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A theory of school-choice lotteries

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We introduce a new notion of ex ante stability (or fairness) that would be desirable for a school-choice mechanism to satisfy. Our criterion stipulates that a mechanism must be stable based solely on the probabilities that each student will be assigned to different schools, i.e., the assignment must be viewed as stable even before students know which school they will end up going to. This is in contrast to much of the existing literature, which has instead focused on ex post stability, meaning that assignments are deemed stable after students are assigned to schools. Armed with this criterion for evaluating mechanisms, we show that one of the mechanisms that has attracted the most attention—deferred acceptance with random tie-breaking—is not ex ante stable and under some circumstances can lead to ex ante discrimination among some students. We then propose two new mechanisms, which satisfy two notions of ex ante stability we introduce—a strong one and a weak one—and we show that these mechanisms are optimal within the class of mechanisms that satisfy these respective criteria.

Keywords. Matching, school choice, deferred acceptance, stability, ordinal efficiency, market design.

JEL classification. C71, C78, D71, D78.

1. Introduction

Following the 1987 decision of the U.S. Court of Appeals, the Boston school district introduced a possibility of "choice" for public schools by relaxing the mandatory zoning policy. In 1989, a centralized clearinghouse, now commonly referred to as the *Boston mechanism* (Abdulkadiroğlu and Sönmez 2003b), was adopted by the district. The Boston

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mechanism remains the most widely used student assignment mechanism in the United States and is currently employed by numerous centralized clearinghouses worldwide.

Beginning with Abdulkadiroğlu and Sönmez (2003b), the literature emphasizes serious flaws associated with the Boston mechanism, mainly rooted in its obvious manipulability. An attractive alternative to the Boston mechanism, the Gale–Shapley student-optimal stable mechanism, was eventually adopted by the Boston and New York City public school systems via the collaborative effort of economists (see Abdulkadiroğlu et al. 2005, 2006).

In school-choice problem, schools' priorities over students constitute the basis for fairness considerations, which the newly adopted Boston/NYC mechanism achieves through a property of "ex post stability." At a *stable* matching, there does not exist any student i who prefers a seat at a different school c than the one he is assigned to such that either (i) school c has not filled its quota or (ii) school c has an enrolled student who has strictly lower priority than i (Gale and Shapley 1962). In practice, there are typically several students who fall in the same priority class at schools, and a common method in dealing with ties within priorities is to use an explicit tie-breaking lottery. A mechanism is *ex post stable* if it induces a lottery over stable matchings (i.e., an ex post stable lottery). Thus, the newly adopted Boston/NYC mechanism is ex post stable.

An important debate in school choice centers around the three-way tension between fairness, efficiency, and incentives. Although under the student-optimal stable mechanism, reporting preferences truthfully is a dominant strategy for each student regardless of the tie-breaking rule used (Roth 1982, Dubins and Freedman 1981), the ensuing assignment can lead to significant welfare losses, both ex ante and ex post. One source of this welfare loss is the inherent incompatibility between ex post stability and ex post Pareto efficiency (Roth 1982). A second source stems from the use of a tie-breaking rule, the effect of which can be further exacerbated when coupled with the first (cf. Erdil and Ergin 2008; Abdulkadiroğlu et al. 2009; Kesten 2010).

Although ex post stability is a meaningful interpretation of fairness for deterministic outcomes, for lottery mechanisms such as those used for school choice, its suitability as the right fairness notion is less clear. To begin with, ex post stability is not defined over stochastic assignments, but rather over deterministic matchings obtained post tie-breaking. And while random tie-breaking conveniently makes the deterministic approaches still applicable, it nevertheless precludes broader views of ex ante fairness and can potentially entail important welfare loss.

In this paper, we present a general model of school choice in which (i) school priorities can be coarse as in real life and (ii) matchings can be random, as opposed to previous models of school choice. Over random matchings, we propose two powerful notions of fairness that are stronger than ex post stability. We say that a random matching causes *ex ante justified envy* if there are two students i and j, and a school c such that student i has strictly higher priority than j for school c but student j can be assigned to school c with positive probability while i can be assigned to a less desirable school for him than c with positive probability (i.e., i has ex ante justified envy toward j). We refer to a random matching as *ex ante stable* if it eliminates ex ante justified envy. This notion can be viewed as the natural analogue of stability when fairness considerations are based solely on the probabilities that each student will be assigned to different schools. We show that (cf. Example 1) the new Boston/NYC mechanism, despite its ex post stability, is not ex ante stable.

Besides its normative support, our ex ante approach has an important practical appeal. Even though ex post stability is normatively appealing as a fairness concept, the set of ex post stable lotteries is highly nontractable for the random matching setup. Indeed, it is difficult to characterize the probability assignment matrix of a generic ex post stable lottery since an ex post unstable lottery may also induce the same matrix as an ex post stable lottery (as demonstrated in our Example 1).[1] In this case, one possible practical solution is approximating ex post stability through ex ante stability.

Coarse priority structures also give rise to natural fairness considerations concerning students who belong to the same priority group for some school. We say that a random matching causes *ex ante discrimination* (among equal-priority students) if there are two students i and j with equal priority for a school c such that j enjoys a higher probability of being assigned to school c than student i even though i suffers from a positive probability of being assigned to a less desirable school for him than school c. We show that the new Boston/NYC mechanism (cf. Example 2) also induces ex ante discrimination between equal-priority students.[2]

We refer to a random matching as *strongly ex ante stable* if it eliminates both ex ante justified envy and ex ante discrimination. Both ex ante stability and strong ex ante stability imply ex post stability. The latter also implies equal treatment of equals. We propose two new mechanisms that select "special" ex ante stable and strongly ex ante stable random matchings.

Our first proposal, the *fractional deferred-acceptance* (FDA) mechanism, selects *the* unique strongly ex ante stable random matching that is ordinally Pareto dominant among all strongly ex ante stable random matchings (Theorems 2 and 3). The algorithm it employs is in the spirit of the deferred-acceptance algorithm of Gale and Shapley (1962), with students applying to schools in an order of decreasing preference and schools tentatively retaining students based on priority. Unlike previous mechanisms, however, the FDA mechanism does not rely on tie-breaking. Loosely, schools always reject lower-priority students in favor of higher-priority students (if shortages arise) as in the deferred-acceptance algorithm. However, whenever there are multiple equal-priority students being considered for assignment to a school for which there is insufficient capacity, the procedure tentatively assigns an equal fraction of these students and rejects the rest of the fractions. These rejected "fractions of students" continue to apply to their next-preferred schools in the usual deferred-acceptance fashion as if they were

[1] A similar difficulty lies in the identification of ex post efficient lotteries as illustrated by Abdulkadiroğlu and Sönmez (2003a).

[2] The elimination of ex ante discrimination is more than a normative concept and is rooted in equal treatment laws. Abdulkadiroğlu and Sönmez (2003b) cite a lawsuit filed by a student against the state of Wisconsin, because the superintendent denied the student entry into a school in a district in which the student did not live based on limited space, while "similar" students were admitted. The Circuit Court reversed the decision as it considered the superintendent's decision "arbitrary" and the Appeals Court affirmed (cf. Michael E. McMorrow, Petitioner–Respondent, v. State superintendant of public instruction, John T. Benson, Respondent–Appellant, 99-1288, Court of Appeals of Wisconsin. Decided on July 25, 2000).

individual students. The procedure iteratively continues to make tentative assignments until one full fraction of each student is assigned to some school. We interpret the assigned fractions of a student at the end of the procedure as his assignment probability to each corresponding school by the FDA mechanism.[3]

Our second proposal, the *fractional deferred-acceptance and trading* (FDAT) mechanism selects an ex ante stable random matching that (i) treats equals equally and (ii) is ordinally Pareto undominated within the set of ex ante stable random matchings (Theorems 4 and 5). It employs a two-stage algorithm that stochastically improves upon the FDA matching. The FDAT mechanism starts from the random-matching outcome of the FDA algorithm and creates a trading market for school-assignment probabilities. In this market, the assignment probability of a student to a school can be traded for an equal amount of probabilities at better schools for the student as long as the trade does not result in ex ante justified envy of some other student. Such trading opportunities are characterized by *stochastic ex ante stable improvement cycles*, i.e., the list of students who can trade fractions of schools among each other without violating any ex ante stability constraints. We show that a random matching is constrained ordinally efficient among ex ante stable random matchings if and only if there is no stochastic ex ante stable improvement cycle (Proposition 5). However, many stochastic ex ante stable improvement cycles can coexist and intersect with each other. To resolve these cycles in a procedurally fair way that preserves equal treatment of equals, in the second stage of the FDAT mechanism, we adapt a combinatorial network-flow algorithm originally proposed by Athanassoglou and Sethuraman (2011) for a problem domain without any priorities, in which ordinal efficiency can be improved by trading fractions of indivisible goods when agents have probabilistic endowments. In our case, endowments correspond to the FDA assignment probabilities.

In indivisible good allocation problems, strategy-proofness is essentially incompatible with ordinal efficiency.[4] We also show that there is no strategy-proof mechanism that is strongly ex ante stable. Due to these tensions, the ultimate cost we pay for superior welfare and stability is in terms of incentives. Neither of our proposals is strategy-proof. However, recent work suggests that these considerations are likely to be mitigated for large populations (cf. Kojima and Manea 2010, Azevedo and Budish 2013). Indeed, we show that in a large market with diverse preference types of students, FDA becomes strategy-proof (Theorem 7).

Our two proposals are practically applicable and enable us to readily contrast their performances with the existing random tie-breaking-based mechanisms. Using the aggregate statistics of the Boston school-choice data from Abdulkadiroğlu et al. (2006),

[3]In contrast with the deferred-acceptance algorithm of Gale and Shapley (1962), the above described procedure may involve rejection cycles that prevent the procedure from terminating in a finite number of steps. Therefore, to obtain a convergent algorithm, we also couple this procedure with a "cycle resolution phase."

[4]More precisely, there is no ordinally efficient and strategy-proof mechanism that satisfies the minimal equity requirement of equal treatment of equals. For example, notwithstanding its appeal in terms of envy-freeness and ordinal efficiency, the probabilistic serial mechanism of Bogomolnaia and Moulin (2001) that has triggered a rapidly growing literature on the random assignment problem is not strategy-proof.

we estimate the actual performance of FDAT and compare it with popular assignment mechanisms from practice and theory. Specifically, we compare the overall efficiency of FDAT with those of the Boston/NYC mechanism and a prominent ex post trading approach from the literature. The latter two mechanisms are only ex post stable, while FDAT is also ex ante stable. Remarkably, we observe that FDAT almost first-order stochastically dominates the Boston/NYC mechanism. We also find that the Boston/NYC and existing theory mechanisms produce very little ex ante justified envy under the aggregate statistics of the Boston data. This suggests that ex ante stability can lend itself to be used as a fairly good approximation of ex post stability, which would in turn enable market designers to include the highly rich set of stochastic mechanisms in their tool kit.

2. Related literature

In dealing with coarse priority structures, previous mechanism design efforts have thus far relied on the deterministic approach to break ties in priorities randomly; cf. Pathak and Sethuraman (2011), Abdulkadiroğlu et al. (2009, forthcoming), Erdil and Ergin (2008), and Ehlers and Westkamp (2011). Our approach does not rely on any form of tie-breaking and enables us to work with a general school-choice model allowing for a rich set of stochastic mechanisms, and doing so we identify and introduce new mechanisms that lead to superior levels of welfare and fairness compared with the deterministic tie-breaking-based mechanisms used in practice and/or proposed in the literature.

There are several strands of literature related to our paper. In the two-sided matching literature, a version of the random-matching problem with strict preferences on both sides of the market was analyzed by Roth et al. (1993).[5] Our ex ante stability and strong ex ante stability concepts are equivalent when school priorities are strict, and they coincide with Roth, Rothblum, and Vande Vate's strong stability concept. Their analysis, however, does not apply to weak priorities, which are inherent features of school-choice problems. The closest attempt to ours is Erdil and Kojima (2007), who independently formulate concepts similar to ours in a school-choice framework. They do not pursue mechanisms satisfying their proposed properties. Echenique et al. (2013) study observable implications of ex post stability for aggregate matchings in nontransferable utility matching markets and propose a condition, similar to our ex-ante stability notion, that is necessary and sufficient for an aggregate matching to be rationalizable.

Alkan and Gale (2003) consider a deterministic two-sided schedule matching model in which the two sides are referred to as firms and workers. In their model, a worker can work for one hour in total, but he can share his time between different firms. A firm can hire fractions of workers that add up to a certain quota of hours. Both firms and workers are equipped with complex preference structures over these fractions. One can interpret a fractional deterministic matching as a random matching. Thus, this similarity creates some overlap between the two models. Alkan and Gale propose a type of stability with respect to these preferences that is stronger than our ex ante stability. They prove that

[5]Also see Teo and Sethuraman (1998) and Manjunath (2013, 2014).

such a set of stable matchings is nonempty provided that certain substitutability and consistency requirements are met, and that this set forms a lattice under certain assumptions over the firm and worker preferences. They do not provide any well defined algorithms to compute the extremal elements of this lattice.

Recently, researchers have started to think about ex ante efficiency in school-choice mechanisms. Specifically, to capture preference intensities, these approaches assume that each student is endowed with cardinal preferences over schools (as opposed to our assumption of ordinal preferences). Abdulkadiroğlu et al. (2011) show that in a highly specialized incomplete information setting, the Boston mechanism's Bayesian equilibria Pareto dominate the dominant-strategy equilibrium outcome of the student-optimal stable Gale–Shapley mechanism. Featherstone and Niederle (2008) show that the Boston mechanism would result in ex ante efficient random matchings in an incomplete information equilibrium when there are no priorities, and they support their finding through experiments.

Another strand of literature deals with the probabilistic assignment of indivisible goods without assuming a priority structure. At least since Hylland and Zeckhauser (1979) and Bogomolnaia and Moulin (2001), it is well known that this approach is superior in terms of efficiency to randomization over priority-based deterministic methods. The latter paper and Katta and Sethuraman (2006) propose ordinally efficient procedures, treating equals equally for the strict and weak preference domains, respectively. Yılmaz (2009, 2010) generalizes these methods to an indivisible good assignment problem, where some agents have initial property rights of some of the goods for the strict and weak preference domains, respectively. Finally, Athanassoglou and Sethuraman (2011) extend these models to a framework where the initial property rights could be over fractions of goods. We also embed one version of their algorithm into the second stage of our fractional deferred-acceptance and trading procedure as a way to achieve "fair" probability trading.

Erdil and Ergin (2008) and Abdulkadiroğlu et al. (2009) have pointed out that the new Boston/NYC mechanism may be subject to welfare losses when ties in priorities are broken randomly. Erdil and Ergin (2007, 2008) propose methods for improved efficiency without violating exogenous stability constraints for school-choice and two-sided matching problems, respectively.[6] All these papers emphasize that random tie-breaking may entail an ex post efficiency loss. We, on the other hand, argue that it may also entail an ex ante fairness loss, both among students with different priorities (ex ante justified envy) and among students with equal priorities (ex ante discrimination).

The current paper generalizes the unrelated approaches summarized in the previous two paragraphs and obtains a unified framework in dealing with school-choice problems in a probabilistic setting with ordinal preferences. Although we do not focus on

[6]Kesten (2010) provides a new mechanism that aims to eliminate the efficiency loss under the Gale–Shapley mechanism by allowing students to give up certain priorities whenever it does not hurt them to do so.

ex post stable mechanisms per se, our analysis also establishes some important infrastructure for studying ex post stable mechanisms in a probabilistic setting where much stronger welfare criteria than ex post efficiency can be conceived. Since the mechanism recently adopted in Boston and New York was chosen instead of a Pareto-efficient alternative[7] due to its superior fairness/stability features, we believe that the stability consideration plays a key role for the practicality of a mechanism in the context of school choice, distinguishing this problem from most other allocation problems. Consequently, our study focuses not only on constrained ordinal efficiency, but also on ex ante fairness.

The rest of the paper is organized as follows. Section 3 formally introduces a general model of school choice. Section 4 discusses desirable properties of mechanisms and introduces the new ex ante stability criteria. Section 5 presents our first proposal, the fractional deferred-acceptance mechanism, and the related results. Section 6 presents our second proposal, the fractional deferred-acceptance and trading mechanism, and the related results. Section 7 inspects the strategic properties of the mechanisms we proposed. Section 8 concludes. The proofs of our main results are relegated to the Appendices.

3. The model

We start by introducing a general model for school choice. A *school-choice problem* is a five-tuple $[I, C, q, P, \succsim]$, where:

- I is a finite set of *students*, each of whom is seeking a seat at a school.
- C is a finite set of *schools*.
- $q = (q_c)_{c \in C}$ is a *quota vector* of schools such that $q_c \in \mathbb{Z}_{++}$ is the maximum number of students who can be assigned to school c. We assume that there is enough space for all students, that is, $\sum_{c \in C} q_c = |I|$.[8]
- $P = (P_i)_{i \in I}$ is a *strict preference profile* for students such that P_i is the strict preference relation of student i over the schools.[9] Let R_i refer to the associated weak preference relation with P_i. Formally, we assume that R_i is a linear order, i.e., a complete, transitive, and antisymmetric binary relation. That is, for any $c, a \in C$, $c\ R_i\ a$ if and only if $c = a$ or $c\ P_i\ a$.

[7] This is the so-called *top trading cycles mechanism* that has been also advocated by Abdulkadiroğlu and Sönmez (2003b) as an attractive replacement. This mechanism is strategy-proof just like the new Boston/NYC mechanism, but is not ex post stable.

[8] If originally $\sum_{c \in C} q_c > |I|$, then we introduce $|I| - \sum_{c \in C} q_c$ additional virtual students, who have the lowest priorities at each school (say, a uniform priority ranking is available among virtual students for all schools and all virtual students have common strict preferences over schools). If originally $\sum_{c \in C} q_c < |I|$, then we introduce a virtual school with a quota $\sum_{c \in C} q_c - |I|$, which is the worst choice of each student, such that all students have equal priority for this school.

[9] For simplicity of exposition, we assume that all schools are acceptable for all students. All of our results are easy to generalize to the setting with unacceptable schools using a null school with quota ∞ and sub-stochastic matrices instead of bi-stochastic matrices.

- $\succsim = (\succsim_c)_{c\in C}$ is a *weak priority structure for schools* such that $\succsim_c$ is the weak priority order of school c over the students. That is, $\succsim_c$ is a reflexive, complete, and transitive binary relation on I. Let $\succ_c$ be the acyclic portion and let $\sim_c$ be the cyclic portion of $\succsim_c$. That is, $i \succsim_c j$ means that student i has at least as high priority as student j at school c, $i \succ_c j$ means that i has strictly higher priority than j at c, and $i \sim_c j$ means that i and j have equal priority at c.

Occasionally, we will fix I, C, and q, and refer to a problem by the strict preference profile of the students and weak priorities of schools, $[P, \succsim]$.

We are seeking matchings such that each student is assigned a seat at a single school and the quota of no school is exceeded. We also allow random (or probabilistic) matchings.

A *random matching* $\rho = [\rho_{i,c}]_{i\in I, c\in C}$ is a real stochastic matrix, i.e., it satisfies (i) $0 \le \rho_{i,c} \le 1$ for all $i \in I$ and $c \in C$, (ii) $\sum_{c\in C} \rho_{i,c} = 1$ for all $i \in I$, and (iii) $\sum_{i\in I} \rho_{i,c} = q_c$ for all $c \in C$. Here $\rho_{i,c}$ represents the probability that student i is being matched with school c. Moreover, the stochastic row vector $\rho_i = (\rho_{i,c})_{c\in C}$ denotes the *random matching (vector) of student i at ρ*, and the stochastic column vector $\rho_c = (\rho_{i,c})_{i\in I}$ denotes the *random matching (vector) of school c at ρ*. A random matching ρ is a *(deterministic) matching* if $\rho_{i,c} \in \{0, 1\}$ for all $i \in I$ and $c \in C$. Let $\mathcal{X}$ be the set of random matchings and let $\mathcal{M} \subseteq \mathcal{X}$ be the set of matchings. We also represent a matching $\mu \in \mathcal{M}$ as the unique nonzero diagonal vector of matrix μ, i.e., as a list $\mu = \left(\begin{smallmatrix} i_1 & i_2 & \cdots & i_{|I|} \\ c_1 & c_2 & \cdots & c_{|I|} \end{smallmatrix}\right)$ such that for each ℓ, $\mu_{i_\ell, c_\ell} = 1$. We interpret each student i_ℓ as matched with school c_ℓ in this list and, with a slight abuse of notation, use μ_{i_ℓ} to denote the match of student i_ℓ.

A *lottery* λ is a probability distribution over matchings. That is, $\lambda = (\lambda_\mu)_{\mu\in\mathcal{M}}$ such that for all $\mu \in \mathcal{M}$, $0 \le \lambda_\mu \le 1$ and $\sum_{\mu\in\mathcal{M}} \lambda_\mu = 1$. Let $\Delta\mathcal{M}$ denote the set of lotteries. For any $\lambda \in \Delta\mathcal{M}$, let ρ^λ be the *random matching of lottery* λ. That is, $\rho^\lambda = [\rho^\lambda_{i,c}]_{i\in I, c\in C} \in \mathcal{X}$ is such that $\rho^\lambda_{i,c} = \sum_{\mu\in\mathcal{M}:\,\mu_i = c} \lambda_\mu$ for all $i \in I$ and $c \in C$. In this case, we say that lottery λ *induces* random matching ρ^λ. Observe that $\rho^\lambda_{i,c}$ is the probability that student i will be assigned to school c under λ. Let $\mathcal{M}(\lambda) \subseteq \mathcal{M}$ be the support of λ, i.e., $\mathcal{M}(\lambda) = \{\mu \in \mathcal{M} : \lambda_\mu > 0\}$.

We state the following theorem whose proof is an extension of the proof of the standard Birkhoff–von Neumann theorem (cf. von Neumann 1953, and see Kojima and Manea 2010 for the school-choice extension).

Theorem 1 (School-choice Birkhoff–von Neumann theorem). *For any random matching $\rho \in \mathcal{X}$, there exists a lottery $\lambda \in \Delta\mathcal{M}$ that induces ρ, i.e., $\rho = \rho^\lambda$.*

Through this theorem's constructive proof and related algorithms in combinational optimization theory, such as the Edmonds (1965) algorithm, one can find a lottery implementing ρ in polynomial time. In general, this lottery need not be unique, i.e., more than one lottery may induce the same random matching.[10] Nevertheless, we will focus on random matchings rather than lotteries in our analysis. Participants in school choice

[10]This is because random matchings are only marginal distributions, whereas lotteries represent joint distributions.

are usually presented only with their own odds of being assigned to different schools, and so it is natural to define notions of fairness and stability in these terms.

A *(school-choice) mechanism* selects a random matching for a given school-choice problem. For problem $[P, \succsim]$, we denote the random matching of a mechanism φ by $\varphi[P, \succsim]$ and denote the random matching vector of a student i by $\varphi_i[P, \succsim]$.

4. Properties

4.1 *Previous notions of fairness*

We first define two previously studied notions that are satisfied by many mechanisms in the literature and real life. Throughout this section, we fix a problem $[P, \succsim]$.

We start with the most standard fairness property in school-choice problems as well as other allocation problems. This weakest notion of fairness is related to the treatment of equal students, i.e., students with the same preferences and priorities. We refer to two students $i, j \in I$ as *equal* if $P_i = P_j$ and $i \sim_c j$ for all $c \in C$. A random matching ρ *treats equals equally* if for any equal student pair $i, j \in I$, we have $\rho_i = \rho_j$, that is, two students with exactly the same preferences and equal priorities at all schools should be guaranteed the same enrollment chance at every school at a matching that treats equals equally. The real-life school-choice mechanism used earlier in Boston as well as the new NYC/Boston mechanism treat equals equally.

Before introducing the second probabilistic fairness property, we define a deterministic fairness notion. A (deterministic) matching μ is *stable* if there is no student pair i, j such that $\mu(j)\ P_i\ \mu(i)$ and $i \succ_{\mu(j)} j$.[11] That is, a matching is stable if there is no student who envies the assignment of a student who has lower priority than he does for that school. Whenever such a student pair exists at a matching, we say that there is *justified envy*. Let $\mathcal{S} \subseteq \mathcal{M}$ be the set of stable matchings. A stable matching always exists (Gale and Shapley 1962).

The second probabilistic fairness property is a direct extension of stability to lottery mechanisms: A random matching ρ is *ex post stable* if it is induced by a lottery whose support includes only stable matchings, i.e., there exists some $\lambda \in \triangle\mathcal{M}$ such that $\mathcal{M}(\lambda) \subseteq \mathcal{S}$ and $\rho = \rho^{\lambda}$.

Since recently introduced real-life mechanisms are ex post stable (and the implemented matchings are stable), ex post stability has been seen as a key property in previous literature. A characterization of ex post stability exists for strict priorities (Roth et al. 1993), yet such a characterization is unknown for weak priorities.

4.2 *A new notion: Ex ante stability*

We now formalize the two fairness notions over random matchings that were informally discussed in the Introduction.

[11] The early literature on college admissions and school choice (e.g., Balinski and Sönmez 1999 and Abdulkadiroğlu and Sönmez 2003b) used the term "fair" instead of "stable." Subsequent studies have used the term "stable" more often based on the connection of their models with the two-sided model of Gale and Shapley (1962). Since we already have several fairness concepts, we have adopted this terminology to avoid confusion.

We say that a random matching $\rho \in \mathcal{X}$ causes ex ante justified envy of $i \in I$ toward (lower-priority student) $j \in I \setminus \{i\}$, with $i \succ_c j$, for $c \in C$ if $\rho_{i,a} > 0$ for some $c \; P_i \; a$ and $\rho_{j,c} > 0$. A random matching is *ex ante stable* if it does not cause any ex ante justified envy.

To elaborate on this choice of definition, this notion stipulates that, being the higher priority student, student i should be granted all the enrollment chance at school c, should he so desire, before student j is given any chance at this school. For example, if, contrary to the requirement, student j were to be assigned to school c with positive probability even though i is assigned to a less desirable school for him than c with positive probability, this would give rise to a possible violation of student i's priority for school c in the final realization of assignments.

Observe that ex ante stability and stability are equivalent concepts for deterministic matchings. Although ex ante stability is appealing, it does not impose any restrictions when dealing with fairness issues regarding students with equal priorities.

We say that a random matching $\rho \in \mathcal{X}$ induces ex ante discrimination (among equal-priority students) $i, j \in I$, with $i \sim_c j$, for $c \in C$, if $\rho_{i,a} > 0$ for some $c \; P_i \; a$ and $\rho_{i,c} < \rho_{j,c}$. Further discussion of our choice of definition may be useful. Given that students i and j have the same priority for school c, it would be natural to assign the two students to this school with the same probability. However, note that the two students may rank c quite differently. Suppose, for example, that i likes many other schools better than c, whereas j likes c best. In this case, giving i a lower shot at c than j would not constitute inducing ex ante discrimination as long as he is given positive probability only at more preferred schools than c. Thus, the definition implies that if both i and j are assigned to schools that they deem inferior to c with positive chance, then they should both be assigned to c with equal probability.

A random matching is *strongly ex ante stable* if it eliminates both ex ante justified envy and ex ante discrimination.

The elimination of ex ante discrimination implies equal treatment of equals. Thus, a strongly ex ante stable random matching satisfies equal treatment of equals. Strong ex ante stability implies ex ante stability, but the converse is not true. Theorem 2 (below) shows that a strongly ex ante stable random matching always exists. For deterministic matchings, elimination of ex ante discrimination among equal-priority students is equivalent to a *no-envy*[12] requirement among students with equal priority (due to a school for which equal priority is shared) and thus may not always be guaranteed.

We compare ex ante (and strong ex ante) stability with the earlier notion, ex post stability. It turns out that ex post stability is weaker than ex ante stability (and strong ex ante stability), while the converse is not true.

Proposition 1. *If a random matching is ex ante stable, then it is also ex post stable. Moreover, any lottery that induces an ex ante stable random matching has a support that includes only stable matchings.*

[12]Given a deterministic matching $\mu \in \mathcal{M}$, there exists no-envy between a pair of students $i, j \in I$ if $\mu_i \; P_i \; \mu_j$ and $\mu_j \; P_j \; \mu_i$.

Proof. We prove the contrapositive of the second part of the proposition. The first part of the proposition follows from the second part. Let a random matching $\rho \in \mathcal{X}$ and a lottery $\lambda \in \Delta\mathcal{M}$ that induces it be given, i.e., $\rho^\lambda = \rho$. Suppose there exists some unstable matching $\mu \in \mathcal{M} \setminus \mathcal{S}$ such that $\lambda_\mu > 0$. Then there exists a blocking pair $(i, c) \in I \times C$ such that $c \; P_i \; \mu_i$ while for some $j \in I$, $\mu_j = c$ with $i \succ_c j$. Since $\lambda_\mu > 0$, we have $\rho_{j,c} > 0$ while $\rho_{i,\mu_i} > 0$, $i \succ_c j$, and $c \; P_i \; \mu_i$, i.e., ρ is not ex ante stable. □

On the other hand, the following example shows that the new NYC/Boston mechanism is not ex ante stable and, hence, an ex post stable lottery can be ex ante unstable, i.e., the converse of the first part of the above proposition does not hold.

Example 1. Consider the following problem with five students $\{1, 2, 3, 4, 5\}$ and four schools $\{a, b, c, d\}$, where each of schools a, b, and c has one seat, and d has two seats. The priority orders and student preferences are as follows, where vertical dots represent arbitrary rankings of remaining students/objects:

$\succsim_a$	$\succsim_b$	$\succsim_c$	$\succsim_d$
5	4, 5	1, 3	$\vdots$
1	$\vdots$	$\vdots$	
2			
$\vdots$			

P_1	P_2	P_3	P_4	P_5
c	a	c	b	b
a	d	d	d	a
d	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$				

Consider the new NYC/Boston mechanism, which uniformly randomly chooses a single tie-breaking order for equal-priority students at each school and then employs the student-proposing deferred-acceptance algorithm using the modified priority structure. It is straightforward to compute that this mechanism implements the lottery

$$\lambda = \frac{1}{4}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ d & d & c & b & a \end{pmatrix}}_{\mu_1} + \frac{1}{4}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & d & c & d & b \end{pmatrix}}_{\mu_2} + \frac{1}{4}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ c & d & d & b & a \end{pmatrix}}_{\mu_3} + \frac{1}{4}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ c & a & d & d & b \end{pmatrix}}_{\mu_4}.$$

The above four deterministic matchings in the support of λ are stable since they are obtained by the student-proposing deferred-acceptance algorithm for tie-breakers $3 \succ_c 1$ and $4 \succ_b 5$; $3 \succ_c 1$ and $5 \succ_b 4$; $1 \succ_c 3$ and $4 \succ_b 5$; $1 \succ_c 3$ and $5 \succ_b 4$, respectively. Thus λ is ex post stable. However, the random matching that lottery λ induces is not ex ante stable because student 1 has ex ante justified envy toward student 2 for school a. Matching μ_1 implies that student 1 suffers from a positive probability of being assigned to school d, while matching μ_4 implies that student 2 enjoys a positive probability of being assigned to school a, for which he has strictly lower priority than 1.

Interestingly, one can find an alternative lottery that, despite being equivalent to λ, is ex post unstable:

$$\lambda' = \frac{1}{4}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ d & a & c & d & b \end{pmatrix}}_{\mu'_1} + \frac{1}{4}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ c & d & d & b & a \end{pmatrix}}_{\mu'_2} + \frac{1}{4}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ c & d & d & b & a \end{pmatrix}}_{\mu'_3} + \frac{1}{4}\underbrace{\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ a & d & c & d & b \end{pmatrix}}_{\mu'_4}.$$

The support of λ' contains an unstable matching, namely μ'_1, since student 1 has schoolwise justified envy toward student 2 at this matching. Lottery λ' exacerbates the justified schoolwise envy situation under λ by transforming it from ex ante to ex post. ◊

Worse still, the new NYC/Boston mechanism may also induce ex ante discrimination.

Example 2. Consider the following problem with three students $\{1, 2, 3\}$ and three schools $\{a, b, c\}$ each with a quota of 1. The priority orders and student preferences are

$\succsim_a$	$\succsim_b$	$\succsim_c$
3	2	2
1, 2	1	1
	3	3

P_1	P_2	P_3
a	a	b
b	c	a
c	b	c

The tie-breaking lottery assigns the second priority at school a to equal-priority students 1 and 2 with equal chances. Then the new NYC/Boston mechanism (which operates on the student-proposing deferred-acceptance algorithm coupled with either strict priority structure) implements the lottery

$$\lambda = \frac{1}{2}\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ a & c & b \end{pmatrix}}_{\mu_1} + \frac{1}{2}\underbrace{\begin{pmatrix} 1 & 2 & 3 \\ b & c & a \end{pmatrix}}_{\mu_2}.$$

Observe that random matching ρ^λ induces ex ante discrimination between students 1 and 2 for school a since matching μ_1 implies that student 1 is given a positive probability of being assigned to school a while student 2 who, despite having equal priority for a, always ends up at school c, which she finds worse than a. In particular, the ex post observation that student 2 has been assigned to school c by this mechanism cannot be attributed to an unlucky lottery draw to determine the priority order at school a. ◊

4.3 *Pareto efficiency*

We define and work with two Pareto efficiency concepts defined over ordinal preferences.

For student $i \in I$, random-matching vector π_i ordinally (Pareto) dominates random-matching vector ρ_i if $\sum_{aR_ic} \pi_{i,a} \geq \sum_{aR_ic} \rho_{i,a}$ for all $c \in C$ and $\sum_{aR_ib} \pi_{i,a} > \sum_{aR_ib} \rho_{i,a}$ for some $b \in C$, i.e., π_i first-order stochastically dominates ρ_i with respect to P_i. A random matching $\pi \in \mathcal{X}$ ordinally (Pareto) dominates $\rho \in \mathcal{X}$ if for all $i \in I$, either π_i ordinally dominates ρ_i or $\pi_i = \rho_i$, and there exists at least one student $j \in I$ such that π_j ordinally dominates ρ_j. We say that a random matching is *ordinally (Pareto) efficient* if there is no random matching that ordinally dominates it.

We refer to ordinally efficient deterministic matchings as *Pareto efficient*. A random matching is *ex post (Pareto) efficient* if there exists a lottery that induces this random matching and has its support only over Pareto efficient matchings.

Ordinal efficiency implies ex post efficiency, while the converse is not true for random matchings (Bogomolnaia and Moulin 2001). It is well known that even with strict school priorities, ex post stability and ex post efficiency are not compatible.

Proposition 2 (Roth 1982). *There does not exist any ex post stable and ex post efficient mechanism.*

Since we take fairness notions as given, we will focus on constrained ordinal efficiency and constrained ordinal dominance as the proper efficiency concepts for mechanisms that belong to a particular class.

5. Strongly ex ante stable school choice

5.1 *Fractional deferred-acceptance mechanism*

Strong ex ante stability is an appealing stability property since (i) it guarantees all the enrollment chances to a higher-priority student at his preferred school before all lower-priority students (i.e., by elimination of ex ante justified envy) thereby also ensuring ex post stability, and (ii) it treats equal-priority students—not only equal students—fairly by giving them equal enrollment chance at "competed"[13] schools (i.e., by elimination of ex ante discrimination). We now introduce the central mechanism in the theory of strongly ex ante stable lotteries. This mechanism employs a fractional deferred-acceptance (FDA) algorithm.

The FDA algorithm is in the spirit of the classical student-proposing deferred-acceptance algorithm of Gale and Shapley (1962). In this algorithm, we talk about a "fraction of a student" applying to, being tentatively assigned to, or being rejected by a school. In using such language, we have in mind that upon termination of the algorithm, the fraction of a student permanently assigned to some school will be interpreted as the assignment probability of the student to that school. Hence, fractions in fact represent enrollment chances. In the FDA algorithm, a student fraction, by applying to a school, may seek a certain fraction of one seat at that school. As a result, depending on its quota

[13] To be precise, we would call a school such as c in the definition of ex ante discrimination a *competed* school. That is, it is not student i's least preferred school among those for which his enrollment chance is positive, i.e., student i is *competing* with student j for school c.

and the priorities of other applicants, the school may *tentatively* assign a certain fraction (possibly less than the fraction the student is seeking) of a seat to the student and reject any remaining fraction of the student. In the algorithm description below, when we say "fraction w of student i applies to school c," this means that at most a fraction w of a seat at school c can be assigned to student i. As an example, suppose fraction $\frac{1}{3}$ of student 1 applies to school c at some step of the algorithm. School c then may, for example, admit $\frac{1}{4}$ of student i and reject the remaining $\frac{1}{12}$ of him. We next give a more precise description.

The FDA Algorithm. *Step 1.* Each student applies to his favorite school. Each school c considers its applicants. If the total number of applicants is greater than q_c, then applicants are tentatively assigned to school c one by one, starting from the highest priority applicants such that equal-priority students, if assigned a fraction of a seat at this school at all, are assigned an equal fraction. Unassigned applicants (possibly some being a fraction of a student) are rejected.

$\vdots$

Step s. In general, each student who has a rejected fraction from the previous step applies to his next-favorite school that has not yet rejected any fraction of him. Each school c considers its tentatively assigned applicants together with the new applicants. Applicant fractions are tentatively assigned to school c starting from the highest-priority applicants as follows: For all applicants of the highest-priority level, increase the tentatively assigned shares from 0 at an equal rate until there is an applicant who has been assigned all of his fraction. In such a case, continue with the rest of the applicants of this priority level by increasing the tentatively assigned shares at an equal rate until there is another applicant who has been assigned all of his fraction. When all applicant fractions of this priority level are served, continue with the next priority level in a similar fashion. If, at some point during the process, the whole quota of school c has been assigned, then reject all outstanding fractions of all applicants.

The algorithm terminates when no unassigned fraction of a student remains. At this point, the procedure is concluded by making all tentative random assignments permanent. We next give a detailed example to illustrate the FDA algorithm.

Example 3 (How does the FDA algorithm work?). Consider the following problem with six students $\{1, 2, 3, 4, 5, 6\}$ and four schools $\{a, b, c, d\}$, two, b and d, with a quota of 1, and the other two, a and c, with a quota of 2:

$\succsim_a$	$\succsim_b$	$\succsim_c$	$\succsim_d$
$\vdots$	6	4	5
$\vdots$	1, 3	2, 3, 5	3, 6
$\vdots$	5	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$

P_1	P_2	P_3	P_4	P_5	P_6
b	c	d	b	c	d
a	a	c	c	d	c
$\vdots$	b	b	$\vdots$	$\vdots$	b
$\vdots$	d	a	$\vdots$	$\vdots$	a

Step 1. Students 1 and 4 apply to school b (with quota 1), which tentatively admits student 1 and rejects student 4. Students 2 and 5 apply to school c (with quota 2), which does not reject any of their fractions. Students 3 and 6 apply to school d (with quota 1), which tentatively admits $\frac{1}{2}$ of 6 and $\frac{1}{2}$ of 3, and rejects the remaining halves.

Step 2. Having been rejected by school d, each outstanding half-fraction of students 3 and 6 applies to the next-favorite school, which is school c. Having been rejected by school b, student 4 applies to his next choice, which is also school c. This means school c considers half-fractions of each of 3 and 6, and one whole of 4 together with one whole of 2 and 5. Among the five students, student 4 has the highest priority and, hence, is tentatively placed at school c. Next in priority are students 2, 3, and 5 with equal priority; thus, $\frac{1}{3}$ of each is tentatively admitted at school c, which exhausts its quota of 2. As a consequence, $\frac{1}{2}$ of student 6, $\frac{1}{6}$ of student 3, and $\frac{2}{3}$ of each of students 2 and 5 are rejected by c.

Step 3. The next choice of student 2 is a, and, hence, the rejected $\frac{2}{3}$ of him applies to a and is tentatively admitted there. The next choice of students 3 and 6 is b, and, hence, $\frac{1}{2}$ of student 6 and $\frac{1}{6}$ of student 3 apply to b, which is currently full and holding the whole of student 1. Since student 6 has higher priority than both 1 and 3, the entire applying fraction of student 6 is tentatively admitted. Since students 1 and 3 share equal priority at b, we gradually increase assigned shares of both students from 0 at an equal rate. This implies that $\frac{1}{6}$ of student 3 and $\frac{1}{3}$ of student 1 are to be tentatively admitted, and the remaining $\frac{2}{3}$ of student 1 is to be rejected. The next choice of student 5 is d, and, hence, $\frac{2}{3}$ of him applies to d, which is currently holding $\frac{1}{2}$ of both student 3 and student 6. Since student 5 has higher priority than students 3 and 6, both of whom have equal priority, the whole $\frac{2}{3}$ of student 5 is tentatively admitted, whereas $\frac{1}{6}$ of each of students 3 and 6 is tentatively admitted, causing the remaining $\frac{1}{3}$ of each student to be rejected.

Step 4. The next choice of student 1 is a; hence, the rejected $\frac{2}{3}$ of him applies to a and is tentatively admitted there. For students 3 and 6, the best choice that has not rejected either is b, and, hence, $\frac{1}{3}$ of each student applies to b. School b is currently full and holding $\frac{1}{2}$ of student 6, $\frac{1}{6}$ of student 3, and $\frac{1}{3}$ of student 1. Since students 1 and 3 have equal but lower priority than student 6 at b, the school holds on to all of the $\frac{1}{2}+\frac{1}{3}=\frac{5}{6}$ fraction of 6, and only $\frac{1}{12}$ of each of students 1 and 3 is tentatively admitted by b, while the remaining $\frac{1}{4}$ of student 1 and $\frac{5}{12}$ of student 3 are rejected.

Step 5. The next choice of students 1 and 3 after b is a, and, hence, $\frac{1}{4}$ of student 1 and $\frac{5}{12}$ of student 3 apply to a, which is not filled yet and can accommodate all of these fractions: It is currently holding $\frac{11}{12}$ of student 1, $\frac{5}{12}$ of student 3, and $\frac{2}{3}$ of student 2. Since there are no further rejections, the algorithm terminates and returns the random matching outcome

	a	b	c	d
1	$\frac{11}{12}$	$\frac{1}{12}$	0	0
2	$\frac{2}{3}$	0	$\frac{1}{3}$	0
3	$\frac{5}{12}$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{1}{6}$
4	0	0	1	0
5	0	0	$\frac{1}{3}$	$\frac{2}{3}$
6	0	$\frac{5}{6}$	0	$\frac{1}{6}$

◊

While the FDA algorithm is intuitive, the computation of its outcome poses a new challenge that did not exist for its deterministic analogue (i.e., the version proposed by Gale and Shapley). It turns out that in the FDA algorithm, a student may end up applying to the same school an infinite number of times. Thus, we next observe that the FDA algorithm as explained above may not converge in a finite number of steps. We illustrate this with a simple example.

Example 4 (The FDA algorithm may not terminate in a finite number of steps). Consider the following simple problem with three students and three schools, each with a quota of 1:

$\succsim_a$	$\succsim_b$	$\succsim_c$
3	1, 2	$\vdots$
1, 2	3	$\vdots$
		$\vdots$

P_1	P_2	P_3
a	a	b
b	c	a
c	b	c

Step 1. Students 1 and 2 apply to school a, and $\frac{1}{2}$ of each is tentatively admitted (while $\frac{1}{2}$ of each is rejected), since they have the same priority. Student 3 applies to b and is tentatively admitted.

Step 2. The rejected $\frac{1}{2}$ of student 2 next applies to school c and is tentatively admitted. The rejected $\frac{1}{2}$ of student 1 applies to school b, where he has higher priority than the currently admitted student 3. Now $\frac{1}{2}$ of student 3 is rejected and $\frac{1}{2}$ of student 1 is tentatively admitted.

Step 3. The rejected $\frac{1}{2}$ of student 3 applies to a, where he has higher priority than both student 1 and student 2. As a result, $\frac{1}{2}$ of student 3 is tentatively admitted whereas $\frac{1}{4}$ of each of students 1 and 2 are rejected.

Step 4. The rejected $\frac{1}{4}$ of student 2 next applies to school c and is tentatively admitted (in addition to the previously admitted $\frac{1}{2}$ of him). The rejected $\frac{1}{4}$ of student 1 applies to school b, where he has higher priority than the currently admitted $\frac{1}{2}$ of student 3. Now $\frac{1}{4}$ of student 3 is rejected and $\frac{1}{4}$ of student 1 is tentatively admitted.

$\vdots$

As the procedure goes on, rejected fractions of student 3 by school b continue to apply to school a in turn, leading fractions of student 3 to accumulate at a and, at the same time, causing (a smaller fraction of) student 1 to be rejected by school a at each application. This, in turn, leads student 1 to apply to school c and cause (the same fraction of) 3 to be further rejected. Consequently, all fractions of student 3 accumulate at school a and all those of student 1 accumulate at school b.

Step ∞. The sum of the admitted fractions of student 3 at school a is 1, which is the sum of the geometric series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots$. The sum of the admitted fractions of student 1 at school b is 1. The sum of the admitted fractions of student 2 at school c is 1. $\Diamond$

Even though the FDA algorithm may not terminate in finite time, the above example suggests that its outcome can be computed without getting lost in infinite loops by examining the rejection cycles that might form throughout the steps of the algorithm.

To define the *finite* version of the FDA algorithm, we need to define a few new concepts. We first define a binary relation between students. Let $i, j \in I$ and $c \in C$. Suppose that at some step s of the FDA algorithm, some fraction of student i is rejected by school c, while he still has some fraction not rejected by c at this step. On the other hand, suppose also that at step s, school c temporarily holds some fraction of some other student j who has not been rejected by c until step s (i.e., not rejected before or at step s). Then we say that i is partially rejected by c in favor of j and denote it by $j \hookrightarrow_c i$. At a later step $r > s$ in the algorithm, if either some fraction of j is rejected by c or all fractions of i get rejected by c, then the above relationship does not hold at step r or at any later step. In this case, we say that $j \hookrightarrow_c i$ is no longer *current*.

A *rejection cycle* is a list of distinct students and schools $(i_1, c_1, \dots, i_m, c_m)$ such that at a step of the algorithm, we have

$$i_1 \hookrightarrow_{c_1} i_2 \hookrightarrow_{c_2} \cdots \hookrightarrow_{c_{m-1}} i_m \hookrightarrow_{c_m} i_1$$

and all partial rejection relations are current.

Observe that at the moment the cycle occurs, student i_1 is partially rejected by school c_m in favor of student i_m. We know that school c_1 has not rejected student i_1 at any fraction; thus, the next available choice for student i_1 is c_1. Therefore, student i_1 applies "again" to school c_1. As a result, student i_2 is partially rejected again, and the same sequence of partial rejections reoccur. That is, the algorithm cycles. We refer to this cycle as a *current* rejection cycle as long as all partial rejection relations are current, and we say that i_1 *induces* this rejection cycle.

Nonetheless, this cycle either converges to a tentative random matching in the limit or, sometimes, in a finite number of steps when some partial rejections turn into full rejections. Thus, once a cycle is detected, it can be solved as a system of linear equations.

We make the following observation, which will be crucial in the definition of the "formal" FDA algorithm.

Observation 1. *If a rejection cycle*

$$i_1 \hookrightarrow_{c_1} i_2 \hookrightarrow_{c_2} \cdots \hookrightarrow_{c_{m-1}} i_m \hookrightarrow_{c_m} i_1$$

is current in the FDA algorithm, then for each student i_ℓ, the best school that has not rejected a fraction of him is school c_ℓ; that is, whenever a fraction of i_ℓ is rejected, he next makes an offer to school c_ℓ.

The outcome of the FDA algorithm described above converges (as the number of steps approaches infinity) to the outcome of the following finite FDA algorithm.

The FDA Algorithm. *Step s.* Fix some student $i_1 \in I$ who has an unassigned fraction from the previous step. He applies to the next best school that has not yet rejected any fraction of him. Let c_1 be this school. Two cases are possible:

(a) If the student i_1 induces a rejection cycle

$$i_1 \hookrightarrow_{c_1} i_2 \hookrightarrow_{c_2} \cdots \hookrightarrow_{c_{m-1}} i_m \hookrightarrow_{c_m} i_1,$$

then we resolve it as follows: For $i_{m+1} \equiv i_1$ and $c_0 \equiv c_m$, c_1 tentatively accepts the maximum possible fraction of i_1 such that each school c_ℓ tentatively accepts

- all fractions of applicants tentatively accepted in the previous step except the ones belonging to the lowest-priority level,
- the total rejected fraction of student i_ℓ from school $c_{\ell-1}$, and
- an equal fraction (if possible) among the lowest-priority applicants tentatively accepted in the previous step (including student $i_{\ell+1}$)

so that it does not exceed its quota q_{c_ℓ}.

(b) If i_1 does not induce a rejection cycle, school c_1 considers its tentatively assigned applicants from the previous step together with the new fraction of i_1. It tentatively accepts these fractions starting from the highest priority. In case its quota is filled in this process, it tentatively accepts an equal fraction (if possible) of all applicants belonging to the lowest accepted priority level. It rejects all outstanding fractions.

We continue until no fraction of a student remains unassigned. At this point, we terminate the algorithm by making all tentative random assignments permanent.

We resolve part (a) of the algorithm by reducing the infinite convergence problem demonstrated in Example 4 to a linear equation system. This is demonstrated in Appendix A, Example 8. We explain this resolution in Appendix B, the proof of Proposition 3, for the general case.

Since we have defined the algorithm in a sequential fashion, it is not clear whether the procedure is independent of the order of the proposing students or which cycle is chosen to be resolved. Corollary 1 (below) shows that this statement is true and, thus, its outcome is unique.[14] We refer to the mechanism whose outcome is found through the above FDA algorithm as the *FDA mechanism*.

[14]This result is analogous to the result regarding the deferred-acceptance algorithm of Gale and Shapley, which can also be executed by students making offers sequentially instead of simultaneously (McVitie and Wilson 1971).

5.2 *Properties of the FDA mechanism*

We next present some desirable properties of the FDA mechanism and its iterative algorithm.

Proposition 3. *The FDA algorithm is well defined and converges to a random matching in a finite number of steps.*

The proof of Proposition 3 is given in Appendix B.

Theorem 2. *An FDA outcome is strongly ex ante stable.*

Proof. Let π be the FDA algorithm's outcome according to some proposal order of students. We first show that this outcome is a well defined random matching. Suppose not. Then there exists a student i who is not matched with probability 1 at π. Thus, π is substochastic and there exists some school c that is undermatched at π, i.e., $\sum_j \pi_{j,c} \leq q_c$. At some step, student i makes an offer to c and some fraction of him is rejected by it, as he ends up with some rejected probability at the end of the algorithm. However, school c only rejects a student if its quota is tentatively filled. Moreover, once it is tentatively filled, it never is undermatched. However, this contradicts the earlier conclusion that its quota was not filled at π. Thus, π is a bi-stochastic matrix, i.e., it is a random matching. Next we show that π is strongly ex ante stable. Since in the algorithm, (i) a student fraction always applies to the best school that has not yet rejected him and (ii) when its quota is filled, a school always prefers higher-priority students to the lower-priority students, a student cannot have ex ante justified envy toward a lower-priority student. If π is not strongly ex ante stable, then it should be the case that there is ex ante discrimination among equal-priority students, i.e., there are $i \sim_c j$ for some school c such that $c \, P_i \, a$, $\pi_{i,a} > 0$, and yet $\pi_{i,c} < \pi_{j,c}$. Consider the first step after which the (tentative) random-matching vector of school c does not change. At this step, some students apply to school c, and in return some fractions of some students with equal priority i' and j' are tentatively accepted and some are rejected. The only way $\pi_{i',c} < \pi_{j',c}$ is if no fraction of i' is ever rejected by school c. Thus, $\pi_{i',a'} = 0$ for all $a' \prec_{i'} c$. This contradicts the claim that such a student i exists. □

Our next result states that from a welfare perspective the FDA outcome is the most appealing strongly ex ante stable matching. This finding can also be interpreted as the random analogue of Gale and Shapley's celebrated result on the constrained Pareto optimality of the student-proposing deferred-acceptance outcome (among stable matchings) for the deterministic two-sided matching context.

Theorem 3. *An FDA outcome ordinally dominates all other strongly ex ante stable random matchings.*

The proof of Theorem 3 is also given in Appendix B. Theorem 3 implies that the FDA mechanism is well defined, i.e., its outcome is unique and independent of the order of students making applications in the algorithm. We invoke a direct proof of this theorem. We can also prove the existence of a student-optimal strongly ex ante stable random matching through Alkan and Gale's (2003) two-sided schedule matching approach. Alkan and Gale introduce a two-sided matching model where each worker can be matched to each firm in fractions (referring to time or probabilities). They show that a worker-optimal stable matching exists when the choice functions of agents over schedule matchings satisfy a standard substitutability requirement and a consistency requirement. It is easy to define an isomorphic Alkan–Gale problem for every school-choice problem of ours in which student and school-choice functions over schedule matchings are both substitutable and consistent such that a school-choice random matching is strongly ex ante stable if and only if its isomorphic Alkan–Gale schedule matching is Alkan–Gale stable. Erdil and Kojima (2007) follow a similar approach to the one described.

Corollary 1. *The FDA algorithm's outcome is independent of the order of students making offers or the rejection cycle chosen to be resolved if more than one is encountered simultaneously and, thus, it is unique.*

6. Ex ante stable school choice

The FDA mechanism satisfies ex ante stability but sacrifices some efficiency at the expense of finding a random matching that treats equal-priority students fairly. Therefore, we next address how we can achieve more efficient outcomes without sacrificing fairness "too much," i.e., by relaxing equal treatment of equal-priority students and thereby allowing for ex ante discrimination but maintaining ex ante stability and equal treatment of equals.

By Proposition 2, we know that there is no mechanism that satisfies ordinal efficiency and ex ante stability. Thus, we define the following constrained efficiency concept: A mechanism φ is constrained ordinally efficient within its class if there exists no mechanism ψ in the same class as φ and no problem $[P, \succsim]$ such that $\psi[P, \succsim]$ ordinally dominates $\varphi[P, \succsim]$.

We now characterize constrained ordinally efficient mechanisms within the class of ex ante stable mechanisms. First, we restate a useful result due to Bogomolnaia and Moulin (2001) that characterizes ordinal efficiency. Fix a problem $[P, \succsim]$. For any random matching $\pi \in \mathcal{X}$, we say that i ex ante envies j for b due to c if $\pi_{j,b} > 0$, $\pi_{i,c} > 0$, and $b\ P_i\ c$.[15] We denote it as

$$(i, c) \triangleright^{\pi} (j, b).$$

A *stochastic improvement cycle* $\mathrm{Cyc} = (i_1, c_1, \ldots, i_m, c_m)$ at π is a list of distinct student–school pairs (i_ℓ, c_ℓ) such that

$$(i_1, c_1) \triangleright^{\pi} (i_2, c_2) \triangleright^{\pi} \cdots \triangleright^{\pi} (i_m, c_m) \triangleright^{\pi} (i_1, c_1).$$

[15] Under this definition, a student will ex ante envy himself if he is assigned fractions from two schools. This is different from the improvement relationship defined by Bogomolnaia and Moulin. Unlike them, we do not rule out this possibility and use it for the constrained efficiency characterization within ex ante stable random matchings.

(We use modulo m whenever it is unambiguous for subscripts, i.e., $m+1 \equiv 1$.) Let $0 < w \leq \min_{\ell \in \{1,\ldots,m\}} \pi_{i_\ell, c_\ell}$. Cycle Cyc is satisfied with fraction w at π if for all $\ell \in \{1,\ldots,m\}$, a fraction w of the school $c_{\ell+1}$ is assigned to student i_ℓ additionally and a fraction w of school c_ℓ is removed from his random matching, while we do not change any of the other matching probabilities at π. Formally, we obtain a new random matching $\rho \in \mathcal{X}$ such that for all $i \in I$ and $c \in C$,

$$\rho_{i,c} = \begin{cases} \pi_{i,c} + w & \text{if } i = i_\ell \text{ and } c = c_{\ell+1} \text{ for some } \ell \in \{1,\ldots,m\} \\ \pi_{i,c} - w & \text{if } i = i_\ell \text{ and } c = c_\ell \text{ for some } \ell \in \{1,\ldots,m\} \\ \pi_{i,c} & \text{otherwise.} \end{cases}$$

The following is a direct extension of Bogomolnaia and Moulin's result to our domain and our definition of the ex ante envy relationship. Therefore, we skip its proof.

Proposition 4 (Bogomolnaia and Moulin 2001). *A random matching is ordinally efficient if and only if it has no stochastic improvement cycle.*

6.1 *Ex ante stability and constrained ordinal efficiency*

Proposition 4 suggests that if a random matching has a stochastic improvement cycle, then one can obtain a new random matching that ordinally dominates the initial one simply by satisfying this stochastic improvement cycle. Observe, however, that satisfying such a cycle may induce ex ante justified envy at the new random matching. Consequently, given that our goal is to maintain ex ante stability, to improve the efficiency of an ex ante stable random matching, we can only work with those stochastic improvement cycles that respect the ex ante stability constraints. For this purpose we introduce an envy relationship as follows.

We say that i ex ante top-priority schoolwise envies j for b due to c, and we denote it as

$$(i, c) \blacktriangleright^{\pi} (j, b)$$

if $(i, c) \succ^{\pi} (j, b)$ and $i \succsim_b k$ for all $(k, a) \in I \times C$ such that $(k, a) \succ^{\pi} (j, b)$. That is, i envies j for b due to c, and i is the highest-priority student who envies j for b.[16]

An *ex ante stable improvement cycle* $(i_1, c_1, \ldots, i_m, c_m)$ at π is a list of distinct student–school pairs (i_ℓ, c_ℓ) such that

$$(i_1, c_1) \blacktriangleright^{\pi} (i_2, c_2) \blacktriangleright^{\pi} \cdots \blacktriangleright^{\pi} (i_m, c_m) \blacktriangleright^{\pi} (i_1, c_1).$$

We state our main result of this subsection below. Although one direction of this result is easy to prove, the proof of the other direction needs extra attention to detail. Our result generalizes Proposition 4 (stated for the equal priority domain by Bogomolnaia and Moulin), and a result by Erdil and Ergin (2008) (stated for the deterministic domain) to the probabilistic school-choice framework.

[16]Like the ex ante envy relationship, a student will ex ante top-priority schoolwise envy himself if he is assigned fractions from two schools, and for the better of the two schools, he is among the highest-priority students ex ante envying.

Proposition 5. *An ex ante stable random matching ρ is not ordinally dominated by any other ex ante stable random matching if and only if there is no ex ante stable improvement cycle at ρ.*

The proof of Proposition 5 is given in Appendix B.

6.2 *Ex ante stable fraction trading*

Motivated by Proposition 5, we shall use the FDA outcome to obtain a constrained ordinally efficient ex ante stable matching. Our second proposal roughly rests on the following intuition: Since the outcome of the FDA mechanism is ex ante stable, if we start initially from this random matching and iteratively satisfy ex ante stable improvement cycles, we should eventually arrive at a constrained ordinally efficient ex ante stable random matching. Though intuitive, this approach need not guarantee equal treatment of equals. Therefore, in what follows, we will also need to pay attention to the ex ante stable improvement cycles that are to be selected.

Our second proposal, the fractional deferred acceptance and trading (FDAT), starts from the FDA outcome and satisfies *all* ex ante stable improvement cycles simultaneously so as to preserve equal treatment of equals to obtain a new random matching. It iterates until there are no new ex ante stable improvement cycles. Before formalizing this procedure, to fix ideas and point out some potential difficulties, we first illustrate our approach with an example.

Example 5 (How does the FDAT algorithm work?). We use the same problem as in Example 3.

Step 0. We have found the FDA outcome in Example 3 as

$$\rho^1 = \begin{array}{c|c|c|c|c} & a & b & c & d \\ \hline 1 & \frac{11}{12} & \frac{1}{12} & 0 & 0 \\ 2 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 3 & \frac{5}{12} & \frac{1}{12} & \frac{1}{3} & \frac{1}{6} \\ 4 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 6 & 0 & \frac{5}{6} & 0 & \frac{1}{6} \end{array}$$

Step 1. We form top-priority schoolwise envy relationships as

$$\begin{aligned}
&(1, a) \blacktriangleright^{\rho^1} (3, b), (6, b), (1, b) \\
&(2, a) \blacktriangleright^{\rho^1} (3, c), (4, c), (5, c), (2, c) \\
&(3, f) \blacktriangleright^{\rho^1} (5, d), (6, d), (3, d) \quad \forall f \in \{a, b, c\} \\
&(3, f) \blacktriangleright^{\rho^1} (2, c), (4, c), (5, c), (3, c) \quad \forall f \in \{a, b\} \\
&(3, a) \blacktriangleright^{\rho^1} (1, b), (6, b), (3, b) \\
&(5, d) \blacktriangleright^{\rho^1} (2, c), (3, c), (4, c), (5, c) \\
&(6, b) \blacktriangleright^{\rho^1} (3, d), (5, d), (6, d).
\end{aligned}$$

There is only one ex ante stable improvement cycle:

$$(3, c) \blacktriangleright^{\rho^1} (5, d) \blacktriangleright^{\rho^1} (3, c).$$

We satisfy this cycle with the maximum possible fraction $\frac{1}{3}$ and obtain

$$\rho^2 = \begin{array}{c|c|c|c|c} & a & b & c & d \\ \hline 1 & \frac{11}{12} & \frac{1}{12} & 0 & 0 \\ 2 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 3 & \frac{5}{12} & \frac{1}{12} & 0 & \frac{1}{2} \\ 4 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 6 & 0 & \frac{5}{6} & 0 & \frac{1}{6} \end{array}$$

Step 2. There are no new top-priority schoolwise envy relationships at ρ^2, and no new ex ante stable improvement cycles; thus, ρ^2 is the outcome of the FDAT algorithm. $\Diamond$

The main difficulty with this approach is determining which ex ante stable improvement cycle to satisfy if there are many. This choice may cause fairness violations regarding the equal treatment of equals or there can be many ways to find a solution respecting equal treatment of equals. Thus, the outcome of the FDAT algorithm as explained above is not uniquely determined. Furthermore, there are also legitimate computational concerns in finding more than one ex ante stable improvement cycle at a time.[17] We overcome these fairness and computational issues by adapting to our domain a fractional trading algorithm, which was introduced in the operations research literature by Athanassoglou and Sethuraman (2011). It is referred to as the *constrained-consumption algorithm* and was introduced to obtain ordinally efficient allocations in house allocation problems with existing tenants (Abdulkadiroğlu and Sönmez 1999). Similar algorithms were also previously introduced by Yılmaz (2009, 2010). Our version, the *ex ante stable consumption* (EASC) algorithm, is embedded in Step $s \geq 1$ of the FDAT algorithm as a way to satisfy ex ante stable improvement cycles simultaneously and equitably. It is explained in detail in Appendix D.

We state the FDAT algorithm formally as follows.

The FDAT Algorithm. *Step 0*. Run the FDA algorithm. Let ρ^1 be its random matching outcome.

⋮

Step s. Let $\rho^s \in \mathcal{X}$ be found at the end of Step $s-1$. If there is an ex ante stable improvement cycle, run the EASC algorithm. Let ρ^{s+1} be the outcome and continue with Step $s+1$. Otherwise, terminate the algorithm with ρ^s as its outcome.

[17] In a worst-case scenario, the number of ex ante stable improvement cycles at an ex ante stable matching grows exponentially with the number of students.

We refer to the mechanism whose outcome is found through this algorithm as the FDAT mechanism.

In Appendix E, Example 9, we illustrate the EASC algorithm to show how the formal FDAT algorithm works for the problem in Example 5. Although the execution of the FDAT algorithm is obvious and simple in this example without the implementation of the EASC algorithm in each step, for expositional purposes we reexecute it with the embedded EASC algorithm.[18]

6.3 *Properties of the FDAT mechanism*

Proposition 6. *The FDAT algorithm is well defined and converges to a random matching in a finite number of steps.*

Proof. We know that Step 0 of the FDA algorithm works in finite steps (by Proposition 3), as well as Step 1, the EASC algorithm (Athanassoglou and Sethuraman 2011).

Next, we prove that the number of steps in FDAT is finite. After each step $t \geq 1$ of the FDAT algorithm, at least one student $i \in I$ leaves a school $c \in C$ with $\rho_{i,c}^{t-1} > 0$ with 0 fraction and gets into better schools, i.e., $\rho_{i,c}^{t} = 0$ and $\sum_{a P_i c} \rho_{i,a}^{t} > \sum_{a P_i c} \rho_{i,a}^{t-1}$. (Otherwise, the same ex ante stable improvement cycle of $\rho_{i,c}^{t-1}$ would still exist at ρ^t, contradicting that the EASC algorithm has converged at Step t.) Thus, the FDAT algorithm converges in no more than $|C||I| + 1$ steps (including Step 0). □

Theorem 4. *The FDAT mechanism is ex ante stable.*

Proof. Consider each step of the FDAT algorithm.

In Step 0, the outcome of the FDA has no schoolwise justified envy toward a lower-priority student by Theorem 2.

In Step 1, students in determined ex ante stable improvement cycles are made better off (in an ordinal dominance sense), while others' welfare is unchanged. Moreover, the students who are made better off are among the highest-priority students who desire a seat at the school where they receive a larger share. That is, for any student i with $\rho_{i,c}^1 > \rho_{i,c}^0$, there is some school b with $c \; P_i \; b$, and $\rho_{i,b}^1 < \rho_{i,b}^0$, and there is no student $j \succ_c i$ such that $\rho_{j,a}^1 > 0$ for some school a with $c \; P_j \; a$. (Otherwise, i would not ex ante top-priority schoolwise envy a student k with $\rho_{k,c}^0 > 0$ for c due to b, since j would do that due to a or a worse school. Moreover, since ρ^0 is ex ante stable, $\rho_{i,c}^0 = 0$. The last two statements would imply $(i, c) \notin \mathcal{A}(\rho^0)$, which in turn implies that $\rho_{i,c}^1 = 0$.) Hence, ρ^1 is ex ante stable.

We repeat this argument for each step. Hence, when the algorithm is terminated, the outcome is ex ante stable. □

Our next result states that from a welfare perspective, the FDAT outcome is among the most appealing ex ante stable random matchings. Improving upon this matching

[18]In general, without the use of the EASC algorithm or a similar well defined technique, Step $s \geq 1$ of the FDAT algorithm may not be well defined.

would necessarily lead to ex ante justified envy. This finding can also be interpreted as the random analogue of the mechanism proposed by Erdil and Ergin (2008) for a deterministic school-choice model with random tie-breaking. In that context, the outcome of the Erdil–Ergin mechanism has been shown to be constrained ex post Pareto efficient among ex post stable matchings.

Theorem 5. *The FDAT mechanism is constrained ordinally efficient within the ex ante stable class.*

Proof. Suppose that the FDAT outcome ρ is ordinally dominated by an ex ante stable random matching for some problem P. By Proposition 5, there exists an ex ante stable improvement cycle at P. Thus, this contradicts the fact that ρ is the FDAT outcome. □

Theorem 6. *The FDAT mechanism treats equals equally.*

Proof. The FDA mechanism treats equals equally as it is strongly ex ante stable (by Theorem 2). Thus, two students with the same preferences and priorities have exactly the same random matching vector under the FDA outcome ρ^0. Let i, j be two equal students. Then $\rho_i^0 = \rho_j^0$ and $(i, c) \in \mathcal{A}(\rho^0)$ if and only if $(j, c) \in \mathcal{A}(\rho^0)$ for any school $c \in C$. By Athanassoglou and Sethuraman (2011), the EASC algorithm treats equals equally. The last two statements imply that outcome of Step 1: ρ^1 treats equals equally. We repeat this argument iteratively for each step, showing that the FDAT outcome treats equals equally. □

6.4 *The FDAT mechanism vs. probabilistic serial mechanism*

The way the FDA and FDAT mechanisms treat equal-priority students resembles the probabilistic serial (PS) mechanism Bogomolnaia and Moulin (2001) proposed for the "random assignment" problem where there are no exogenous student priorities. Loosely speaking, within any given step of the PS algorithm, those students who compete for the available units of the same object are allowed to consume equal fractions until the object is exhausted. Similarly, within any given step of the FDA algorithm, those equal-priority students who have applied to the same school are also treated equally in very much the same way. Despite such similarity, the two procedures are indeed quite different in general. The difference between the two algorithms comes from the fact that the PS algorithm makes permanent random matchings within each step, whereas the FDA algorithm always makes tentative random matchings until the last step. We can expect to have some efficiency loss due to FDA's strong ex ante stability property, while the PS mechanism is not strongly ex ante stable. Even if FDA and PS outcomes are different, one may think that starting from the FDA outcome, fractional trading will somehow establish the equivalence with the PS outcome. However, as the following example shows, the PS outcome does not necessarily ordinally dominate the FDA outcome; hence, the FDAT outcome, which ordinally dominates the FDA outcome, and the PS outcome are not the same either.

Example 6 (Neither FDA nor FDAT is equivalent to the PS mechanism when all students have the same priority). Assume there are four students $\{1, 2, 3, 4\}$ and four schools $\{a, b, c, d\}$ each with a quota of 1. All students have equal priorities at all schools. The students' preferences are given as

P_1	P_2	P_3	P_4
d	a	d	c
c	d	c	b
a	c	b	d
b	b	a	a

The FDA, FDAT, and PS outcomes are

$$\rho^{\text{FDA}} = \begin{array}{c|c|c|c|c} & a & b & c & d \\ \hline 1 & \frac{1}{3} & 0 & \frac{1}{3} & \frac{1}{3} \\ 2 & \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 3 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 4 & 0 & \frac{2}{3} & \frac{1}{3} & 0 \end{array} \qquad \rho^{\text{FDAT}} = \begin{array}{c|c|c|c|c} & a & b & c & d \\ \hline 1 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 2 & 1 & 0 & 0 & 0 \\ 3 & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ 4 & 0 & \frac{2}{3} & \frac{1}{3} & 0 \end{array}$$

$$\rho^{\text{PS}} = \begin{array}{c|c|c|c|c} & a & b & c & d \\ \hline 1 & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & \frac{1}{2} \\ 2 & \frac{5}{6} & \frac{1}{6} & 0 & 0 \\ 3 & 0 & \frac{1}{3} & \frac{1}{6} & \frac{1}{2} \\ 4 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \end{array}$$

Observe that ρ^{FDAT} and ρ^{PS} are both ordinally efficient. Moreover, ρ^{PS} does not ordinally dominate ρ^{FDA} (e.g., contrast student 2's random matching vectors under ρ_2^{FDA} vs. ρ_2^{PS}). ◊

7. Simulations

We ran simulations to estimate the performance of the FDAT mechanism and contrast it with those of the NYC/Boston deferred-acceptance algorithm with a single tie-breaking lottery (DA henceforth) and Erdil and Ergin (2008) (EE henceforth) mechanisms in problems that approximately match the main characteristics of the Boston data from 2008 to 2011 (Abdulkadiroğlu et al. 2006). EE dominates DA, as it starts from a deterministic DA outcome using some ex ante tie-breaking at each instance, and finds deterministic stable improvement cycles randomly and satisfies them. There is no clear theoretical efficiency comparison between FDAT and EE (or between FDAT and DA). FDAT is a constrained ordinally efficient mechanism within the ex ante stable class, while EE is a constrained ex post efficient mechanism within the ex post stable class. Ordinal efficiency is a stronger efficiency notion; however, the ex post stable class is larger than the ex ante stable class. Even in instances where the outcome of the EE is ex ante stable, FDAT may not dominate it.

In our simulations, we randomly generated 100 markets, each with $|S|$ schools and $|I|$ students, and computed the corresponding outcomes of FDAT, DA, and EE, where 100 random tie-breaking priority orderings were additionally generated for the latter two mechanisms. More specifically, we assumed that students were zoned in n neighborhoods, $|S|/n$ schools per each neighborhood. Students were grouped in these neighborhoods such that $|I|/n$ students were assumed to be living in each neighborhood. Also, s students in each neighborhood were assumed to have elder siblings attending high school, some attending a neighborhood school, and others a nonneighborhood school. As in Boston, the priorities at each school were generated to prioritize the neighborhood students with siblings attending the school first, nonneighborhood students with siblings attending the school second, neighborhood students without siblings at the school third, and nonneighborhood students without siblings attending the school last. We generated student preferences using the following randomization process: Each student had p_{sn} probability to first-rank a particular neighborhood school that a sibling is attending, p_s probability to first-rank a particular nonneighborhood school that a sibling is attending, p_n probability to first-rank a neighborhood school that a sibling is attending, and the remaining probability was divided up equally for each nonneighborhood school to determine its probability to be ranked first. If the student did not have a sibling, then p_s and p_{sn} were ignored; if the student had a sibling attending a neighborhood school, then p_s was ignored; and if the student had a sibling attending a nonneighborhood school, then p_{sn} was ignored when generating the first choice. Once the first-choice school is randomly determined, the conditional probabilities for the remaining schools were updated and then second choice was randomly generated. The remaining choices were determined sequentially and randomly after updating the probabilities for remaining schools after each selection. We chose the above preference parameters roughly based on real preference summary statistics of Boston high school applicants in years 2008–2011 (Abdulkadiroğlu et al. 2006).

The data suggested that 60% of the students have siblings in the system, although we do not have data on how many of them are older siblings who generate sibling priorities for the students at their schools. There were, on average, 26 schools and 2705 students per year applying for high school. Each school had neighborhood priority (either with or without siblings) for 208 students on average, and there were, on average, two schools located in each neighborhood. Based on the aggregate statistics, students ranked a nonpriority school 64% of the time, a sibling's nonneighborhood school 3% of the time, a sibling's neighborhood school 3% of the time, and a neighborhood school without a sibling priority 30% of the time. Observe that the latter includes all students and does not distinguish among student types with or without siblings. Also, we did not have reliable data on quotas of schools. Our simulation statistics assumed that half of the students with siblings have older siblings in high school (a total of 30%) so that 10% of the students have older in-walk-zone siblings and 20% of the students have older siblings attending nonneighborhood schools. Using a back-of-the-envelope calculation for our simulations to approximately match the preference characteristics of data with our preference generation process, we chose $p_{sn} = 0.3$, $p_s = 0.15$, and $p_n = 0.15$. We also chose the number of schools comparable in size to the sample: we had $|S| = 20$ and

Average fraction of students

	FDAT f.o.s.d. DA	DA f.o.s.d. FDAT	FDAT = DA	Not comparable	Overall
Average fraction:	70.7% (3.4%)	1.4% (0.8%)	19.5% (2.7%)	8.5% (2.7%)	100%

Average fractions of students getting in their various choices in each category

Choices	FDAT	DA	FDAT	DA	FDAT = DA	FDAT	DA	FDAT	DA
1st	78.9%	52.6%	47.6%	56.0%	100%	57.0%	51.4%	80.7%	61.7%
2nd	15.5%	28.5%	51.2%	43.2%		27.4%	31.1%	14.0%	23.4%
3rd	4.8%	11.2%	1.1%	0.8%		13.5%	12.1%	4.5%	9.0%
4th	0.8%	4.5%				2.0%	3.5%	0.7%	3.5%
5th	0.1%	1.8%				0.2%	1.1%	0.1%	1.4%
6th		0.8%					0.2%		0.6%
7th		0.3%					0.1%		0.2%
8th		0.1%					0.1%		0.1%
9th		0.1%							0.0%

Average fraction of justifiably ex ante envious students in DA: 0.08% (0.48%)

Table 1. Comparison of FDAT with DA in the simulations (the sample standard errors of the fractions of students are given in parentheses after the averages).

$n = 10$ neighborhoods, so that there were two schools per neighborhood. However, to have a manageable simulation (as we ran DA and EE 10,000 times, and ran FDAT 100 times), we chose $|I| = 200$ total students, hence, 20 students per walk zone instead of 208. We report the results of these simulations below.[19]

Tables 1 and 2 show the average allocation of different types of students in the simulations. Table 1 compares the DA outcome with that of FDAT. Although there is no domination relationship between the two mechanisms in theory, FDAT does extremely well for almost all students with respect to DA. The first row in Table 1 shows the average proportion of students for whom FDAT first-order stochastically dominates (f.o.s.d. for short) DA and DA f.o.s.d. FDAT. DA and FDAT outcomes are the same and there is no comparison with f.o.s.d. among the two outcomes. A supermajority of students, 70.7%, prefer FDAT over DA (with a low standard error of 3.4%). Among these students, the fraction of students receiving their first-choice schools is 78.9%, while this fraction is 52.6% under DA. While only 1.4% prefer DA over FDAT (with a standard error of 0.8%), the average fractions of students being assigned to their various choices are very similar under both mechanisms and only slightly favorable for the DA. The probability for the first two choices is 98.8% for FDAT and 99.2% for DA. On the other hand, the students who get the same outcome under FDAT and DA always get their first choices, and these students comprise 19.5% of the whole sample (with a standard error of 2.7%). For 8.5% of the sample, the FDAT and DA outcomes are not comparable in the f.o.s.d. sense (with a standard error of 2.7%), and the average allocation probabilities for their choices are similar.

[19]The reported results are highly robust to changes in in-walk-zone/out-of-walk-zone sibling ratios.

Average fraction of students

	FDAT f.o.s.d. EE	EE f.o.s.d. FDAT	FDAT = EE	Not comparable	Overall
Average fraction:	36.5% (4.6%)	18.1% (3.8%)	24.8% (2.9%)	20.6% (3.2%)	100%

Average fractions of students getting in their various choices in each category

Choices	FDAT	EE	FDAT	EE	FDAT = EE	FDAT	EE	FDAT	EE
1st	82.9%	74.0%	65.7%	77.7%	100%	66.9%	71.2%	80.7%	80.5%
2nd	14.6%	19.8%	25.4%	18.3%		19.6%	17.9%	14.0%	14.3%
3rd	2.1%	3.8%	7.5%	3.6%		11.6%	7.3%	4.5%	3.5%
4th	0.3%	1.5%	1.3%	0.5%		1.8%	2.2%	0.7%	1.1%
5th		0.5%	0.1%			0.1%	0.9%	0.1%	0.4%
6th		0.2%					0.3%		0.1%
7th		0.1%					0.1%		0.0%

Average fraction of justifiably ex ante envious students in EE: 0.03% (0.16%)

Table 2. Comparison of FDAT with EE in the simulations (the sample standard errors of the fractions of students are given in parentheses after the averages).

The second table reports the results of similar comparisons between FDAT and EE. Note that in theory, there is no overall domination relation between these mechanisms. Students seem overall much better off under EE with respect to DA, but FDAT outcomes appear to be more favorable for a higher percentage of students than EE: 36.5% of the students unambiguously prefer FDAT over EE (with a standard error of 4.6% for the sample) while only 18.1% of the students prefer EE over FDAT (with a standard error of 3.8%). Of all the students, 24.8% receive the same allocation under both mechanisms, at which each of them receives his first choice with probability 1. For 20.6% of the students, the outcomes are not comparable with respect to f.o.s.d. (with a standard error of 3.2% for the sample).

Although neither EE nor DA is ex ante stable, we observe a very small percentage of agents having ex ante justified envy. Hence, under these preferences and priorities, both mechanisms almost behave like ex ante stable mechanisms (the last row in both tables). Although in theory, there can be stable and more efficient mechanisms than FDAT, as the two stability concepts seem to be close under realistic simulations, FDAT's superior performance with respect to both mechanisms is not surprising.

8. Incentives

Strategic issues regarding lottery matching mechanisms, in general, have not been well understood. A mechanism is *strategy-proof* if, for each agent, his random matching vector obtained through the mechanism via his truth-telling behavior ordinally dominates or is equal to the one obtained via his revelation of any untruthful ranking. In the context of one-sided matching (i.e., the special case of our model where all students have equal priority at all schools), strategy-proofness is essentially incompatible with ordinal efficiency. Therefore, notwithstanding its appeal in terms of various properties including ordinal efficiency, the probabilistic serial mechanism of Bogomolnaia and Moulin (2001) is not strategy-proof. In the context of school choice, due to the well known three-way tension among stability, efficiency, and incentives, strategy-proof and stable mechanisms are necessarily inefficient (cf. Erdil and Ergin 2008, Abdulkadiroğlu et al. 2009, and Kesten 2010). The current NYC/Boston mechanism, which is strategy-proof, is the most efficient stable mechanism (Gale and Shapley 1962) when priorities are strict. However, in the school-choice problem with weak priorities, it is not even ex post efficient within the ex post stable class of mechanisms. Moreover, it has been shown empirically (Abdulkadiroğlu et al. 2009) and theoretically (Kesten 2010) to be subject to significant and large welfare losses. As a result of this observation, nonstrategy-proof mechanisms have been proposed and advocated in the recent literature on school choice (cf. Erdil and Ergin 2008, Kesten 2010, and Abdulkadiroğlu et al. forthcoming).

Given the negative results outlined above regarding different fairness and efficiency properties, it is probably not surprising that the two mechanisms proposed in this paper are not strategy-proof. This observation follows from the following two impossibility results regarding the existence of strategy-proof mechanisms in our problem domain. We state these observations in the next two remarks. The first remark is a reformulation of a result due to Bogomolnaia and Moulin (2001) for the present context.

Remark 1. When $|I| \geq 4$, there is no strategy-proof, ex ante stable, and constrained ordinally efficient mechanism that also respects equal treatment of equals.

The next remark shows the incompatibility between strategy-proofness and strong ex ante stability. Its proof is given in Appendix B.

Remark 2. When $|I| \geq 3$, there is no strategy-proof and strongly ex ante stable mechanism.

However, in sufficiently large markets, nonstrategy-proof mechanisms of small markets can turn out to be strategy-proof (cf. Kojima and Manea 2010, Azevedo and Budish 2013). Indeed, in a large market with diverse preference types of students, FDA is strategy-proof. We prove this result in the following subsection.

8.1 *Incentives under FDA in a large market*

A *continuum school-choice problem* is denoted by a seven-tuple $[I, T, \tau, C, q, P, \succsim]$, where I is a Lebesgue-measurable continuum set of students, each of whom is seeking a seat at a school; T is a finite set of priority types of students; $\tau : I \to T$ is a type-specification function for students; C is a finite set of schools; $q = (q_c)_{c \in C}$ is a quota vector of schools such that $q_c \in \mathbb{Z}_{++}$ is the maximum Lebesgue measure of students who can be assigned to school c; $P = (P_i)_{i \in I}$ is a strict preference profile for students; and $\succsim = (\succsim_c)_{c \in C}$ is a weak priority structure for schools over T. Let $|J|$ denote the Lebesque measure of student subset $J \subseteq I$. Let $|I| > 0$.[20] We assume that there are enough spots

[20]See Che and Kojima (2010) for a foundational exercise in modeling continuum matching problems.

for all students, that is, $\sum_{c\in C} q_c = |I|$. We also assume that if $t \in \tau(I)$, then $\tau^{-1}(t)$ has a positive Lebesgue measure, $|\tau^{-1}(t)| > 0$. In particular, for all possible preference relations P_j over schools, there exists a positive Lebesgue measure of students in $\tau^{-1}(t)$ with the same preference relation P_j, that is, $|\{i \in \tau^{-1}(t) : P_i = P_j\}| > 0$ for all $t \in \tau(I)$ and preference relation P_j.[21] The ordinally Pareto-dominant strongly ex ante stable random matching still exists in this framework for each problem.[22] Let FDA be defined through the mechanism that picks this random matching. In this framework, we can state the following result.

Theorem 7. *In continuum school-choice problems as specified above, FDA is strategy-proof; that is, for any student, it is a weakly ordinally dominant strategy to reveal his true preferences.*[23]

Proof. Suppose a student i of type t, instead of revealing his true preference P_i, reveals some other student's preference P_j, where j is also of type t (for every manipulation of student i, such a student j exists by the assumptions). Now the outcome of the FDA mechanism is the same under both problems, with truthful revelation of i and with i pretending to be a student identical to j. This is true as the set of strongly ex ante stable random matchings will not change without changing the measures of types of agents existing in the problem. Suppose ρ is this outcome. All we need to show is that ρ_i ordinally dominates ρ_j under P_i or $\rho_i = \rho_j$ (suppose we denote this relationship by $\rho_i \succeq_i \rho_j$). We assume that $a_1\ P_i\ a_2\ P_i \cdots P_i\ a_n$ denotes the preference relation of i. There exists some a_k with $k \geq 1$ such that $\rho_{i,a_k} > 0$. Suppose a_k is the lowest ranked school in P_i with this property. Then by elimination of ex ante discrimination among equal priority students i and j under ρ, we have $\rho_{i,a_\ell} \geq \rho_{j,a_\ell}$ for all $\ell < k$. Therefore, for all $\ell < k$, $\sum_{m=1}^{\ell} \rho_{i,a_m} \geq \sum_{m=1}^{\ell} \rho_{j,a_m}$. Moreover, we have $\sum_{m=1}^{k} \rho_{i,a_m} = 1$. Hence, $\sum_{m=1}^{\ell} \rho_{i,a_m} \geq \sum_{m=1}^{\ell} \rho_{j,a_m}$ for all ℓ, showing that $\rho_i \succeq_i \rho_j$.[24] □

It is known that PS and DA are equivalent in a continuum economy when school priorities are the same over all students (cf. Che and Kojima 2010). One can wonder whether the above result is a corollary of a possible equivalence between FDA and DA mechanisms in the continuum model. A counterexample, however, shows that this is not the case. Indeed, FDA is fairer than DA and, hence, potentially less efficient. Such an equivalence between FDAT and DA does not hold either.

[21] Note that we do not assume that all prioritizations of different preference types are possible in a given problem. The possible prioritizations are given through the fixed priority profile $\succsim$.

[22] Establishing this fact requires a little more formal work, but we skip it for brevity and refer our reader to the corresponding result in a finite problem.

[23] It can be shown that there is no strategy-proof and constrained-efficient ex ante stable mechanism that also satisfies equal treatment of equals in the continuum school-choice problems. Hence, FDAT is also not strategy-proof in such a model.

[24] It can be shown that there is no strategy-proof and constrained-efficient ex ante stable mechanism that also satisfies equal treatment of equals in the continuum school-choice problems. Hence, FDAT is also not strategy-proof in such a model.

Example 7. Consider a continuum economy where there are three schools a, b, c, with $q_a = q_b = 1$, and $q_c = 1 + 4\delta$ for $\delta > 0$. All students have equal priority at all schools. There are six types of students with positive measure partitioned as I_1, I_2, I_3, I_4, I_5, I_6 with preferences

P_1	P_2	P_3	P_4	P_5	P_6
a	b	a	b	c	c
b	a	c	c	a	b
c	c	b	a	b	a

for all students $\mathbf{i} \in I_i$ for all $i \in \{1, 2, \dots, 6\}$. These type sets have measures $|I_1| = 2$, $|I_2| = 1$, and $|I_3| = |I_4| = |I_5| = |I_6| = \delta$. The DA outcome of this problem is equal to the PS outcome as proven by Che and Kojima (2010). DA, FDA, and FDAT outcomes are given as

$$\rho^{\mathrm{DA}} = \begin{array}{c|c|c|c} & a & b & c \\ \hline 1 & \frac{1}{2+\delta} & \frac{1}{(2+\delta)(3+\delta)} & \frac{2+4\delta+\delta^2}{(2+\delta)(3+\delta)} \\ 2 & 0 & \frac{4+\delta}{(2+\delta)(3+\delta)} & \frac{2+4\delta+\delta^2}{(2+\delta)(3+\delta)} \\ 3 & \frac{1}{2+\delta} & 0 & \frac{1+\delta}{2+\delta} \\ 4 & 0 & \frac{4+\delta}{(2+\delta)(3+\delta)} & \frac{2+4\delta+\delta^2}{(2+\delta)(3+\delta)} \\ 5 & 0 & 0 & 1 \\ 6 & 0 & 0 & 1 \end{array} \qquad \rho^{\mathrm{FDA}} = \begin{array}{c|c|c|c} & a & b & c \\ \hline 1 & \frac{1}{3+\delta} & \frac{1}{3+\delta} & \frac{1+\delta}{3+\delta} \\ 2 & \frac{1}{3+\delta} & \frac{1}{3+\delta} & \frac{1+\delta}{3+\delta} \\ 3 & \frac{1}{3+\delta} & 0 & \frac{2+\delta}{3+\delta} \\ 4 & 0 & \frac{1}{3+\delta} & \frac{2+\delta}{3+\delta} \\ 5 & 0 & 0 & 1 \\ 6 & 0 & 0 & 1 \end{array}$$

$$\rho^{\mathrm{FDAT}} = \begin{array}{c|c|c|c} & a & b & c \\ \hline 1 & \frac{2}{3(3+\delta)} & \frac{1}{3(3+\delta)} & \frac{1+\delta}{3+\delta} \\ 2 & 0 & \frac{2}{3+\delta} & \frac{1+\delta}{3+\delta} \\ 3 & \frac{1}{3+\delta} & 0 & \frac{2+\delta}{3+\delta} \\ 4 & 0 & \frac{1}{3+\delta} & \frac{2+\delta}{3+\delta} \\ 5 & 0 & 0 & 1 \\ 6 & 0 & 0 & 1 \end{array}$$

for all students $\mathbf{i} \in I_i$.

Hence $\rho^{\mathrm{FDA}} \neq \rho^{\mathrm{DA}} \neq \rho^{\mathrm{FDAT}} \neq \rho^{\mathrm{FDA}}$ for generic $\delta > 0$. $\diamondsuit$

9. Concluding comments

In this paper, we have established a framework that generalizes one-to-many two-sided and one-sided matching problems. Such a framework enables the mechanism designer to achieve strong and appealing ex ante efficiency properties when students are endowed with ordinal preferences as exemplified in the pioneering work of Bogomolnaia and Moulin (2001). Alternatively, fairness considerations play a crucial role in the design of practical school-choice mechanisms since school districts are vulnerable to possible legal action resulting from a violation of student priorities. We have formulated two natural and intuitive ex ante fairness notions called strong ex ante stability and ex ante stability, and have shown that they are violated by prominent school-choice mechanisms such as the current Boston/NYC mechanism. We have proposed two mechanisms that stand out as attractive members of their corresponding classes. Our proposals are practically applicable and can be roughly described to families similarly to the way one would describe the usual DA algorithm, the difference being that, in contrast to the current practice, our proposals defer the lottery phase to after all the preference and priority information about students has been processed to compute all possible assignment chances of each student to each school.

The research on school-choice lotteries is a relatively new area in market design theory, and there are many remaining open questions. One important question is about the characterization of ex post stability when matchings are allowed to be random. Similar to the results we have established for strong ex ante stability (Theorem 3) and ex ante stability (Theorem 5), a characterization of constrained ordinally efficient and ex post stable random matchings currently remains an important future issue.

Appendix A: How does the FDA algorithm work when there is a rejection cycle?

Example 8 (How does the finite FDA algorithm work?). Assume there are four students $\{1, 2, 3, 4\}$ and four schools, $\{a, b, c, d\}$ each with a quota of 1. The priorities and preferences are given as

$\succ_a$	$\succ_b$	$\succ_c$	$\succ_d$
4	1, 2	2	$\vdots$
2	3, 4	1, 3	
$\vdots$		4	

P_1	P_2	P_3	P_4
c	a	b	b
b	c	c	a
d	b	d	$\vdots$
a	d	a	$\vdots$

Students propose according to the order 1, 2, 3, 4.

Step 1. Student 1 applies to school c and is tentatively admitted.

Step 2. Student 2 applies to school a and is tentatively admitted.

Step 3. Student 3 applies to school b and is tentatively admitted.

Step 4. Student 4 applies to school b. The applicants of school b are students 3 and 4 (who have equal priority). Since applications exceed the quota, $\frac{1}{2}$ of each of 3 and 4 is rejected by b, while $\frac{1}{2}$ of each of 3 and 4 is tentatively admitted.

Step 5. Student 3 has an outstanding fraction of $\frac{1}{2}$ and applies to his next best school, c. The applicants of c are student 1 with a whole fraction and student 3 with fraction $\frac{1}{2}$. Each has equal priority at school c whose quota has been exceeded. Thus $\frac{1}{2}$ of each of students 1 and 3 is tentatively admitted at school c, while $\frac{1}{2}$ of student 1 is rejected. Since student 1 is partially rejected in favor of student 3 by c, we have $3 \hookrightarrow_c 1$.

Step 6. Student 1 has an outstanding fraction of $\frac{1}{2}$ and applies to his next best school, b. School b has three applicants, $\frac{1}{2}$ of student 3, $\frac{1}{2}$ of student 4, and $\frac{1}{2}$ of student 1. Since the quota of the school is exceeded and student 1 has the highest priority among the three applicants, $\frac{1}{2}$ of student 1 is tentatively admitted, while $\frac{1}{4}$ of each of students 3 and 4 is tentatively admitted, and $\frac{1}{4}$ of each of students 3 and 4 is rejected. We have $1 \hookrightarrow_b 4$, and $1 \hookrightarrow_b 3$; hence, there is a rejection cycle $(3, c, 1, a)$. The resolution of this cycle is trivial, since once student 3 applies to school c again with his outstanding fraction $\frac{1}{4}$, all of this is rejected by c since both student 1 and student 3 have equal priority at c and they already have a fraction of $\frac{1}{2}$ each at c. Thus, it is no longer true that $3 \hookrightarrow_c 1$, and the cycle is resolved.

Step 7. Student 3 has an outstanding fraction of $\frac{1}{4}$ and applies to his next best school, d. This is tentatively accepted by d.

Step 8. Student 4 has an outstanding fraction of $\frac{3}{4}$ and applies to his next best school, a. School a has two applicants, a whole fraction of student 2 and $\frac{3}{4}$ of student 4. Since the quota of a is only 1, and student 4 has higher priority than student 2 at a, then $\frac{3}{4}$ of student 4 and $\frac{1}{4}$ of student 2 is tentatively admitted to a, while $\frac{3}{4}$ of student 2 is rejected. We have $4 \hookrightarrow_a 2$.

Step 9. Student 2 has an outstanding fraction of $\frac{3}{4}$ and applies to his next best school, c. School c has three applicants: student 1 with fraction $\frac{1}{2}$, student 3 with fraction $\frac{1}{2}$, and student 2 with fraction $\frac{3}{4}$. Since the quota of c, which is 1, has been exceeded, and since student 2 has higher priority than each of students 1 and 3, who have equal priority, $\frac{3}{4}$ of student 2 and $\frac{1}{8}$ of each of students 1 and 3 are tentatively admitted to c, while $\frac{3}{8}$ of each of students 1 and 3 is rejected. We have $2 \hookrightarrow_c 1$ and $2 \hookrightarrow_c 3$. The former relation induces a new cycle $(1, b, 4, a, 2, c)$. This cycle is not trivial. We use a simple system of equations to resolve this cycle with unknowns y_1, y_4, and y_2 as the eventual limit rejected fractions from c, b, a and tentatively admitted fractions to b, a, c of students 1, 4, and 2, respectively:

$$\begin{aligned} y_1 + \omega_1 &= \max\{y_4, 0\} + \max\{\phi_{3,b} - (\phi_{4,b} - y_4), 0\}, \\ y_4 &= \max\{y_2, 0\}, \\ y_2 &= \max\{y_1, 0\} + \max\{\phi_{3,c} - (\phi_{1,c} - y_1), 0\}, \end{aligned}$$

where $\omega_1 = \frac{3}{8}$ is the fraction of student 1 that will be tentatively admitted to b when the cycle is initiated, $\phi_{4,b} = \phi_{3,b} = \frac{1}{4}$ are the fractions of students 4 and 3 currently tentatively admitted to b when the cycle is initiated, and $\phi_{3,c} = \phi_{1,c} = \frac{1}{8}$ are the fractions of students 3 and 1 currently tentatively admitted to c. Observe that these unknowns can be solved through the linear system

$$y_1 + \tfrac{3}{8} = 2y_4, \qquad y_4 = y_2, \qquad y_2 = 2y_1.$$

By solving them, we obtain

$$y_4 = \tfrac{1}{4}, \qquad y_2 = \tfrac{1}{4}, \qquad y_1 = \tfrac{1}{8}.$$

As these rejected fractions are all less than or equal to the initially admitted fractions of students 4, 2, and 1 to b, a, and c, respectively, indeed it is possible to resolve this cycle with these fractions.[25] At this point, the tentative random matchings of students are a

[25]If we had a situation such that $y_{i_\ell} > \phi_{i_\ell, c_{\ell-1}}$ for some i_ℓ in the cycle where $\phi_{i_\ell, c_{\ell-1}}$ is the initially tentatively admitted fraction of i_ℓ at $c_{\ell-1}$, from which he is being rejected by repeated applications of $i_{\ell-1}$, then

whole fraction of student 1 at b, a whole fraction of student 4 at a, and a whole fraction of student 2 at c. From the previous step, we also have $\frac{1}{4}$ of student 3 tentatively admitted at d.

Step 10. Student 3 has an outstanding fraction of $\frac{3}{4}$, with which he applies to his best school that has not rejected him yet, d. Now, the whole fraction of student 3 is applying to d, which tentatively admits him.

There are no outstanding student fractions left. The algorithm terminates with the outcome

	a	b	c	d
1	0	1	0	0
2	0	0	1	0
3	0	0	0	1
4	1	0	0	0

◊

Appendix B: Proofs of the results regarding the FDA mechanism

Proof of Proposition 3. First, we prove that a rejection cycle can be resolved in finite time. Suppose a rejection cycle occurs at a step when i_1 applies to c_1 with fraction ω_1 after this fraction is rejected from c_m:

$$i_1 \hookrightarrow_{c_1} i_2 \hookrightarrow_{c_2} \cdots \hookrightarrow_{c_{m-1}} i_m \hookrightarrow_{c_m} i_1.$$

At this point, for each s, let ϕ_{i,c_s} be the fraction of student i tentatively assigned to school c_s. Note that among the students $i_1, \dots, i_m$, only i_1 has a positive fraction, ω_1, that is tentatively unassigned at this time. Let ϕ_{i,c_s} be the fraction of student $i \in I$ tentatively held at each c_s at this point. We will place ω_1 in c_1 if we can. If not, i_1 will be rejected by c_1 and the cycle will be resolved. Suppose ω_1 can be tentatively placed in c_1.

Let y_{s+1} be the rejected fraction of i_{s+1} from c_s when the cycle is resolved. This fraction will be tentatively held by school c_{s+1} at this stage. Observe that all rejected fractions from c_s belong to the students who are at the same priority level with i_{s+1}. We need to make sure that each student i at the same priority level as i_{s+1} is held at the same fraction $\phi_{i_{s+1},c_s} - y_{s+1}$ at c_s, unless i did not have that much of a fraction to start with. The sum of all tentatively accepted fractions at the resolution of the cycle will be q_{c_s}.

A simple way to solve these equations is as follows: Let $M_s = \{i \sim_{c_s} i_{s+1} \mid \phi_{i,c_s} > 0\}$ for all s. Observe that if we did have sufficient fractions already held at the school c_s of i_{s+1}, then we can iteratively solve for y_s as

$$y_s = \sum_{i \in M_s} \max\{\phi_{i,c_s} - (\phi_{i_{s+1},c_s} - y_{s+1}), 0\} \quad \forall s \in \{2, \dots, m\}, \tag{1}$$

as when y_s of i_s is admitted at c_s, it causes the fraction $[\phi_{i,c_s} - (\phi_{i_{s+1},c_s} - y_{s+1})]$ to be rejected for each student $i \in M_s$ whenever this fraction is greater than 0 to start with

---

we would set $y_{i_\ell} = \phi_{i_\ell,c_{\ell-1}}$ and solve the other equations. See the proof of Proposition 3 in Appendix B for a generalization of this method.

(where $m+1=1$ in modulo m). For c_1, we have

$$y_1+\omega_1=\sum_{i\in M_1}\max\{\phi_{i,c_1}-(\phi_{i_2,c_1}-y_2),0\}, \tag{2}$$

as $y_1+\omega_1$ is the total admitted fraction of i_1 at school c_1. Then we can solve these m equations in m unknowns using a number of linear equation systems.

At the determined $\{y_t\}$ vector satisfying (1) and (2), a student i_{s+1} may have $y_{s+1}>\phi_{i_{s+1},c_s}$, i.e., we cannot reject y_{s+1} fraction of i_{s+1} from c_s. Then we try setting $y_{s+1}=\phi_{i_{s+1}}$; otherwise the equations for the other $y_t\neq y_{s+1}$ are given as in (1) and (2). We can similarly solve this system. Each y_t decreases, as y_{s+1} decreased and y_s and y_{s+2} are positively correlated with y_{s+1} and so on and so forth for all the other y_t's. If we still have at the new vector $\{y_t\}$ a student i_{u+1} such that $y_{u+1}>\phi_{i_{u+1},c_u}$, we set $y_{u+1}=\phi_{u+1,c_u}$ and all other $y_t\neq y_{u+1}$ are given as in (1) and (2). We solve the new system. As $\{y_t\}$ decreases again, we have $y_{s+1}<\phi_{i_{s+1},c_s}$ and, hence, the problem for the first student i_{s+1} is resolved. We continue iteratively as above for all students i_{t+1}, $y_{t+1}\leq\phi_{i_{t+1},c_t}$. We are done.

If a cycle does not occur, similarly the step of the algorithm can be resolved easily.

Observe that in each step of the FDA algorithm, students get weakly worse off, since they only make proposals to a school that has not rejected a fraction of themselves. After all $|I|$ students make offers, at least one student is rejected by one school and has an outstanding fraction, or the algorithm converges, whether or not a cycle occurs. Since there are $|C|$ schools, the algorithm converges in at most $|I||C|$ steps. □

Proof of Theorem 3. We argue by contradiction. Suppose this is not true for some school-choice problem. Fix a problem $[P,\succsim]$. Let $\pi\in\mathcal{X}$ be the FDA algorithm's outcome random matching for some order of students making offers and let $\rho\in\mathcal{X}$ be a strongly ex ante stable random assignment that is not stochastically dominated by π. This means that

$$\begin{aligned}&\text{there exist } i_0\in I \text{ and } a_0\in C \text{ such that } 0\neq\rho_{i_0,a_0}>\pi_{i_0,a_0}\\ &\qquad\qquad\text{where } a_0\,P_{i_0}\,e_0 \text{ for some } e_0\in C \text{ with } 0\neq\pi_{i_0,e_0}>\rho_{i_0,e_0}.\end{aligned} \tag{3}$$

We will construct a finite sequence of student–school pairs as follows.

Construction of a trading cycle from π to ρ: Statement (3) implies that there exists $i_1\in I\setminus\{i_0\}$ such that $\rho_{i_1,a_0}<\pi_{i_1,a_0}\neq 0$. Then strong ex ante stability of the FDA outcomes implies that $i_1\succsim_{a_0} i_0$, for otherwise π would have induced ex ante justifiable envy of i_1 toward i_0 for a_0 (in case $i_1\succ_{a_0} i_0$) or π would have ex ante discriminated between i_0 and i_1 at a_0 (in case $i_1\sim_{a_0} i_0$). Then, since $\rho_{i_1,a_0}<\pi_{i_1,a_0}$, $\rho_{i_0,a_0}>\pi_{i_0,a_0}$, and ρ is strongly ex ante stable, for ρ not to have ex ante justified envy of i_1 toward i_0 for a_0 (in case $i_1\succ_{a_0} i_0$) and ρ not to have ex ante discrimination between i_0 and i_1 for a_0 (in case $i_1\sim_{a_0} i_0$), there must exist some $a_1\in C\setminus\{a_0\}$ such that $0\neq\rho_{i_1,a_1}>\pi_{i_1,a_1}$, where $a_1\,P_{i_1}\,a_0$. More precisely, there are two cases.

Case 1: $i_1\succ_{a_0} i_0$. Suppose by contradiction that for all $b\in C$ with $b\,P_{i_1}\,a_0$, we have $\rho_{i_1,b}\leq\pi_{i_1,b}$. Then by feasibility there is $c\in C$ with $a_0\,P_{i_1}\,c$ and $\rho_{i_1,c}>\pi_{i_1,c}$. But then i_1 would ex ante justifiably envy i_0 for a_1 at ρ, contradicting ρ is strongly ex ante stable.

Case 2: $i_1 \sim_{a_0} i_0$. Since $\pi_{i_0,e_0} \neq 0$ and $a_0 \, P_{i_0} \, e_0$ (by statement (3) above), we have $0 \neq \pi_{i_1,a_0} \leq \pi_{i_0,a_0} \neq 0$. Thus, $\rho_{i_1,a_0} < \pi_{i_1,a_0}$ and $\rho_{i_0,a_0} > \pi_{i_0,a_0}$ imply that $\rho_{i_1,a_0} < \rho_{i_0,a_0}$. Then no ex ante discrimination at ρ between i_0 and i_1 for a_0 implies that there is no $d \in C$ where $a_0 \, P_{i_1} \, d$ with $\rho_{i_1,d} \neq 0$. Then such an a_1 should exist for i_1.

Observe that i_1 satisfies the same statement (3) above as i_0 does using a_1 instead of a_0, using a_0 instead of e_0, and using $i_1 \succsim_{a_0} i_0$, i.e.,

$$\rho_{i_1,a_1} > \pi_{i_1,a_1}, \quad a_1 \, P_{i_1} \, a_0 \text{ with } \rho_{i_1,a_0} < \pi_{i_1,a_0} \neq 0 \text{ and } i_1 \succsim_{a_0} i_0.$$

Thus, as we continue iteratively, we obtain a finite sequence of students and schools such that each pair (a_{s-1}, i_s) (subscripts are modulo $n+1$, so that $n+1 \equiv 0$) appears only once in the sequence,

$$e_0, i_0, a_0, i_1, a_1, \ldots, i_n, a_n$$

and each i_s satisfies condition (3), replacing i_s with i_0, a_s with a_1, and a_{s-1} with e_0, and additionally satisfying $i_s \succsim_{a_{s-1}} i_{s-1}$, i.e.,

$$\rho_{i_s,a_s} > \pi_{i_s,a_s}, \quad a_s \, P_{i_s} \, a_{s-1} \text{ with } \rho_{i_s,a_{s-1}} < \pi_{i_s,a_{s-1}} \neq 0 \text{ and } i_s \succsim_{a_{s-1}} i_{s-1}, \tag{4}$$

and, finally, by finiteness of schools and students, we have

$$a_n \equiv e_0 \quad \text{and yet} \quad i_n \neq i_0,$$

where e_0 can be chosen as defined in condition (3). This sequence describes a special probability trading cycle from π to ρ for some better schools, so that ρ cannot be ordinally dominated by π.

Observe that there can be many such cycles, some of them overlapping. And each such cycle has at least two agents and two schools. Suppose there are m^* such cycles $\text{Cyc}^1, \ldots, \text{Cyc}^m, \ldots, \text{Cyc}^{m^*}$, and let $I^1, I^2, \ldots, I^m, \ldots, I^{m^*}$ be the sets of students and $C^1, C^2, \ldots, C^m, \ldots, C^{m^*}$ be the corresponding sets of schools in these cycles, respectively. Let I^* be the union of all above student sets and let C^* be the union of all above school sets. We will prove some claims that will facilitate the proof of the theorem.

Claim 1. *Take a cycle* $\text{Cyc}^m = (i_0, a_0, \ldots, i_n, a_n)$. *There is no* $a_s \in C^m$ *and no* $b \in C$ *such that for student* i_{s+1}, *we have* $a_s \, P_{i_{s+1}} \, b$ *and* $\rho_{i_{s+1},b} \neq 0$.

Proof. Suppose, to the contrary, there are $a_s \in C^m$ and $b \in C$ such that $a_s \, P_{i_{s+1}} \, b$ and $\rho_{i_{s+1},b} \neq 0$. We also have $\rho_{i_{s+1},a_s} < \pi_{i_{s+1},a_s} \neq 0$ by construction of cycle Cyc^m (see statement (4) above). We also have by construction $0 \neq \rho_{i_s,a_s} > \pi_{i_s,a_s}$, $a_s \, P_{i_s} \, a_{s-1}$, $\rho_{i_s,a_{s-1}} < \pi_{i_s,a_{s-1}} \neq 0$, and, finally, $i_{s+1} \succsim_{a_s} i_s$ (see statement (4) above). Consider two cases:

Case 1: $i_{s+1} \succ_{a_s} i_s$. Since $\rho_{i_s,a_s} \neq 0$, $\rho_{i_{s+1},b} \neq 0$, and $a_s \, P_{i_{s+1}} \, b$, student i_{s+1} ex ante justifiably envies i_s for a_s at ρ, contradicting that ρ is strongly ex ante stable.

Case 2: $i_{s+1} \sim_{a_s} i_s$. By the strong ex ante stability of π, there is no ex ante discrimination between i_{s+1} and i_s for a_s at π. Since $a_s \, P_{i_s} \, a_{s-1}$ and $\pi_{i_s,a_{s-1}} \neq 0$, we must have $\pi_{i_s,a_s} \geq \pi_{i_{s+1},a_s}$. Then we have $\rho_{i_s,a_s} > \pi_{i_s,a_s} \geq \pi_{i_{s+1},a_s} > \rho_{i_{s+1},a_s}$. Recall that $\rho_{i_{s+1},b} \neq 0$ for

$a_s \, P_{i_{s+1}} \, b$. The last two statements imply that ρ ex ante discriminates between i_s and i_{s+1} at a_s, contradicting that ρ is strongly ex ante stable. $\lhd$

Consider the sequence of offers and rejections in the FDA algorithm that leads to π. Let $i_s \in I^m$ for a cycle Cyc^m (without loss of generality, let $(i_0, a_0, i_1, a_1, \ldots, i_n, a_n)$ be this cycle) be the *last* student in I^* to apply and get a positive fraction under π from the next school in his cycle (i.e., for i_s, this school is $a_s \in C^m$). Let t be this step of the algorithm. We prove the following claim.

Claim 2. *The total sum of student fractions that school a_{s-1} has tentatively accepted until the beginning of step t of the FDA algorithm is equal to its quota, i.e., school a_{s-1} is filled at the beginning of step t.*

Proof. Consider agent i_{s-1}. We have $\pi_{i_{s-1},a_{s-2}} > 0$ by construction of Cyc^m. By the choice of student i_s, student i_{s-1} should have applied to school a_{s-2} at some step $p < t$. We also have $a_{s-1} \, P_{i_{s-1}} \, a_{s-2}$ by construction of Cyc^m. Then, in the FDA algorithm, i_{s-1} should have applied to a_{s-1} first at some step $r < p$. This is true as he can apply to a_{s-2} in the algorithm only after having been rejected by school a_{s-1}. A school can reject a student only if it has tentatively accepted student fractions adding up to its quota. Since a_{s-1} remains to be filled after it becomes filled in the algorithm, the claim follows. $\lhd$

Thus, by Claim 2, a_{s-1} is full at the beginning of step t just before student i_s applies. Then there exists some student $j \in I$ with $i_s \succsim_{a_s} j$, such that some fraction of j was tentatively accepted by school a_s before step t and some fraction of j is kicked out of school a_s at the end of step t (so that by the choice of i_s, some fraction of his gets in a_{s-1}). Since the FDA algorithm converges to a well defined random matching, there is some $b \in C$ such that $a_{s-1} \, P_j \, b$ and $\pi_{j,b} \neq 0$. We prove the following claim.

Claim 3. *We have $j \notin I^*$.*

Proof. Suppose not, i.e., j is in some cycle. By the choice of student i_s, the ordered fourtuple $b \, (= a_{s-2})$, $j \, (= i_{s-1})$, a_{s-1}, i_s cannot be part of Cyc^m, i.e., j cannot be accepted by b after being rejected by a_{s-1} in the FDA algorithm and yet $\rho_{j,b} < \pi_{j,b}$ (i.e., see statement (4) for the construction of a cycle). But then by the choice of school b, $\rho_{j,b} \geq \pi_{j,b} \neq 0$. However, Claim 1 applied for school a_{s-1} and student $j \, (= i_{s-1})$, and the fact that $a_{s-1} \, P_j \, b$ together imply that $\rho_{j,b} = 0$, contradicting the previous statement. Thus, $j \notin I^*$. $\lhd$

We are ready to finish the proof of the theorem. Since school a_{s-1} is full at the beginning of step t (by Claim 2), there is student $i_{s-1} \in I^m \setminus \{i_s\}$, i.e., preceding a_{s-1} in Cyc^m, with $0 \neq \rho_{i_{s-1},a_{s-1}} > \pi_{i_{s-1},a_{s-1}}$ who applied to school $a_{s-2} \prec_{i_{s-1}} a_{s-1}$ after being rejected by school a_{s-1}. Moreover, by the choice of i_s, student i_{s-1} applies to a_{s-2} before step t (for the last time) and, hence, he was rejected by a_{s-1} before step t. Moreover, $i_{s-1} \neq j$ (by Claim 3). Thus, $j \succsim_{a_{s-1}} i_{s-1}$. We will establish a contradiction and complete the proof of the theorem. Two cases are possible.

Case 1: $j \succ_{a_{s-1}} i_{s-1}$. Since $\rho_{j,b} \neq 0$, strong ex ante stability of ρ implies that $\rho_{i_{s-1},a_{s-1}} = 0$, leading to a contradiction to the fact that $0 \neq \rho_{i_{s-1},a_{s-1}}$.

Case 2: $j \sim_{a_{s-1}} i_{s-1}$. Recall again that $\pi_{i_{s-1},a_{s-2}} > 0$ and $a_{s-1}\ P_{i_{s-1}}\ a_{s-2}$, $\pi_{j,b} > 0$, and $a_{s-1}\ P_j\ b$. But then i_{s-1} is rejected by a_{s-1} at the FDA algorithm at the same step as j is rejected with some fraction, which is step t (since ρ does not ex ante discriminate between j and i_{s-1} at a_{s-1}, they should have equal fractions at a_{s-1} prior to step t), and thus i_{s-1} applies to school a_{s-2} after step t, contradicting the choice of student i_s. □

Proof of Remark 2. Let φ be a strongly ex ante stable mechanism. Consider the following problem with three students 1, 2, 3, and three schools a, b, c, each with quota 1:

P_1	P_2	P_3
a	a	b
b	c	a
c	b	c

$\succsim_a$	$\succsim_b$	$\succsim_c$
3	1	$\vdots$
1, 2	$\vdots$	

There is a unique strongly ex ante stable random matching that is given as

$$\rho = \begin{array}{c|c|c|c} & a & b & c \\ \hline 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 \end{array}$$

Thus, $\varphi[P, \succsim] = \rho$.

However, if student 1 submits the preferences $P'_1 = (acb)$ instead of P_i, then the unique strongly ex ante stable random matching will be

$$\rho' = \begin{array}{c|c|c|c} & a & b & c \\ \hline 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 2 & \frac{1}{2} & 0 & \frac{1}{2} \\ 3 & 0 & 1 & 0 \end{array}$$

Hence, $\varphi[(P'_1, P_{-1}), \succsim] = \rho'$. Observe that there can be von Neumann–Morgenstern utility functions of student 1 that may make ρ'_1 more desirable than ρ_1.

If $|I| > 3$, we can make the example hold by embedding it in a problem with $I = \{1, 2, \dots, |I|\}$ and $C = \{a, b, c, d_4, \dots, d_{|I|}\}$, where each student $i \in \{4, \dots, |I|\}$ is ranking school d_i as his first choice and each student $i \in \{1, 2, 3\}$ is ranking each school d_i lower than schools a, b, c. Under any ex ante strongly stable matching, each $i \in \{4, 5, \dots, |I|\}$ will be matched with d_i and $\{1, 2, 3\}$ will be mapped with $\{a, b, c\}$. □

Appendix C: Proof of Proposition 5

Proof of Proposition 5. *"Only if."* Let ρ be an ex ante stable random matching with an ex ante stable improvement cycle $\mathrm{Cyc} = (i_1, a_1, \dots, i_m, a_m)$. Let $i_{m+1} \equiv i_1$ and $a_{m+1} \equiv a_1$. Let π be the random matching obtained by satisfying this cycle with some feasible fraction. Then π ordinally dominates ρ. Each student i_s envies student i_{s+1} for a_{s+1} due to a_s at ρ, and i_s is a highest a_l-priority student ex ante envying a student with a positive probability at school a_{s+1}. Thus, either (i) i_s is at the same priority level with i_{s+1} for a_{s+1} or (ii) i_s is at a lower-priority level than i_{s+1} for a_{s+1} but any $i \sim_{a_{s+1}} i_{s+1}$ does not ex ante envy himself or i_{s+1} for a_{s+1} at ρ; that is, i is not assigned with a positive probability to a worse school than a_{s+1} at ρ. Thus, when we satisfy the cycle Cyc, there will be no ex ante justified envy toward a lower-priority student, and π is ex ante stable.

"*If.*" Let ρ be an ex ante stable random matching. Let $\pi \neq \rho$ be an ex ante stable random matching that ordinally dominates ρ. We will construct a particular ex ante stable improvement cycle at ρ.

Let $I' = \{i \in I : \rho_i \neq \pi_i\}$. Clearly, $I' \neq \varnothing$. Note that for all $i' \in I'$, $\pi_{i'}$ stochastically dominates $\rho_{i'}$. Thus, whenever $\pi_{i',a} > \rho_{i',a}$ for some $i' \in I'$ and $a \in C$, then there is $j' \in I'$ with $\pi_{j',a} < \rho_{j',a}$; moreover, since $\pi_{j'}$ stochastically dominates $\rho_{j'}$, there is $b \in C$ with $b \, P_{j'} \, a$ and $\pi_{j',b} > \rho_{j',b}$. Let $C' = \{c \in C : \pi_{i,c} > \rho_{i,c}$ for some $i \in I'\}$. Clearly, $C' \neq \varnothing$.

Consider the following directed graph: Each student–school pair $(i, c) \in I' \times C'$ with $\rho_{i,c} \neq 0$ is represented by a node. Fix a school $c \in C'$. Let each student–school pair (i, c) in this graph containing school c be pointed to by every student–school pair containing a student who schoolwise-envies student i for school c and has the highest priority among such schoolwise-envying students in I'. We repeat this for each $c \in C'$.

Note that no student–school pair in the resulting graph points to itself, and each student–school pair in this graph is pointed to by at least one other student–school pair. Moreover, each student–school pair (i, c) in this graph can only be pointed to by a student–school pair that contains a different school than c. Then there is at least one cycle of student–school pairs $\text{Cyc} = (i_1, a_1, i_2, a_2, \ldots, i_m, a_m)$ with $(i_m, a_m) \equiv (i_0, a_0)$ and $m \geq 2$. By construction, we have $(i_s, a_s) \gtrdot^{\rho} (i_{s+1}, a_{s+1})$ for $s = 0, \ldots, m-1$. Note also that cycle Cyc contains at least two distinct students. Then cycle Cyc is a stochastic improvement cycle.

Now consider school a_{s+1} of the pair (i_{s+1}, a_{s+1}) in cycle Cyc. Suppose, for a contradiction, that student i_s does not ex ante top-priority schoolwise-envy i_{s+1} for a_{s+1} due to a_s. Then there is a student–school pair (j, d) with $j \notin I'$, which is not represented in our graph, such that $(j, d) \blacktriangleright^{\rho} (i_{s+1}, a_{s+1})$. In particular, $j \succ_{a_{s+1}} i$ for any $i \in I'$ such that $(i, d) \gtrdot^{\rho} (i_{s+1}, a_{s+1})$ for any $d \in C'$. Let $k \in I'$ such that $\pi_{k,a_{s+1}} > \rho_{k,a_{s+1}}$. Since $j \succ_{a_{s+1}} k$ and $\rho_{j,d} = \pi_{j,d}$, student j justifiably ex ante envies k at π. This contradicts the ex ante stability of π. □

Appendix D: The EASC algorithm

The description of the algorithm mostly follows the constrained consumption algorithm of Athanassoglou and Sethuraman (2011) with a few modifications.

Given an ex ante stable matching ρ, we first define the set

$$\mathcal{A}(\rho) = \{(i, c) \in I \times C : \rho_{i,c} > 0 \text{ or } (i, a) \blacktriangleright^{\rho} (j, c) \text{ for some } j \in I \text{ and } a \in C\}.$$

Given an initial ex ante stable random matching ρ, our adaptation of the constrained consumption algorithm finds a random matching π such that (i) π ordinally dominates or is equal to ρ and (ii) $(i, c) \notin \mathcal{A}(\rho) \Rightarrow \pi_{i,c} = 0$.

It is executed through a series of *flow networks*, each of which is a directed graph from an artificial *source node* to an artificial *sink node*, denoted as σ and τ, respectively. We will carry the assignment probabilities from source to sink over this flow network, so that the eventual flow will always determine a feasible random matching. The initial network is constructed as follows.

The nodes of the network are (i) source σ and sink τ, (ii) each school $c \in C$, and (iii) for each $i \in I$ and $\ell \in \{1, \dots, |C|\}$, $i_{(\ell)} \in I \times \{1, \dots, |C|\}$; i.e., the ℓth node of student i is a node, where this node corresponds to the ℓth choice of student i among the schools.

Let $N = I \times \{1, \dots, |C|\} \cup C \cup \{\sigma, \tau\}$ be the set of nodes of the network.

An *arc* from node x to node y is represented as $x \to y$. Let $\omega_{x\to y}$ be the *capacity* of arc $x \to y$.[26] The arcs have the following load capacities:

1. Each arc $\sigma \to i_{(\ell)}$ has the capacity $\rho_{i,c}$, where school c is the ℓth choice of student i.
2. Each arc $i_{(\ell)} \to c$ has the capacity ∞ if $(i, c) \in \mathcal{A}(\rho)$ and c is ranked ℓth or better at the student i's preferences, and is 0 otherwise.
3. Each arc $c \to \tau$ has the capacity q_c, the quota of school c.
4. Any arc between any other two nodes has capacity zero.

Thus, the arcs with positive load capacities are directed from the source σ to the student nodes, from the student nodes to feasible school nodes with respect to $\mathcal{A}(\rho)$, and from school nodes to the sink τ.

Let $\Gamma = \langle N, \omega\rangle$ denote this network. We define additional concepts for such a network.

A *cut* of the network is a subset of nodes $K \subseteq N$ such that $\sigma \in K$ and $\tau \in N \setminus K$. The capacity of a cut K is the sum of the capacities of the arcs that are directed from nodes in K to nodes in $N\setminus K$, and it is denoted as $\Omega(K)$, that is, $\Omega(K) = \sum_{x\in K, y\in N\setminus K} \omega_{x\to y}$. A *minimum cut* K^* is a minimum capacity cut, i.e., $K^* \in \arg\min_{\{\sigma\}\subseteq K\subseteq N\setminus\{\tau\}} \Omega(K)$. A *flow* of the network is a list $\phi = (\phi_{x\to y})_{x,y\in N}$ such that (i) for each $x, y \in N$, $\phi_{x\to y} \le \omega_{x\to y}$, i.e., the flow cannot exceed the capacity, and (ii) for all $x \in N \setminus \{\sigma, \tau\}$, $\sum_{y\in N} \phi_{y\to x} = \sum_{y\in N} \phi_{x\to y}$, i.e., total incoming flow to a node should be equal to the total outgoing flow. Let Φ be the set of flows. The *value* of a flow ϕ is the total outgoing flow from the source, i.e., $\Omega(\phi) = \sum_{y\in N} \phi_{\sigma\to y}$. A *maximum flow* ϕ^* is a flow with the highest value, i.e., $\phi^* \in \arg\max_{\phi\in\Phi} \Omega(\phi)$. Observe that in our network Γ, the maximum flow value is equal to $|I|$.

The algorithm solves iterative maximum flow–minimum cut problems, a powerful tool in graph theory and linear programming. The corresponding duality theorem is stated as follows.

Theorem 8 (Ford and Fulkerson 1956; maximum flow–minimum cut theorem). *The value of the maximum flow is equal to the capacity of a minimum cut.*

[26]Without loss of generality, we focus on rational numbers as load capacities, since only rational numbers appear as input to both FDA and FDAT.

There are various polynomial-time algorithms, such as the Edmonds and Karp (1972) algorithm, that can determine a minimum cut and maximum flow.

The ex ante stable consumption algorithm updates the network starting from Γ by updating the capacity of some of the source arcs $\omega_{\sigma\to i_{(\ell)}}$ over time, which is a continuous parameter $t\in[0,1]$. It starts from $t=0$ and increases up to $t=1$. Thus, let us relabel the source arc weights as a function of time t as $\omega^t_{\sigma\to i_{(\ell)}}$ by setting $\omega^0_{\sigma\to i_{(\ell)}}\equiv\omega_{\sigma\to i_{(\ell)}}$ for each arc $\sigma\to i_{(\ell)}$. No other arc capacity is updated. Let Γ^t be the corresponding flow network at time t.

There will also be iterative steps in the algorithm with start times $t^1=0\le t^2\le\cdots\le t^n\le 1=t^{n+1}$, for steps $1,\ldots,n$, respectively. All assignment activity in step m occurs in the time interval $(t^m,t^{m+1}]$.

This algorithm is in the class of eating algorithms introduced by Bogomolnaia and Moulin (2001), and t also represents the assigned fraction of each student, since each student is assumed to be assigned at a uniform speed of 1. This activity is referred to as *eating a school*. Each school is assumed to be a perfectly divisible object with q_c copies.

We update the feasible assignment set $\mathcal{A}(\rho)$ in each step. Let $\mathcal{A}^m(\rho)$ be the feasible student–school pairs at step $m=1,\ldots,n$. We have $\mathcal{A}^1(\rho)=\mathcal{A}(\rho)\supseteq\mathcal{A}^2(\rho)\supseteq\cdots\supseteq\mathcal{A}^n(\rho)$.

At each step m, let $b_i\in C$ be the *best feasible* school for student i, that is, $(i,b_i)\in\mathcal{A}^m(\rho)$ and $b_i\,R_i\,c$ for all c with $(i,c)\in\mathcal{A}^m(\rho)$. Also, let $e_i\in C$ be the *endowment* school of student i, that is, if $R_i(c)$ is the rank of school c for i, then $\omega^{t^m}_{\sigma\to i_{(R_i(e_i))}}>0$ and $b_i\,P_i\,e_i\,R_i\,c$ for all c with $\omega^{t^m}_{\sigma\to i_{(R_i(c))}}>0$. Observe that e_i may not exist for a student i, which case is denoted as $e_i=\varnothing$. In the algorithm we describe here, each student consumes the best school feasible for him at t while his endowment of a worse school decreases.

We are ready to state the algorithm, a slightly modified version of the Athanassoglou and Sethuraman (2011) algorithm.

The EASC Algorithm. Suppose that until Step $m\ge 1$, we determined t^m, $\{\omega^{t^m}_{\sigma\to i_{(\ell)}}\}_{i\in I,\ell\in\{1,\ldots,|C|\}}$, and $\mathcal{A}^m(\rho)$.

Step m: We determine t^{m+1}, $\{\omega^t_{\sigma\to i_{(\ell)}}\}_{i\in I,\ell\in\{1,\ldots,|C|\}}$ for all $t\in(t^m,t^{m+1}]$, and $\mathcal{A}^{m+1}(\rho)$ as follows: Initially time satisfies $t=t^m$. Let $\{b_i,e_i\}_{i\in I}$ be determined given $\mathcal{A}^m(\rho)$ and $\{\omega^{t^m}_{\sigma\to i_{(\ell)}}\}$.

Then t continuously increases. At t, the arc capacities $\omega^{(t)}_{\sigma\to i_{(\ell)}}$ are updated for each $i\in I$ and $c\in C$ as

$$\omega^t_{\sigma\to i_{(R_i(c))}}:=\begin{cases}\max\{t-\sum_{\ell=1}^{R_i(b_i)-1}\omega^{t^m}_{\sigma\to i_{(\ell)}},\,\omega^{t^m}_{\sigma\to i_{(R_i(b_i))}}\} & \text{if } c=b_i \text{ and } e_i\neq\varnothing\\ \min\{\sum_{\ell=1}^{R_i(b_i)}\omega^{t^m}_{\sigma\to i_{(\ell)}}+\omega^{t^m}_{\sigma\to i_{(R_i(e_i))}}-t,\,\omega^{t^m}_{\sigma\to i_{(R_i(e_i))}}\} & \text{if } c=e_i\\ \omega^{t^m}_{\sigma\to i_{(R_i(c))}} & \text{otherwise.}\end{cases}$$

That is, each student i consumes his best feasible school b_i with uniform speed by trading away fractions from his endowment school e_i if it exists and the consumption fraction of the best school exceeds his initial consumption of $\omega^{t^m}_{\sigma\to i_{(R_i(b_i))}}$.

Time t increases until one of the following two events occurs:

- We have $t < 1$, and yet
  - the endowment school fraction endowed to some student reaches zero, i.e., $\omega^t_{\sigma \to e_i} = 0$ for some $i \in I$: We update

$$t^{m+1} := t$$
$$\mathcal{A}^{m+1}(\rho) := \mathcal{A}^m(\rho);$$

  or

  - any further increase in t will cause the maximum flow capacity in the network to fall, i.e., for $t^\varepsilon > t$ and arbitrarily close to t, the network $\Gamma^{(t^\varepsilon)}$ has a maximum flow capacity less than that of $\Gamma^{(t)}$ (maximum flow of any $\Gamma^{(s)}$ can be determined by an algorithm such as Edmonds–Karp): This means that if some student were to consume his best feasible school more, the resulting outcome will not be first-order stochastically improving for all students. Let K be a minimum cut of Γ^{t^ε}. Any student i, who satisfies $i_{(R_i(b_i))} \in K$ and $i_{(R_i(e_i))} \notin K$, has an endowment that is not in high demand. Alternatively, he wants to consume more of a school that is in high demand. Thus, if he consumes more of his best school, then eventually this will cause some student's final assignment not to first-order stochastically dominate his endowment, contradicting the goal of achieving stochastic Pareto improvement. We refer to the set of such students as the set of bottleneck students. Thus, we update

$$t^{m+1} := t$$
$$\mathcal{A}^{m+1}(\rho) := \mathcal{A}^m(\rho) \setminus \{(i, b_i) : e_i \neq \varnothing, i_{(R_i(b_i))} \in K, \text{ and } i_{(R_i(e_i))} \notin K\}.$$

  We continue with Step $m + 1$.

- $t = 1$: The algorithm terminates. The outcome of the algorithm $\pi \in \mathcal{X}$ is found as follows: Let ϕ be a maximum flow of the network Γ^1 (i.e., the final network at time $t = 1$). Then, we set for all $i \in I$ and $c \in C$,

$$\pi_{i,c} := \sum_{\ell=1}^{|C|} \phi_{i_{(\ell)} \to c},$$

  i.e., the total flow from student i to school c.

Athanassoglou and Sethuraman (2011) proved that this algorithm with $\mathcal{A}(\rho) = I \times C$ (i.e., the case in which all schools are feasible to be assigned to each student) converges to a unique ordinally efficient random matching such that it treats equals equally whenever ρ treats equals equally; and it Pareto dominates or is equal to ρ. Their statements can be generalized to the case in which $\mathcal{A}(\rho) \subseteq I \times C$ such that the outcome of the above algorithm π is constrained ordinally efficient in the class of random matchings $\chi \in \mathcal{X}$ satisfying $\chi_{i,c} > 0 \Rightarrow (i, c) \in A(\rho)$. Moreover, π is also ex ante stable whenever ρ is ex ante stable by the construction of $A(\rho)$; it ordinally dominates or is equal to ρ; and it treats equals equally whenever ρ treats equals equally. We skip these proofs for brevity.

Appendix E: How is the EASC algorithm embedded in the FDAT algorithm?

Example 9. We illustrate the functioning of the FDAT algorithm with the EASC algorithm using the same problem in Example 3 (and Example 5).

Step 0. We found the FDA outcome in Example 3 as

$$\rho^{[1]} = \begin{array}{c|c|c|c|c} & a & b & c & d \\ \hline 1 & \frac{11}{12} & \frac{1}{12} & 0 & 0 \\ 2 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 3 & \frac{5}{12} & \frac{1}{12} & \frac{1}{3} & \frac{1}{6} \\ 4 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & \frac{1}{3} & \frac{2}{3} \\ 6 & 0 & \frac{5}{6} & 0 & \frac{1}{6} \end{array}$$

Step 1. We form the feasible student–school pairs for matching as

$$\mathcal{A}^1(\rho^{[1]}) = \{(1,a),(1,b),(2,a),(2,c),(3,a),(3,b),(3,c),(3,d),$$
$$(4,c),(5,c),(5,d),(6,a),(6,b),(6,c),(6,d)\}.$$

We execute the EASC algorithm as follows.

Step 1.1. Time is set as $t^1 = 0$. Given that $i_{(\ell)}$ represents the ℓth choice school of student i, we form the flow network with the positive weights obtained from the endowment random matching $\rho^{[1]}$ as for all $i \in I$ and for all schools $f \in C$, we set the arc capacities

$$\omega^0_{\sigma \to i_{(R_i(f))}} = \rho^{[1]}_{i,f},$$

where $R_i(f)$ is the ranking of school f in i's preferences. Next, for all $i \in I$ and $f \in C$, if $(i,f) \in A^1(\rho^{[1]})$, we set the arc capacities of the flow network as

$$\omega^0_{i_{(\ell)} \to f} = \infty$$

for all ranks $\ell \le R_i(f)$. Finally, for all $f \in C$, we set the arc capacities

$$\omega^0_{f \to \tau} = q_f.$$

Figure 1 shows this network for $t \in [0, \frac{1}{12}]$.

Moreover, given these constraints, the best available schools and endowment schools are

Students (i)	Best school (b_i)	Endowment school (e_i)
1	b	a
2	c	a
3	d	c
4	c	$\emptyset$
5	c	d
6	d	b

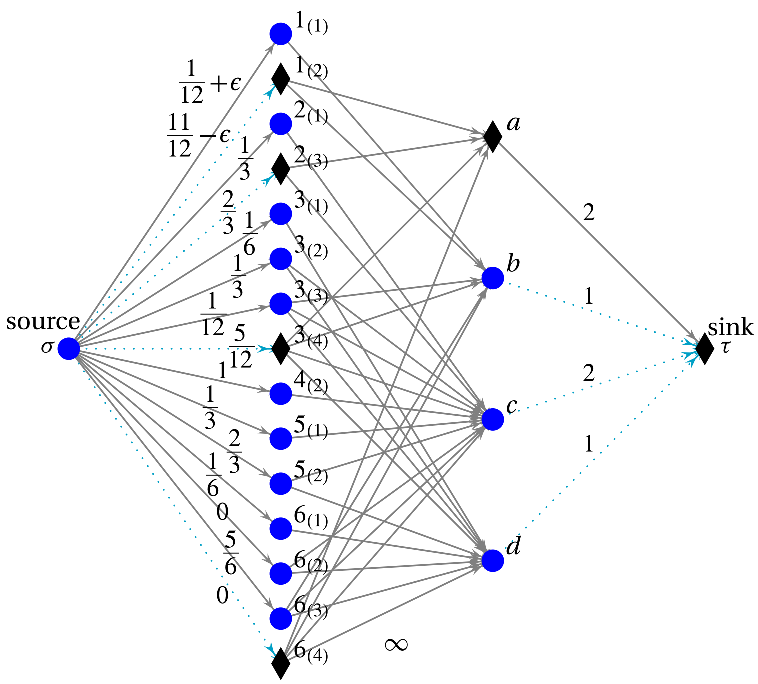

FIGURE 1. The consumption network for Example 9 at Step 1.1 for times $t = \frac{1}{12} + \epsilon$ for $\epsilon \geq 0$ sufficiently small. A minimum cut at time t is denoted by circular nodes in the graph as $K = \{\sigma, 1_{(1)}, 2_{(1)}, 3_{(1)}, 3_{(2)}, 3_{(3)} 4_{(2)}, 5_{(1)}, 5_{(2)}, 6_{(1)}, 6_{(2)}, 6_{(3)}, b, c, d\}$. This results with the bottleneck set of students $J = \{1, 2\}$. The dotted edges show the arcs going out of the minimum cut with the total capacity $6 - \epsilon$, which is equal to the maximum flow. Moreover, for all $t \in [0, \frac{1}{12}]$ maximum flow is 6. Hence $t^2 = \frac{1}{12}$.

We start increasing time t starting from $t^1 = 0$; thus, each student starts consuming his best available school by trading away from his endowment school (whenever $e_i \neq \varnothing$). That is, the capacity of each arc $\sigma \to i_{(R_i(b_i))}$ is updated as

$$\omega^t_{\sigma \to i_{(R_i(b_i))}} = \max\left\{ t - \sum_{\ell=1}^{R_i(b_i)-1} \omega^0_{\sigma \to i_{(\ell)}}, \omega^0_{\sigma \to i_{(R_i(b_i))}} \right\}$$

and the capacity of each arc $\sigma \to i_{(R_i(e_i))}$ is updated as

$$\omega^t_{\sigma \to i_{(R_i(e_i))}} = \min\left\{ \sum_{\ell=1}^{R_i(b_i)} \omega^0_{\sigma \to i_{(\ell)}} + \omega^0_{\sigma \to i_{(R_i(e_i))}} - t, \omega^0_{\sigma \to i_{(R_i(e_i))}} \right\}$$

as long as a feasible random assignment can be obtained in the network, i.e., the value of the maximum flow of the network is $|I| = 6$ or the capacity of the endowment school arc does not go to zero. The first condition is satisfied at $t = \frac{1}{12}$: If t increases above $\frac{1}{12}$, the value of the maximum flow falls below 6 because of the bottleneck set of agents

$$J = \{1, 2\}$$

for the minimum cut

$$K = \{\sigma, 1_{(1)}, 2_{(1)}, 3_{(1)}, 3_{(2)}, 3_{(3)}, 4_{(2)}, 5_{(1)}, 5_{(2)}, 6_{(1)}, 6_{(2)}, 6_{(3)}, b, c, d\}.$$

At this t, there is an excess demand for student 1's and student 2's best schools, but other agents do not demand student 1's and student 2's endowment school. Thus, students 1 and 2 can no longer trade their endowment school in exchange for a fraction of their best schools. To see that $\{1, 2\}$ is a bottleneck set, we find a minimum cut K as seen in Figure 1 for the network at $t = \frac{1}{12}$. Each student's representative nodes for his best school and his endowment school are in K, except for students 1 and 2. Their nodes for best schools are in K, but not their nodes for endowment schools. Also their endowment school a is not in K. Thus, Step 1.1 ends, and students 1 and 2 can no longer consume their best schools b and c, respectively. (Observe that the network at $t = \frac{1}{12}$ is identical to the network at $t = 0$.) We set

$$t^2 = \tfrac{1}{12}$$
$$\mathcal{A}^2(\rho^{[1]}) = \mathcal{A}^1(\rho^{[1]}) \setminus \{(1, b), (2, c)\}.$$

Step 1.2. Time is set as $t^2 = \frac{1}{12}$. The best and endowment schools are updated as

Students (i)	Best school (b_i)	Endowment school (e_i)
1	a	$\varnothing$
2	a	$\varnothing$
3	d	c
4	c	$\varnothing$
5	c	d
6	d	b

Time increases until $t = \frac{1}{6}$, when there is a new bottleneck set of students with minimum cut

$$K = \{\sigma, 2_{(1)}, 3_{(1)}, 3_{(2)}, 4_{(2)}, 5_{(1)}, 5_{(2)}, 6_{(1)}, c, d\}.$$

Since $6_{(R_6(b_6))} = 6_{(R_6(d))} = 6_{(1)} \in K$ and $6_{(R_6(e_6))} = 6_{(R_6(b))} = 6_{(3)} \notin K$, and there is no other student such that his node for his best (available) school is in K while his node for his endowment school is not, we determine the new bottleneck set as

$$J = \{6\}.$$

Thus, we update

$$t^3 = \tfrac{1}{6}$$
$$\mathcal{A}^3(\rho^{[1]}) = \mathcal{A}^2(\rho^{[1]}) \setminus \{(6, d)\}.$$

At this point the capacities of the source–agent nodes are set still as their initial values at ω^0 (seen in Figure 1).

Step 1.3. Time is set as $t^3 = \frac{1}{6}$. The best and endowment schools are updated as

Students (i)	Best school (b_i)	Endowment school (e_i)
1	a	$\varnothing$
2	a	$\varnothing$
3	d	c
4	c	$\varnothing$
5	c	d
6	b	$\varnothing$

At this step, we observe actual trading of fractions of schools c and d between students 3 and 5, since all other students have no endowment schools to trade: time t increases until $\frac{1}{2}$ at which point only the following arc capacities are changing, while the others are still at the ω^0 level:

$$
\begin{aligned}
\omega^{1/2}_{\sigma\to 3_{(R_3(b_3))}} &= \omega^{1/2}_{\sigma\to 3_{(R_3(d))}} \\
&= \omega^{1/2}_{(\sigma\to 3_{(1)})} \\
&= \max\left\{t - \sum_{\ell=1}^{R_3(b_3)-1} \omega^{1/6}_{\sigma\to 3_{(\ell)}}, \omega^{1/6}_{\sigma\to 3_{(R_3(b_3))}}\right\} \\
&= \max\left\{\tfrac{1}{2} - 0, \tfrac{1}{6}\right\} = \tfrac{1}{2}; \\
\omega^{1/2}_{\sigma\to 3_{(R_3(e_3))}} &= \omega^{1/2}_{\sigma\to 3_{(R_3(c))}} \\
&= \omega^{1/2}_{\sigma\to 3_{(2)}} \\
&= \min\left\{\sum_{\ell=1}^{R_3(b_3)} \omega^{1/6}_{\sigma\to 3_{(\ell)}} + \omega^{1/6}_{\sigma\to 3_{(R_3(e_3))}} - t, \omega^{1/6}_{\sigma\to 3_{(R_3(e_3))}}\right\} \\
&= \min\left\{\tfrac{1}{6} + \tfrac{1}{3} - \tfrac{1}{2}, \tfrac{1}{3}\right\} = 0; \\
\omega^{1/2}_{\sigma\to 5_{(R_5(b_5))}} &= \omega^{1/2}_{\sigma\to 5_{(R_5(c))}} \\
&= \omega^{1/2}_{\sigma\to 5_{(1)}} \\
&= \max\left\{t - \sum_{\ell=1}^{R_5(b_5)-1} \omega^{1/6}_{\sigma\to 5_{(\ell)}}, \omega^{1/6}_{\sigma\to 5_{(R_3(b_3))}}\right\} \\
&= \max\left\{\tfrac{1}{2} - 0, \tfrac{1}{3}\right\} = \tfrac{1}{2}; \\
\omega^{1/2}_{\sigma\to 5_{(R_5(e_5))}} &= \omega^{1/2}_{\sigma\to 5_{(R_5(d))}} \\
&= \omega^{1/2}_{\sigma\to 5_{(2)}}
\end{aligned}
$$

$$
\begin{aligned}
&= \min\left\{\sum_{\ell=1}^{R_5(b_5)} \omega^{1/6}_{\sigma\to 5_{(\ell)}} + \omega^{1/6}_{\sigma\to 5_{(R_5(e_5))}} - t, \omega^{1/6}_{\sigma\to 5_{(R_5(e_5))}}\right\} \\
&= \min\left\{\tfrac{1}{3} + \tfrac{2}{3} - \tfrac{1}{2}, \tfrac{2}{3}\right\} = \tfrac{1}{2}.
\end{aligned}
$$

Since the endowment school's matching probability reaches zero for student 3, the step ends and we update:

$$
\begin{aligned}
t^4 &= \tfrac{1}{2} \\
\mathcal{A}^4(\rho^{[1]}) &= \mathcal{A}^3(\rho^{[1]}).
\end{aligned}
$$

Step 1.4. Time is set to $t^4 = \frac{1}{2}$ and only student 3's endowment school changed as $e_3 = b$. But at this time, there is a minimum cut

$$K = \{\sigma, 2_{(1)}, 3_{(1)}, 4_{(2)}, 5_{(1)}, 5_{(2)}, 6_{(1)}, c, d\}.$$

Since $3_{(R_3(b_3))} = 3_{(R_3(d))} = 3_{(1)} \in K$ and $3_{(R_3(e_3))} = 3_{(R_3(b))} = 3_{(3)} \notin K$, and there is no other student with this property, the bottleneck set is

$$J = \{3\}.$$

Thus, we set

$$
\begin{aligned}
t^5 &= \tfrac{1}{2} \\
\mathcal{A}^5(\rho^{[1]}) &= \mathcal{A}^4(\rho^{[1]}) \setminus \{(3, d)\}.
\end{aligned}
$$

Step 1.5. Time is set to $t^5 = \frac{1}{2}$ and only student 3's best school changed as $e_3 = c$. But at this time, there is a minimum cut

$$K = \{\sigma, 2_{(1)}, 3_{(1)}, 4_{(2)}, 5_{(1)}, 5_{(2)}, 6_{(1)}, c, d\}.$$

Since $3_{(R_3(b_3))} = 3_{(R_3(c))} = 3_{(1)} \in K$ and $3_{(R_3(e_3))} = 3_{(R_3(b))} = 3_{(3)} \notin K$, and there is no other student with this property, the bottleneck set is

$$J = \{3\}.$$

Thus, we set

$$
\begin{aligned}
t^5 &= \tfrac{1}{2} \\
\mathcal{A}^5(\rho^{[1]}) &= \mathcal{A}^4(\rho^{[1]}) \setminus \{(3, c)\}.
\end{aligned}
$$

Step 1.6. Time is set to $t^6 = \frac{1}{2}$, student 3's best school changed as $b_3 = b$, and his endowment school changed as $e_3 = a$. Time t increases until $\frac{7}{12}$, when further increasing t would create a bottleneck set of students with minimum cut

$$K = \{\sigma, 1_{(1)}, 2_{(1)}, 3_{(1)}, 3_{(3)}, 4_{(2)}, 5_{(1)}, 5_{(2)}, 6_{(1)}, 6_{(3)}, b, c, d\}.$$

Since $3_{(R_3(b_3))} = 3_{(R_3(b))} = 3_{(3)} \in K$ and $3_{(R_3(e_3))} = 3_{(R_3(a))} = 3_{(4)} \notin K$, and there is no other student with this property, we have the bottleneck set at

$$J = \{3\}.$$

Observe that in the interval $t \in (\frac{1}{2}, \frac{7}{12}]$, student 3 does not consume his best school more than his capacity. This interval serves as the continuation of the trading between students 3 and 5 regarding schools c and d that started at Step 1.3. Although student 3 has already traded all his endowment of $(\frac{1}{3})c$ away in return to get $(\frac{1}{12})d$, student 5 has not fully gotten $(\frac{1}{3})d$ and traded away $(\frac{1}{3})c$. Thus, the market has not yet cleared. Increase in t helps the market to clear, since now we have

$$\begin{aligned}
\omega^{7/12}_{\sigma\to 5_{(R_5(b_5))}} &= \omega^{7/12}_{\sigma\to 5_{(R_5(c))}}\\
&= \omega^{7/12}_{\sigma\to 5_{(1)}}\\
&= \max\left\{t - \sum_{\ell=1}^{R_5(b_5)-1} \omega^{1/2}_{\sigma\to 5_{(\ell)}}, \omega^{1/2}_{\sigma\to 5_{(R_3(b_3))}}\right\}\\
&= \max\left\{\tfrac{7}{12} - 0, \tfrac{1}{2}\right\} = \tfrac{7}{12};\\
\omega^{7/12}_{\sigma\to 5_{(R_5(e_5))}} &= \omega^{7/12}_{\sigma\to 5_{(R_5(d))}}\\
&= \omega^{7/12}_{\sigma\to 5_{(2)}}\\
&= \min\left\{\sum_{\ell=1}^{R_5(b_5)} {\omega_{\sigma\to 5_{(\ell)}}}^{1/2} + \omega^{1/2}_{\sigma\to 5_{(R_5(e_5))}} - t, \omega^{1/2}_{\sigma\to 5_{(R_5(e_5))}}\right\}\\
&= \min\left\{\tfrac{1}{2} + \tfrac{1}{2} - \tfrac{7}{12}, \tfrac{1}{2}\right\} = \tfrac{5}{12},
\end{aligned}$$

while all other arc capacities remain the same. We update as

$$t^7 = \tfrac{7}{12}$$
$$\mathcal{A}^7(\rho^{[1]}) = \mathcal{A}^6(\rho^{[1]}) \setminus \{(3, b)\}.$$

Step 1.7. Time is set to $t^7 = \frac{7}{12}$, student 3's best school changed as $b_3 = a$, and he no longer has an endowment school, i.e., $e_3 = \varnothing$. Time t increases until $\frac{2}{3}$, when further increasing t would create a bottleneck set of students with minimum cut

$$K = \{\sigma, 4_{(2)}, 5_{(1)}, c\}.$$

Since $5_{(R_5(b_5))} = 5_{(R_3(c))} = 5_{(1)} \in K$ and $5_{(R_5(e_5))} = 5_{(R_5(d))} = 5_{(2)} \notin K$, and there is no other student with this property, we have the bottleneck set as

$$J = \{5\}.$$

Similar to Step 1.6, trade of c from student 3 to student 5 has continued at this step in return of d, and it can be verified that the only updated arc capacities are

$$\omega^{2/3}_{\sigma\to 5_{(1)}} = \frac{2}{3}$$
$$\omega^{2/3}_{\sigma\to 5_{(2)}} = \frac{1}{3}.$$

We update as

$$t^8 = \frac{2}{3}$$
$$\mathcal{A}^8(\rho^{[1]}) = \mathcal{A}^7(\rho^{[1]}) \setminus \{(5, c)\}.$$

Step 1.8. Time is set to $t^8 = \frac{2}{3}$, student 5's best school is updated as $b_5 = d$, and he no longer has an endowment school, i.e., $e_5 = \varnothing$. Since no student has any endowment school, no more trade takes place in this step, time t increases to 1, and the ex ante stable consumption algorithm terminates with

$$\rho^{[2]} = \begin{array}{c|c|c|c|c} & a & b & c & d \\ \hline 1 & \frac{11}{12} & \frac{1}{12} & 0 & 0 \\ 2 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 3 & \frac{5}{12} & \frac{1}{12} & 0 & \frac{1}{2} \\ 4 & 0 & 0 & 1 & 0 \\ 5 & 0 & 0 & \frac{1}{6} & \frac{1}{3} \\ 6 & 0 & \frac{5}{6} & 0 & \frac{1}{6} \end{array}$$

Step 2. We have the new feasible student–school set

$$\mathcal{A}^1(\rho^{[2]}) = \{(1, a), (1, b), (2, a), (2, c), (3, a), (3, b), (3, c), (3, d), (4, c), (5, c), (5, d), (6, b), (6, d)\}.$$

It is easy to check that there are no feasible ex ante stable improvement cycles, and the FDAT algorithm terminates with outcome ρ^2. ◊

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