$.

Definition 9. A rule f on the classical domain is a *generalized Vickrey rule* if for each classical preference profile R, $f^x(v'(R)) \in \arg\max_{(x_1,\dots,x_n)\in X} \sum_i v_i(x_i; R_i)$, and for each $i \in N$, $f_i^t(v'(R)) = \sigma'_{-i}(v'(R)) - \sigma_{-i}(v'(R))$.

A classical preference R_i is *object-blind* if for each $x, y \in M$ and each $t \in \mathbb{R}$, $(x, t)\ I_i\ (y, t)$. We call the class of object-blind preferences the *object-blind domain*. The object-blind domain is a subset of the classical domain. On the object-blind domain, Saitoh and Serizawa (2008) and Sakai (2008) characterize the generalized Vickrey rules.

Fact 7 (Saitoh and Serizawa 2008, Sakai 2008). *Let $n > m$. A rule on the object-blind domain satisfies strategy-proofness, efficiency, individual rationality, and no subsidy if and only if it is a generalized Vickrey rule.*[42]

On the quasi-linear domain, the classes of Vickrey rules, generalized Vickrey rules, and MPW rules coincide. Fact 7 suggests that the generalized Vickrey rules are natural generalizations of the Vickrey rules on the object-blind domain. On the object-blind domain, the classes of generalized Vickrey rules and MPW rules also coincide. However, these two classes of rules differ outside the above two domains, as explained in Section 6.2. Thus, Fact 7 does not imply Theorem 2. Since the object-blind domain is smaller than the classical domain, Theorem 2 does not imply Fact 7 either. Therefore, the two results are mathematically independent.

6.2 *Generalized Vickrey rule vs. minimum price Walrasian rule*

Notice that, in example of Section 3.2 (Figure 1), agent 2's payment in the MPWE allocation z cannot be computed from the compensated valuations $v_i(\cdot; R_i)$, $i = 1, 2, 3$, from the origin $\mathbf{0}$. Payments of the MPW rule depend on the compensated valuations from various points. It is worthwhile to mention that for the preference profile in Figure 1, it is agent 1's preference R_1 that determines whether agent 2 or agent 3 receives a real object in the MPWE allocation. In Figure 1, agent 1 prefers $(A, CV_3(A; \mathbf{0}))$

[42] It is straightforward that on the object-blind domain, strategy-proofness, efficiency, individual rationality, and no subsidy for losers imply no subsidy.

to $(B, CV_2(B; \mathbf{0}))$, and agent 2 receives a real object. However, if agent 1 prefers $(B, CV_2(B; \mathbf{0}))$ to $(A, CV_3(A; \mathbf{0}))$, agent 3 instead receives a real object. Object allocations of the MPW rule also depend on the compensated valuations from various points. Thus, the MPWE allocation z is not the outcome of the generalized Vickrey rule. Accordingly, the MPW rule does not coincide with the generalized Vickrey rule.[43]

One can easily check that the generalized Vickrey rule is neither efficient nor strategy-proof on the classical domain with heterogeneous objects. To check this fact, let $R_1 \in \mathcal{R}^C$, $R_2 \in \mathcal{R}^Q$, and $R_3 \in \mathcal{R}^Q$ be such that $CV_1(A; \mathbf{0}) = 9$, $CV_1(B; \mathbf{0}) = 10$, $(A, 6)\ P_1\ (B, 5)$, $CV_2(A; \mathbf{0}) = 3$, $CV_2(B; \mathbf{0}) = 5$, $CV_3(A; \mathbf{0}) = 6$, and $CV_3(B; \mathbf{0}) = 2$. The indiffence curves of Figure 1 illustrate those preferences. The outcome of the generalized Vickrey rule for R is $z \equiv ((B, 5), (0, 0), (A, 4))$. Let $z' \equiv ((A, 6), (B, 5), (0, -2))$. Then z' Pareto-dominates z, a violation of efficiency. Let $R'_1 \in \mathcal{R}^Q$ be such that $CV'_1(A; \mathbf{0}) = 8$ and $CV'_1(B; \mathbf{0}) = 5$. Then under the generalized Vickrey rule, the bundle that agent 1 obtains for (R'_1, R_{-1}) is $(A, 6)$. Since $(A, 6)\ P_1\ (B, 5)$, the generalized Vickrey rule violates strategy-proofness.

The generalized Vickrey rule employs only a small part of the information about agents' preferences (i.e., compensated valuations from the origin). On the other hand, the MPW rule employs other information (i.e., compensated valuations from various points). As we stated in Section 4, only the MPW rule satisfies strategy-proofness, efficiency, individual rationality, and no subsidy for losers on the domain including non-quasi-linear preferences. Thus, the information about compensated valuations from various points is necessary to design rules satisfying the above four properties on this domain. Proposition 1 states that the SA auction achieves the same outcome as the MPW rule.

7. Concluding remarks

In this article, we mainly focus on the analysis of rules that allocate objects efficiently, and we show that only the MPW rules are desirable based on the four properties strategy-proofness, efficiency, individual rationality, and no subsidy for losers. It would be also important to investigate rules that produce more revenues for the auctioneer. An interesting question relating to this issue is whether there are strategy-proof, efficient, and individually rational rules that produce greater revenues than the MPW rule for each preference profile. We hope that the results and techniques developed in this article will be useful for the study of this research topic.

Appendixes: Proofs

In this appendix, we provide the proofs of all results in the article. In Appendix A, we prove Theorem 1 and Corollaries 1 and 2. In Appendix B, we give the proofs of the main results (Fact 6 and Theorem 2). Appendix C gives the proof of Proposition 1. The proofs of Facts 4 and 5 appear in the supplementary file on the journal website.

[43] When agent 1 prefers $(B, CV_2(B; \mathbf{0}))$ to $(A, CV_3(A; \mathbf{0}))$, the MPWE allocation $z^* \equiv (z_1^*, z_2^*, z_3^*)$ is $z^* \equiv ((B, CV_2(B; \mathbf{0})), (0, 0), (A, CV_1(A; z_1^*)))$. Thus, unless $CV_1(A; z_1^*) = CV_2(B; \mathbf{0}) + CV_1(A; \mathbf{0}) - CV_1(B; \mathbf{0})$, the MPWE allocation z^* does not coincide with the outcome of the generalized Vickrey rule.

Appendix A: Proofs for Section 3 (Theorem 1, and Corollaries 1 and 2)

Let $\mathcal{R} \subseteq \mathcal{R}^E$ in this section.

Lemma 1. *Let $R \in \mathcal{R}^n$, $(z, p) \in W(R)$, and R' be a d-truncation of R such that for each $i \in N$ with $x_i \neq 0$, $d_i \leq -CV_i(0; z_i)$, and for each $i \in N$ with $x_i = 0$, $d_i \geq 0$. Then $(z, p) \in W(R')$.*

Proof. Since $(z, p) \in W(R)$, (z, p) satisfies (WE-i) and (WE-ii) for R. Since (WE-ii) is independent of preferences, we show only (WE-i) for R', that is, that for each $i \in N$ and each $y \in L$, $(x_i, p^{x_i})\, R'_i\, (y, p^y)$. Let $i \in N$ and $y \in L$.

Case 1. $x_i \neq 0$. If $y \neq 0$, then by Remark 1(i), $(x_i, p^{x_i})\, R'_i\, (y, p^y)$. If $y = 0$, then by $d_i \leq -CV_i(0; z_i)$, $(x_i, p^{x_i})\, R'_i\, \mathbf{0} = (y, p^y)$.

Case 2. $x_i = 0$. If $y = 0$, then by $(y, p^y) = \mathbf{0} = (x_i, p^{x_i})$, $(x_i, p^{x_i})\, R'_i\, (y, p^y)$. If $y \neq 0$, then by $(x_i, p^{x_i})\, R_i\; y(y, p^y)$ and $d_i \geq 0$, $(x_i, p^{x_i})\, R'_i\, (y, p^y)$. □

Lemma 2. *Let $i \in N$, $R_i \in \mathcal{R}$, $d_i \in \mathbb{R}$, and R'_i be the d_i-truncation of R_i. Let $p, q \in \mathbb{R}^m_+$, $x \in M$, and $y \in L$ be such that $x \in D(R_i, p)$ and $y \in D(R'_i, q)$.*

(i) If $q^x < p^x$ and $y \in M$, then $(y, q^y)\, P_i\, (x, p^x)$ and $q^y < p^y$.

(ii) If $q^x < p^x$ and $d_i \leq -CV_i(0; (x, p^x))$, then $y \in M$, $(y, q^y)\, P_i\, (x, p^x)$, and $q^y < p^y$.

Proof. (i) Let $q^x < p^x$ and $y \in M$. By $y \in D(R'_i, q)$, $(y, q^y)\, R'_i\; y(x, q^x)$. Since R'_i is the d_i-truncation of R_i, by Remark 1(i), $(y, q^y)\, R_i\, (x, q^x)$. Then

$$(y, q^y)\, R_i\, (x, q^x) \underset{q^x < p^x}{P_i} (x, p^x) \underset{x \in D(R_i, p)}{R_i} (y, p^y).$$

Thus, $(y, q^y)\, P_i\, (x, p^x)$. Also, $(y, q^y)\, P_i\, (y, p^y)$ implies $q^y < p^y$.

(ii) Let $q^x < p^x$ and $d_i \leq -CV_i(0; (x, p^x))$. Then $CV'_i(0; (x, p^x)) \leq 0$ and so $(x, p^x)\, R'_i\, \mathbf{0}$. Thus,

$$(y, q^y) \underset{y \in D(R'_i, q)}{R'_i} (x, q^x) \underset{q^x < p^x}{P'_i} (x, p^x)\, R'_i\, \mathbf{0}.$$

Then $(y, q^y)\, P'_i\, \mathbf{0}$ implies $y \in M$. Thus, by Lemma 2(i), $(y, q^y)\, P_i\, (x, p^x)$ and $q^y < p^y$. □

Proof of Theorem 1. *"If."* Assume that no set is overdemanded, and no set is weakly underdemanded at p for R. Then, by Fact 4, $p \in P(R)$. Suppose that there is $q \in P(R)$ such that $q \leq p$ and $q \neq p$. Without loss of generality, assume that for each $x \in M'$, $q^x < p^x$, and for each $x \notin M'$, $q^x = p^x$, where $M' \equiv \{1, \ldots, m'\}$ and $1 \leq m' \leq m$.

Since M' is not weakly underdemanded at p for R, there is $N' \subseteq N$ such that $|N'| > |M'|$ and for each $i \in N'$, $D(R_i, p) \cap M' \neq \varnothing$. For each $i \in N'$, let $y_i \in D(R_i, p) \cap M'$. Since for each $x \in M'$, $q^x < p^x$, and for each $x \notin M'$, $q^x = p^x$, it follows that for each $i \in N'$ and each $x \notin M'$, $(y_i, q^{y_i})\, P_i\, (y_i, p^{y_i})\, R_i\, (x, p^x) = (x, q^x)$. Thus, for each $i \in N'$, $D(R_i, q) \subseteq M'$. By $|N'| > |M'|$, M' is overdemanded at q. Since $q \in P(R)$, this contradicts Fact 4.

"*Only if.*" Let $p \equiv p_{\min}(R)$. Then, by Fact 4, no set is overdemanded and no set is underdemanded at p for R. We show that no set is weakly underdemanded at p for R. Suppose that there is a set M' that is weakly underdemanded at p for R, i.e., for each $x \in M'$, $p^x > 0$, and $|\{i \in N : D(R_i, p) \cap M' \neq \varnothing\}| \leq |M'|$. Let $N' \equiv \{i \in N : D(R_i, p) \cap M' \neq \varnothing\}$. Without loss of generality, assume that M' is minimal: no proper subset of M' is weakly underdemanded at p for R. Since $p \in P(R)$, there is $z \in Z$ such that for each $i \in N$, $x_i \in D(R_i, p)$ and $t_i = p^{x_i}$. Since no set is underdemanded at p for R, $|N'| = |M'|$. Without loss of generality, let $M' \equiv \{1, \dots, m'\}$ and $N' \equiv \{1, \dots, m'\}$.

Step 1. *For each $i \in N'$, $x_i \in M'$.*

Proof. Since for each $x \in M'$, $p^x > 0$, (WE-ii) implies that for each $x \in M'$, there is $i(x) \in N'$ such that $x_{i(x)} = x$. Then, by $|N'| = |M'|$, for each $i \in N'$, $x_i \in M'$. $\lhd$

For each $x \in M'$, let $q^x \equiv \max\{CV_j(x; z_j) : j \in N \setminus N'\} \cup \{0\}$. Then, for each $x \in M'$, $q^x < p^x$.[44] Let $R'_{m'+1} \in \mathcal{R}$ be such that for each $x \in M'$, if $q^x > 0$, then $CV'_{m'+1}(x; \mathbf{0}) = q^x$, and if $q^x = 0$, then $CV'_{m'+1}(x; \mathbf{0}) \in (0, p^x)$. We consider the economy E' with object set M' and agent set $N'' \equiv N' \cup \{m' + 1\}$.

Let $W^{M',N''}(\bar{R}_{N''})$ and $W^{M',N''}_{\min}(\bar{R}_{N''})$ be the sets of Walrasian and minimum price Walrasian equilibria of the economy with object set M' and agent set N'' with preference $\bar{R}_{N''}$, and let $P^{M',N''}(\bar{R}_{N''})$ and $p^{M',N''}_{\min}(\bar{R}_{N''})$ be the set of Walrasian prices and the minimum Walrasian equilibrium price vector of the economy, respectively. Let $z_{m'+1} \equiv \mathbf{0}$ and $z_{N''} \equiv (z_{N'}, z_{m'+1})$.

Step 2. *We have $(z_{N''}, p^{M'}) \in W^{M',N''}_{\min}(R_{N'}, R'_{m'+1})$.*

Proof. Let $(\tilde{z}_{N''}, \tilde{p}^{M'}) \in W^{M',N''}_{\min}(R_{N'}, R'_{m'+1})$. Since $(z_{N''}, p^{M'}) \in W^{M',N''}(R_{N'}, R'_{m'+1})$, we have $\tilde{p}^{M'} \leq p^{M'}$. Let $M^- \equiv \{x \in M' : \tilde{p}^x < p^x\}$. We show $M^- = \varnothing$. Suppose $M^- \neq \varnothing$. Let $N^- \equiv \{i \in N' : D(R_i, p^{M'}) \cap M^- \neq \varnothing\}$.

Step 2.1. *For each $i \in N^-$, $\tilde{x}_i \in M^-$.*

Proof. Let $i \in N^-$. Then there is $x \in D(R_i, p^{M'}) \cap M^-$. Thus, $x \in M'$ and $\tilde{p}^x < p^x$. Since $(\tilde{z}_{N''}, \tilde{p}^{M'}) \in W^{M',N''}_{\min}(R_{N'}, R'_{m'+1})$, we have $\tilde{x}_i \in D(R_i, \tilde{p}^{M'})$. Then, by Lemma 2(ii), $\tilde{x}_i \in M'$ and $\tilde{p}^{\tilde{x}_i} < p^{\tilde{x}_i}$. Thus, $\tilde{x}_i \in M^-$. $\lhd$

Step 2.2. *We have $M^- = M'$, $N^- = N'$, and $|M^-| = |N^-|$.*

Proof. Since no two agents in N^- receive the same object, Step 2.1 implies $|M^-| \geq |N^-|$.

[44] To see this, suppose that for some $x \in M'$, $q^x \geq p^x$. Then there is $j \in N \setminus N'$ such that $(x, p^x)\, R_j\, z_j$. Since $x_j \in D(R_j, p)$, we have $x \in D(R_j, p)$. Thus, $j \in N'$. This contradicts $j \in N \setminus N'$.

Suppose $M^- \neq M'$. Then since $M^- \subsetneq M'$ and M' is a minimal weakly underdemanded set at p for R, M^- is not weakly underdemanded at $p^{M'}$ for $(R_{N'}, R'_{m'+1})$.[45] Thus, since for each $x \in M^-$, $p^x > 0$, we have $|N^-| \geq |M^-| + 1$. This contradicts $|M^-| \geq |N^-|$. Thus, $M^- = M'$.

By the definition of N^-, $M^- = M'$ implies $N^- = N'$. Since M' is weakly underdemanded, we have $|N'| = |M'|$. From the above results, $|M^-| = |M'| = |N'| = |N^-|$. ◁

Step 2.3. *For each $x \in M'$, $\tilde{p}^x \geq q^x$.*

Proof. Suppose that there is $x \in M'$ such that $\tilde{p}^x < q^x$. Then $q^x > 0$. Note that by $\tilde{x}_{m'+1} \in D(R'_{m'+1}, \tilde{p}^{M'})$ and $\tilde{p}^x < q^x = CV'_{m'+1}(x; \mathbf{0})$, $\tilde{x}_{m'+1} \in M'$. By $M^- = M'$ and $N^- = N'$ (Step 2.2), Step 2.1 implies that for each $i \in N'$, $\tilde{x}_i \in M'$. This contradicts $|M'| = m'$. ◁

Let $(\bar{z}, \bar{p}) \in Z \times \mathbb{R}^m_+$ be such that $\bar{z}_{N'} = \tilde{z}_{N'}$, $\bar{z}_{-N'} = z_{-N'}$, $\bar{p}^{M'} = \tilde{p}^{M'}$, and $\bar{p}^{-M'} = p^{-M'}$.

Step 2.4. *The pair $(\bar{z}, \bar{p})$ is a Walrasian equilibrium of the original economy, i.e., $(\bar{z}, \bar{p}) \in W(R)$.*

Proof. By Step 2.3, for each $y \in M'$, $\tilde{p}^y \geq q^y$. Let $h \in N \setminus N'$. Then, for each $y \in L$, if $y \notin M'$, then

$$(\bar{x}_h, \bar{p}^{\bar{x}_h}) \underset{h \notin N'}{=} (x_h, p^{x_h}) \underset{x_h \in D(R_h, p)}{R_h} (y, p^y) \underset{y \notin M'}{=} (y, \bar{p}^y),$$

and if $y \in M'$, then

$$(\bar{x}_h, \bar{p}^{\bar{x}_h}) \underset{h \notin N'}{=} (x_h, p^{x_h}) \underset{\text{def. of } q^y}{R_h} (y, q^y) \underset{q^y \leq \tilde{p}^y = \bar{p}^y}{R_h} (y, \bar{p}^y).$$

Thus, for each $h \in N \setminus N'$, $\bar{x}_h \in D(R_h, \bar{p})$.

Let $h \in N'$. Then, for each $y \in L$, if $y \notin M'$, then

$$(\bar{x}_h, \bar{p}^{\bar{x}_h}) \underset{h \in N'}{=} (\tilde{x}_h, \tilde{p}^{\tilde{x}_h}) \underset{\tilde{x}_h \in D(R_h, \tilde{p}^{M'})}{R_h} (x_h, \tilde{p}^{x_h}) \underset{\tilde{p}^{M'} \leq p^{M'}}{R_h} (x_h, p^{x_h}) \underset{x_h \in D(R_h, p)}{R_h} (y, p^y) \underset{y \notin M'}{=} (y, \bar{p}^y),$$

and if $y \in M'$, then

$$(\bar{x}_h, \bar{p}^{\bar{x}_h}) \underset{h \in N'}{=} (\tilde{x}_h, \tilde{p}^{\tilde{x}_h}) \underset{\tilde{x}_h \in D(R_h, \tilde{p}^{M'})}{R_h} (y, \tilde{p}^y) \underset{y \in M'}{=} (y, \bar{p}^y).$$

Thus, for each $h \in N'$, $\bar{x}_h \in D(R_h, \bar{p})$. Since (z, p) and $(\tilde{z}_{N''}, \tilde{p})$ satisfy (WE-ii), so does $(\bar{z}, \bar{p})$. ◁

[45] By $M^- \subseteq M'$, $\{i \in N : D(R_i, p) \cap M^- \neq \varnothing\} \subseteq \{i \in N : D(R_i, p) \cap M' \neq \varnothing\} = N'$. Thus, $\{i \in N' : D(R_i, p^{M'}) \cap M^- \neq \varnothing\} = \{i \in N : D(R_i, p) \cap M^- \neq \varnothing\}$. Hence,

$$|N^-| = \left|\{i \in N' : D(R_i, p^{M'}) \cap M^- \neq \varnothing\}\right| = \left|\{i \in N : D(R_i, p) \cap M^- \neq \varnothing\}\right| > |M^-|.$$

Recall that $p = p_{\min}(R)$. However, since $M^- \neq \varnothing$, we have $\bar{p} \leq p$ and $\bar{p} \neq p$. This is a contradiction. Thus, $M^- = \varnothing$. This completes the proof of Step 2. $\lhd$

Step 3. *There is a d_1-truncation R_1' of R_1 such that $CV_1'(x_1; \mathbf{0}) < p^{x_1}$ and agent 1 obtains a real object in an MPWE for $(R_1', R_{m'+1}', R_{N'\setminus\{1\}})$, i.e., for $(\hat{z}_{N''}, \hat{p}^{M'}) \in W_{\min}^{M',N''}(R_1', R_{m'+1}', R_{N'\setminus\{1\}})$, $\hat{x}_1 \neq 0$.*

Proof. Note that d_1 needs to be large enough so that $CV_1'(x_1; \mathbf{0}) < p^{x_1}$, but at the same time, d_1 needs to be small enough so that $\hat{x}_1 \neq 0$, that is, $CV_1'(x_1; \mathbf{0})$ needs to be close to p^{x_1}. To analyze how $CV_1'(x_1; \mathbf{0})$ needs to be close to p^{x_1}, we introduce the concept of assignment sequence as follows:

Let Π denote the set of permutations of M' and denote by $\{x(k)\}_{k=1}^{m'}$ its generic element. Given $\{x(k)\}_{k=1}^{m'} \in \Pi$, let $\{i(k)\}_{k=1}^{m'}$ be the permutation of agents in N' defined by $x_{i(1)} = x(1)$, $x_{i(2)} = x(2)$, $\ldots$, $x_{i(m')} = x(m')$, and let $\{t(k)\}_{k=1}^{m'}$ be a sequence of the payments of agents in N' such that $t(1) \leq CV_{m'+1}'(x(1); \mathbf{0})$, $t(2) \leq CV_{i(1)}(x(2); z_0(1))$, $\ldots$, $t(m') \leq CV_{i(m'-1)}(x(m'); z_0(m'-1))$, where for each $k \in \{1, \ldots, m'\}$, $z_0(k) \equiv (x(k), t(k))$. We call such a pair $\{z_0(k), i(k)\}_{k=1}^{m'}$ an *assignment sequence.*

Step 3.1. *There is an upper bound $b < p^{x_1}$ such that for any assignment sequence $\{z_0(k), i(k)\}_{k=1}^{m'}$ constructed as above and for k with $x(k) = x_1$, $t(k) < b$.*

Proof. Given $\{x(k)\}_{k=1}^{m'} \in \Pi$, let $\{z_0(k), i(k)\}_{k=1}^{m'}$ be the assignment sequence such that $t(1) = CV_{m'+1}'(x(1); \mathbf{0})$, $t(2) = CV_{i(1)}(x(2); z_0(1))$, $\ldots$, and $t(m') = CV_{i(m'-1)}(x(m'); z_0(m'-1))$. Since $CV_{m'+1}'(x(1); \mathbf{0}) < p^{x(1)}$, the following relation holds inductively: for each $k \geq 2$,

$$
\begin{aligned}
(x(k), t(k)) \;& I_{i(k-1)} \; z_0(k-1) \quad \text{(by def. of } t(k)\text{)} \\
& P_{i(k-1)} \; (x(k-1), p^{x(k-1)}) \quad \text{(by } t(k-1) < p^{x(k-1)}\text{)} \\
& R_{i(k-1)} \; (x(k), p^{x(k)}) \quad \text{(by } x(k-1) \in D(R_{i(k-1)}, p)\text{)}
\end{aligned}
$$

and $t(k) < p^{x(k)}$. Note that for any assignment sequence $\{z_0'(k), i'(k)\}_{k=1}^{m'}$ associated with the same $\{x(k)\}_{k=1}^{m'}$, for each k, $t'(k) \leq t(k)$ and thus, for k with $x(k) = x_1$, $t'(k) < p^{x_1}$.

Since the cardinality of Π is finite $(m'!)$, the conclusion of the above paragraph implies that there is $b < p^{x_1}$ such that for any assignment sequence $\{z_0(k), i(k)\}_{k=1}^{m'}$ and for k with $x(k) = x_1$, $t(k) < b$. $\lhd$

Step 3.2. *Let R_1' be a d_1-truncation of R_1 such that $b < CV_1'(x_1; \mathbf{0}) < p^{x_1}$. Let $(\hat{z}_{N''}, \hat{p}^{M'}) \in W_{\min}^{M',N''}(R_1', R_{m'+1}', R_{N'\setminus\{1\}})$. Then $\hat{x}_1 \neq 0$.*

Proof. Suppose that $\hat{x}_1 = 0$. We use Claim 1 below. It implies that m' agents (agents 2, $\ldots$, $m'+1$) receive m' different objects in $M' \setminus \{x_1\}$. By $|M'| = m'$, this is a contradiction. Thus, proving Claim 1 completes the proof of Step 3.2. $\lhd$

Claim 1. *Construct the assignment sequences* $\{z_0(k), i(k)\}_{k=1}^{m'}$ *as follows:* $x(1) \equiv \hat{x}_{m'+1}$, $x_{i(1)} = x(1)$, *and* $t(1) \equiv \hat{p}^{x(1)}$, *and for each* $k \in \{2, \dots, m'\}$, $x(k) \equiv \hat{x}_{i(k-1)}$, $x_{i(k)} = x(k)$, *and* $t(k) \equiv \hat{p}^{x(k)}$. *Then, for each* $k \in \{1, \dots, m'\}$, $x(k) \neq 0$, $x(k) \neq x_1$, *and* $\hat{p}^{x(k)} < p^{x(k)}$.

Proof. The proof is by induction.

Induction base. First, we show $x(1) \equiv \hat{x}_{m'+1} \neq 0$. Suppose $\hat{x}_{m'+1} = 0$. Then since two agents (1 and $m'+1$) in N'' receive the null object and $|N''| = |M'| + 1$, there is $x \in M$ such that for each $h \in N''$, $\hat{x}_h \neq x$. By (WE-ii), $\hat{p}^x = 0$. Since $CV'_{m'+1}(x; \mathbf{0}) > 0$, we have $(x, \hat{p}^x) P'_{m'+1} \mathbf{0}$. This is a contradiction since $\hat{x}_{m'+1} = 0$ and $(\hat{z}_{N''}, \hat{p}^{M'}) \in W_{\min}^{M',N''}(R'_1, R'_{m'+1}, R_{N' \setminus \{1\}})$. Thus, $x(1) \neq 0$.

Note that by Step 1, $x(1) \neq 0$ implies that agent $i(1)$ with $x_{i(1)} = x(1)$ uniquely exists. Thus, $x(1)$, $i(1)$, and $t(1)$ are well defined.

Second, by $x(1) \equiv \hat{x}_{m'+1} \in D(R'_{m'+1}, \hat{p}^{M'})$, $\hat{p}^{x(1)} \leq CV'_{m'+1}(x(1); \mathbf{0}) < p^{x(1)}$.

Third, we show $x(1) \neq x_1$. Suppose $x(1) = x_1$. Then, by Step 3.1 and the definition of R'_1, $\hat{p}^{x_1} \equiv t(1) < b < CV'_1(x_1; \mathbf{0})$, that is, $(x_1, \hat{p}^{x_1})\ P'_1\ \mathbf{0}$. Thus, by $\hat{x}_1 = 0$, $\hat{x}_1 \notin D(R'_1, \hat{p}^{M'})$. However, since $(\hat{z}_{N''}, \hat{p}^{M'}) \in W_{\min}^{M',N''}(R'_1, R'_{m'+1}, R_{N' \setminus \{1\}})$, this is a contradiction. Therefore, $x(1) \neq x_1$.

Induction argument. Let $k \in \{2, \dots, m'\}$. Assume that Claim 1 holds until $k-1$. Since $x(k-1) \in D(R_{i(k-1)}, p)$, $\hat{x}_{i(k-1)} \in D(R_{i(k-1)}, \hat{p}^{M'})$, and $\hat{p}^{x(k-1)} < p^{x(k-1)}$, Lemma 2(ii) implies that $x(k) \equiv \hat{x}_{i(k-1)} \neq 0$ and $\hat{p}^{x(k)} < p^{x(k)}$.

Note that by Step 1, $x(k) \neq 0$ implies that agent $i(k)$ with $x_{i(k)} = x(k)$ uniquely exists. Thus, $x(k)$, $i(k)$, and $t(k)$ are well defined.

We can show $x(k) \neq x_1$ similarly to the induction base. If $x(k) = x_1$, then $\hat{p}^{x_1} = t(k) < b < CV'_1(x_1; \mathbf{0})$, which implies $\hat{x}_1 \notin D(R'_1, \hat{p}^{M'})$, contradicting $(\hat{z}_{N''}, \hat{p}^{M'}) \in W_{\min}^{M',N''}(R'_1, R'_{m'+1}, R_{N' \setminus \{1\}})$. ◁

Step 4. *Concluding that no set is weakly underdemanded at* p *for* R.

Note that $CV'_1(x_1; \mathbf{0}) < p^{x_1}$ (Step 3) implies $d_1 > 0$. By $(\hat{z}_{N''}, \hat{p}^{M'}) \in W_{\min}^{M',N''}(R'_1, R'_{m'+1}, R_{N' \setminus \{1\}})$, Step 2 and Fact 5 imply that $\hat{p}^{M'} \leq p^{M'}$. Note that

$$(\hat{x}_1, \hat{p}^{\hat{x}_1}) \underset{\hat{x}_1 \in D(R'_1, \hat{p}^{M'})}{R'_1} \mathbf{0} \underset{\text{def. of } R'_1}{I'_1} (x_1, CV'_1(x_1; \mathbf{0})) \underset{CV'_1(x_1; \mathbf{0}) < p^{x_1}}{P'_1} (x_1, p^{x_1}).$$

By Steps 1 and 3, $x_1 \neq 0$ and $\hat{x}_1 \neq 0$. By the definition of R'_1 and Remark 1(i), $(\hat{x}_1, \hat{p}^{\hat{x}_1})\ P'_1\ (x_1, p^{x_1})$ implies $(\hat{x}_1, \hat{p}^{\hat{x}_1})\ P_1\ (x_1, p^{x_1})$. Therefore,

$$(\hat{x}_1, \hat{p}^{\hat{x}_1})\ P_1\ (x_1, p^{x_1}) \underset{x_1 \in D(R_1, p)}{R_1} (\hat{x}_1, p^{\hat{x}_1}).$$

This implies $\hat{p}^{\hat{x}_1} < p^{\hat{x}_1}$. By the definition of R'_1, R_1 is the $(-d_1)$-truncation of R'_1 and $-d_1 \leq 0 \leq -CV'_1(0; \hat{z}_1)$. Thus, Lemma 1 implies $\hat{p}^{M'} \in P^{M',N''}(R_{N'}, R'_{m'+1})$.

However, by Step 2, $p^{M'} = p_{\min}^{M',N''}(R_{N'}, \hat{R}_{m'+1})$. Since $\hat{p}^{M'} \leq p^{M'}$ and $\hat{p}^{\hat{x}_1} < p^{\hat{x}_1}$, this is a contradiction. □

Proof of Corollary 1. Suppose that for each $i \in N$, $p_{\min}^{x_i}(R) > 0$. Then, for each $i \in N$, $x_i \neq 0$. Let $\bar{M} \equiv \{x_1, \dots, x_n\}$. Then $|\bar{M}| = |N|$. Since $|\bar{M}| = |\{i \in N : D(R_i, p) \cap \bar{M} \neq \varnothing\}|$, $\bar{M}$ is weakly underdemanded at p for R. This contradicts Theorem 1. □

Proof of Corollary 2. Let $x \in M$ be such that $p^x > 0$. We construct the sequence of agents in two steps.

Step 1. By (WE-ii) in Definition 3, there is $j_1 \in N$ such that $x_{j_1} = x$. By Theorem 1, the set $\{x\}$ is demanded at p by at least two agents. Thus, there is $j_2' \in N \setminus \{j_1\}$ such that $x \in D(R_{j_2'}, p)$. Let N_2 be the set of such agents. If $x_{j_2'} = 0$ or $p^{x_{j_2'}} = 0$ for some agent $j_2' \in N_2$, then let $j_2 = j_2'$ and go to Step 2. If $x_{j_2'} \neq 0$ and $p^{x_{j_2'}} > 0$ for each $j_2' \in N_2$, pick arbitrarily an agent $j_2 \in N_2$.

By Theorem 1, the set $\{x_{j_1}, x_{j_2}\}$ is demanded at p by at least three agents. Thus, there is $j_3' \in N \setminus \{j_1, j_2\}$ such that $D(R_{j_3'}, p) \cap \{x_{j_1}, x_{j_2}\} \neq \varnothing$. Let N_3 be the set of such agents. If $x_{j_3'} = 0$ or $p^{x_{j_3'}} = 0$ for some agent $j_3' \in N_3$, then let $j_3 = j_3'$ and go to Step 2. If $x_{j_3'} \neq 0$ and $p^{x_{j_3'}} > 0$ for each $j_3' \in N_3$, pick arbitrarily an agent $j_3 \in N_3$.

Since m is finite, proceeding inductively, there are $K' \leq m$ and a sequence $\{j_k\}_{k=1}^{K'}$ of K' distinct agents such that (a) $x_{j_{K'}} = 0$ or $p^{x_{j_{K'}}} = 0$, (b) for each $k \in \{2, \dots, K'-1\}$, $x_{j_k} \neq 0$ and $p^{x_{j_k}} > 0$, (c) $x_{j_1} = x$, and (d) for each $k \in \{2, \dots, K'\}$, $\{x_{j_1}, \dots, x_{j_{k-1}}\} \cap D(R_{j_k}, p) \neq \varnothing$. Then go to Step 2.

Step 2. Let $i_1 \equiv j_{K'}$. By $(z, p) \in W(R)$, $x_{i_1} \in D(R_{i_1}, p)$. By (d), there is $i_2 \in \{j_1, \dots, j_{K'-1}\}$ such that $x_{i_2} \in D(R_{i_1}, p)$. By $(z, p) \in W(R)$, $x_{i_2} \in D(R_{i_2}, p)$. By (d), there is $i_3 \in \{j_1, \dots, j_{k'-1}\}$ such that $x_{i_3} \in D(R_{i_2}, p)$, where k' is such that $j_{k'} = i_2$. By $(z, p) \in W(R)$, $x_{i_3} \in D(R_{i_3}, p)$. Proceeding inductively, we have some K such that $i_K = j_1$. Then the sequence $\{i_k\}_{k=1}^{K}$ of K distinct agents satisfies (i), (ii), (iii), and (iv). □

Appendix B: Proofs for Section 4 (main results: Fact 6 and Theorem 2)

Proof of Fact 6. Let $\mathcal{R} \subseteq \mathcal{R}^E$. Let g be an MPW rule on $\mathcal{R}^n$. By contradiction, suppose that there are $R \in \mathcal{R}^n$, $N' \subseteq N$, and $R'_{N'} \in \mathcal{R}^{|N'|}$ such that for each $i \in N'$, $g_i(R'_{N'}, R_{-N'}) \; P_i \; g_i(R)$. Let $z \equiv g(R)$ and $z' \equiv g(R'_{N'}, R_{-N'})$, with associated equilibrium prices p and p'. Without loss of generality, let $N' = \{1, \dots, n'\}$. Let $M^+ \equiv \{x \in M : 0 < p^x\}$ and $m^+ \equiv |M^+|$. Note that if $n > m$, then $n > m^+$, and if $n \leq m$, then by Corollary 1, $m^+ \leq n - 1 < n$.

In this paragraph, we show that for each $i \in N'$, $x_i' \neq 0$ and $p'^{x_i'} < p^{x_i'}$. Let $i \in N'$. Note that

$$(x_i', p'^{x_i'}) \underset{z_i' P_i z_i}{P_i} (x_i, p^{x_i}) \underset{x_i \in D(R_i, p)}{R_i} \mathbf{0}.$$

Thus, $x'_i \neq 0$. Also,

$$(x'_i, p'^{x'_i})\ P_i\ (x_i, p^{x_i}) \underset{x_i \in D(R_i, p)}{R_i} (x'_i, p^{x'_i}).$$

Thus, $(x'_i, p'^{x'_i})\ P_i\ (x'_i, p^{x'_i})$ implies that $p'^{x'_i} < p^{x'_i}$.

For each $i \in N'$, since $0 \leq p'^{x'_i} < p^{x'_i}$, $x'_i \in M^+$. Then if $m^+ < n'$, more than m^+ agents receive the objects in M^+, which is a contradiction. Thus, assume that $m^+ \geq n'$. By Theorem 1, there is $i' \in N \setminus N'$ such that $D(R_{i'}, p) \cap \{x'_1, \dots, x'_{n'}\} \neq \varnothing$. Without loss of generality, let $i' \equiv n' + 1$. By Lemma 2(ii), $x'_{n'+1} \neq 0$ and $0 \leq p'^{x'_{n'+1}} < p^{x'_{n'+1}}$. Thus, $x'_{n'+1} \in M^+$. Then, by Theorem 1, there is $i'' \in N \setminus \{1, \dots, n'+1\}$ such that $D(R_{i''}, p) \cap \{x'_1, \dots, x'_{n'+1}\} \neq \varnothing$. Without loss of generality, let $i'' \equiv n' + 2$. Thus, by Lemma 2(ii), $x'_{n'+2} \neq 0$ and $0 \leq p'^{x'_{n'+2}} < p^{x'_{n'+2}}$. Thus, $x'_{n'+2} \in M^+$. Repeat this argument $(m^+ - n' + 1)$ times. Then more than m^+ agents receive objects in M^+. This is impossible. □

Next, we prove Theorem 2. Let $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$. Let f be a rule on $\mathcal{R}^n$. Since the "if" part of Theorem 2 follows from the discussion in Section 4.1, we only give the proof of the "only if" part of the theorem.

B.1 *Difficulties and overview of the proof of Theorem 2*

We explain the difficulties of the proof, compared to the previous works and give an overview of the proofs.

First, we discuss the difficulties of our proof compared to the literature assuming quasi-linearity such as Holmström (1979). As emphasized in the Introduction, without quasi-linearity, efficient allocations of objects depend on agents' payments. Thus, it is difficult to identify the object allocations of the rules satisfying our desirable properties without knowing how much agents pay. At the same time, it is also difficult to identify the payments of the rules satisfying our properties without knowing how objects are allocated. Therefore, without quasi-linearity, we need to identify simultaneously the object allocation and payments of the rules. This is similar to solving simultaneous equations, and is much more difficult than identifying only the payments by assuming quasi-linearity.

Second, we discuss the difficulties of our proof, compared to Miyake (1998), who shows that only the MPW rule satisfies strategy-proofness among Walrasian rules. Notice that the Walrasian rules are a small part of the class of allocation rules satisfying efficiency, individual rationality, and no subsidy for losers. Our proof is to investigate how the four properties restrict the possibility of allocation changes when an agent changes his preference. If the rule is assumed to be among Walrasian rules as in Miyake (1998), the possibility of allocation is limited to a small class from the beginning, and so this investigation is relatively easy. Since we establish the uniqueness of the rules satisfying the four properties without confinement to Walrasian rules, our proof is much more difficult.

Third, we discuss how we overcome the above difficulties. The widely employed methods to solve simultaneous equations comprise constructing algorithms to reach the solutions step by step. In our proof, we employ a similar method, that is, we analyze how the above four properties restrict the possibility of allocations step by step. Lemma 6 states that payments are bounded below by the $(m+1)$th highest compensated valuations from the origin. Lemmas 9 and 10 restrict the possibility of object allocations in turn, i.e., Lemma 9 restricts the candidates who obtain a real object; Lemma 10 gives a sufficient condition that an agent obtains a specific real object. These lemmas enable us to prove that agents pay at most the minimum Walrasian prices (Proposition 2).

As Corollary 2 states, the MPWE allocation has a structure, called demand connectedness. Lemma 12 states that the allocation chosen by the rules satisfying our four properties has a similar structure for special preference profiles. Lemma 14 states that if an agent obtains an object but pays less than the minimum Walrasian price, whenever the object is connected to the origin by the demands of agents who pay the minimum Walrasian price, there is a Pareto improvement. These lemmas enable us to prove that agents pay at least the minimum Walrasian prices (Proposition 3 and the proof of Theorem 2).

Our proof has four parts.

Part 1. The following six lemmas are used in the proof.

First, under individual rationality and no subsidy for losers, whenever an agent receives the null object, he pays nothing.

Lemma 3 (Zero payment for losers). *Let f satisfy individual rationality and no subsidy for losers. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $f_i^x(R) = 0$. Then $f_i^t(R) = 0$.*

Under efficiency, individual rationality, and no subsidy for losers, each real object should be assigned to someone.

Lemma 4 (No remaining object). *Let f satisfy efficiency, individual rationality, and no subsidy for losers. Let $R \in \mathcal{R}^n$ and $x \in M$. Then there is $i \in N$ such that $f_i^x(R) = x$.*

Given an allocation and a pair $\{i, j\}$ of agents such that agent i receives a real object and prefers his assignment at least as desirable as j's, but j prefers i's assignment to his own, if the difference between j's payment and i's compensated valuation (CV) of j's assignment of objects from i's assignment is less than the difference between i's payment and j's CV of i's assignment of objects from j's assignment, then a Pareto improvement is possible.

Lemma 5 (Sufficient condition for a Pareto improvement to be possible). *Let $R \in \mathcal{R}^n$, $i, j \in N$, and $z \in Z$ be such that $x_i \neq 0$, $z_i \, R_i \, z_j$, and $z_i \, P_j \, z_j$. Assume that (a) $t_j - CV_i(x_j; z_i) < CV_j(x_i; z_j) - t_i$. Then there is $z' \in Z$ that Pareto-dominates z at R.*

We introduce additional notations. Given $R \in \mathcal{R}^n$, $x \in M$, and $z \in [L \times \mathbb{R}]^n$, let $\pi^x(R) \equiv (\pi_1^x(R), \ldots, \pi_n^x(R))$ be the permutation on N such that $CV_{\pi_n^x(R)}(x; z_{\pi_n^x(R)}) \leq$

$\cdots \leq CV_{\pi_1^x(R)}(x; z_{\pi_1^x(R)})$. For each $k \in N$, let $C^k(R, x; z) \equiv CV_{\pi_k^x(R)}(x; z_{\pi_k^x(R)})$. That is, $C^k(R, x; z)$ is the *kth highest compensated valuation (CV) of x from z*. We simply write $C^k(R, x; (\mathbf{0}, \ldots, \mathbf{0}))$ as $C^k(R, x)$.

Under the four axioms of Theorem 2, if an agent receives $x \in M$, then he pays at least the $(m+1)$th highest CV of x from the origin. Thus, the $(m+1)$th highest CV of each object from $\mathbf{0}$ is a lower bound for the payment of the agent who obtains the object.

Lemma 6 (Payment lower bound). *Let f satisfy the four axioms of Theorem 2. Let $R \in \mathcal{R}^n$, $i \in N$, and $x \in M$. If $f_i^x(R) = x$, then $f_i^t(R) \geq C^{m+1}(R, x)$.*

No subsidy is implied by our four axioms.

Lemma 7 (No subsidy). *The four axioms of Theorem 2 imply no subsidy.*

Hereafter, we use this implication repeatedly.

Given $z_i \equiv (x_i, t_i) \in M \times \mathbb{R}$, let $\mathcal{R}_{\text{NCV}}(z_i)$ be the set of preferences $R_i' \in \mathcal{R}$ such that for each $y \in L \setminus \{x_i\}$, $CV_i'(y; z_i) < 0$, that is, for each object except for x_i, the compensated valuation of R_i' from z_i is negative. We refer to the preferences in $\mathcal{R}_{\text{NCV}}(z_i)$ as *z_i-favoring*.

Under strategy-proofness and no subsidy for losers, given $R \in \mathcal{R}^n$, for each agent who receives a real object, if the agent's preference is changed to a preference that is $f_i(R)$-favoring, then his assignment remains the same.

Lemma 8 (Invariance property). *Let f satisfy strategy-proofness and no subsidy. Let $R \in \mathcal{R}^n$ and $i \in N$ be such that $f_i^x(R) \neq 0$. Let $R_i' \in \mathcal{R}_{\text{NCV}}(f_i(R))$. Then $f_i(R_i', R_{-i}) = f_i(R)$.*

Part 2. The next proposition says that for each preference profile, the allocation chosen by a rule satisfying the four axioms of Theorem 2 should (weakly) dominate the MPWE allocations from the bidders' perspectives. This implies that for a rule satisfying our properties, the agent who receives $x \in M$ pays at most the minimum Walrasian price p^x. Thus, Proposition 2 implies stringent upper bounds for payments even without knowing how objects are allocated.

Proposition 2.[46] *Let f satisfy the four axioms of Theorem 2. Let $R \in \mathcal{R}^n$ and $z \in W_{\min}(R)$. Then, for each $i \in N$, $f_i(R)\, R_i\, z_i$.*

We introduce two lemmas to prove Proposition 2. Hereafter, we maintain the assumption that f satisfies the four axioms of Theorem 2. By Lemma 7, f also satisfies no subsidy.

From Lemma 6, we deduce that if an agent receives $x \in M$, then his CV of x from $\mathbf{0}$ is no less than the mth highest CV of x from $\mathbf{0}$. For each $x \in M$, Lemma 9 restricts who can obtain x without knowing how much agents pay.

Lemma 9 (Necessary condition for receiving $x \in M$). *Let $R \in \mathcal{R}^n$, $i \in N$, and $x \in M$. If $f_i^x(R) = x$, then $CV_i(x; \mathbf{0}) \geq C^m(R, x)$.*

[46] This result also holds for any Walrasian equilibrium allocation z.

By Lemma 9, assumption (a) of Lemma 10 implies that for any real object other than $x \in M$, if an agent's CV from $\mathbf{0}$ is less than the mth highest, then he never receives a real object other than x. Together with this assumption, (b) and (c) of Lemma 10 guarantee that agent i receives x.

Given $R \in \mathcal{R}^n$, let $Z^{\mathrm{IR}}(R)$ be the set of individually rational allocations, that is, $Z^{\mathrm{IR}}(R) \equiv \{z \in Z : \text{for each } i \in N, z_i \, R_i \, \mathbf{0}\}$.

Lemma 10 (Sufficient condition for receiving $x \in M$). *Let $R \in \mathcal{R}^n$, $x \in M$, $i \in N$, and $z \in Z^{\mathrm{IR}}(R)$. Assume that (a) for each $y \in M \setminus \{x\}$, $CV_i(y; \mathbf{0}) < C^m(R, y)$, (b) for each $j \in N \setminus \{i\}$, $f_j(R) \, R_j \, z_j$, and (c) $CV_i(x; \mathbf{0}) > C^1(R_{-i}, x; z)$. Then $f_i^x(R) = x$.*

Part 3. Given $R \in \mathcal{R}^n$ and $z \in Z_{\min}(R)$, let $\mathcal{R}^I(z)$ be the set of preferences $R_i' \in \mathcal{R}$ such that for each pair $i, j \in N$, $z_i \, I_i' \, z_j$, that is, all the assignments under z are indifferent. We refer to the preferences in $\mathcal{R}^I(z)$ as *z-indifferent*.

Proposition 3 says that given $(z^*, p) \in W_{\min}(R)$ and a preference profile such that a group N' of agents have z^*-indifferent preferences, if for any z^*-indifferent preferences of N' and each $x \in M$, the agent outside N' who obtains x pays at least p^x, then for each $x \in M$, the agent in N' who obtains x pays at least p^x. Thus, although in a limited pattern, this proposition implies lower bounds for payments even without knowing how objects are allocated.

Proposition 3. *Let $R \in \mathcal{R}^n$, $(z^*, p) \in W_{\min}(R)$, and $N' \subseteq N$. Assume that (3-i) for each $\bar{R}_{N'} \in \mathcal{R}^I(z^*)^{|N'|}$, each $i \in N \setminus N'$, and each $x \in M$, if $f_i^x(\bar{R}_{N'}, R_{-N'}) = x$, then $f_i^t(\bar{R}_{N'}, R_{-N'}) \geq p^x$. Let $R'_{N'} \in \mathcal{R}^I(z^*)^{|N'|}$. Then, for each $i \in N'$ and each $x \in M$, if $f_i^x(R'_{N'}, R_{-N'}) = x$, then $f_i^t(R'_{N'}, R_{-N'}) \geq p^x$.*

We introduce four lemmas to prove Proposition 3.

Given $(z^*, p) \in W_{\min}(R)$, if a group of agents change their preferences to z^*-indifferent preferences, then for the new preference profile, (a) (z^*, p) is also an MPWE and (b) the allocation chosen by the rule f (weakly) dominates z^* from the bidders' viewpoints.

Lemma 11. *Let $R \in \mathcal{R}^n$, $(z^*, p) \in W_{\min}(R)$, $N' \subseteq N$, $R'_{N'} \in \mathcal{R}^I(z^*)^{|N'|}$, and $R' \equiv (R'_{N'}, R_{-N'})$. Then (a) $(z^*, p) \in W_{\min}(R')$ and (b) for each $i \in N$, $f_i(R') \, R_i' \, z_i^*$.*

Given $p \in \mathbb{R}^m_{++}$ and $R \in \mathcal{R}^n$, let $N(R, p)$ denote the set of demanders of the real objects at prices p, that is, $N(R, p) \equiv \{i \in N : D(R_i, p) \cap M \neq \varnothing\}$.

Lemma 12 states a limited pattern of demand connectedness. Given $R \in \mathcal{R}^n$ and $(z^*, p) \in W_{\min}(R)$, when some agents' preferences are changed to z^*-indifferent preferences, under the assumptions of Lemma 12, the object obtained by some z^*-indifferent agent is connected to the null object by the demands of non-z^*-indifferent agents.

Lemma 12. *Let $R \in \mathcal{R}^n$ and $(z^*, p) \in W_{\min}(R)$. Let $N' \subseteq N$ with $1 \leq |N'|$, $R'_{N'} \in \mathcal{R}^I(z^*)^{|N'|}$, $R' \equiv (R'_{N'}, R_{-N'})$, and $N'' \equiv N(R, p) \setminus N'$. Assume that (12-i) for each $i \in N \setminus N'$, and each*

$x \in M$, if $f_i^x(R') = x$, then $f_i^t(R') \geq p^x$, and (12-ii) for each $j \in N'$, $f_j^x(R') \neq 0$.[47] *Then there is a sequence $\{i_k\}_{k=1}^K$ of K distinct agents such that*

(i) $K \geq 2$

(ii) $f_{i_1}^x(R') = 0$

(iii) for each $k \in \{2, \ldots, K\}$, $f_{i_k}^x(R') \neq 0$

(iv) for each $k \in \{1, \ldots, K-1\}$, $i_k \in N''$ *and* $i_K \in N'$

(v) for each $k \in \{1, \ldots, K-1\}$, $\{f_{i_k}^x(R'), f_{i_{k+1}}^x(R')\} \subseteq D(R_{i_k}, p)$.

When an agent i receives $x \in M$ and his CV of the null object from his assignment is negative, for each agent $j \neq i$, if j's CV of x from $\mathbf{0}$ is greater than the difference between what i pays and i's CV of the null object from his assignment, then j receives a real object.

Lemma 13. *Let $R \in \mathcal{R}^n$, $i \in N$, and $x \in M$ be such that $f_i^x(R) = x$ and $CV_i(0; f_i(R)) < 0$. Let $j \in N \setminus \{i\}$. Assume that (13-i) $f_i^t(R) - CV_i(0; f_i(R)) < CV_j(x; \mathbf{0})$. Then $f_j^x(R) \neq 0$.*

Given a preference profile such that an object x obtained by an agent j who pays less than p^x is connected to the null object by the demands of the agents who pay prices p, if p^x is greater than the difference between what j pays and his CV of the null object from his assignment, then a Pareto improvement is possible.

Lemma 14. *Let $R \in \mathcal{R}^n$ and $(z^*, p) \in W_{\min}(R)$. For each $i \in N$, let $x_i \equiv f_i^x(R)$. Assume that there is a sequence $\{i_k\}_{k=1}^K$ of K distinct agents such that (a) $2 \leq K \leq m+1$, (b) $x_{i_1} = 0$, (c) for each $k \in \{1, \ldots, K-1\}$, $\{x_{i_k}, x_{k+1}\} \subseteq D(R_{i_k}, p)$ and $f_{i_k}^t(R) = p^{x_i}$, and (d) $f_{i_K}^t(R) < p^{x_{i_K}}$ and $f_{i_K}^t(R) - CV_{i_K}(0; f_{i_K}(R)) < p^{x_{i_K}}$. Then there is $z' \in Z$ that Pareto-dominates $f(R)$ at R.*

Part 4. By applying the above lemmas and propositions, we complete the proof of Theorem 2.

Let $R \in \mathcal{R}^n$ and $(z^*, p) \in W_{\min}(R)$. Let $R' \in \mathcal{R}^n$ be a profile of z^*-indifferent preferences. Then, for each $x \in M$, the $(m+1)$th highest CV of x from the origin is equal to p^x. Thus, by Lemma 6, for each $x \in M$, the agent who obtains x pays at least p^x. We replace the preferences in R' by the original preferences in R one by one, and inductively show that for each $x \in M$, the agent who obtains x pays at least p^x.

Step 1. We replace the preference R_i' in R' of agent $i \in N$ by his original preference R_i. Then if agent i obtains x at the new profile (R_i, R_{-i}'), then $f_i^t(R_i, R_{-i}') \geq p^x$. For otherwise, since R_i' is z^*-indifferent, $f_i(R_i, R_{-i}') \, P_i' \, f_i(R')$, contradicting strategy-proofness. Then, by Proposition 3, for each $x \in M$, the remaining agent who obtains x pays also at least p^x.

[47] Assumption (12-ii) implies $|N'| \leq m$.

Step 2. We replace the preference R'_j in (R_i, R'_{-i}) of agent $j \neq i$ by his original preference R_j. Then if agent i obtains x at the new profile $(R_{i,j}, R'_{-i,j})$, then $f_i^t(R_{i,j}, R'_{-i,j}) \geq p^x$. For otherwise, since R'_i is z^*-indifferent, Step 1 implies $f_i(R_{i,j}, R'_{-i,j})\ P'_i\ f_i(R_j, R'_{-j})$, contradicting strategy-proofness. Similarly, if agent j obtains x at $(R_{i,j}, R'_{-i,j})$, then $f_j^t(R_{i,j}, R'_{-i,j}) \geq p^x$. Then, by Proposition 3, for each $x \in M$, the remaining agent who obtains x pays also at least p^x.

Proceeding inductively, we conclude that at R, for each $x \in M$, the agent who obtains x pays at least p^x. Then, from Proposition 2, we deduce that each agent is assigned an object in his demand set at prices p and pays its price.[48] Thus, (WE-i) in Definition 3 holds. Since $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$, for each $x \in M$, $p^x > 0$. By Lemma 4, each real object is assigned to someone. Thus, (WE-ii) in Definition 3 also holds. Since $p = p_{\min}(R)$, we have $f(R) \in W_{\min}(R)$.

B.2 *Formal proofs of lemmas and propositions for Theorem 2*

Part 1

Proof of Lemma 3. By no subsidy for losers, $f_i^t(R) \geq 0$. By individual rationality, $f_i^t(R) \leq 0$. Thus, $f_i^t(R) = 0$. □

Proof of Lemma 4. By contradiction, suppose that for each $i \in N$, $f_i^x(R) \neq x$. Then, by $n > m$, there is $j \in N$ such that $f_j^x(R) = 0$. By Lemma 3, $f_j^t(R) = 0$. Let $z' \in Z$ be defined by setting $z'_j \equiv (x, 0)$ and for each $i \in N \setminus \{j\}$, $z'_i \equiv f_i(R)$. Then, since $(x, 0)\ P_j\ (0, 0)$, we have $z'_j\, P_j\, f_j(R)$. Note that for each $i \in N \setminus \{j\}$, $z'_i\, I_i\, f_i(R)$ and $\sum_{i \in N} t'_i = \sum_{i \in N} f_i^t(R)$. Thus, z' Pareto-dominates $f(R)$ at R, which contradicts efficiency. □

Proof of Lemma 5. Let $d \equiv t_j - CV_i(x_j; z_i)$ and let $z' \in Z$ be defined by setting $z'_i \equiv (x_j, t_j - d)$, $z'_j \equiv (x_i, t_i + d)$, and for each $k \in N \setminus \{i, j\}$, $z'_k \equiv z_k$. By $z'_i = (x_j, CV_i(x_j; z_i))$, $z'_i\ I_i\ z_i$. By (a) and $z'_j = (x_i, t_i + t_j - CV_i(x_j; z_i))$, $z'_j\ P_j\ (x_i, CV_j(x_i; z_j))\ I_j\ z_j$. For each $k \in N \setminus \{i, j\}$, $z'_k\ I_k\ z_k$ and $\sum_{k \in N} t'_k = t_j - d + t_i + d + \sum_{k \neq i,j} t_k = \sum_{k \in N} t_k$. Thus, z' Pareto-dominates z at R. □

Proof of Lemma 6. First, for each $y \in M$ and each $i \in N$, $(y, 0)\ P_i\ (0, 0)$. Thus, for each $y \in M$, $C^{m+1}(R, y) > 0$. To the contrary, suppose that $f_i^x(R) = x$ and $f_i^t(R) < C^{m+1}(R, x)$. Let $R'_i \in \mathcal{R}^Q$ be such that for each $y \in M$, $0 < CV'_i(y; \mathbf{0}) < C^{m+1}(R, y)$ and $f_i^t(R) < CV'_i(x; \mathbf{0})$. Let $y' \equiv f_i^x(R'_i, R_{-i})$. Then, by strategy-proofness, $f_i^t(R'_i, R_{-i}) \leq CV'_i(y'; f_i(R))$. Since $CV'_i(0; f_i(R)) < 0$, no subsidy for losers implies $y' \neq 0$.

Since $|\{j \in N \setminus \{i\} : CV_j(y'; \mathbf{0}) \geq C^{m+1}(R, y')\}| \geq m$, there is $j \in N \setminus \{i\}$ such that $CV_j(y'; \mathbf{0}) \geq C^{m+1}(R, y')$ and $f_j^x(R'_i, R_{-i}) = 0$. By Lemma 3, $f_j^t(R'_i, R_{-i}) = 0$.

Let $z'_i \equiv (0, CV'_i(0; f_i(R'_i, R_{-i}))$, $z'_j \equiv (y', CV'_i(y'; \mathbf{0}))$, and for each $k \neq i, j$, $z'_k \equiv f_k(R'_i, R_{-i})$. Then $z'_i\, I'_i\, f_i(R'_i, R_{-i})$, and for each $k \neq i, j$, $z'_k\ I_k\ f_k(R'_i, R_{-i})$. By $CV_j(y'; \mathbf{0}) >$

[48]To see this, let $i \in N$ and $y \equiv f_i^x(R)$. By Proposition 2, $f_i^t(R) \leq CV_i(y; z_i^*)$. By $x_i^* \in D(R_i, p)$, $CV_i(y; z_i^*) \leq p^y$, where $p^y = 0$ if $y = 0$. If $y = 0$, $p^y = 0 = f_i^t(R)$. If $y \neq 0$, $p^y \leq f_i^t(R)$. Thus, by $f_i^t(R) \leq CV_i(y; z_i^*) \leq p^y \leq f_i^t(R)$, $CV_i(y; z_i^*) = p^y = f_i^t(R)$. Then $f_i(R)\ I_i\ z_i^*$. Since $x_i^* \in D(R_i, p)$, for each $z'_i \in B(p)$, $f_i(R)\ I_i\ z_i^*\ R_i\ z'_i$. By $f_i^t(R) = p^y$, $f_i^x(R) \equiv y \in D(R_i, p)$.

$CV'_i(y';\mathbf{0})$, $z'_j\ P_j\ f_j(R'_i, R_{-i})$. By $R'_i \in \mathcal{R}^Q$, $CV'_i(0; f_i(R'_i, R_{-i})) = f^t_i(R'_i, R_{-i}) - CV'_i(y';\mathbf{0})$. Thus, $t'_i + t'_j = CV'_i(0; f_i(R'_i, R_{-i})) + CV'_i(y';\mathbf{0}) = f^t_i(R'_i, R_{-i})$. By $f^t_j(R'_i, R_{-i}) = 0$, $\sum_{k\in N} t'_k = \sum_{k\in N} f^t_k(R'_i, R_{-i})$. Thus, z' Pareto-dominates $f(R'_i, R_{-i})$ at (R'_i, R_{-i}), which contradicts efficiency. □

Proof of Lemma 7. Let f satisfy the four axioms of Theorem 2 on $\mathcal{R}^n$. Let $R \in \mathcal{R}^n$, $i \in N$, and $x \equiv f^x_i(R)$. If $x = 0$, Lemma 7 follows from no subsidy for losers. Thus, suppose that $x \neq 0$. Then, by Lemma 6, $f^t_i(R) \geq C^{m+1}(R, x)$. Since for each $y \in M$ and each $i \in N$, $(y, 0)\ P_i\ (0, 0)$, for each $y \in M$, $C^{m+1}(R, y) > 0$. Thus, $f^t_i(R) > 0$. □

Proof of Lemma 8. First, we show $f^x_i(R'_i, R_{-i}) = f^x_i(R)$. Suppose not. Let $x \equiv f^x_i(R'_i, R_{-i})$. By strategy-proofness, $f_i(R'_i, R_{-i})\ R'_i\ f_i(R)$ and so $f^t_i(R'_i, R_{-i}) \leq CV'_i(x; f_i(R))$. Since $R'_i \in \mathcal{R}_{\mathrm{NCV}}(f_i(R))$, $CV'_i(x; f_i(R)) < 0$. Thus, $f^t_i(R'_i, R_{-i}) < 0$, contradicting no subsidy.

Next, we show $f^t_i(R'_i, R_{-i}) = f^t_i(R)$. Suppose that $f^t_i(R'_i, R_{-i}) < f^t_i(R)$. (The opposite case can be treated symmetrically.) Then $f_i(R'_i, R_{-i})\ P_i\ f_i(R)$, contradicting strategy-proofness. □

Part 2: Proof of Proposition 2

Proof of Lemma 9. By contradiction, suppose that $f^x_i(R) = x$ and $CV_i(x;\mathbf{0}) < C^m(R, x)$. Then, by Lemma 6, $C^{m+1}(R, x) \leq f^t_i(R)$. By individual rationality, $f^t_i(R) \leq CV_i(x;\mathbf{0})$. Then, by $CV_i(x;\mathbf{0}) \leq C^{m+1}(R, x)$, $f^t_i(R) = CV_i(x;\mathbf{0})$. Since $|\{j \in N : CV_j(x;\mathbf{0}) \geq C^m(R, x)\}| = m$, there is $j \in N \setminus \{i\}$ such that $CV_j(x;\mathbf{0}) \geq C^m(R, x)$ and $f^x_j(R) = 0$. By Lemma 3, $f^t_j(R) = 0$. Then, by $CV_i(0; f_i(R)) = 0$ and $f^t_i(R) = CV_i(x;\mathbf{0}) < C^m(R, x) \leq CV_j(x;\mathbf{0})$, $f^t_j(R) - CV_i(0; f_i(R)) < CV_j(x; f_j(R)) - f^t_i(R)$. Note that $f_i(R)\ I_i\ f_j(R)$ and $f_i(R)\ P_j\ f_j(R)$. Thus, by $x \neq 0$ and Lemma 5, there is $z' \in Z$ that Pareto-dominates $f(R)$ at R, which contradicts efficiency. □

Proof of Lemma 10. (Figure 2.) By contradiction, suppose that $f^x_i(R) \neq x$. Then, by Lemma 4, there is $j \in N \setminus \{i\}$ such that $f^x_j(R) = x$. By (b) and (c), $f^t_j(R) \leq CV_j(x; z_j) < CV_i(x;\mathbf{0})$. Since $z \in Z^{\mathrm{IR}}(R)$, for each $y \in M$, $CV_j(y; z_j) \leq CV_j(y;\mathbf{0})$. Let $R'_j \in \mathcal{R}_{\mathrm{NCV}}(f_j(R))$ be such that (i) $-CV'_j(0; f_j(R)) < CV_i(x;\mathbf{0}) - f^t_j(R)$ and (ii) for each $y \in M \setminus \{x\}$, $CV'_j(y;\mathbf{0}) = CV_j(y;\mathbf{0})$. Then, by Lemma 8, $f_j(R'_j, R_{-j}) = f_j(R)$. Since $f^x_j(R'_j, R_{-j}) = x$, $f^x_i(R'_j, R_{-j}) \neq x$.

Next, we show $f^x_i(R'_j, R_{-j}) \notin M \setminus \{x\}$. Suppose there is $y \in M \setminus \{x\}$ such that $f^x_i(R'_j, R_{-j}) = y$. By (ii), $C^m(R'_j, R_{-j}, y) = C^m(R, y)$. By (a), $CV_i(y;\mathbf{0}) < C^m(R'_j, R_{-j}, y)$, which contradicts Lemma 9. Thus, $f^x_i(R'_j, R_{-j}) = 0$. By Lemma 3, $f^t_i(R'_j, R_{-j}) = 0$. Then, by (i) and $f_j(R'_j, R_{-j}) = f_j(R)$, $f^t_i(R'_j, R_{-j}) - CV'_j(0; f_j(R'_j, R_{-j})) < CV_i(x; f_i(R'_j, R_{-j})) - f^t_j(R'_j, R_{-j})$. Note that $f_j(R'_j, R_{-j})\ P'_j\ f_i(R'_j, R_{-j})$ and $f_j(R'_j, R_{-j})\ P_i\ f_i(R'_j, R_{-j})$. By $x \neq 0$ and Lemma 5, there is $z' \in Z$ that Pareto-dominates $f(R'_j, R_{-j})$ at (R'_j, R_{-j}), which contradicts efficiency. □

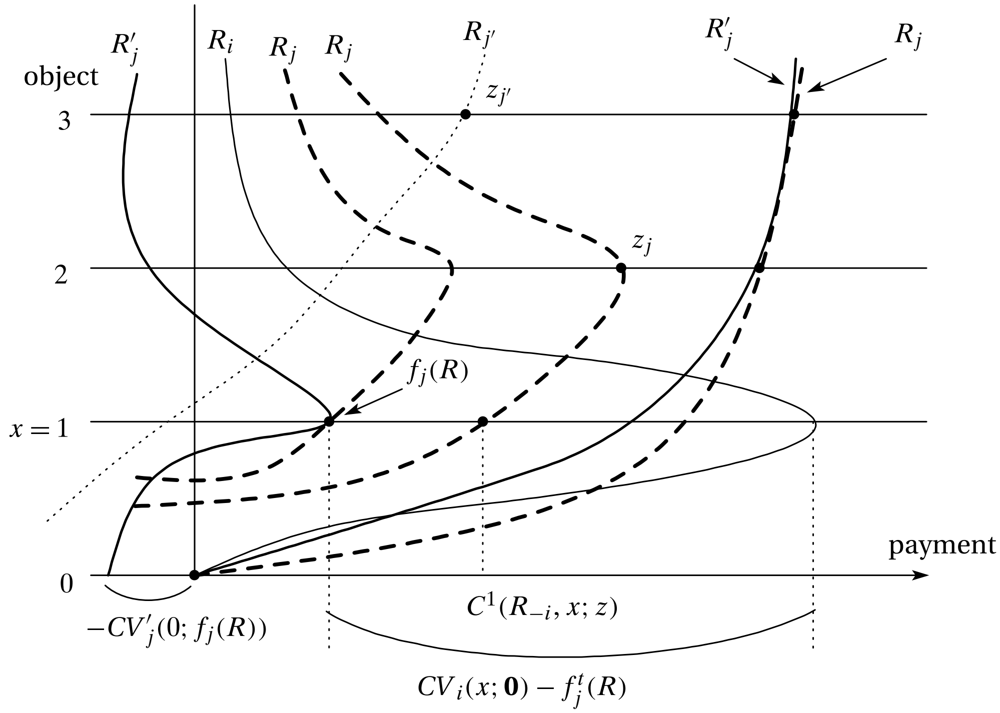

Figure 2. Illustration of the proof of Lemma 10.

Proof of Proposition 2. We only show $f_1(R)\ R_1\ z_1$, since the other agents can be treated in the same way. If $x_1 = 0$, then $z_1 = \mathbf{0}$ and so, by individual rationality, $f_1(R)\ R_1\ z_1$. Thus, assume that $x_1 \neq 0$. Let $N^+ \equiv \{j \in N : x_j \neq 0\}$. Note that $|N^+| = m$.

By contradiction, suppose that $z_1\ P_1\ f_1(R)$. We prove Claim 2 by induction. Part (iv) of Claim 2 induces a contradiction by the finiteness of N^+.

Claim 2. *For each $k \geq 0$, there exist a set $N(k+1)$ of $k+1$ distinct agents, say $N(k+1) \equiv \{1, \ldots, k+1\}$, and $R'_{N(k+1)} \in \mathcal{R}^{k+1}$ such that*

(i) $z_{k+1}\ P_{k+1}\ f_{k+1}(R'_{N(k)}, R_{-N(k)})$

(ii) for each $j \in N(k+1)$ and each $y \in M \setminus \{x_j\}$,

$$CV'_j(y; \mathbf{0}) < C^n(R'_{\{1,\ldots,j-1\}}, R_{-\{1,\ldots,j-1\}}, y)$$

(iii) $t_{k+1} < CV'_{k+1}(x_{k+1}; \mathbf{0}) < CV_{k+1}(x_{k+1}; f_{k+1}(R'_{N(k)}, R_{-N(k)}))$

(iv) $N(k+1) \subsetneq N^+$,

where $N(k) \equiv \{1, \ldots, k\}$.

Figure 3 illustrates (i), (ii), and (iii) for $k = 1$.

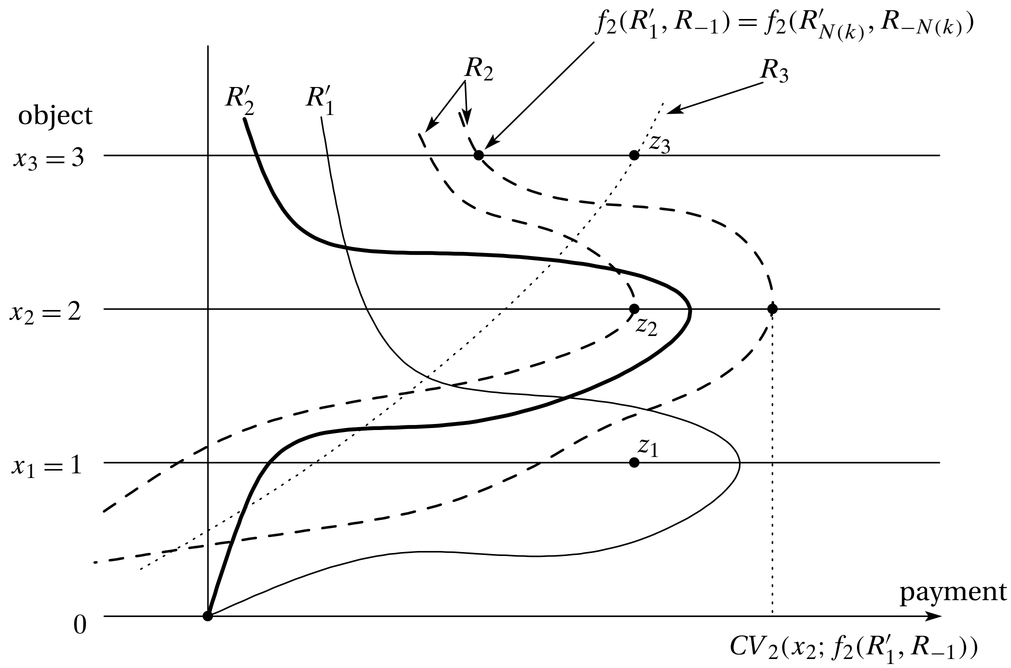

FIGURE 3. Illustration of (i), (ii), and (iii) in the proof of Proposition 2 for $k = 1$.

PROOF.

STEP 1. Let $k = 0$ and $N(1) \equiv \{1\}$. By $z_1\ P_1\ f_1(R)$, (i) holds and so $t_1 < CV_1(x_1; f_1(R))$. Note that for each $y \in M$, $C^n(R, y) > 0$. Thus, there is $R'_1 \in \mathcal{R}$ such that (ii) for each $y \in M \setminus \{x_1\}$, $CV'_1(y; \mathbf{0}) < C^n(R, y)$, and (iii) $t_1 < CV'_1(x_1; \mathbf{0}) < CV_1(x_1; f_1(R))$.

Note that $\{1\} \subseteq N^+$. Suppose $\{1\} = N^+$. Since $|N^+| = m$, $m = 1$. Thus, by $x_1 \neq 0$, for each $j \in N \setminus \{1\}$, $z_j = \mathbf{0}$. Since $z \in W(R)$, for each $j \in N \setminus \{1\}$, $z_j\ R_j\ z_1$ and so $CV_j(x_1; \mathbf{0}) \leq t_1$. Thus, by (iii), $C^1(R_{-1}, x_1; z) \leq t_1 < CV'_1(x_1; \mathbf{0})$. By individual rationality, for each $j \in N \setminus \{1\}$, $f_j(R'_1, R_{-1})\ R_j\ \mathbf{0} = z_j$. Since $z \in Z^{\mathrm{IR}}(R'_1, R_{-1})$, Lemma 10 implies $f_1^x(R'_1, R_{-1}) = x_1$. By individual rationality, $f_1^t(R'_1, R_{-1}) \leq CV'_1(x_1; \mathbf{0})$. However, by (iii), $f_1^t(R'_1, R_{-1}) < CV_1(x_1; f_1(R))$. Thus, $f_1(R'_1, R_{-1})\ P_1\ f_1(R)$, contradicting strategy-proofness. Thus, (iv) $\{1\} \subsetneq N^+$.

STEP 2 (INDUCTION ARGUMENT). Let $k \geq 1$. As induction hypothesis, assume that there exist a set $N(k) \supseteq N(1)$ of k distinct agents, say $N(k) \equiv \{1, \ldots, k\}$, and $R'_{N(k)} \in \mathcal{R}^k$ such that

(i-k) $z_k\ P_k\ f_k(R'_{N(k)\setminus\{k\}}, R_{-N(k)\setminus\{k\}})$

(ii-k) for each $j \in N(k)$ and each $y \in M \setminus \{x_j\}$, $CV'_j(y; \mathbf{0}) < C^n(R'_{\{1,\ldots,j-1\}}, R_{-\{1,\ldots,j-1\}}, y)$

(iii-k) $t_k < CV'_k(x_k; \mathbf{0}) < CV_k(x_k; f_k(R'_{N(k)\setminus\{k\}}, R_{-N(k)\setminus\{k\}}))$

(iv-k) $N(k) \subsetneq N^+$.

By (iv-k), $N^+ \setminus N(k) \neq \varnothing$. The proof has two steps.

Step 2.1. *There is $i \in N^+ \setminus N(k)$ such that $z_i \; P_i \; f_i(R'_{N(k)}, R_{-N(k)})$.*

Proof. By contradiction, suppose that for each $j \in N^+ \setminus N(k)$, $f_j(R'_{N(k)}, R_{-N(k)}) \; R_j \; z_j$.

First, we show that $f_k^x(R'_{N(k)}, R_{-N(k)}) = x_k$. By (ii-$k$), for each $y \in M \setminus \{x_k\}$,

$$
\begin{aligned}
CV'_k(y; \mathbf{0}) &< C^n(R'_{N(k)\setminus\{k\}}, R_{-N(k)\setminus\{k\}}, y) \\
&= C^{n-1}(R'_{N(k)}, R_{-N(k)}, y) \\
&\leq C^m(R'_{N(k)}, R_{-N(k)}, y).
\end{aligned}
$$

Let $z' \in Z$ be defined by setting for each $j \in N \setminus N(k)$, $z'_j \equiv z_j$, and for each $j \in N(k)$, $z'_j \equiv \mathbf{0}$. Then $z' \in Z^{\mathrm{IR}}(R'_{N(k)}, R_{-N(k)})$. By the supposition of Step 2.1, for each $j \in N^+ \setminus N(k)$, $f_j(R'_{N(k)}, R_{-N(k)}) \, R_j \, z_j \equiv z'_j$. By individual rationality, for each $j \in N(k) \cup (N \setminus N^+)$, $f_j(R'_{N(k)}, R_{-N(k)}) \, R_j \, \mathbf{0} = z'_j$.

Since $z \in W(R)$, for each $j \in N \setminus N(k)$, $CV_j(x_k; z'_j) = CV_j(x_k; z_j) \leq t_k$. By (ii-$k$), for each $j \in N(k) \setminus \{k\}$,

$$
CV'_j(x_k; z'_j) = CV'_j(x_k; \mathbf{0}) < C^n(R'_{\{1,\dots,j-1\}}, R_{-\{1,\dots,j-1\}}, x_k) \leq C^n(R, x_k) \leq t_k.
$$

Thus, by (iii-k), $C^1(R'_{N(k)\setminus\{k\}}, R_{-N(k)}, x_k; z') \leq t_k < CV'_k(x_k; \mathbf{0})$.

Since the assumptions of Lemma 10 hold for $(R'_{N(k)}, R_{-N(k)})$ as above, Lemma 10 implies that $f_k^x(R'_{N(k)}, R_{-N(k)}) = x_k$. By individual rationality, $f_k^t(R'_{N(k)}, R_{-N(k)}) \leq CV'_k(x_k; \mathbf{0})$. However, (iii-$k$) implies that $f_k^t(R'_{N(k)}, R_{-N(k)}) < CV_k(x_k; f_k(R'_{N(k)\setminus\{k\}}, R_{-N(k)\setminus\{k\}}))$.

Thus, $f_k(R'_{N(k)}, R_{-N(k)}) \; P_k \; f_k(R'_{N(k)\setminus\{k\}}, R_{-N(k)\setminus\{k\}})$, contradicting strategy-proofness. ◁

Step 2.2. *We complete the proof of Claim 2.*

Proof. Without loss of generality, let $i \equiv k+1$ and $N(k+1) \equiv N(k) \cup \{k+1\}$.

Then $N(k+1) \supsetneq N(k)$, and (i) follows from $z_i \; P_i \; f_i(R'_{N(k)}, R_{-N(k)})$. By (i), $t_{k+1} < CV_{k+1}(x_{k+1}; f_{k+1}(R'_{N(k)}, R_{-N(k)}))$. Also, for each $y \in M$, $C^n(R'_{N(k)}, R_{-N(k)}, y) > 0$. Thus, there is $R'_{k+1} \in \mathcal{R}$ such that

$$
t_{k+1} < CV'_{k+1}(x_{k+1}; \mathbf{0}) < CV_{k+1}(x_{k+1}; f_{k+1}(R'_{N(k)}, R_{-N(k)})),
$$

and for each $y \in M \setminus \{x_{k+1}\}$, $CV'_{k+1}(y; \mathbf{0}) < C^n(R'_{N(k)}, R_{-N(k)}, y)$. Let $R'_{N(k+1)} \equiv (R'_{N(k)}, R'_{k+1})$. Then (ii) and (iii) follow from (ii-k).

By (iv-k) and $\{k+1\} \subseteq N^+$, $N(k+1) \subseteq N^+$.

Finally, we show (iv): $N(k+1) \subsetneq N^+$. Suppose that $N(k+1) = N^+$. Then $|N(k+1)| = |N^+| = m$. Thus, for each $j \in N \setminus N(k+1)$, $z_j = \mathbf{0}$.

By (ii), for each $y \in M \setminus \{x_{k+1}\}$,

$$
\begin{aligned}
CV'_{k+1}(y; \mathbf{0}) &< C^n(R'_{N(k)}, R_{-N(k)}, y) \\
&= C^{n-1}(R'_{N(k+1)}, R_{-N(k+1)}, y) \\
&\leq C^m(R'_{N(k+1)}, R_{-N(k+1)}, y).
\end{aligned}
$$

Let $z' \in Z$ be defined by setting for each $j \in N$, $z'_j \equiv \mathbf{0}$. Then $z' \in Z^{\mathrm{IR}}(R'_{N(k+1)}, R_{-N(k+1)})$.

By individual rationality, for each $j \in N \setminus \{k+1\}$, $f_j(R'_{N(k+1)}, R_{-N(k+1)})\ R_j\ \mathbf{0} = z'_j$. Since $z \in W(R)$, for each $j \in N \setminus N(k+1)$, $CV_j(x_{k+1}; z'_j) = CV_j(x_{k+1}; z_j) \le t_{k+1}$. By (ii), for each $j \in N(k+1) \setminus \{k+1\}$,

$$CV'_j(x_{k+1}; z'_j) = CV'_j(x_{k+1}; \mathbf{0}) < C^n(R'_{\{1,\dots,j-1\}}, R_{-\{1,\dots,j-1\}}, x_{k+1}) \le C^n(R, x_{k+1}) \le t_{k+1}.$$

Thus, by (iii), $CV'_{k+1}(x_{k+1}; \mathbf{0}) > t_{k+1} \ge C^1(R'_{N(k)}, R_{-N(k+1)}, x_{k+1}; z')$, and the assumptions of Lemma 10 hold for $(R'_{N(k+1)}, R_{-N(k+1)})$. By Lemma 10, $f^x_{k+1}(R'_{N(k+1)}, R_{-N(k+1)}) = x_{k+1}$. By individual rationality, $f^t_{k+1}(R'_{N(k+1)}, R_{-N(k+1)}) \le CV'_{k+1}(x_{k+1}; \mathbf{0})$. However, (iii) implies that $f^t_{k+1}(R'_{N(k+1)}, R_{-N(k+1)}) < CV_{k+1}(x_{k+1}; f_{k+1}(R'_{N(k)}, R_{-N(k)}))$.

Thus, $f_{k+1}(R'_{N(k+1)}, R_{-N(k+1)})\ P_{k+1}\ f_{k+1}(R'_{N(k)}, R_{-N(k)})$, contradicting strategy-proofness. □

Part 3: Proof of Proposition 3

Proof of Lemma 11. First, we show (a). Let $M' \subseteq M$. Since $z^* \in Z_{\min}(R)$, by Theorem 1, (i) $|\{i \in N : D(R_i, p) \subseteq M'\}| \le |M'|$ and (ii) $|\{i \in N : D(R_i, p) \cap M' \neq \varnothing\}| > |M'|$. Note that for each $i \in N'$, $D(R'_i, p) = L$ and for each $j \in N \setminus N'$, $D(R'_j, p) = D(R_j, p)$. Thus, for each $i \in N'$, $D(R'_i, p) \nsubseteq M'$ and $D(R'_i, p) \cap M' \neq \varnothing$. Then

$$\bigl|\{i \in N : D(R'_i, p) \subseteq M'\}\bigr| \le \bigl|\{i \in N : D(R_i, p) \subseteq M'\}\bigr| \le |M'| \quad \text{and}$$
$$\bigl|\{i \in N : D(R'_i, p) \cap M' \neq \varnothing\}\bigr| \ge \bigl|\{i \in N : D(R_i, p) \cap M' \neq \varnothing\}\bigr| > |M'|.$$

That is, no set is overdemanded or weakly underdemanded at p for R'. Thus, (a) follows from Theorem 1. Then (b) follows from Proposition 2. □

Proof of Lemma 12.

Step 1. We show that for each $j \notin N(R, p) \cup N'$, $f^x_j(R') = 0$.

Suppose that for some $j \notin N(R, p) \cup N'$, $x \equiv f^x_j(R') \neq 0$. By $D(R'_j, p) = 0$, individual rationality implies $f^t_j(R') \le CV_j(x; \mathbf{0}) < p^x$. This contradicts (12-i).

Step 2. Let $z' \equiv (x', t')$ be such that for each $i \in N'$, $x'_i \equiv f^x_i(R')$ and $t'_i \equiv p^{x'_i}$, and for each $i \in N \setminus N'$, $z'_i \equiv f_i(R')$. We show $z' \in Z_{\min}(R')$.

First, we show that z' satisfies (WE-i) in Definition 3 for R'. Note that by $R'_{N'} \in \mathcal{R}^I(z^*)^{|N'|}$, for each $i \in N'$ and each $z''_i \in B(p)$, $z'_i\, I'_i\, z''_i$. Thus, for each $i \in N'$, $x'_i \in D(R'_i, p)$. By the definition of z', for each $i \in N'$, $t'_i = p^{x'_i}$. Therefore, it is sufficient to prove that for each $i \in N \setminus N'$, $f^x_i(R') \in D(R_i, p)$ and $f^t_i(R') = p^y$, where $y = f^x_i(R')$. Let $i \in N \setminus N'$ and $y \equiv f^x_i(R')$. By $(z^*, p) \in W_{\min}(R)$, $R'_{N'} \in \mathcal{R}^I(z^*)^{|N'|}$, and by Lemma 11(b), $f^t_i(R') \le CV_i(y; z^*_i)$. By $x^*_i \in D(R_i, p)$, $CV_i(y; z^*_i) \le p^y$, where $p^y = 0$ if $y = 0$. If $y = 0$, by Lemma 3, $f^t_i(R') = 0 = p^y$. If $y \neq 0$, by (12-i), $p^y \le f^t_i(R')$. Then, by $f^t_i(R') \le CV_i(y; z^*_i) \le$

$p^y \leq f_i^t(R')$, $CV_i(y; z_i^*) = p^y = f_i^t(R')$. Thus, $f_i(R')\, I_i\, z_i^*$. By $x_i^* \in D(R_i, p)$, for each $z_i'' \in B(p)$, $f_i(R')\, I_i\, z_i^*\, R_i\, z_i''$. Then, by $f_i^t(R') = p^y$, $f_i^x(R') \in D(R_i, p)$.

Next, we show that z' satisfies (WE-ii) in Definition 3. Since $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$, for each $x \in M$, $p^x > 0$. By Lemma 4, for each $x \in M$, there is $i \in N$ such that $x_i' \equiv f_i^x(R') = x$.

Thus, $(z', p) \in W(R')$. By $(z^*, p) \in W_{\min}(R)$, $R'_{N'} \in \mathcal{R}^I(z^*)^{|N'|}$, and Lemma 11(a), $p = p_{\min}(R')$. Hence, $z' \in Z_{\min}(R')$.

Step 3. Let $M' \equiv \{x \in M : \text{for some } j \in N', f_j^x(R') = x\}$. By (12-ii) and $1 \leq |N'|$, $M' \neq \varnothing$. Let $x \in M'$. Since $\mathcal{R} \equiv \mathcal{R}^C$ and $n > m$, for each $y \in M$, $p^y > 0$. Then, by Step 2 and Corollary 2, there is a sequence $\{i_k\}_{k=1}^{K'}$ of K' distinct agents such that (i) $x'_{i_1} = 0$ or $p^{x'_{i_1}} = 0$, (ii) for each $k \in \{2, \ldots, K'-1\}$, $x'_{i_k} \neq 0$ and $p^{x'_{i_k}} > 0$, (iii) $x'_{i_{K'}} = x$, and (iv) for each $k \in \{1, \ldots, K'-1\}$, $\{x'_{i_k}, x'_{i_{k+1}}\} \subseteq D(R'_{i_k}, p)$. Since for each $y \in M$, $p^y > 0$, then $x'_{i_1} = 0$. By (12-ii), $i_1 \notin N'$. By Step 1, for each $k \in \{1, \ldots, K'\}$, $i_k \in N'' \cup N'$. Note that by $i_{K'} \in N'$, $\{k : i_k \in N'\} \neq \varnothing$. Let $K \equiv \min\{k : i_k \in N'\}$. Then, for each $k \in \{1, \ldots, K-1\}$, $i_k \in N''$ and $i_K \in N'$. Thus, the sequence $\{i_k\}_{k=1}^{K}$ satisfies (i), (ii), (iii), (iv), and (v) of Lemma 12. □

Proof of Lemma 13. Suppose that $f_j^x(R) = 0$. By Lemma 3, $f_j^t(R) = 0$. By (13-i), $f_j^t(R) - CV_i(0; f_i(R)) < CV_j(x; f_j(R)) - f_i^t(R)$ and $f_i(R)\, P_j\, f_j(R)$. By $CV_i(0; f_i(R)) < 0$, $f_i(R)\, P_i\, f_j(R)$. Then, by $x \neq 0$ and Lemma 5, there is $z' \in Z$ that Pareto-dominates $f(R)$ at R, which contradicts efficiency. □

Proof of Lemma 14. Let $z' \in Z$ be defined by setting $z'_{i_K} \equiv (0, CV_{i_K}(0; f_{i_K}(R)))$, $z'_{i_{K-1}} \equiv (x_{i_K}, f_{i_K}^t(R) - CV_{i_K}(0; f_{i_K}(R)))$ for each $k \in \{1, \ldots, K-1\}$, $z'_{i_k} \equiv f_{i_{k+1}}(R)$, and for each $i \in N \setminus \{i_k\}_{k=1}^{K}$, $z_i' \equiv f_i(R)$. (See Figure 4.)

We show that z' Pareto-dominates $f(R)$ at R. By the definition of $CV_{i_K}(0; f_{i_K}(R))$, $z'_{i_K}\, I_{i_K}\, f_{i_K}(R)$. Also,

$$z'_{i_{K-1}} \underset{\text{def. of } z'_{i_{K-1}} \text{ and (d)}}{P_{i_{K-1}}} (x_{i_K}, p^{x_{i_K}}) \underset{\text{(c)}}{I_{i_{K-1}}} f_{i_{K-1}}(R).$$

For each $k \in \{1, \ldots, K-2\}$, by (c) and $z'_{i_k} = f_{i_{k+1}}(R)$, $z'_{i_k}\, I_{i_k}\, f_{i_k}(R)$. Note that

$$\begin{aligned}
\sum_{i \in N} t_i' &= CV_{i_K}(0; f_{i_K}(R)) + f_{i_K}^t(R) - CV_{i_K}(0; f_{i_K}(R)) + \sum_{k=1}^{K-2} f_{i_{k+1}}^t(R) + \sum_{i \in N \setminus \{i_k\}_{k=1}^{K}} f_i^t(R) \\
&= f_{i_K}^t(R) + \sum_{k=2}^{K-1} f_{i_k}^t(R) + \sum_{i \in N \setminus \{i_k\}_{k=1}^{K}} f_i^t(R) \\
&= \sum_{i \in N} f_i^t(R) \quad \text{by (b).}
\end{aligned}$$

Thus, z' Pareto-dominates $f(R)$ at R. □

Proof of Proposition 3. Let $R' \equiv (R'_{N'}, R_{-N'})$. Without loss of generality, let $N' \equiv \{1, 2, \ldots, n'\}$. We only show that if $f_1^x(R') = x \in M$, then $f_1^t(R') \geq p^x$ since we can treat

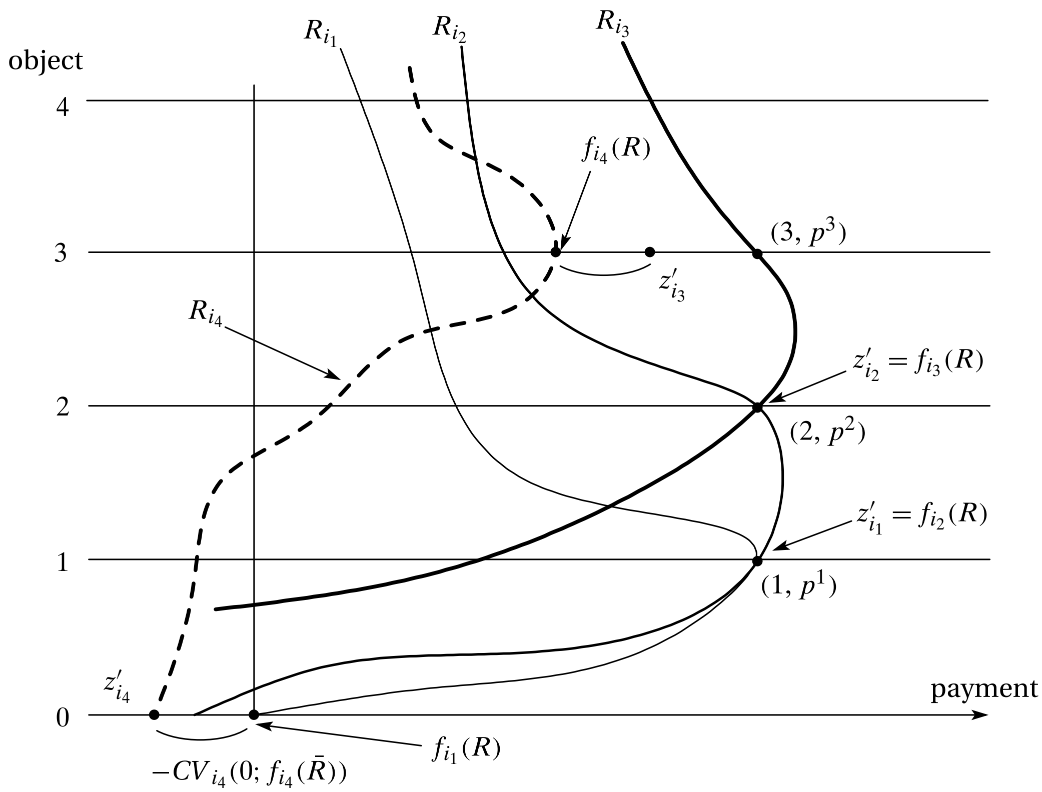

FIGURE 4. Illustration of z' in Lemma 14 for $K = 4$.

similarly the other agents in N'. Let $f_1^x(R') \equiv x \in M$. By contradiction, suppose that $f_1^t(R') < p^x$. Let $N'' \equiv N(R, p) \setminus N'$.

We derive a contradiction in two steps.

STEP 1. *There is $\bar{R}_{N'} \in \mathcal{R}^I(z^*)^{|N'|}$ such that the following statements hold:*

(a) For each $i \in N'$ and each $z_i \equiv (y, t) \in M \times \mathbb{R}$ with $t < p^y$, $-\overline{CV}_i(0; z_i) < p^y - t$.

(b) For some $j \in N'$, $f_j^x(\bar{R}_{N'}, R'_{-N'}) \equiv y \neq 0$ and $f_j^t(\bar{R}_{N'}, R'_{-N'}) < p^y$.

PROOF. We replace the preference R'_i of each agent i in N' with $\bar{R}_i$, inductively. Since $f_1^x(R') \equiv x \neq 0$ and $f_1^t(R') < p^x$, there is $\bar{R}_1 \in \mathcal{R}^I(z^*) \cap \mathcal{R}_{\mathrm{NCV}}(f_1(R'))$ such that

$$\text{for each } z_1 \equiv (y, t) \in M \times \mathbb{R} \text{ with } t < p^y,\ -\overline{CV}_1(0; z_1) < p^y - t.$$

Then, by Lemma 8, $f_1(\bar{R}_1, R'_{-1}) = f_1(R')$. Since $f_1^x(R') \equiv x \neq 0$ and $f_1^t(R') < p^x$, $f_1^x(\bar{R}_1, R'_{-1}) = x \neq 0$ and $f_1^t(\bar{R}_1, R'_{-1}) < p^x$.

INDUCTION ARGUMENT. Let $s \leq |N'| - 1$. As induction hypothesis, assume that there are $S \subsetneq N'$ and $\bar{R}_S \in \mathcal{R}^I(z^*)^{|S|}$ such that $|S| = s$.

(3-a) For each $i \in S$ and each $z_i \equiv (y, t) \in M \times \mathbb{R}$ with $t < p^y$, $-\overline{CV}_i(0; z_i) < p^y - t$.

(3-b) For some $j_s \in S$, $f^x_{j_s}(\bar{R}_S, R'_{-S}) \equiv y_s \neq 0$ and $f^t_{j_s}(\bar{R}_S, R'_{-S}) < p^{y_s}$.

Then, by (3-a), (3-b), and Lemma 13, for each $i \in N'$, $f^x_i(\bar{R}_S, R'_{-S}) \neq 0$. Thus, (12-ii) of Lemma 12 holds for $(\bar{R}_S, R'_{-S})$. By (3-i) of Proposition 3, (12-i) of Lemma 12 holds for $(\bar{R}_S, R'_{-S})$. Thus, by Lemma 12, there is a sequence $\{i_k\}_{k=1}^K$ of K distinct agents such that (i) $2 \leq K$, (ii) $f^x_{i_1}(\bar{R}_S, R'_{-S}) = 0$, (iii) for each $k \in \{2, \dots, K\}$, $f^x_{i_k}(\bar{R}_S, R'_{-S}) \neq 0$, (iv) for each $k \in \{1, \dots, K-1\}$, $i_k \in N''$ and $i_K \in N'$, and (v) for each $k \in \{1, \dots, K-1\}$, $\{f^x_{i_k}(\bar{R}_S, R'_{-S}), f^x_{i_{k+1}}(\bar{R}_S, R'_{-S})\} \subseteq D(R'_{i_k}, p)$.

We show $f^t_{i_K}(\bar{R}_S, R'_{-S}) < p^{\bar{x}_{i_K}}$, where $\bar{x}_{i_K} \equiv f^x_{i_K}(\bar{R}_S, R'_{-S})$. If $i_K = j_s$, $f^t_{i_K}(\bar{R}_S, R'_{-S}) < p^{\bar{x}_{i_K}}$ follows from (3-b). Thus, let $i_K \neq j_s$. By Lemma 11(b), $f^t_{i_K}(\bar{R}_S, R'_{-S}) \leq p^{\bar{x}_{i_K}}$. By contradiction, suppose that $f^t_{i_K}(\bar{R}_S, R'_{-S}) = p^{\bar{x}_{i_K}}$. Then we construct a sequence of agents satisfying the assumption of Lemma 14 by adding agent j_s to the above sequence $\{i_k\}_{k=1}^K$ as the $(K+1)$th agent. Thus, there is an allocation z' that Pareto-dominates $f(\bar{R}_S, R'_{-S})$ at $(\bar{R}_S, R'_{-S})$. This contradicts efficiency. Thus, $f^t_{i_K}(\bar{R}_S, R'_{-S}) < p^{\bar{x}_{i_K}}$.

Next, we show $i_K \notin S$. By contradiction, suppose that $i_K \in S$. Then, by $f^t_{i_K}(\bar{R}_S, R'_{-S}) < p^{\bar{x}_{i_K}}$, the above sequence $\{i_k\}_{k=1}^K$ satisfies the assumptions of Lemma 14. Thus, there is an allocation z' that Pareto-dominates $f(\bar{R})$ at $\bar{R}$, which contradicts efficiency. Thus, $i_K \notin S$ and so $i_K \in N' \setminus S$.

Let $j_{s+1} \equiv i_K$ and $S' \equiv S \cup \{j_{s+1}\}$. Then, by $i_K \notin S$, $|S'| = s+1$, and $f^x_{j_{s+1}}(\bar{R}_S, R'_{-S}) \equiv y_{s+1} \neq 0$ and $f^t_{j_{s+1}}(\bar{R}_S, R'_{-S}) < p^{y_{s+1}}$. Note that $f^t_{j_{s+1}}(\bar{R}_S, R'_{-S}) < p^{y_{s+1}}$ implies that there is $\bar{R}_{j_{s+1}} \in \mathcal{R}^I(z^*) \cap \mathcal{R}_{\text{NCV}}(f_{j_{s+1}}(\bar{R}_S, R'_{-S}))$ such that

$$\text{for each } z_{j_{s+1}} \equiv (y, t) \in M \times \mathbb{R} \text{ with } t < p^y,\ -\overline{CV}_{j_{s+1}}(0; z_{j_{s+1}}) < p^y - t.$$

Thus, by (3-a),

$$\text{for each } i \in S' \text{ and each } z_i \equiv (y, t) \in M \times \mathbb{R} \text{ with } t < p^y,\ -\overline{CV}_i(0; z_i) < p^y - t.$$

By Lemma 8, $f_{j_{s+1}}(\bar{R}_{S'}, R'_{-S'}) = f_{j_{s+1}}(\bar{R}_S, R'_{-S})$.

Since $f^x_{j_{s+1}}(\bar{R}_S, R'_{-S}) \equiv y_{s+1} \neq 0$ and $f^t_{j_{s+1}}(\bar{R}_S, R'_{-S}) < p^{y_{s+1}}$, we have $f^x_{j_{s+1}}(\bar{R}_{S'}, R'_{-S'}) \equiv y_{s+1} \neq 0$ and $f^t_{j_{s+1}}(\bar{R}_{S'}, R'_{-S'}) < p^{y_{s+1}}$. $\triangleleft$

Step 2. *Concluding that* $f^t_1(R') \geq p^x$.

By (a) and (b) of Step 1, and Lemma 13, for each $i \in N'$, $f^x_i(\bar{R}_{N'}, R'_{-N'}) \neq 0$. Then it follows from (3-i) of Proposition 3 that (12-i) and (12-ii) of Lemma 12 hold for the profile $(\bar{R}_{N'}, R'_{-N'})$. Thus, by Lemma 12, there is a sequence $\{i_k\}_{k=1}^K$ of K distinct agents such that (i) $2 \leq K$, (ii) $f^x_{i_1}(\bar{R}_{N'}, R'_{-N'}) = 0$, (iii) for each $k \in \{2, \dots, K\}$, $f^x_{i_k}(\bar{R}_{N'}, R'_{-N'}) \neq 0$, (iv) for each $k \in \{1, \dots, K-1\}$, $i_k \in N''$ and $i_K \in N'$, and (v) for each $k \in \{1, \dots, K-1\}$, $\{f^x_{i_k}(\bar{R}_{N'}, R'_{-N'}), f^x_{i_{k+1}}(\bar{R}_{N'}, R'_{-N'})\} \subseteq D(R'_{i_k}, p)$. Therefore, similarly to Step 1, we can show $i_K \notin N'$, which contradicts $i_K \in N'$. $\square$

Appendix C: Proof for Section 5 (Proposition 1)

Proof of Proposition 1. Let $\mathcal{R} \subseteq \mathcal{R}^E$ and $R \in \mathcal{R}^n$. Consider the simultaneous ascending (SA) auction defined in Section 5. By the definition of the auction, the price path $p(\cdot)$ is nondecreasing with respect to time. For each $x \in M$, let $\bar{p}^x > C^1(R, x)$. Then, for each $x \in M$, each p^{-x}, and each $i \in N$, $x \notin D(R_i, (\bar{p}^x, p^{-x}))$. This implies that each $x \in M$ is not in a minimal overdemanded set whenever its price is $\bar{p}^x$. Thus, the price path $p(\cdot)$ is bounded above by $\bar{p}$, that is, for each time $s \in \mathbb{R}_+$, $p(s) \leq \bar{p}$. Note that prices are raised at a speed at least $d > 0$. Thus, there is a price vector p^* such that the price path $p(\cdot)$ converges to p^* in a finite time.

Let T be the termination time of the SA auction. We show that the final price $p(T) = p_{\min}(R)$. By the definition of the SA auction, no overdemanded set exists at the price $p(T)$. If no weakly underdemanded set exists at $p(T)$, then the desired conclusion follows from Theorem 1. Thus, we show that no weakly underdemanded set exists at $p(T)$. The proof is in two steps.

Step 1. *Let time $s' \in (0, T]$. Assume that there is a set M' that is weakly underdemanded at $p(s')$. Let $N' \equiv \{i \in N : D(R_i, p(s')) \cap M' \neq \varnothing\}$. Then (1-a) $|N'| \geq 2$ and (1-b) there exist time $s'' \in (0, s')$ and $M'' \subsetneq M'$ such that $N'' \equiv \{i \in N : D(R_i, p(s'')) \cap M'' \neq \varnothing\} \subsetneq N'$ and M'' is underdemanded at $p(s'')$.*

Proof. Since M' is weakly underdemanded at $p(s')$, for each $x \in M'$, $p^x(s') > 0$ and $|N'| \leq |M'|$. For each $i \in N$, let $x'_i \in D(R_i, p(s'))$ and $z'_i \equiv (x'_i, p^{x'_i}(s'))$. Note that for each $i \in N \setminus N'$ and each $x \in M'$, $CV_i(x; z'_i) < p^x(s')$. For each $x \in M'$, let $q^x \equiv \max\{CV_i(x; z'_i) : i \in N \setminus N'\} \cup \{0\}$. Let $e > 0$ be such that for each $x \in M'$, $q^x < p^x(s') - e \equiv p^x$. Let $s'' \equiv \max\{s \in \mathbb{R}_+ : \text{for some } x \in M', p^x(s) \leq p^x\}$. Then there is $x' \in M'$ such that $dp^{x'}(s'')/ds > 0$ and $p^{x'}(s'') = p^{x'}$. By $dp^{x'}(s'')/ds > 0$, there is a minimal overdemanded set $\hat{M}$ at $p(s'')$ that includes x'.

Let $\hat{M}' \equiv \hat{M} \cap M'$ and $\hat{M}'' \equiv \hat{M} \setminus M'$. Then (i) $\hat{M}'' \cup \hat{M}' = \hat{M}$ and $\hat{M}'' \cap \hat{M}' = \varnothing$. By $x' \in M'$, (ii) $\hat{M}' \neq \varnothing$. Let

$$\hat{N}' \equiv \{i \in N : D(R_i, p(s'')) \cap \hat{M}' \neq \varnothing \text{ and } D(R_i, p(s'')) \subseteq \hat{M}\},$$

and let $\hat{N}'' \equiv \{i \in N : D(R_i, p(s'')) \subseteq \hat{M}''\}$. Then, by (i), we have (iii) $\hat{N}'' \cap \hat{N}' = \varnothing$.

Note that for each $x \in M'$ and each $i \in N \setminus N'$, $CV_i(x; z'_i) \leq q^x < p^x \leq p^x(s'')$, and for each $x \in M'$, $p^x(s'') \leq p^x(s')$. Thus, for each $i \in N \setminus N'$ and each $x \in M'$,

$$(x'_i, p^{x'_i}(s'')) \underset{p^{x'_i}(s'') \leq p^{x'_i}(s')}{R_i} (x'_i, p^{x'_i}(s')) = z'_i \underset{CV_i(x; z'_i) \leq q^x}{R_i} (x, q^x) \underset{q^x < p^x(s'')}{P_i} (x, p^x(s'')).$$

By $\hat{M}' \subseteq M'$, this implies that (iv) for each $i \in N \setminus N'$, $D(R_i, p(s'')) \cap \hat{M}' = \varnothing$. Thus, $\hat{N}'' = \{i \in N : D(R_i, p(s'')) \cap \hat{M}' = \varnothing \text{ and } D(R_i, p(s'')) \subseteq \hat{M}\}$ and (v) $\hat{N}' \subseteq N'$. The former implies (vi) $\hat{N}'' \cup \hat{N}' = \{i \in N : D(R_i, p(s'')) \subseteq \hat{M}\}$.

Note that

$$\begin{aligned}|\hat{N}''| + |\hat{N}'| &= |\{i \in N : D(R_i, p(t'')) \subseteq \hat{M}\}| \quad \text{(by (iii) and (vi))} \\ &> |\hat{M}| \quad \text{(since } \hat{M} \text{ is overdemanded at } p(s'')) \\ &= |\hat{M}''| + |\hat{M}'| \quad \text{(by (i))}.\end{aligned}$$

Since (i) and (ii) imply $\hat{M}'' \subsetneq \hat{M}$, and since $\hat{M}$ is a minimal overdemanded set at $p(s'')$, $\hat{M}''$ is not overdemanded at $p(s'')$, that is, $|\hat{N}''| \leq |\hat{M}''|$. Thus, by $|\hat{N}''| + |\hat{N}'| > |\hat{M}''| + |\hat{M}'|$, we have (vii) $|\hat{N}'| > |\hat{M}'|$.

Notice that $1 \underset{\text{(ii)}}{\leq} |\hat{M}'| \underset{\text{(vii)}}{<} |\hat{N}'| \underset{\text{(v)}}{\leq} |N'|$. Thus, (1-a) holds.

Next, we establish (1-b). Let $M'' \equiv M' \setminus \hat{M}'$. Since $\hat{M}' \subsetneq M'$,[49] $M'' \neq \varnothing$. By (ii), $M'' \subsetneq M'$. First, we show that $N'' \subseteq N' \setminus \hat{N}'$, that is, for each $i \in N''$, $i \in N'$, and $i \notin \hat{N}'$. Let $i \in N''$. Then $D(R_i, p(s'')) \cap M'' \neq \varnothing$. By (iv), this implies $i \in N'$. Since $\hat{M}' = M' \cap \hat{M}$ implies $M'' = M' \setminus \hat{M}$, $D(R_i, p(s'')) \cap M'' \neq \varnothing$ implies $D(R_i, p(s'')) \setminus \hat{M} \neq \varnothing$. Since $\hat{N}' \subseteq \{j \in N : D(R_j, p(s'')) \subseteq \hat{M}\}$, this implies $i \notin \hat{N}'$. Therefore, $N'' \subseteq N' \setminus \hat{N}'$.

Since (ii) and (vii) imply $|\hat{N}'| > |\hat{M}'| \geq 1$, we have $|\hat{N}'| \geq 2$, and so $N'' \subsetneq N'$. Finally, it follows from the inequalities below that M'' is underdemanded at $p(s'')$.

$$\begin{aligned}|N''| &\leq |N'| - |\hat{N}'| \quad \text{(by (v))} \\ &< |N'| - |\hat{M}'| \quad \text{(by (vii))} \\ &\leq |M'| - |\hat{M}'| \quad \text{(by } |N'| \leq |M'|) \\ &= |M''| \quad \text{(by } M'' = M' \setminus \hat{M}').\end{aligned}$$ ◁

Step 2. *There is no weakly underdemanded set at $p(T)$.*

Proof. By contradiction, suppose that there is a set M_1 that is weakly underdemanded at $p(T)$. Let $N_1 \equiv \{i \in N : D(R_i, p(T)) \cap M_1 \neq \varnothing\}$. Then, by Step 1, $|N_1| \geq 2$, and there exist time $s_1 < T$ and $M_2 \subsetneq M_1$ such that $N_2 \equiv \{i \in N : D(R_i, p(s_1)) \cap M_2 \neq \varnothing\} \subsetneq N_1$ and M_2 is underdemanded at $p(s_1)$. Since M_2 is underdemanded at $p(s_1)$, Step 1 also implies that $|N_2| \geq 2$, and there exist time $s_2 < s_1$ and $M_3 \subsetneq M_2$ such that $N_3 \equiv \{i \in N : D(R_i, p(s_2)) \cap M_3 \neq \varnothing\} \subsetneq N_2$ and M_3 is underdemanded at $p(s_2)$. Proceeding with this argument inductively, we obtains a sequence $\{N_k\} \subsetneq N_1$ such that for each $k \geq 2$, $|N_k| < |N_{k-1}|$ and $|N_k| \geq 2$. However, since N_1 is finite and for each $k \geq 2$, $N_k \subsetneq N_1$, this is a contradiction. □

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