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Incentive-compatible voting rules with positively correlated beliefs

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We study the consequences of positive correlation of beliefs in the design of voting rules in a model with an arbitrary number of voters. We propose a notion of positive correlation, based on the likelihood of agreement of the k-best alternatives (for any k) of two orders called top-set (TS) correlation. We characterize the set of ordinal Bayesian incentive compatible (OBIC) (d'Aspremont and Peleg 1988) voting rules with TS-correlated beliefs and additionally satisfy robustness with respect to local perturbations. We provide an example of a voting rule that satisfies OBIC with respect to *all* TS-correlated beliefs. The generally positive results contrast sharply with the negative results obtained for the independent case by Majumdar and Sen (2004) and parallel similar results in the auction design model (Crémer and McLean 1988).

Keywords. Voting rules, ordinal Bayesian incentive compatibility, positive correlation, robustness with respect to beliefs.

JEL classification. C70, C72.

1. Introduction

It seems reasonable to believe that the difficulties associated with satisfactory group decision-making will diminish as the opinions of the members of the group become "more similar." In the limit, if all agents have the same objectives, all conflicts of interest disappear and we may expect a trivial resolution of the problem. In mechanism design

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theory, agents have private information about their objectives or preferences (referred to as types); the theory seeks to analyze collective (or social) goals (referred to as social choice functions (SCFs)), which are attainable subject to the constraint that all agents have the incentive to reveal their private information truthfully (referred to as incentive compatibility). Here too, if the private information of all agents is perfectly correlated, the issue of incentives can typically be resolved.[1] More interestingly, an extensive literature initiated by Crémer and McLean (1988) has pointed out that in environments where monetary compensation is feasible and preferences are quasi-linear (i.e., preferences over money are not dependent on type), even a little correlation in the beliefs over types leads to a dramatic enlargement of the class of incentive-compatible SCFs.

In this paper we explore the issue of correlated beliefs in the design of voting rules. In this environment, voters have opinions or preferences on the ranking of a finite number of candidates. These preferences (types, in this model), expressed as linear orders over the set of candidates, are private information. A SCF or voting rule is a mapping that associates a candidate with a collection of types, one for each voter. The goal of the theory is to identify SCFs that induce voters to reveal their types truthfully for every conceivable realization of these types.

We consider the plausible case where beliefs over types are positively correlated. To do so, we need to interpret positive correlation between distributions over linear orders. To the best of our knowledge, this specific question has not received attention in the literature. We therefore propose a notion of our own that suits our purpose. The particular notion of positive correlation that we have is based on the likelihood of other voters top k alternatives (for any k) agreeing with one's own opinion of the top k alternatives. To illustrate this notion, consider the case of voting for the annual Chess Oscar award by chess journalists and experts. Assume that the three players in serious contention are Aronian (A), Carlsen (C), and Kramnik (K). Assume that a voter's opinion is C followed by K followed by A. Then she believes that the event where all other voters rank C best is strictly more likely than the event where the best alternative for *all* other voters is either K or A. In addition, she believes that the event where the best two players for *all* other voters is $\{C, K\}$ is more likely than the event where the best two alternatives for *all* other voters is either $\{C, A\}$ or $\{K, A\}$. We call this notion of positive correlation top-set (TS) correlation. We note that the requirements for TS correlation are weak in the sense that the conditions for positive correlation apply only in "exceptional" circumstances (all other voters are unanimous about their best k alternatives for any k). Our choice of definition is deliberate because as we shall see, even this weak notion leads to a dramatic increase in the possibilities for the design of incentive-compatible social choice functions, at least in certain circumstances.

We consider ordinal Bayesian incentive compatible (OBIC) voting rules introduced in d'Aspremont and Peleg (1988). This requires the probability distribution over outcomes obtained by truth-telling to first-order stochastically dominate the distribution from misreporting for every voter type. These distributions are obtained from a voter's

[1] The problem is still nontrivial because the mechanism designer may be ignorant of the common type realized; this is the complete information implementation studied in Maskin (1999).

beliefs about the types of the other voters and the assumption that the other voters are telling the truth. The condition is equivalent to requiring that truth-telling be optimal in terms of expected utility for all possible utility functions that represent the voter's type.

In addition to OBIC, we consider local robustness conditions of the mechanism with respect to beliefs. The local robustness requires the mechanism to remain incentive-compatible if voter beliefs are perturbed slightly. The notion of positive correlation, i.e., TS correlation, then leads to the notion of TS-local robustness (TS-LOBIC). The motivation of imposing robustness requirements on beliefs is the well known Wilson doctrine (Wilson 1987). Robust mechanisms have the attractive feature that they continue to implement the objectives of the mechanism designer even if the designer or the voters make errors in their beliefs.

Our results are as follows. We characterize the class of TS-LOBIC SCFs subject to the weak requirement of unanimity. In particular, we provide a necessary and sufficient condition that a SCF needs to satisfy so that there exists some neighborhood of TS-correlated beliefs such that the SCF is OBIC with respect to all beliefs in the neighborhood. It is clear that if truth-telling for a particular type is weakly dominated by a misreport for a SCF, then the SCF cannot be locally robust incentive-compatible with respect to any class of beliefs. We show that a minor modification of this condition to take into account the ordinal nature of OBIC is also *sufficient* if TS correlation is considered. We also prove a general possibility result in this regard. We show that any SCF satisfying the property of neutrality and elementary monotonicity (a large class, including, for instance, SCFs derived from scoring correspondences) is TS-LOBIC. Moreover, they are incentive-compatible in a neighborhood of the *uniform prior*. The overall picture that emerges from the analysis is that it is not difficult to find SCFs satisfying LOBIC requirements.

Two observations are in order regarding our LOBIC results. The first is that our results may not be compatible with the requirement of a *common prior* for all agents (Example 5). However, this drawback is not serious in our opinion. We believe this because the constructed beliefs are nevertheless compatible with *approximately* common priors valid up to *any* degree of approximation.

There is a second, perhaps more serious criticism of the formulation of our problem. A more direct approach would be to address the following question: fix a positively correlated prior and characterize the class of incentive-compatible (or robustly incentive-compatible) SCFs with respect to this prior. Unfortunately, we are not able to provide a general answer to such a question: indeed, there is no reason to believe that a general "interesting" answer exists. For any belief, the set of inequalities describing the incentive-compatible SCFs can be written down without much difficulty. However, characterizing such SCFs is hard without imposing restrictions on the positively correlated beliefs considered. Our Theorems 1 and 2 exactly constitute such efforts, with the former dealing with beliefs "near unanimity" and the latter "near uniformity." In addition, we have several examples that explicitly identify the beliefs for which particular SCFs are LOBIC.

Our results contrast sharply with the negative results obtained in Majumdar and Sen (2004) for the case of independent beliefs. In this case, there is a generic set of beliefs for each voter such that OBIC with respect to *any* belief in this set is equivalent to dictatorship where truth-telling is, of course, a weakly dominant strategy. There are beliefs such as the uniform prior with respect to which a wide class of SCFs are OBIC. However, even local robustness cannot be satisfied for any nondictatorial SCF because of the generic impossibility result. Alternatively, in the positively correlated case, we demonstrate significant possibility results with local robustness.

Our results are in the same spirit as the possibility results in auction design theory with correlated valuations (Crémer and McLean 1988). However, our results and arguments bear no resemblance to their auction theory counterparts because of at least two significant differences between the models. The first is that monetary transfers, which are at the heart of the possibility results in the auction model, are not permitted in the voting model. The second is that the nature of types in the voting model (linear orders) is very different from its counterpart in the auction model (a nonnegative real number or vector). The notion of correlation in the voting model is, therefore, more delicate. Several alternative approaches and definitions are possible and the results depend on the choices made. Finally, our results are different because we address a different question from that in (Crémer and McLean 1988). We investigate the structure of social choice functions that satisfy certain robustness properties with respect to beliefs in addition to standard incentive-compatibility requirements.

The paper is organized as follows. In Section 2, we try to explain why correlation of types may help in mechanism design in our model. Section 3 introduces the notations and definitions. Section 4 discusses alternative notions of positive correlation, while Section 5 deals with incentive compatibility with local robustness. Section 6 concludes.

2. Why does correlation of types help in mechanism design in voting models?

Consider the case where there are two voters 1 and 2, and three alternatives a, b, and c from which to choose. A voter's type is one of the six orderings of the alternatives. These types will be represented by $abc\cdots$, which signifies "a is preferred to b is preferred to $c\cdots$." A social choice function or voting rule is a 6×6 matrix where each entry in the matrix is an alternative.

Consider a "partial" social choice function described in the array (1). Thus, if the row voter's type is abc, the outcome is a if the column voter's type is abc, bca, or cab. If the row voter's type is acb, the outcome is a if the column voter's type is bac or bca.

$$\begin{array}{ccccccc} & abc & acb & bac & bca & cab & cba \\ abc & a & . & . & a & a & . \\ acb & . & . & a & a & . & . \end{array} \quad (1)$$

Suppose the row voter's type is abc. Suppose further that she has a cardinal representation of her type where the utility of alternative a is 1, that of c is 0, and that of b is arbitrarily close to 0. What is the expected utility of this voter from truth-telling, assuming that the column voter tells the truth? It clearly depends on the prior beliefs of the row voter of type abc about the type of the column voter. It is, in fact,

$$\mu_1(abc|abc) + \mu_1(bca|abc) + \mu_1(cab|abc),$$

where $\mu_1(abc|abc)$ is the row voter's belief that the column voter's type is abc conditional on the row voter's type being abc, etc. By deviating to acb, the row voter of type abc will obtain the expected utility

$$\mu_1(bac|abc) + \mu_1(bca|abc).$$

Incentive compatibility will then require

$$\mu_1(abc|abc) + \mu_1(cab|abc) \geq \mu_1(bac|abc). \tag{2}$$

Now consider a row voter of type acb with a utility representation where the utility of a is 1, that of b is 0, and that of c arbitrarily close to 0. For this type not to deviate to abc, we require

$$\mu_1(bac|acb) \geq \mu_1(abc|acb) + \mu_1(cab|acb). \tag{3}$$

If the row voter's beliefs are *independent*, then the probabilities are not conditional on her type realization. Removing the dependence of beliefs on the row voter's types, inequalities (2) and (3) yield the equality

$$\mu_1(bac) = \mu_1(abc) + \mu_1(cab). \tag{4}$$

Observe that the equality (4) cannot hold for a "generic" belief over the column voter's type. If it does for some belief, a small "perturbation" will destroy it. The only way for incentive compatibility to be maintained for a generic belief is for all the a's to line up along the same column, i.e., if the outcome is a when the row voter's type is abc and the column voter's type is t_2, then the outcome is also a when the row and column voter's types are acb and t_2, respectively. In fact, there are several restrictions of this sort implied by the independence and genericity assumptions. Majumdar and Sen (2004) demonstrate that if there are at least three alternatives, incentive compatibility implies dictatorship.

A critical observation is that if beliefs are correlated, then the distribution of the column voter's type, conditional on different realizations of the row voter's types, is *distinct*. Hence inequalities such as (2) and (3) can hold without precipitating a restriction such as (4). Consequently, a much wider class of social choice functions are incentive-compatible. The rest of the paper explores the class of incentive-compatible social choice functions under different notions of positive correlation.

3. Notation and definitions

The set of voters is $N = \{1, \ldots, n\}$. Individual voters are denoted by i, j, etc. The set of outcomes is the set A with $|A| = m$. Elements of A will be denoted by $a, b, c, d, \ldots$. Let

$\mathbb{P}$ denote the set of strict orderings[2] of the elements of A. A typical preference ordering or type for a voter will be denoted by P_i, and for all $a, b \in A$ and $a \neq b$, $a\, P_i\, b$ will be interpreted as "a is strictly better than b according to P_i." A preference profile is an element of the set $\mathbb{P}^n$. Preference profiles will be denoted by $P, \bar{P}, P', \ldots$ and their ith components as $P_i, \bar{P}_i, P'_i, \ldots$, respectively, with $i \in N$.

For all $P_i \in \mathbb{P}$ and $k = 1, \ldots, M$, let $r_k(P_i)$ denote the kth-ranked alternative in P_i, i.e., $r_k(P_i) = a$ implies that $|\{b \neq a | b\, P_i\, a\}| = k - 1$. For all $i \in N$, for any $P_i \in \mathbb{P}$, and for any $a \in A$, let $B(a, P_i) = \{b \in A | b\, P_i\, a\} \cup \{a\}$. Thus $B(a, P_i)$ is the set of alternatives that are weakly preferred to a under P_i. For any $k = 1, \ldots, m$, $B(r_k(P_i), P_i)$ is the set of alternatives that are ranked k or higher in the ordering P_i. To economize on notation, we shall denote $B(r_k(P_i), P_i)$ simply as $B_k(P_i)$.

Definition 1. A *social choice function* (SCF) f is a mapping $f: \mathbb{P}^n \to A$.

We now state some familiar axioms on SCFs that we will use at various places in the paper.

Definition 2. A SCF f is unanimous or satisfies unanimity if $f(P) = a_j$ whenever $a_j = r_1(P_i)$ for all voters $i \in N$.

The axiom states that in any situation where all individuals agree on some alternative as the best, the SCF must respect this consensus. A stronger requirement than unanimity is the notion of Pareto efficiency or simply efficiency. This requires that it should not be possible to make all voters better off relative to the outcome of the SCF at any preference profile.

Definition 3. A SCF f is efficient or satisfies efficiency if for all profiles $P \in \mathbb{P}^n$, there does not exist an alternative $x \in A$ such that $x\, P_i\, f(P)$ for all $i \in N$.

A dictatorial SCF picks a particular voter's best alternative at every preference profile.

Definition 4. A SCF f is dictatorial if there exists a voter $i \in N$ such that for all profiles $P \in \mathbb{P}^n$, $f(P) = r_1(P_i)$.

The fundamental assumption in strategic voting theory is that a voter's preference ordering is her private information. The objective of a mechanism designer is to design SCFs that provide appropriate incentives for voters to reveal their private information. A standard requirement (for example, Gibbard 1973 and Satterthwaite 1975) is for SCFs to be dominant strategy incentive-compatible or strategy-proof. In such a SCF, no voter can profitably misrepresent her preferences irrespective of what (the) other voter(s) reveal as their preferences.

Definition 5. A SCF f is dominant strategy incentive-compatible or strategy-proof if, for all $P_i, P'_i \in \mathbb{P}$ and for all $P_{-i} \in \mathbb{P}^{n-1}$, either $f(P_i, P_{-i}) = f(P'_i, P_{-i})$ or $f(P_i, P_{-i})\, P_i\, f(P'_i, P_{-i})$ holds.

[2]A strict ordering is a complete, transitive, and antisymmetric binary relation.

Gibbard (1973) and Satterthwaite (1975) show that if $|A| \geq 3$, every strategy-proof SCF satisfying unanimity is dictatorial. We employ a weaker notion of incentive compatibility.

Definition 6. A *belief* for voter i is a probability distribution on the set $\mathbb{P}^n$, i.e., it is a map $\mu_i : \mathbb{P}^n \to [0, 1]$ such that $\sum_{P \in \mathbb{P}^n} \mu_i(P) = 1$.

Clearly μ_i belongs to the unit simplex of dimension $m!^n - 1$. For all μ_i, for all $(P_i, P_{-i}) \in \mathbb{P}^n$, we shall let $\mu_i(P_{-i}|P_i)$ denote the conditional probability of P_{-i} given P_i. A *belief system* is a n-tuple of beliefs $(\mu_1, \dots, \mu_n)$, one for each voter.

Definition 7. The utility function $u : A \to \Re$ represents $P_i \in \mathbb{P}$ if and only if for all $a, b \in A$, we have $a\, P_i\, b \Leftrightarrow u(a) > u(b)$.

d'Aspremont and Peleg (1988) introduced the following notion of ordinal Bayesian incentive compatibility.

Definition 8. A SCF f is ordinal Bayesian incentive compatible (OBIC) with respect to the belief system $(\mu_1, \dots, \mu_n)$ if for all $i \in N$, for all $P_i, P'_i \in \mathbb{P}$, for all u representing P_i, we have

$$\sum_{P_{-i} \in \mathbb{P}^{n-1}} u(f(P_i, P_{-i}))\mu_i(P_{-i}|P_i) \geq \sum_{P_{-i} \in \mathbb{P}^{n-1}} u(f(P'_i, P_{-i}))\mu_i(P_{-i}|P_i).$$

Suppose f is a SCF that is OBIC with respect to the belief system $(\mu_1, \dots, \mu_n)$. Consider voter i with preference P_i. Then reporting truthfully is optimal in the sense that it yields a higher *expected* utility than that obtained by any misrepresentation. In computing this expected utility, it is assumed that voters other than i will reveal truthfully so that a probability distribution over outcomes is induced by f and voter i's beliefs, conditional on P_i, i.e., $\mu_i(\cdot|P_i)$. Furthermore, higher expected utility from truth-telling occurs for *all* representations of the true preference P_i. An equivalent way to state the same requirement is that truth-telling is a Bayes–Nash equilibrium of the revelation game induced by f for all possible utility representations of true preferences.

OBIC is a natural way to relax the requirement of truth-telling as a dominant strategy in the (standard) ordinal voting model. A fairly obvious relationship between OBIC and dominant strategies is as follows.

Observation 1. Suppose f is OBIC with respect to all belief systems $(\mu_1, \dots, \mu_n)$. Then f is strategy-proof.

Definition 9 below provides an equivalent stochastic dominance statement of OBIC without explicit reference to utility functions.

Definition 9. The SCF f is OBIC with respect to the belief system $(\mu_1, \dots, \mu_n)$ if for all $i \in N$, for all integers $k = 1, \dots, m$, and for all P_i and P'_i,

$$\mu_i\big(\{P_{-i} | f(P_i, P_{-i}) \in B_k(P_i)\}|P_i\big) \geq \mu_i\big(\{P_{-i} | f(P'_i, P_{-i}) \in B_k(P_i)\}|P_i\big).$$

Suppose f satisfies OBIC with respect to $(\mu_1, \dots, \mu_n)$. Consider voter i with preferences P_i. Then the aggregate probability induced by f on the first k alternatives of her true preference P_i for any $k = 1, \dots, m$ is maximized by truth-telling.

We now turn our attention to the issue of positively correlated beliefs.

4. Positive correlation

We wish to capture the notion of positive correlation between linear orders defined on the elements of an arbitrary set. There is a large literature in statistics on positive interdependence between random variables of various kinds (see, for instance, Drouet Mari and Kotz (2001)). However, to the best of our knowledge, no existing notion is directly applicable to our model. We therefore develop a concept suitable for our purpose.

Consider a voter with beliefs μ_i and type P_i, and consider the set of k-best alternatives in P_i: $B_k(P_i)$ for some $k = 1, \dots, m$. Let $D \subset A$ be such that $|D| = k$ and $D \neq B_k(P_i)$. Now consider the following two events:

- Event I. The k-best alternatives for all voters $j \neq i$ is $B_k(P_i)$.
- Event II. The k-best alternatives for all voters $j \neq i$ is D.

The belief μ_i is TS- (top-set) correlated if Event I is strictly more likely than Event II according to the conditional distribution $\mu_i(\cdot|P_i)$.

Definition 10 (TS Correlation). A belief for voter i, μ_i is positively TS-correlated if for all P_i and for all $k = 1, \dots, m-1$,

$$\sum_{\{P_{-i}|B_k(P_j)=B_k(P_i)\ \forall j\neq i\}} \mu(P_{-i}|P_i) > \sum_{\{P_{-i}|B_k(P_j)=D\ \forall j\neq i\}} \mu(P_{-i}|P_i),$$

where $D \subset A$, $D \neq B_k(P_i)$, and $|D| = k$.

We denote by TS* the set of all μ satisfying TS correlation.

The following examples illustrate TS correlation.

Example 1. Let $N = \{1, 2\}$ and $A = \{a, b, c\}$. Consider the following belief μ_i, which generates the conditional beliefs $\mu_i(\cdot|abc)$ specified[3] as

$$\begin{array}{ccccccc} & abc & acb & bac & bca & cab & cba \\ abc & \mu_i^1 & \mu_i^2 & \mu_i^3 & \mu_i^4 & \mu_i^5 & \mu_i^6 \end{array},$$

where $\mu_i^1 = \mu_i(abc|abc), \dots, \mu_i^6 = \mu_i(cba|abc)$.

Observe that

$$\mu_i \in TS^* \quad \Rightarrow \quad \begin{cases} \mu_i^1 + \mu_i^2 > \mu_i^3 + \mu_i^4 \\ \mu_i^1 + \mu_i^2 > \mu_i^5 + \mu_i^6 \\ \mu_i^1 + \mu_i^3 > \mu_i^2 + \mu_i^5 \\ \mu_i^1 + \mu_i^3 > \mu_i^4 + \mu_i^6 \end{cases}. \tag{5}$$

◊

[3]Here abc denotes the ordering "a is preferred to b is preferred to $c, \dots$."

We note that other notions of positive correlation in this model can be proposed. For instance, we can define a dual of TS correlation where a voter believes that her k worst-ranked alternatives are most likely to be the k worst-ranked alternatives of the other voter. Other definitions can be built using classical concepts in statistics such as Spearman's coefficient of rank correlation. We do not pursue these lines of research any further since TS correlation is plausible and offers rich and interesting possibilities.

5. Incentive compatibility with local robustness

In this section, we explore incentive-compatible SCFs that satisfy an additional *local robustness* property. The latter requires the SCF to remain incentive-compatible if the belief of each voter is slightly perturbed. Successful information revelation occurs in such SCFs even if the mechanism designer makes "small mistakes" in his assessment of voter beliefs.

Definition 11. A SCF f is locally robust OBIC (LOBIC) with respect to the belief system μ if there exists $\epsilon > 0$ such that f is OBIC with respect to all μ' such that $\mu' \in B_\epsilon(\mu)$.[4]

Definition 12. A SCF f is TS-locally robust OBIC (TS-LOBIC) with respect to the belief system μ if

(i) $\mu_i \in \mathrm{TS}^*$ for all i and

(ii) f is LOBIC with respect to μ.

Consider a belief system μ where μ_i is TS-correlated for each voter i. Then f is TS-LOBIC with respect to μ if f is OBIC with respect to every belief system in some sufficiently small neighborhood of μ. In fact, all the perturbed beliefs are also TS-correlated.

Example 2. Let $N = \{1, 2\}$ and $A = \{a, b, c\}$. Let f^1 be the scoring rule with score vector $(2, 1.5, 0)$[5] with tie-breaking in favor of agent 1. This SCF is described with voter 1 and 2's preference orderings represented by rows and columns, respectively:

	abc	acb	bac	bca	cab	cba
abc	a	a	a	b	a	b
acb	a	a	a	c	a	c
bac	b	a	b	b	a	b
bca	b	c	b	b	c	b
cab	a	c	a	c	c	c
cba	b	c	b	c	c	c

[4]The function $B_\epsilon(\mu_i)$ denotes the open ball of radius ϵ centered at μ_i.

[5]A score vector s is an m-tuple of real numbers $(s_1, \ldots, s_m)$, $s_1 \geq s_2 \geq \cdots \geq s_m$ and $s_1 > s_m$. The score assigned to alternative a by individual i in profile P is s_k if $r_k(P_i) = a$. The aggregate score of a in P is the sum of the individual scores. Let $W_s(P)$ be the set of alternatives whose scores in P are maximal. A SCF f is a scoring rule with respect to score vector s if it selects an element of W_s for every P.

We claim that f^1 is TS-LOBIC. It is easy to verify that truth-telling is weakly dominant for voter 1 of all types. In the case of voter 2, the following inequalities for μ_2 are necessary and sufficient so that f^1 will be OBIC with respect to the belief pair $(\cdot, \mu_2)$: $\mu_2(cab|abc) > \mu_2(bac|abc)$, $\mu_2(bac|acb) > \mu_2(cab|acb)$, $\mu_2(cba|bac) > \mu_2(abc|bac)$, $\mu_2(abc|bca) > \mu_2(cba|bca)$, $\mu_2(bca|cab) > \mu_2(acb|cab)$, and $\mu_2(acb|cba) > \mu_2(bca|cba)$.

It can be easily verified that these inequalities are easily satisfied by a belief μ_2 satisfying TS correlation. Hence f^1 is TS-LOBIC. ♢

There are SCFs that are not TS-LOBIC as the next example demonstrates.

Example 3. Let $A = \{a, b, c\}$ and $N = \{1, 2\}$. Consider the SCF f^2 as shown in

	abc	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*b*	*a*	*b*
acb	*a*	*a*	*a*	*c*	*a*	*c*
bac	*c*	*a*	*b*	*b*	*a*	*b*
bca	*b*	*b*	*b*	*b*	*c*	*b*
cab	*a*	*b*	*a*	*c*	*c*	*c*
cba	*c*	*c*	*b*	*c*	*c*	*c*

Consider voter 2, the column voter, with preference *abc* who considers reporting *acb* instead of her true preference. Then she will lose by misreporting if voter 1 has preference *cab* by getting c instead of b; she will gain if voter 1's preference is *bac* by getting a instead of c. Suppose f^2 is OBIC with respect to some belief pair (μ_1, μ_2). By virtue of the robustness criterion, we can assume $\mu_2(bac|abc), \mu_2(cab|abc) > 0$. Now pick a utility representation u of *abc* such that $u(a) = 1$, $u(b) = \alpha$, and $u(c) = 0$, where $0 < \alpha < 1$. The difference in expected utility between truth-telling and lying is $\Delta = (1 - \alpha)\mu_2(cab|abc) - \mu_2(bac|abc)$. Since $\mu_2(cab|abc), \mu_2(bac|abc) > 0$, Δ can be made strictly less than 0 by choosing α sufficiently close to 1. This contradicts the assumption that f^2 is OBIC with respect to (μ_1, μ_2). ♢

The example above suggests a necessary condition that a TS-LOBIC SCF must satisfy. Since all conditional probabilities can be assumed to be nonzero by local robustness, expected utility for a type cannot be maximized by truth-telling if misrepresentation weakly dominates truth-telling. However, in addition, the gain from truth-telling cannot be "washed out" relative to the gain from misrepresentation by picking a different utility representation. We formalize this notion below.

Definition 13. A SCF $f : \mathbb{P}^n \to A$ satisfies ordinal nondomination (OND) if for all i, for all P_i, P'_i, and P_{-i} such that $f(P'_i, P_{-i})\, P_i\, f(P_i, P_{-i})$, there exists P'_{-i} such that one of the following statements holds:

(i) Either $f(P_i, P'_{-i}) = f(P'_i, P_{-i})$ or $f(P_i, P'_{-i})\, P_i\, f(P'_i, P_{-i})$.

(ii) Either $f(P_i, P_{-i}) = f(P'_i, P'_{-i})$ or $f(P_i, P_{-i})\, P_i\, f(P'_i, P'_{-i})$.

Consider the SCF f^2 in Example 3. Observe that voter 2 strictly prefers $f^2(bac, acb) = a$ to $c = f^2(bac, abc)$ under abc. According to OND, there must exist another preference ordering for voter 1 where voter 2 does strictly better by reporting abc rather than acb. The only candidate for such an ordering for voter 1 is cab. However, $f^2(cab, acb)$ is strictly preferred to $f^2(bac, abc)$, violating part (i) of the OND condition (Definition 13). The example clearly shows how OBIC will now fail: by choosing a suitable utility representation, the gain from telling the truth when voter 1's report is cab can be made arbitrarily small relative to the gain from lying when voter 1's report is bac. The necessity of part (ii) of the OND condition can be demonstrated similarly.

The OND condition is weak, as the following example suggests.

Example 4. Let $A = \{a, b, c\}$ and $N = \{1, 2, 3\}$. Let f^p be the *plurality rule* with voter 1 as the tie-breaker. In other words, the outcome at any profile is the alternative that is ranked first by the largest number of voters. In case of a tie, voter 1's best alternative is selected.

Observe that voter 1 has a dominant strategy to be truthful. Suppose voter 2's type is abc. She can profitably deviate from truth-telling only when voter 1's and voter 3's best alternatives are c and b, respectively. Then voter 2 obtains c by telling the truth and obtains b by deviating to a type where b is the best alternative. Alternatively, if voter 1's and voter 3's best alternatives are c and a, respectively, then voter 2 obtains a by truth-telling and obtains c when deviating to a type where b is the best alternative. It is easy to verify that these profiles and outcomes satisfy the requirements of OND. An identical argument holds for voter 3. ◊

More examples of SCFs satisfying OND will be provided later in the section. We now show that OND is necessary and almost sufficient for the TS-LOBIC property to hold.

Theorem 1. *If a SCF is TS-LOBIC, it satisfies OND. If a SCF satisfies unanimity and OND, it is TS-LOBIC.*

Proof. We first prove that if a SCF is TS-LOBIC it satisfies OND.

Let f be a TS-LOBIC SCF. Then, for all i, there exists $\mu_i \in TS^*$ such that for all P_i, P'_i, and u representing P_i, we have

$$\sum_{P_{-i} \in \mathbb{P}^{n-1}} \mu_i(P_{-i}|P_i)\big[u(f(P_i, P_{-i}), P_i) - u(f(P'_i, P_{-i}), P_i)\big] \geq 0. \tag{6}$$

Moreover, inequality (6) holds for all μ'_i in a neighborhood of μ_i. Hence we can assume without loss of generality that $\mu_i(P_{-i}|P_i) > 0$ in inequality (6). Suppose that there exists P_i, P'_i, and P_{-i} such that $f(P'_i, P_{-i})\ P_i\ f(P_i, P_{-i})$, i.e., $u(f(P'_i, P_{-i})) > u(f(P_i, P_{-i}))$ for all u representing P_i. Since $\mu_i(P_{-i}|P_i) > 0$, there must exist P'_{-i} such that $u(f(P_i, P'_{-i})) > u(f(P'_i, P'_{-i}))$, i.e., $f(P_i, P'_{-i})\ P_i\ f(P'_i, P'_{-i})$, so that inequality (6) holds. Let L denote the set of all such P'_{-i}'s.

Now suppose $f(P'_i, P_{-i})\ P_i\ f(P_i, P'_{-i})$ holds for all $P'_{-i} \in L$. Then we can choose a utility representation $\hat{u}$ of P_i such that $\hat{u}(f(P'_i, P_{-i}))$ is arbitrarily close to 1, and

$\hat{u}(f(P_i, P'_{-i}))$, $\hat{u}(f(P_i, P_{-i}))$, and $\hat{u}(f(P'_i, P'_{-i}))$ are all arbitrarily close to 0. Then the left-hand side of (6) for the utility function $\hat{u}$ can be made arbitrarily close to $-\mu_i(P_{-i}|P_i) < 0$, violating inequality (6).

Now suppose $f(P'_i, P'_{-i})\ P_i\ f(P_i, P_{-i})$ holds. Then we can choose a utility representation $\tilde{u}$ of P_i such that $\tilde{u}(f(P'_i, P_{-i}))$, $\tilde{u}(f(P_i, P'_{-i}))$, and $\tilde{u}(f(P'_i, P'_{-i}))$ are arbitrarily close to 1, and $\tilde{u}(f(P_i, P_{-i}))$ is arbitrarily close to 0. Once again the left-hand side of (6) for the utility function $\tilde{u}$ can be made arbitrarily close to $-\mu_i(P_{-i}|P_i) < 0$, violating inequality (6).

Hence f satisfies OND.

For the second part of the proof, suppose that f satisfies unanimity and OND. We will construct an open set of beliefs for each voter satisfying TS correlation and such that f is OBIC with respect to all beliefs in this set.

Pick a voter i and an ordering P_i. For any $k \in \{1, \dots, m\}$, define $A_k^f(P_i) = \{P_{-i}|f(P_i, P_{-i}) = r_k(P_i)\}$. Thus $A_k^f(P_i)$ is the set of preferences for voters other than i that gives, under f, the kth-ranked alternative of voter i as the outcome. Define by P^0_{-i} the preference profile for voters other than i, where each voter $j \neq i$ has the preference ordering P_i. Since f satisfies unanimity, $P^0_{-i} \in A_1^f(P_i)$.

Let $\mathcal{C}_i^*$ denote the set of probability distributions over $\mathbb{P}^n$ such that for each $\mu_i^* \in \mathcal{C}_i^*$ and P_i, the conditional distribution $\mu_i^*(\cdot|P_i)$ satisfies the following properties:

(i) We have $\mu_i^*(P_{-i}|P_i) > 0$ for all P_{-i}.

(ii) We have $\mu_i^*(P^0_{-i}|P_i) > \sum_{P_{-i} \neq P^0_{-i}} \mu_i^*(P_{-i}|P_i)$.

(iii) For all $P_{-i} \neq P^0_{-i}$, $\mu_i^*(P_{-i}|P_i) > \sum_{P'_{-i} \in \bigcup_{r=k+1}^{r=m} A_r^f(P_i)} \mu_i^*(P'_{-i}|P_i)$, where $P_{-i} \in A_k^f(P_i)$.

Suppose $f(P_i, P_{-i})$ is the kth-ranked alternative in P_i. Then the conditional probability $\mu_i^*(P_{-i}|P_i)$ exceeds the sum of the conditional probabilities of realizing a profile P'_{-i}, where the outcome $f(P_i, P'_{-i})$ is strictly worse than the kth-ranked alternative in P_i. In addition, the conditional probability of realizing the profile P^0_{-i} exceeds the sum of the conditional probabilities of realizing any other ordering. There are clearly no difficulties in defining $\mathcal{C}_i^*$. Moreover, since the restrictions on the conditional probabilities are described by strict inequalities, it follows that $\mathcal{C}_i^*$ is an open set in the unit simplex of dimension $m!^n - 1$.

We claim that $\mathcal{C}_i^* \subset TS^*$. This is easily verified by noting that the term $\mu_i^*(P^0_{-i}|P_i)$ appears in the left-hand side of every inequality in the system of inequalities (5) that define TS correlation while it does not appear on the right-hand side of any one of them. To complete the proof, we will show that f is OBIC with respect to all beliefs $(\mu_1^*, \dots, \mu_n^*)$, where $\mu_i^* \in \mathcal{C}_i^*$, $i \in N$.

Pick an arbitrary voter i, orderings P_i, P'_i, and a utility function u representing P_i. Let $G = \{P_{-i}|f(P'_i, P_{-i})\ P_i\ f(P_i, P_{-i})\}$ and $L = \{P_{-i}|f(P_i, P_{-i})\ P_i\ f(P'_i, P_{-i})\}$. Pick an arbitrary $\mu_i^* \in \mathcal{C}_i^*$. For OBIC to be satisfied with respect to μ_i^*, we must have

$$\sum_{P_{-i} \in L} \mu_i^*(P_{-i}|P_i)\beta(P_{-i}) - \sum_{P_{-i} \in G} \mu_i^*(P_{-i}|P_i)\gamma(P_{-i}) \geq 0, \tag{7}$$

where

$$\beta(P_{-i}) = \big[u(f(P_i, P_{-i})) - u(f(P'_i, P_{-i}))\big] \quad \text{and} \quad \gamma(P_{-i}) = \big[u(f(P'_i, P_{-i})) - u(f(P_i, P_{-i}))\big].$$

If $G = \varnothing$, inequality (7) is clearly satisfied. Suppose, therefore, that $G \neq \varnothing$. We claim that for all $P_{-i} \in G$, there exists $P'_{-i} \in L$ satisfying

(i) $\beta(P'_{-i}) > \gamma(P_{-i})$

(ii) $\mu_i^*(P'_{-i}|P_i) > \sum_{\{\tilde{P}_{-i}|f(P_i,P'_{-i})P_i f(P_i,\tilde{P}_{-i})\}} \mu_i^*(\tilde{P}_{-i}|P_i)$,

where (i) follows from the assumption that f satisfies OND and (ii) follows from properties (ii) and (iii) in the specification of μ_i^*.

Let $\sigma : G \to L$ be a map such that for all $P_{-i} \in G$, $\sigma(P_{-i})$ is the $P'_{-i} \in L$ satisfying conditions (i) and (ii) above. Let P'_{-i} be an arbitrary element in the range of σ and let $Q(P'_{-i}) = \{P_{-i}|\sigma(P_{-i}) = P'_{-i}\}$. A critical observation is that for all $P_{-i} \in Q$, OND implies $f(P_i, P'_{-i})\, P_i\, f(P_i, P_{-i})$, i.e., $Q(P'_{-i}) \subset \{\tilde{P}_{-i}|f(P_i, P'_{-i})\, P_i\, f(P_i, \tilde{P}_{-i})\}$. Hence condition (ii) above implies $\mu_i^*(P'_{-i}|P_i) > \sum_{P_{-i}\in Q(P'_{-i})} \mu_i^*(P_{-i}|P_i)$. Moreover, using condition (i) above, we have $\mu_i^*(P'_{-i}|P_i)\beta(P'_{-i}) > \sum_{P_{-i}\in Q(P'_{-i})} \mu_i^*(P_{-i}|P_i)\gamma(P_{-i})$. Now summing over all P'_{-i} in L and noting that OND implies that $G \subset \bigcup_{P'_{-i}\in L} Q(P'_{-i})$, we obtain inequality (7). □

The proof of the first part of Theorem 1 shows that OND is a necessary condition for locally robust OBIC with respect to *any* subset of prior beliefs. It applies equally to beliefs that are restricted to lie in the set of TS- or K-correlated beliefs or in the set of independent beliefs, for that matter, to some subset of negative correlated beliefs, however defined. It is an inescapable consequence of local robustness. The sufficiency part of Theorem 1 is that TS correlation leads to the most permissive result for incentive compatibility subject to the very mild requirement that the SCFs under consideration satisfy unanimity.

We now show that if SCFs satisfy two additional restrictions, they are TS-LOBIC with respect to beliefs that are arbitrarily close to the *uniform prior*. These additional restrictions were introduced in Majumdar and Sen (2004).

Definition 14. Let $\sigma : A \to A$ be a permutation of A. Let P^σ denote the profile $(P_1^\sigma, \dots, P_n^\sigma)$, where for all i and $a, b \in A$,

$$a\, P_i\, b \quad \Rightarrow \quad \sigma(a)\, P_i^\sigma\, \sigma(b).$$

The SCF f satisfies neutrality if, for all profiles P and for all permutation functions σ, we have

$$f(P^\sigma) = \sigma(f(P)).$$

Neutrality is a standard axiom for social choice functions that ensures that alternatives are treated symmetrically.

Let P_i be an ordering and let $a \in A$. We say that P'_i represents an *elementary a-improvement of P_i* if the following statements hold:

- For all $x, y \in A \setminus \{a\}$, $[x\, P_i\, y] \Leftrightarrow [x\, P_i'\, y]$.
- We have $[a = r_k(P_i)] \Rightarrow [a = r_{k-1}(P_i')]$ if $k > 1$.
- We have $[a = r_1(P_i)] \Rightarrow [a = r_1(P_i')]$.

Definition 15. The SCF f satisfies elementary monotonicity if, for all i, P_i, P_i', and P_{-i} such that P_i' represents an elementary a-improvement of P_i,

$$[f(P_i, P_{-i}) = a] \quad \Rightarrow \quad [f(P_i', P_{-i}) = a].$$

Elementary monotonicity requires that if an alternative a is the outcome at a particular profile, then it is also the outcome at the profile where a single voter makes a single upward local switch of a. Once again, this is a weak requirement and is discussed at greater length in Majumdar and Sen (2004).

Finally, let $\bar{\mu}$ denote the uniform prior system of beliefs, i.e., $\bar{\mu}_i(P_{-i}|P_i) = 1/m!^{n-1}$ for all P_i.

According to our next result, any neutral SCF satisfying elementary monotonicity is TS-LOBIC with respect to a TS-correlated prior that can be chosen arbitrarily close to the uniform prior.

Theorem 2. *Let f be a neutral SCF satisfying elementary monotonicity and unanimity. Then there exists $\epsilon > 0$ and a belief system $\mu \in B_\epsilon(\bar{\mu}) \cap TS^\star$ such that f is TS-LOBIC with respect to μ.*

Proof. Pick a voter i and an ordering P_i. As in the proof of Theorem 1, let P^0_{-i} be the preference profile for voters other than i where each voter $j \neq i$ has the preference ordering P_i. Let $K \subset \{1, \dots, m\}$ be such that (i) if $k \in K$ and $k \geq 2$, then $A^f_k(P_i) = \{P_{-i}|f(P_i, P_{-i}) = r_k(P_i)\} \neq \varnothing$, and (ii) if $k = 1$ and $k \in K$, then $A^f_1(P_i) = \{P_{-i}|f(P_i, P_{-i}) = r_1(P_i)\} \setminus P^0_{-i} \neq \varnothing$. In other words, if $k \in K$ and $k \geq 2$, there exists an $n-1$ voter profile P_{-i} such that $f(P_i, P_{-i})$ is the kth-ranked alternative in P_i. The index 1 is included in K if there exists a profile P_{-i} *distinct from* P^0_{-i} such that $f(P_i, P_{-i})$ is the first-ranked alternative in P_i.

Without loss of generality, let $K = \{k_1, \dots, k_L\}$ such that $k_1 < k_2 \cdots < k_L$. For each $l = 1, \dots, L$, pick $\delta_l > 0$ satisfying $1/m!^{n-1} > \delta_0 = \sum_{l=1}^L \delta_{k_l}$ and $\delta_1 < \delta_2 < \cdots < \delta_L$.

Define the conditional beliefs $\mu_i^*(\cdot|P_i)$ as

$$\mu_i^*(P_{-i}|P_i) = \begin{cases} \frac{1}{m!^{n-1}} + \delta_0 & \text{if } P_{-i} = P^0_{-i} \\ \frac{1}{m!^{n-1}} - \frac{\delta_l}{|A^f_{k_l}(P_i)|} & \text{if } P_{-i} \in A^f_{k_l}(P_i) \text{ for some } l = 1, \dots, L. \end{cases}$$

It is clear that by choosing the δ's sufficiently small, we can generate a belief system μ^* arbitrarily close to $\bar{\mu}$.

We claim that $\mu_i^*(\cdot|P_i)$ is TS-correlated for all P_i. Consider an arbitrary inequality in (5), for instance, for some $k = 1, \dots, m$ and a set $|B| = k$ with $B \neq B_{k_l}(P_i)$. Note that $|\{P_{-i} : B_k(P_j) = B\ \forall j \neq i\}| = |\{P_{-i} : B_k(P_j) = B_k(P_i)\ \forall j \neq i\}|$. Therefore, the number of terms in the left- and right-hand sides of every inequality in (5) has the same number of terms. Each of these terms contains $1/m!^{n-1}$, which can be canceled with each other. Now consider the inequality after canceling these terms. The term δ_0 appears on the left-hand side but not on the right-hand side. Therefore, a lower bound for the left-hand side is when all the terms other than P^0_{-i} belong to $A^f_{k_L}(P_i)$. Hence a lower bound for the left-hand side is $\delta_0 - \delta_{k_L} > 0$ by construction. On the right-hand side, all terms are strictly negative (in fact, the maximum value it can attain is $-\delta_1$). Clearly the left-hand side is strictly greater than the right-hand side so that $\mu^*_i(\cdot|P_i)$ is TS-correlated. Note that there exists a neighborhood of μ^*_i where all beliefs are TS-correlated.

We now show that for any voter i and type P_i, f satisfies incentive compatibility with respect to all priors chosen in a suitable neighborhood of $\mu^*_i(\cdot|P_i)$. Pick an arbitrary P'_i and an integer $k \in \{1,\dots,m-1\}$. Let $\sigma: A \to A$ be a permutation such that $r_l(P_i) = r_{\sigma(l)}(P'_i)$ for all $l = 1,\dots,m$. Let $\mathbb{P}_{-i} = \{P_{-i}: f(P) \in B_k(P_i)\}$ and $\mathbb{P}^\sigma_{-i} = \{P^\sigma_{-i}: f(P) \in \sigma^{-1}(B_k(P_i))\}$[6]. For each $P_{-i} \in \mathbb{P}_{-i}$, let $s(P_{-i}) \in \{k_1,\dots,k_L\}$ be such that $f(P_i, P_{-i}) = r_{s(P_{-i})}(P_i)$. Let

$$\Delta(\mathbb{P}_{-i}) = \sum_{P_{-i}\in\mathbb{P}_{-i}} \frac{\delta_{s(P_{-i})}}{|A^f_{s(P_{-i})}(P_i)|}.$$

Similarly, let

$$\Delta(\mathbb{P}^\sigma_{-i}) = \sum_{P_{-i}\in\mathbb{P}^\sigma_{-i}} \frac{\delta_{s(P_{-i})}}{|A^f_{s(P_{-i})}(P_i)|}.$$

Therefore, we have

$$\mu^*_i\big(\{P_{-i}|f(P_i, P_{-i}) \in B_k(P_i)\}|P_i\big) = \frac{1}{m!^{n-1}}|\mathbb{P}_{-i}| + \delta_0 - \Delta(\mathbb{P}_{-i}) \tag{8}$$

$$\mu^*_i\big(\{P_{-i}|f(P^\sigma_i, P_{-i}) \in B_k(P_i)\}|P_i\big) = \frac{1}{m!^{n-1}}|\mathbb{P}^\sigma{}_{-i}| + \delta_0\mathbb{I}_{\{r_1(P_i)\in\sigma^{-1}(B_k(P_i))\}} - \Delta(\mathbb{P}^\sigma_{-i}). \tag{9}$$

Here $\mathbb{I}$ is the indicator function, i.e., $\mathbb{I}_{\{r_1(P_i)\in\sigma^{-1}(B_k(P_i))\}} = 1$ if $r_1(P_i) \in \sigma^{-1}(B_k(P_i))$ and 0 otherwise.

Majumdar and Sen (2004) prove that if f satisfies elementary monotonicity and neutrality, then

(i) $|A^f_k(P_i)| \geq |A^f_t(P_i)|$ whenever $k < t$

(ii) $|\mathbb{P}_{-i}| \geq |\mathbb{P}^\sigma_{-i}|$.

Fix an arbitrary $k \in \{1,\dots,m-1\}$. We wish to compare the right-hand sides of (8) and (9). Consider the following cases.

Case I: $m!^{n-1} > |\mathbb{P}_{-i}| = |\mathbb{P}^\sigma_{-i}|$. Let $T^0 = \{P_{-i}: P_{-i} \in \mathbb{P}_{-i} \setminus \mathbb{P}^\sigma_{-i}\}$ and $T^1 = \{P_{-i}: P_{-i} \in \mathbb{P}^\sigma_{-i} \setminus \mathbb{P}_{-i}\}$. In view of item (ii) above, $|T^0| \geq |T^1| \neq 0$. Pick an arbitrary $P_{-i} \in T^1$. Since $P_{-i} \notin \mathbb{P}_{-i}$, it follows that $s(P_{-i}) > k$. On the other hand, for all $P_{-i} \in T^0$, we have

[6] We have $\sigma^{-1}(B_k(P_i) = \{a \in A: \sigma(a) \in B_k(P_i)\}$.

$s(P_{-i}) \le k$. Note that for all P_{-i}, P'_{-i} if $s(P_i) > s(P'_{-i})$, then $\delta_{s(P_{-i})} > \delta_{s(P'_{-i})}$ by construction and $|A^f_{s(P_{-i})}| \le |A^f_{s(P'_{-i})}|$, so that

$$\frac{\delta_{s(P_{-i})}}{|A^f_{s(P_{-i})}|} > \frac{\delta_{s(P'_{-i})}}{|A^f_{s(P'_{-i})}|}.$$

Therefore,

$$\Delta(\mathbb{P}^{\sigma}_{-i}) - \Delta(\mathbb{P}_{-i}) = \sum_{P_{-i}\in T^1} \frac{\delta_{s(P_{-i})}}{|A^f_{s(P_{-i})}|} - \sum_{P_{-i}\in T^0} \frac{\delta_{s(P_{-i})}}{|A^f_{s(P_{-i})}|} > 0.$$

Hence,

$$\begin{aligned} &\mu^*_i(\{P_{-i}|f(P_i, P_{-i}) \in B_k(P_i)\}|P_i) - \mu^*_i(\{P_{-i}|f(P^{\sigma}_i, P_{-i}) \in B_k(P_i)\}|P_i) \\ &\qquad \ge \Delta(\mathbb{P}^{\sigma}_{-i}) - \Delta(\mathbb{P}_{-i}) \\ &\qquad > 0. \end{aligned}$$

Case II: $m!^{n-1} > |\mathbb{P}_{-i}| > |\mathbb{P}^{\sigma}_{-i}|$. We claim that $\Delta(\mathbb{P}^{\sigma}_{-i}) - \Delta(\mathbb{P}_{-i}) < \delta_0$. Note that $\Delta(\mathbb{P}^{\sigma}_{-i}) - \Delta(\mathbb{P}_{-i}) = \sum_{l=1}^{L} \delta_{k_l} = \delta_0$ only if either $\mathbb{P}_{-i}$ or $\mathbb{P}^{\sigma}_{-i}$ is the set of all $n-1$ voter profiles, i.e., either $|\mathbb{P}_{-i}| = m!^{n-1}$ or $|\mathbb{P}^{\sigma}_{-i}| = m!^{n-1}$. However, both cases contradict underlying assumptions for Case II to hold. Consequently, $\Delta(\mathbb{P}^{\sigma}_{-i}) - \Delta(\mathbb{P}_{-i}) < \delta_0$. Thus,

$$\begin{aligned} &\mu^*_i(\{P_{-i}|f(P_i, P_{-i}) \in B_k(P_i)\}|P_i) - \mu^*_i(\{P_{-i}|f(P^{\sigma}_i, P_{-i}) \in B_k(P_i)\}|P_i) \\ &\qquad \ge \frac{1}{m!^{n-1}}(|\mathbb{P}_{-i}| - |\mathbb{P}^{\sigma}_{-i}|) + \Delta(\mathbb{P}^{\sigma}_{-i}) - \Delta(\mathbb{P}_{-i}) \\ &\qquad > \frac{1}{m!^{n-1}} - \delta_0 \\ &\qquad > 0. \end{aligned}$$

Observe that if either Case I or Case II holds, then incentive-compatibility conditions hold with *strict* inequality with respect to μ^*_i. Therefore, they will continue to hold in a neighborhood of μ^*_i. The only remaining case (in view of item (ii)) is when $|\mathbb{P}_{-i}| = m!^{n-1}$. In this case, $\mu^*_i(\{P_{-i}|f(P_i, P_{-i}) \in B_k(P_i)\}|P_i) = 1$ so that incentive-compatibility conditions will continue to hold for all beliefs.

We have established that f is TS-LOBIC at μ^* as required. □

Observation 2. Majumdar and Sen (2004) show that a large class of "well behaved" SCFs satisfy neutrality and elementary monotonicity. These include all *scoring rules* with neutral tie-breaking rules (for instance, by picking the maximal element among all alternatives with the highest score according to a fixed voter's ordering). Note that these SCFs must also satisfy the OND condition.

5.1 *Common priors*

A feature of the proofs of Theorems 1 and 2 is that the conditional beliefs constructed may not be compatible with a common prior for all voters. Unfortunately, this drawback cannot be easily remedied as the example below demonstrates.

Example 5. Let $N = \{1, 2\}$ and $A = \{a, b, c\}$. Consider the SCF defined as

	abc	*acb*	*bac*	*bca*	*cab*	*cba*
abc	*a*	*a*	*a*	*b*	*a*	*a*
acb	*a*	*a*	*b*	*a*	*a*	*a*
bac	*b*	*a*	*b*	*b*	*b*	*b*
bca	*a*	*b*	*b*	*b*	*b*	*b*
cab	*a*	*a*	*b*	*b*	*c*	*c*
cba	*a*	*a*	*b*	*b*	*c*	*c*

The SCF satisfies OND. We claim that there does not exist any *nonindependent common* prior with respect to which the SCF is strictly OBIC (i.e., the OBIC inequalities are satisfied as strict inequalities so that the SCF is LOBIC). Suppose such a common prior μ exists.[7]

Strict OBIC implies

$$\begin{aligned} \mu(abc, bac) &> \mu(abc, bca) \\ \mu(acb, bca) &> \mu(acb, bac). \end{aligned} \tag{10}$$

Note that (10) is obtained from the incentive requirements of the row voter when she is of type *abc* and *acb*, respectively. Similar arguments for the column voter of type *bac* and *bca* yield

$$\begin{aligned} \mu(acb, bac) &> \mu(abc, bac) \\ \mu(abc, bca) &> \mu(acb, bca). \end{aligned} \tag{11}$$

Combining (10) and (11), we get,

$$\begin{aligned} &\mu(acb, bca) > \mu(acb, bac) > \mu(abc, bac) > \mu(abc, bca) \\ &\qquad \Rightarrow \quad \mu(acb, bca) > \mu(abc, bca). \end{aligned} \tag{12}$$

Inequality (12) contradicts inequality (11). ◊

Example 5 demonstrates that OND is not sufficient for a SCF to be strictly OBIC if we insist on common priors. Moreover, identifying the precise necessary and sufficient condition for this purpose appears to be cumbersome.

[7]Note that μ here refers to the common *joint* probability distribution over $\mathbb{P}^n$, while the earlier definitions of OBIC were in terms of conditional distributions.

We claim, however, that the problem is not as serious as it appears. This is because voter priors can be made *arbitrarily close to each other* (without being identical to each other) in the proofs of the two theorems. The following version of the second part of Theorem 1 holds: Let the SCF f satisfy OND and unanimity, and let $\epsilon > 0$. Then (i) there exists a belief system $(\mu_1, \dots, \mu_n)$ with the distance (in the sup norm, for instance) between (μ_i, μ_j) being less than ϵ for all i, j, and (ii) f is strictly OBIC with respect to $(\mu_1, \dots, \mu_n)$, i.e., f is TS-LOBIC. A similar variant for Theorem 2 holds; we omit its statement.

5.2 *Discussion and interpretation*

Theorems 1 and 2 stand in sharp contrast to results in Majumdar and Sen (2004) for the independent beliefs case. Their main result says the that if beliefs are independent, there exists a set that is generic such that OBIC with respect to *any* beliefs in this set implies that truth-telling must be a dominant strategy. It may be possible to find nondictatorial SCFs for very special beliefs such as the uniform prior. However, if beliefs are picked from a slightly perturbed set, the class of incentive-compatible SCFs immediately shrinks to the dictatorial class. In contrast, if beliefs are TS- or K-correlated, it is possible to find SCFs that are incentive-compatible with respect to *all* beliefs in some neighborhood of beliefs.

Theorem 1 provides a very general answer to the question of what SCFs are TS-LOBIC. The proof of the second part of the theorem constructs a class of conditional beliefs for each voter with respect to which a SCF satisfying OND and unanimity is TS-LOBIC. These beliefs depend on the SCF and are constructed as follows: a voter i with type i puts high weight on all other voters' types being P_i (i.e., coinciding with her own); in addition, she puts higher weight on voters types being P_{-i} instead of P'_{-i} if $f(P_i, P_{-i})$ is strictly better than $f(P_i, P'_{-i})$ according to P_i. In general, one may say that voters are optimistic in their beliefs in the sense that they assign "much higher" probabilities to more favorable events. In this case, these events are realizations of the other voter's types that lead to better outcomes through the SCF. Loosely speaking, this is in accordance with the general intuition regarding why positive correlation may ameliorate the problems of designing incentive-compatible SCFs.

Theorem 2 shows that SCFs satisfying neutrality and elementary monotonicity are TS-LOBIC. In addition, the neighborhood of beliefs with respect to which the SCFs are LOBIC, can be chosen to be arbitrarily close to the uniform prior, i.e., at the center of the simplex. In contrast, the neighborhood of beliefs constructed in the proof of Theorem 1 was near the vertex of the simplex where an agent believes that all other agents have the same type that she does. In the case of Theorem 2 as earlier, beliefs are constructed assuming that voters are optimistic about their beliefs about the types of other voters. However, it suffices for their optimism to be "small."

6. Conclusion

In this paper, we have explored the problem of mechanism design in a voting environment with an arbitrary number of voters where a voter's belief about the type of the other voters is positively correlated with her own type. Our general conclusion is that the prospects for constructing incentive-compatible social choice functions in this environment are significantly improved relative to the independent case. In this respect, our results parallel those in environments with transfers and quasi-linear utility such as Crémer and McLean (1988). However, the reasons behind the enhanced possibilities in the voting environment are quite different from the quasi-linear context.

In future research, we hope to extend our analysis to other notions of correlation and to understand better the relationship between the structure of beliefs and incentive-compatible social choice functions.

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