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Monotonic redistribution of performance-based allocations: A case for proportional taxation

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Within a simple setup, we show that proportional taxation is implied by three properties: efficiency, symmetry, and monotonicity. Efficiency: redistribution has no cost. Symmetry: members of the society with the same performance obtain the same reward after redistribution. Monotonicity: whenever both the performance of a certain member of the society as well as the overall performance of the society do not decrease, then this member's reward after redistribution does not decrease.

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> The moment you abandon the cardinal principle of exacting from all individuals the same proportion of their income or of their property, you are at sea without rudder or compass, and there is no amount of injustice and folly you may not commit.
> —McCulloch (1975, p. 174)

1. Introduction

In 1845, McCulloch, author of the most extensive and systematic treatment of public finance in the classical literature, made the above case for proportional taxation. Later on, notable others joined him, for example, Mill (1848, Chapter 2), Hayek (1960, Chapter 20), Friedman (1962, Chapter X), and more recently Hall and Rabushka (1985) and Hall (1996).[1]

In this paper, we make an axiomatic case for proportional taxation[2] via monotonic redistribution of performance-based allocations.[3] Particularly, we consider a society in which its members first are rewarded based on their individual contributions to the society's wealth (individual performance). Modern societies, however, base the allocation

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[1] For a general treatment of taxation, see Musgrave and Musgrave (1976) or Stiglitz (2000), for example.

[2] Fleubaey and Maniquet (2011, Chapters 10 and 11), for example, provide a survey of axiomatic foundations of taxation.

[3] Lambert (2002) provides an overview on the formal treatment of redistribution.

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of wealth among their members not only on individual performance but also on egalitarian or solidarity principles. This leads to the question of how individual contributions should be redistributed within a society.

We consider three properties of redistribution rules: efficiency, symmetry, and monotonicity. For the sake of simplicity, we assume that redistribution has no cost, i.e., redistribution is efficient. Moreover, members of the society with the same performance obtain the same reward after redistribution, i.e., redistribution is symmetric. Our third property—monotonicity—requires that whenever both the performance of a certain member of the society as well as the overall performance of the society do not decrease, then this member's reward after redistribution should not decrease.

For societies comprising more than two members, it turns out that the redistribution rules satisfying these properties are of a particularly simple form. First, the individual performance-based allocations are *taxed proportionally* at a certain rate, and second, the overall tax revenue is distributed equally within the society.

The next section gives a formal account of this result. Some remarks conclude the paper. The Appendix contains the proof of our result.

2. Monotonic redistribution rules and proportional taxation

In this paper, we consider a particularly simple model of a society. Its members are distinguished only by their individual contributions to the society's wealth in a certain period of time (for short, performance). In reality, the individual gross income may be viewed as an indicator for these individual contributions. Technically, we consider the n-member society $\mathbb{N}_n := \{1, \dots, n\}$, $n \in \mathbb{N}$, i.e., the society's members are represented by natural numbers. The individual performances are given by a vector $x \in \mathbb{R}^n$, i.e., we allow for negative performances. This can be justified by considering a period of time in which a member of the society or the entire society does not fare that well.

In real-life societies, members are not just rewarded according to their performances. A considerable amount of the society's wealth is redistributed via taxation and public spending. In our simple model, this is reflected by redistribution rules. A *redistribution rule* for an n-person society is a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$. For $x \in \mathbb{R}^n$ and $i \in \mathbb{N}_n$, $f_i(x)$ denotes the reward of member i of the society after redistribution.

In this framework, the properties of redistribution rules advanced in the Introduction can be formalized as follows.

Efficiency (E). For all $x \in \mathbb{R}^n$, we have $\sum_{\ell \in \mathbb{N}_n} f_\ell(x) = \sum_{\ell \in \mathbb{N}_n} x_\ell$.

The very idea of *re*distribution suggests that the sum of individual rewards after redistribution should not be greater than before. In addition, efficiency requires that redistribution comes at no cost. While in real life, redistribution is costly, efficiency might be acceptable in our simple and abstract framework.

Symmetry (S). For all $x \in \mathbb{R}^n$ and $i, j \in \mathbb{N}_n$ such that $x_i = x_j$, we have $f_i(x) = f_j(x)$.

In our simple framework, the society's members are fully described by their performance. Therefore, rewards after redistribution should be the same for members with the same productivity.

MONOTONICITY (M). For all $x, y \in \mathbb{R}^n$ and $i \in \mathbb{N}_n$ such that $\sum_{\ell \in \mathbb{N}_n} x_\ell \geq \sum_{\ell \in \mathbb{N}_n} y_\ell$ and $x_i \geq y_i$, we have $f_i(x) \geq f_i(y)$.

This property relates rewards of a member of the society when individual performances change in a monotonic way. Whenever both the society's overall performance *and* the performance of a particular member of the society do not decrease, then this member's reward after redistribution does not decrease. The intuition behind this property is as follows. Nondecreasing overall performance guarantees that no member of the society *necessarily has to be* rewarded less than before the change. For a member whose performance does not decrease at the same time, monotonicity ensures that she actually *is* not awarded less, which seems to be desirable.

The following theorem shows that redistribution rules that obey the above three properties entail proportional taxation for societies comprising more than two members, in a sense. Its proof is deferred to the Appendix. Of course, the one-member case is trivial.

THEOREM 1. *Let $n \neq 2$. A redistribution rule $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies efficiency (E), symmetry (S), and monotonicity (M) if and only if there exists some $\tau \in [0, 1]$ such that*

$$f_i(x) = (1 - \tau) \cdot x_i + \tau \cdot \sum_{\ell \in \mathbb{N}_n} \frac{x_\ell}{n} \quad \textit{for all } x \in \mathbb{R}^n \textit{ and } i \in \mathbb{N}_n. \tag{1}$$

It is immediate that the redistribution rules (1) satisfy all the properties, even for $n = 2$. Moreover, one might have the strong feeling that there cannot be other redistribution rules that do so. Yet, the theorem fails for $n = 2$. It is straightforward to show that the redistribution rule $f^\heartsuit : \mathbb{R}^2 \to \mathbb{R}^2$ given by $f^\heartsuit(x) = (x_1, x_2)$ if $x_1 > x_2$ and $f^\heartsuit(x) = (\frac{x_1+x_2}{2}, \frac{x_1+x_2}{2})$ if $x_1 \leq x_2$ for all $x \in \mathbb{R}^2$ satisfies the three properties.

To see how we can interpret formula (1) in terms of proportional taxation, note that performance is taxed at a rate of τ. This way, member $i \in \mathbb{N}_n$ keeps an amount of $(1 - \tau) \cdot x_i$ from his performance x_i. In addition, member i obtains one nth of the overall tax revenue of $\tau \cdot \sum_{\ell \in \mathbb{N}_n} x_\ell$. The latter can be interpreted as that all members of the society benefit equally from public spending, which fits our assumption that the society's members are equal up to performance.

Note that Theorem 1 leaves the tax rate indeterminate. Monotonic redistribution only requires proportional taxation at a certain rate. Hence, the tax rate has to be determined by other criteria, for example, optimality considerations.

3. CONCLUDING REMARKS

In this paper, we consider the possibly simplest meaningful framework to study redistribution within a society. We show that three intuitive and plausible properties of redistribution rules—efficiency, symmetry, and monotonicity—jointly entail proportional taxation of performance-based allocations.

In our simple framework, however, we cannot distinguish between income tax and consumption tax. Hence, Theorem 1 implies proportional *overall* taxation. In view of the regressive effect usually attributed to consumption tax, our findings do not rule out progressive taxation of income.

Appendix: Proof of Theorem 1

It is straightforward to show that the redistribution rules in formula (1) obey E, S, and M. Let $f:\mathbb{R}^n \to \mathbb{R}^n$ meet E, S, and M. For $n=1$, the second part of the claim drops from E.

Let now $n>2$. By M, there are mappings $F_i:\mathbb{R}^2 \to \mathbb{R}$, $i \in \mathbb{N}_n$ such that

$$f_i(x) = F_i\Bigl(x_i, \sum_{\ell\in\mathbb{N}_n} x_\ell\Bigr) \quad \text{for all } x \in \mathbb{R}^n \text{ and } i \in \mathbb{N}_n. \tag{A.1}$$

Next, we show that $F_i = F_j =: F$ for all $i, j \in \mathbb{N}_n$. Let $(a,c) \in \mathbb{R}^2$ and $i, j, k \in \mathbb{N}_n$, $i \neq j \neq k \neq i$. Let $y, z \in \mathbb{R}^n$ be given by

$$y_i = a \quad \text{and} \quad y_\ell = \frac{c-a}{n-1} \quad \text{for all } \ell \in \mathbb{N}_n \setminus \{i\} \tag{A.2}$$

and

$$z_j = a \quad \text{and} \quad z_\ell = \frac{c-a}{n-1} \quad \text{for all } \ell \in \mathbb{N}_n \setminus \{j\}. \tag{A.3}$$

We have

$$\begin{aligned} F_i(a,c) &\overset{\text{(A.1)}}{=} f_i(y) \overset{\text{(A.2)},\text{E},\text{S}}{=} c-(n-1)\cdot f_k(y) \\ &\overset{\text{(A.2)},\text{(A.3)},\text{M}}{=} c-(n-1)\cdot f_k(z) \overset{\text{(A.3)},\text{E},\text{S}}{=} f_j(z) \overset{\text{(A.1)}}{=} F_j(a,c). \end{aligned}$$

By E and M, the mapping F has the following properties.

Efficiency (E*). For all $a \in \mathbb{R}^n$, we have $\sum_{\ell\in\mathbb{N}_n} F(a_\ell, \sum_{k\in\mathbb{N}_n} a_k) = \sum_{\ell\in\mathbb{N}_n} a_\ell$.

Monotonicity (M*). For all $a, a', c, c' \in \mathbb{R}$ such that $a \geq a'$ and $c \geq c'$, we have $F(a,c) \geq F(a',c')$.

For $c \in \mathbb{R}$, let the mapping $\Phi^c:\mathbb{R}\to\mathbb{R}$ be given by

$$\Phi^c(a) := F(a,c) - F(0,c) \quad \text{for all } a \in \mathbb{R}. \tag{A.4}$$

For $a, b, c \in \mathbb{R}$, we have

$$\begin{aligned} &F(a,c) + F(b,c) + (n-2)\cdot F\Bigl(\frac{c-a-b}{n-2}, c\Bigr) \\ &\quad \overset{\text{E}^*}{=} F(a+b,c) + F(0,c) + (n-2)\cdot F\Bigl(\frac{c-a-b}{n-2}, c\Bigr). \end{aligned} \tag{A.5}$$

By (A.4) and (A.5), we obtain

$$\Phi^c(a) + \Phi^c(b) = \Phi^c(a+b) \quad \text{for all } a, b, c \in \mathbb{R}.$$

This already entails

$$\Phi^c(\rho \cdot a) = \rho \cdot \Phi^c(a) \quad \text{for all } a, c \in \mathbb{R} \text{ and } \rho \in \mathbb{Q}. \tag{A.6}$$

By M*, Φ^c is monotonic, i.e., $\Phi^c(a) \geq \Phi^c(b)$ for all $a, b \in \mathbb{R}$ such that $a \geq b$. Since $\mathbb{Q}$ is a dense subset of $\mathbb{R}$, (A.6) entails

$$\Phi^c(\rho \cdot a) = \rho \cdot \Phi^c(a) \quad \text{for all } a, c, \rho \in \mathbb{R}. \tag{A.7}$$

For all $c \in \mathbb{R}$, set

$$\alpha_c := F(1, c) - F(0, c). \tag{A.8}$$

By (A.4), (A.7), and (A.8), we have

$$F(a, c) = \alpha_c \cdot a + F(0, c) \quad \text{for all } a, c \in \mathbb{R}. \tag{A.9}$$

Moreover, we obtain

$$\frac{c}{n} \overset{\text{E}^*}{=} F\left(\frac{c}{n}, c\right) \overset{\text{(A.9)}}{=} \alpha_c \cdot \frac{c}{n} + F(0, c),$$

i.e., $F(0, c) = (1 - \alpha_c) \cdot \frac{c}{n}$ for all $c \in \mathbb{R}$ and, therefore,

$$F(a, c) = \alpha_c \cdot a + (1 - \alpha_c) \cdot \frac{c}{n} \quad \text{for all } a, c \in \mathbb{R}. \tag{A.10}$$

Now, we show that α_c does not depend on c. Let $c, c' \in \mathbb{R}$, $c > c'$. Suppose $\alpha_c \neq \alpha_{c'}$. By (A.10), we have

$$F(a, c) - F(a, c') = (\alpha_c - \alpha_{c'}) \cdot a + (1 - \alpha_c) \cdot \frac{c}{n} - (1 - \alpha_{c'}) \cdot \frac{c'}{n} \quad \text{for all } a \in \mathbb{R},$$

i.e., one can find some $a^* \in \mathbb{R}$ such that $F(a^*, c') > F(a^*, c)$, contradicting M*. Thus, $\alpha_c = \alpha_{c'} =: \alpha$ for all $c, c' \in \mathbb{R}$.

By (A.8) and M*, we have $\alpha \geq 0$. Moreover, we have

$$0 \overset{\text{(A.10)}}{=} F(0, 0) \overset{\text{M}^*}{\leq} F(0, 1) \overset{\text{(A.10)}}{=} \frac{1 - \alpha}{n},$$

i.e., $1 \geq \alpha$. Finally, by (A.1), (A.10), (1), and our findings on α, f is as in formula (1). □

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